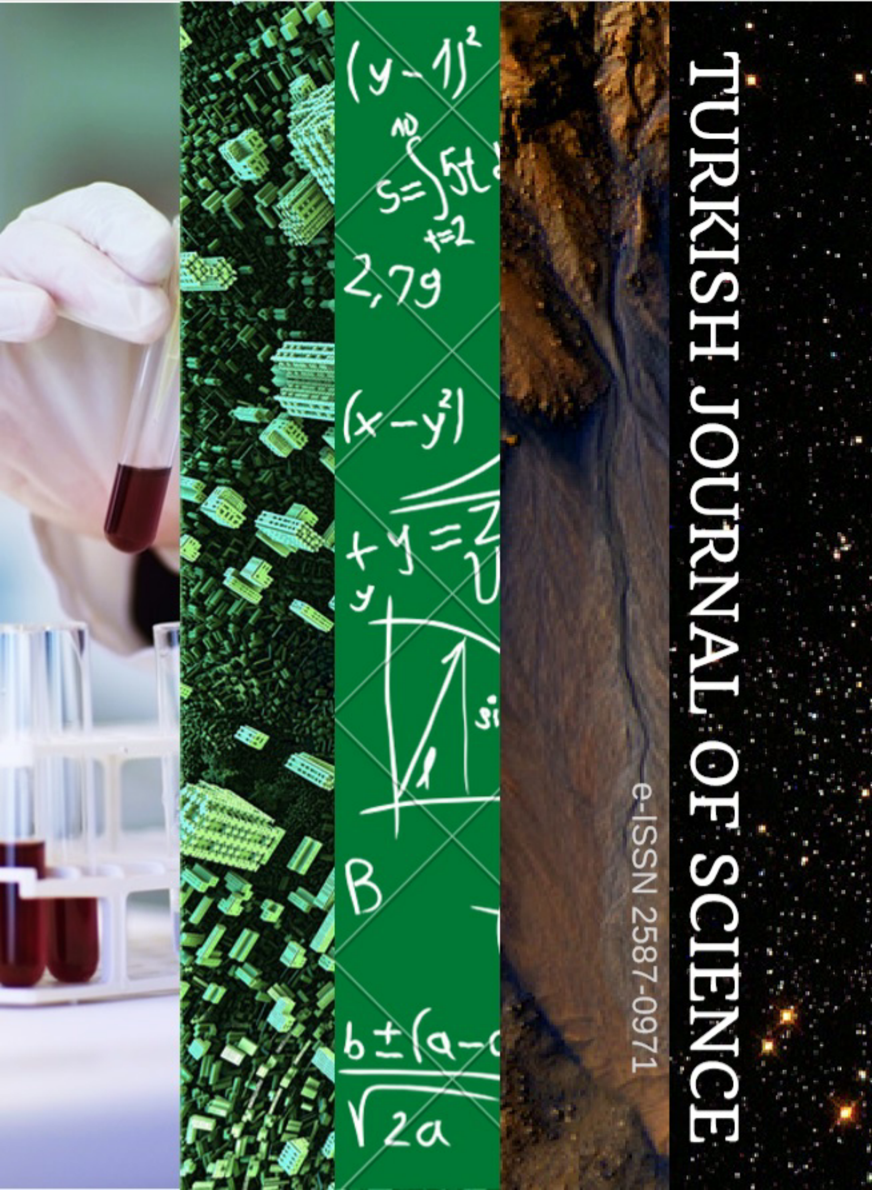
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CONTENTS

TURKISH JOURNAL OF SCIENCE
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Numerical Solution of Volterra Integral Equations Using Hosoya Polynomial

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Abstract. In this study, Volterra integral equation is solved by Hosoya Polynomials. The solutions obtained with Hosoya method were compared on the figure and table. And error analysis was done. Matlab package programming has been used to obtain results, tables and error analysis.

1. Introduction

Many mathematical models in disciplines such as engineering, physics and chemistry consist of integral equations [1]. Integral equations are equations in which the unknown function is under the integral sign [9]. Integral equations has been used in various applications such as geophysics, electricity and magnetism, kinetic theory of gases, regeneration theory, quantum mechanics, radiation, optimization, optimal control systems, mathematical economics, mathematical problems of radiative equilibrium, fluid mechanics, steady state heat [11]. One of most important integral equation is Volterra integral equation. Recently, Volterra integral equations have been increasingly used in engineering and applied mathematics studies. This equation has been studied in many fields of study such as Banach space, Haar functions problems, potential theory and Dirichlet problems, spectral methods, numerical computational problems and computer science problems [10]. In addition, the method studied in this paper was applied to the Volterra integral equation.

2. Volterra Integral Equations

The third kind of Volterra integral equations is of the form

$$u(x)h(x) = f(x) + \lambda \int_{\alpha}^{x} K(x,t)u(t)dt \tag{1}$$

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where the limits of integration are function of x and the unknown function $u(x)$ appears linearly under the integral sign. Second kind of Volterra integral equations is of the form

$$u(x) = f(x) + \lambda \int_{\alpha}^{x} K(x,t)u(t)dt \tag{2}$$

where $h(x) = 1$. First kind of Volterra integral equations is of the form

$$f(x) = \lambda \int_{\alpha}^{x} K(x,t)u(t)dt \tag{3}$$

3. Hosoya Polynomials

The Hosoya polynomial was initiated in 1988 by Haruo Hosoya [5, 8]. Hosoya polynomials count the distance between vertices of the path graph [12]. It is obtained from path graphs of certain pairs of graphs [3, 4]. Studies such as obtaining the physical and chemical properties of organic molecules with the Hosoya polynomial of the graph were carried out [12]. For a path graph with the Hosoya polynomial is described as,

$$H(P,\delta) = \sum_{l\geq 0} d(P,l)\delta^{l} \tag{4}$$

where $d(P,l)$ is the distance between vertex pairs in the path graph [6, 7]. Sum of the path graph vertices m with $1, 2, ..., m$ are multipled δ parameter. Then Hosoya values are calculated based on m vertex values [13]. For m integer values we represent path as ρ_m , then Hosoya polynomial of path compute as:

$$\begin{aligned}
&H(\rho_1,\delta) = \sum_{l\geq 0} d(\rho_1,l)\delta^{l} = 1\\
&H(\rho_2,\delta) = \sum_{l\geq 0} d(\rho_2,l)\delta^{l} = \delta + 2\\
&H(\rho_3,\delta) = \sum_{l\geq 0} d(\rho_3,l)\delta^{l} = \delta^2 + 2\delta + 3\\
&\vdots\\
&H(\rho_m,\delta) = m + (m-1)\delta + (m-2)\delta^2 + ...\\
&\ + (m-(m-2))\delta^{m-2} + (m-(m-1))\delta^{m-1}
\end{aligned}$$

A function $w(x) \in L_2[0;\ 1]$ is dilated as:

$$w(x) = \sum_{i=1}^{n} z_i H(\rho_i, x) = Z^T H_\rho(x), \tag{5}$$

where Z and $H_\rho(x)$ are $m \times 1$ matrices shown as:

$$Z = [z_1, z_2, z_3, ..., z_m]^T \tag{6}$$

and

$$H_\rho(x) = [H(\rho_1, x), H(\rho_2, x), ..., H(\rho_m, x)]^T. \tag{7}$$

4. Hosoya Polynomial Method

Consider The Volterra integral equation

$$y(x) = w(x) + \int_{1}^{x} L(x,t)y(t)dt, 0 \leq x, t \leq 1 \tag{8}$$

to solve equation (8), the method is as follows:

1. First we define $y(x)$ as defined in Equation (5). This equation is,

$$y(x) = Z^T H_\rho(x) \tag{9}$$

2. Then using place of (9) in (8), we get,

$$Z^T H_\rho(x) = w(x) + \int_{1}^{x} L(x,t)\left[Z^T H_\rho(t)\right] dt \tag{10}$$

3. Replacing the collocation point $x_j = \frac{j-0.5}{m}, j = 1, 2, \cdots, m$ in Equation (10). Then we get,

$$Z^T H_\rho(x_j) = w(x_j) + Z^T \left[\int_{1}^{x} L(x_j,t)H_\rho(t)dt\right] \tag{11}$$

$$Z^T(H_\rho(x_j) - Y) = w$$

where

$$Y = \int_{1}^{x} L(x_j,t)H_\rho(t)dt$$

4. In the last step, we get the conclutions of unknown Hosoya values,

$$Z^T L = w$$

where

$$L = H_\rho(x_j) - Y$$

solving this system of equations we get coefficients Z and then use in place of these coefficients in (9), we obtain the necessary result of (8) [2].

5. Numerical Example

5.1. Example

Consider Volterra integral equation,

$$u(x) = x + \int_{1}^{x} (t-x)u(t)dt \tag{12}$$

which has the exact solution $u(x) = \sin(x)$. First we substitute $u(x) = Z^T H_\rho(x)$ in equation (12). We get,

$$Z^T H_\rho(x) = x + \int_1^x (t - x)\left[Z^T H_\rho(t)\right] dt \tag{13}$$

Because of that reason for $m = 3$,

$$\begin{aligned} &Z_1[H_1(x) - \left(\int_1^x tH_1(t)dt - \int_1^x xH_1(t)dt\right)] \\ &+ Z_2[H_2(x) - \left(\int_1^x tH_2(t)dt - \int_1^x xH_2(t)dt\right)] \\ &+ Z_3[H_3(x) - \left(\int_1^x tH_3(t)dt - \int_1^x xH_3(t)dt\right)] = x \end{aligned} \tag{14}$$

Next, we achieve the Hosoya polynomials as

$$\begin{aligned} &Z_1[1 - \left(\int_1^x tdt - \int_1^x xdt\right)] \\ &+ Z_2[(x+2) - \left(\int_1^x t(t+2)dt - \int_1^x x(t+2)dt\right)] \\ &+ Z_3[(x^2+2x+3) - \left(\int_1^x t(t^2+2t+3)dt - \int_1^x x(t^2+2t+3)dt\right)] = x \end{aligned} \tag{15}$$

Next,

$$\begin{aligned} &Z_1[\frac{3}{2} + \frac{x^2}{2} - x] \\ &+ Z_2[\frac{x^3}{6} + x^2 - \frac{3x}{2} + \frac{10}{3}] \\ &+ Z_3[5 - \frac{x^2}{2} + \frac{53x}{12} - \frac{x^4}{4} - \frac{2x^3}{3}] = x \end{aligned} \tag{16}$$

If it is compute as $x_j = \frac{j-0.5}{m}$ and putting instead of the collocation points x_1, x_2, x_3 , we get the system of three equations with three unknowns as,

$$
\begin{aligned}
&Z_1[\frac{3}{2} + \frac{x_1{}^2}{2} - x_1] \\
&+ Z_2[\frac{x_1{}^3}{6} + x_1{}^2 - \frac{3x_1}{2} + \frac{10}{3}] \\
&+ Z_3[5 - \frac{x_1{}^2}{2} + \frac{53x_1}{12} - \frac{x_1{}^4}{4} - \frac{2x_1{}^3}{3}] = x_1 \\
\\
&Z_1[\frac{3}{2} + \frac{x_2{}^2}{2} - x_2] \\
&+ Z_2[\frac{x_2{}^3}{6} + x_2{}^2 - \frac{3x_2}{2} + \frac{10}{3}] \\
&+ Z_3[5 - \frac{x_2{}^2}{2} + \frac{53x_2}{12} - \frac{x_2{}^4}{4} - \frac{2x_2{}^3}{3}] = x_2 \\
\\
&Z_1[\frac{3}{2} + \frac{x_3{}^2}{2} - x_3] \\
&+ Z_2[\frac{x_3{}^3}{6} + x_3{}^2 - \frac{3x_3}{2} + \frac{10}{3}] \\
&+ Z_3[5 - \frac{x_3{}^2}{2} + \frac{53x_3}{12} - \frac{x_3{}^4}{4} - \frac{2x_3{}^3}{3}] = x_3
\end{aligned}
\tag{17}
$$

resolving these systems we get the three unknown Hosoya values,

$$Z_1 = 0.5012, Z_2 = 0.8672, Z_3 = \text{-}0.4101$$

putting back with these coefficients in the approximation, we get

$$u(x) = Z_1[H_1(x)] + Z_2[H_2(x) + Z_3[H_3(x)]$$

If in (17) is written in place of the x_1, x_2, x_3 values, approximate solutions are achieved.

$$
\begin{aligned}
u_1(x) &= Z_1[H_1(x_1)] + Z_2[H_2(x_1) + Z_3[H_3(x_1)] \\
u_2(x) &= Z_1[H_1(x_2)] + Z_2[H_2(x_2) + Z_3[H_3(x_2)] \\
u_3(x) &= Z_1[H_1(x_3)] + Z_2[H_2(x_3) + Z_3[H_3(x_3)]
\end{aligned}
\tag{18}
$$

We get the approximate values,

$$u_1 = \text{-}0.0682718, u_2 = 0.397826, u_3 = 0.820127$$

Maximum Error analyzed for $m = 3$ is,

$$
\begin{aligned}
E_{\max} = \sqrt{\sum_{i=1}^{m} (u_e(x_i) - u_a(x_i))^2} = \\
\sqrt{(x_1 - u_1)^2 + (x_2 - u_2)^2 + (x_3 - u_3)^2} = 0.2605
\end{aligned}
\tag{19}
$$

and for $m = 3, 8, 10$ are shown in the Tables 1, 2, 3 and Figures 1, 2, 3.

Table 1: Conclution of Hosoya Polynomial Method, for $m = 3$

x	Hosoya Polynomial Method	Exact Solution
0.1667	0.1659	-0.0682718
0.5	0.4794	0.397826
0.8333	0.7402	0.820127

Figure 1: Example 5.1 for $m = 3$

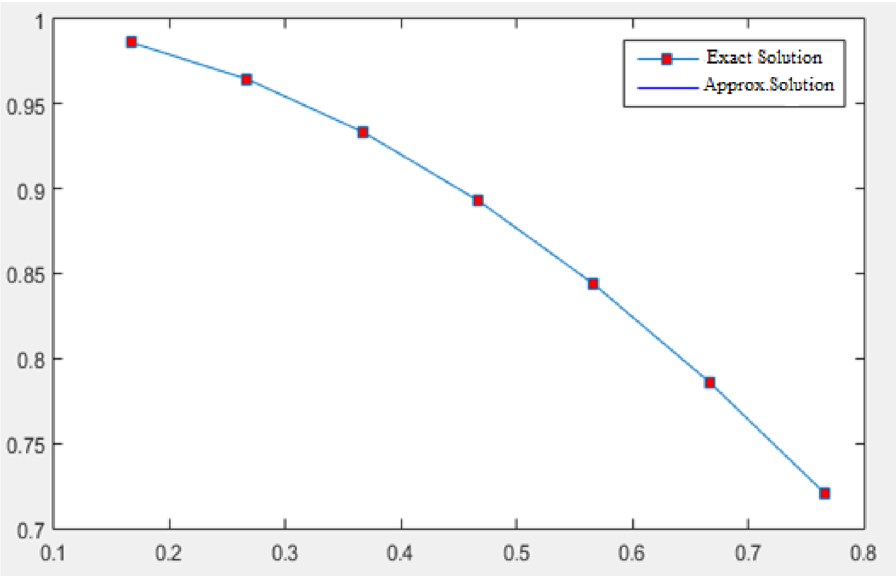


Table 2: Conclution of Hosoya Polynomial for $m = 8$

x	Hosoya Polynomial Method	Exact Solution
0.0625	0.5851	-0.214276
0.1875	0.1864	-0.0383231
0.3125	0.3125	0.138228
0.4375	0.3074	0.312622
0.5625	0.5333	0.482137
0.6875	0.6875	0.644129
0.8125	0.6346	0.79607
0.9375	0.9361	0.935588

Figure 2: Example 5.1 for $m = 8$

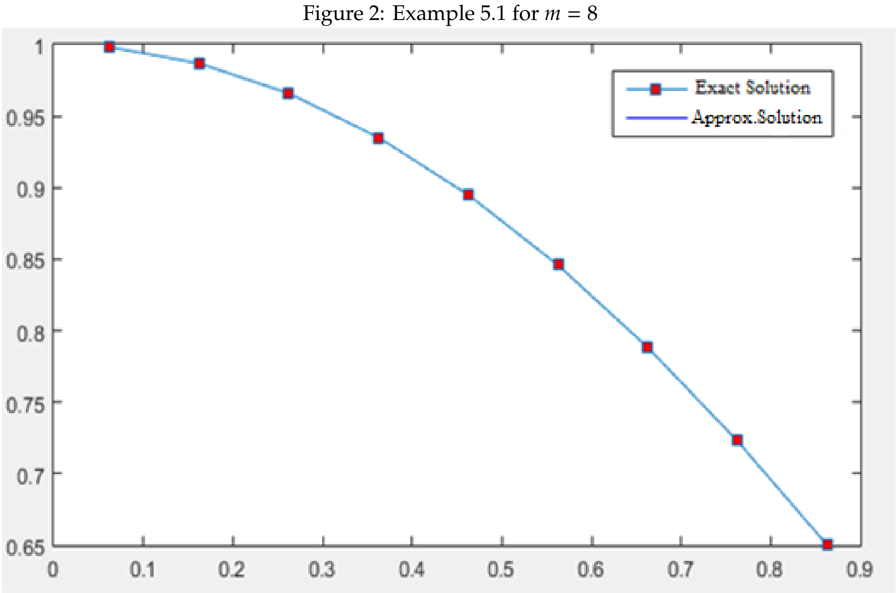
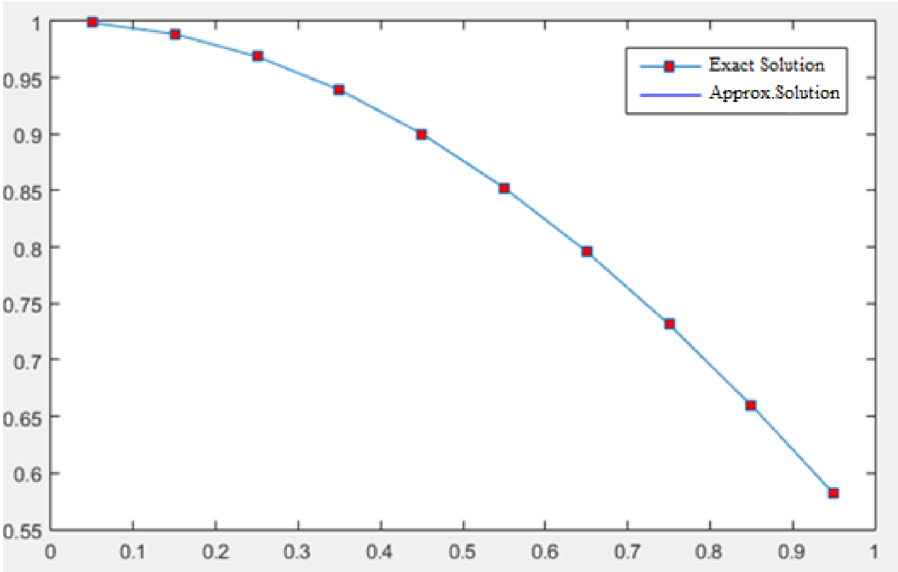


Table 3: . Conclution of Hosoya Polynomial Method for $m = 10$

x	Hosoya Polynomial Method	Exact Solution
0.050	0.050	-0.231732
0.150	0.1494	-0.0912973
0.250	0.2474	0.0500501
0.350	0.3429	0.190897
0.450	0.4350	0.329837
0.550	0.5227	0.465482
0.650	0.6052	0.596475
0.750	0.6816	0.721508
0.850	0.7513	0.839333
0.950	0.9134	0.948771

Figure 3: Example 5.1 for $m = 10$



6. Conclution

In this paper, the solution of Volterra integral equations with Hosoya method is examined. The method was applied to test problem in the matlab achieved with a certain algorithm. The method is solved for $m = 3$, $m = 8$, $m = 10$ values. The maximum error analysis was obtained according to the results exact and approximate solutions. The results exact and approximate solutions are shown with tables and figures. When the achieved conclutions are analyzed, it is seen that the Hosoya method is an useful method for solving the Volterra integral equation.

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Structure of nearly $\alpha-$cosymplectic manifolds

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Abstract. The main purpose of this paper is to study the structure of nearly $\alpha-$ cosymplectic manifolds and some basic curvature relations of this manifolds satisfying some conditions where α is real defined.

1. Introduction

In more recent times,the geometry of cosymplectic manifolds has an increasing interest. The topology of cosymplectic manifolds and curvature properties of almost cosymplectic manifolds have been examined by Blair and Goldberg[1], Yano[14], Olszak[6], Kirichenko[17], Endo[9] and others. The category of almost cosymplectic manifolds is much wider than other structures. Many other authors also have applications to characterize and analize the properties of almost cosymplectic manifolds (see [18, 22, 25]).

In addition to geometric studies of cosymplectic manifolds, recent interest in the subject of the geometry of nearly contact structures has become favorite. Many mathematicians have began to examine nearly structures on various manifolds by examining new curvature properties. Some of these are nearly Kaehler, nearly Sasakian, nearly Kenmotsu and nearly cosymplectic manifolds etc. Now we will try to give some of these works in a chronological order.

Nearly Kaehler manifolds are presented by Gray in [4, 5]. Blair et al. has introduced nearly Sasakian manifolds [2] and also Olszak has improved this kind of manifolds [7]. In another study of Olzsak, the properties of five dimensional nearly Sasakian and non-Sasakian manifolds have been given [8]. Parallel to Olszak's works, Endo has analyzed and has studied the geometry and curvature properties of nearly cosymplectic manifolds [10]. In addition to these important works, nearly cosymplectic manifolds and some curvature conditions on nearly cosymplectic structures have been studied by many authors and they have also introduced some of the remarkable properties of nearly cosymplectic structures [16, 20, 23, 24].

Starting from the previous studies, in this study we define nearly $\alpha-$ cosymplectic manifolds and obtain some basic curvature properties of nearly $\alpha-$cosymplectic manifolds. By means of this paper, we will elaborate on the subject using the notations and terminology of nearly $\alpha-$cosymplectic manifolds.

2. Preliminaries

Throughout this study, M is considered as C^∞ class manifolds and we accept $X, Y, U', V', U, V \in \chi(M)$ as vector fields unless otherwise stated.

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Let (M, ϕ, ξ, η, g) be $(2n+1)-$ dimensional differentiable almost contact metric manifold with $(1,1)-$tensor field ϕ, a characteristic vector field ξ, $1-$form η and the Riemannian metric g. M, with this structure (ϕ, ξ, η, g) is called an almost contact metric structure. By the way, an almost contac metric structure satisfies the following conditions here with [1];

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1}$$

$$\begin{aligned} \phi^2 X &= -X + \eta(X)\xi, \\ \eta(\xi) &= 1, \\ rank\phi &= 2n. \end{aligned}$$

where X ,$Y \in \chi(M)$. Also an almost contact metric structure (ϕ, ξ, η, g) satisfies;

$$\begin{aligned} \eta(X) &= g(X, \xi), \\ \phi(\xi) &= 0, \\ \eta(\phi X) &= 0, \end{aligned}$$

$$g(X, \phi Y) + g(Y, \phi X) = 0. \tag{2}$$

In the above equations, ϕ is skew-symmetric operator with respect to g and Φ is the bilininear fundamental $2-$form such that $\Phi(X, Y) = g(X, \phi Y)$ on M [15]. An almost contact metric manifold with $d\eta = 2\Phi$ is called a contact metric manifold on M. Moreover, almost contact metric manifolds in which both Φ and η are closed are called almost cosymplectic manifolds with $d\eta = 0$ and $d\Phi = 0$, where d is the exterior differential operator. Finally, a normal almost cosymplectic manifold is called a cosymplectic manifold (see [1–3] for further details).

By the way, Kenmotsu manifolds, as it is named, were defined and studied by Katsueı Kenmotsu in 1972 [13]. Later, nearly Kenmotsu manifolds were studied by Shukla [11]. Shukla, A. defined an almost contact manifold (M, ϕ, ξ, η, g) as a nearly Kenmotsu manifold with the following relation;

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = -\eta(Y)\phi X - \eta(X)\phi Y \tag{3}$$

where ∇ is Levi-Civita connection of g.

Recently, many other authors [12, 19, 21] have studied the geometric properties of nearly Kenmotsu manifolds. If we mentioning about nearly Kenmotsu manifolds briefly, we can describe the skew-symmetric $(1,1)-$tensor field H, with $d\eta(X, Y) = g(HX, Y)$. When $H = 0$, M is said to be a nearly Kenmotsu manifold. Now, it is easy to see that every Kenmotsu manifold is nearly Kenmotsu manifold but converse is not true.

On the other hand, a nearly cosymplectic manifold is an almost contact metric manifold (M, ϕ, ξ, η, g) such that

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = 0, \quad X, Y \in \chi(M),$$

where ∇ denote the Levi-Civita connection with respect to the Riemannian metric g on M [10]. For a nearly cosymplectic manifold, the vector field ξ is Killing and satisfies $\nabla_\xi \xi = 0$ and $\nabla_\xi \eta = 0$ conditions.

As we know that with the normality condition $([\phi, \phi] + 2\eta d \otimes \xi) = 0)$, a nearly cosymplectic structure is a cosymplectic structure [10].

Beside, for an $\alpha-$cosymplectic manifold the following condition holds [22];

$$(\nabla_X \phi)Y = \alpha \left[g(\phi X, Y)\zeta - \eta(Y)\phi X \right] \tag{4}$$

for any vector field X and Y on M.

Now from the equation above, by the sum of $(\nabla_X \phi)Y$ and $(\nabla_Y \phi)X$, we define a nearly $\alpha-$cosymplectic manifold (M, ϕ, ξ, η, g) , with the following definition;

Definition 2.1. *Let (M, ϕ, ξ, η, g) be $(2n+1)-$ dimensional differentiable almost contact metric manifold with $(1,1)-$tensor field ϕ, a characteristic vector field ξ, $1-$form η and the Riemannian metric g.Then if M satisfies the following relation ;*

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = \alpha[-\eta(Y)\phi X - \eta(X)\phi Y] \tag{5}$$

then, M is a said to be a nearly $\alpha-$cosymplectic manifold where ∇ is Levi-Civita connection of g and $\alpha \in \mathbb{R}$.

3. Nearly $\alpha-$cosymplectic manifolds

In this section, for a nearly $\alpha-$cosymplectic manifold (M, ϕ, ξ, η, g), some basic structures are given.

Proposition 3.1. *For a nearly $\alpha-$cosymplectic manifold (M, ϕ, ξ, η, g) we have;*

$$\begin{aligned} g(\nabla_{U'}\xi, V') + g(U', \nabla_{V'}\xi) &= 2\alpha g(\phi U', \phi V'), \\ \nabla_{U'}\xi &= -\alpha\phi^2 U' + HU', \end{aligned} \tag{6}$$

$$\begin{aligned} \phi H + H\phi &= 0, \\ \nabla_\xi \phi &= \phi H, \\ H\xi &= 0, \\ \nabla_\xi \xi &= 0, \end{aligned} \tag{7}$$

where H is the skew-symmetric $(1,1)$-tensor field.

Proof. By (5), $(\nabla_\xi \phi)\xi = -\phi(\nabla_\xi \xi) = 0$, hence $\nabla_\xi \xi = 0$ and $\nabla_\xi \eta = 0$. Now by making use of equation (1) we have

$$\begin{aligned} 0 &= g((\nabla_\xi \phi)U', \phi V') + g((\nabla_\xi \phi)V', \phi U') \\ &= -g((\nabla_{U'}\phi)\xi, \phi V') - g((\nabla_{V'}\phi)\xi, \phi U') - 2g(\phi U', \phi V') \\ &= g(\nabla_{U'}\xi, V') + g(\nabla_{V'}\xi, U') - 2\alpha g(\phi U', \phi V'). \end{aligned}$$

With help of definition of H, we get $\nabla_{U'}\xi = -\alpha\phi^2 U' + HU'$.
By $\phi\xi = 0$ and $\eta(\phi U') = 0$,we have

$$0 = (\nabla_{U'}\phi)\xi + \phi\nabla_{U'}\xi = -(\nabla_\xi \phi)U' + \phi HU', \tag{8}$$

$$\begin{aligned} 0 &= \eta((\nabla_{U'}\phi)V') + \eta((\nabla_{V'}\phi)U') \\ &= -g(U', (\nabla_{V'}\phi)\xi) - g(V', (\nabla_{U'}\phi)\xi) \\ &= g((\nabla_\xi \phi)U', V') + g((\nabla_\xi \phi)V', U') \\ &= g(U', \phi HY) + g(V', \phi HU') \\ &= g((\phi HU' + H\phi U'), V'). \end{aligned}$$

□

4. Curvature properties of nearly $\alpha-$cosymplectic manifolds

In this section, for a nearly $\alpha-$cosymplectic manifold (M, ϕ, ξ, η, g), some curvature relations are given. R is the Riemannian curvature tensor and it is defined by

$$R(U', V')U = (\nabla^2_{U',V'}U) - (\nabla^2_{V',U'}U) = [\nabla_{U'}, \nabla_{V'}]U - \nabla_{[U',V']}U.$$

At the same time, the $(0,4)-$type tensor field is defined as

$$R(U', V', U, V) = g(R(U', V')U, V).$$

Theorem 4.1. *For a nearly $\alpha-$cosymplectic manifold (M, ϕ, ξ, η, g), following curvature relations are hold;*

$$R(U', V', \phi U, V) + R(U', V', U, \phi V) + R(U', \phi V', U, V) + R(\phi U', V', U, V) = 0, \tag{9}$$

$$\begin{aligned} R(\xi, U', V', U) &= \alpha\left[-2\eta(U')g(U, HY) + \eta(V')g(HU, U') - \eta(U)g(HY, U')\right] \\ &+ \alpha^2\left[\eta(V')g(U', U) - \eta(U)g(V', U')\right] - g((\nabla_{U'}H)V', U), \end{aligned} \tag{10}$$

$$R(\phi U', \phi V', U, V) = R(U', V', \phi U, \phi V), \tag{11}$$

$$R(\phi U', \phi V', \phi U, \phi V) = R(U', V', U, V) - \eta(U')R(\xi, V', U, V) + \eta(V')R(\xi, U', U, V). \tag{12}$$

Proof. Let define a $(1,3)-$type tensor field T_s as follow

$$(\nabla^2_{U',V'}\phi)U - (\nabla^2_{U',U}\phi)V' = T_s(U', V', U), \tag{13}$$

which satisfies $T_s(U', V', U) = T_s(U', U, V')$. To put it simples, we can write the $(0,4)-$type tensor field T_s, with respect to g, as follows;

$$T_s(U', V', U, V) = g(T_s(U', V', U), V).$$

If we use the Ricci identity, then we obtain

$$0 = R(U', V', U, \phi V) - R(U', V', V, \phi U) - g((\nabla^2_{U',V'}\phi)U, V) + g(((\nabla^2_{V',U'}\phi)U, V).$$

Also by the first Bianchi identity and (13), we get

$$\begin{aligned} R(U', V', U, \phi V) &= R(U', V', V, \phi U) + g((\nabla^2_{U',V'}\phi)U, V) - g(((\nabla^2_{V',U'}\phi)U, V) \\ &= R(U', V', V, \phi U) - g((\nabla^2_{U',U}\phi)V', V) + g((\nabla^2_{V',U}\phi)U', V) \\ &+ T_s(U', U, V', V) - T_s(V', U, U', V), \end{aligned}$$

and thus, we have

$$\begin{aligned} R(U', V', U, \phi V) &= R(U', U, V', \phi V) - R(V', U, U', \phi V) \\ &= R(U', U, V', \phi V) - R(V', U, V, \phi U') \\ &+ g((\nabla^2_{U,V'}\phi)U', V) - g((\nabla^2_{V',U}\phi)U', V), \end{aligned}$$

If we equalize the right sides of equations above, we get

$$\begin{aligned} &R(U', V', U, \phi V) - R(U', V', V, \phi U) - R(V', U, V, \phi U') + g((\nabla^2_{U',U}\phi)V', V) \\ &+g((\nabla^2_{U,V'}\phi)U', V) + T_s(V', U, U', V) - T_s(U', U, V', V) = 2g((\nabla^2_{V',U}\phi)U', V). \end{aligned} \tag{14}$$

and we note that

$$g((\nabla^2_{U',U}\phi)V',V)+g((\nabla^2_{U,V'}\phi)U',V)=R(U',U,V',\phi V)-R(U',U,V,\phi V')+T_s(U,U',V',V),$$

$$g((\nabla^2_{V',U}\phi)U',V)=g((\nabla^2_{V',V}\phi)U,U')-T_s(V',V,U,U').$$

By considering this in (14), we have

$$\begin{aligned}&2R(U',U,V',\phi V)-R(U',V',V,\phi U)-R(V',U,V,\phi U')-R(U',U,V,\phi V')\\&-T_s(U',V',U,V)+T_s(V',U,U',V)+T_s(U,U',V',V)+2T_s(V',V,U,U')\\&=2g((\nabla^2_{V',V}\phi)U,U').\end{aligned} \tag{15}$$

If we apply (5) to (15), we obtain

$$\begin{aligned}T_s(U',V',U,V)=&\ \alpha(-g(V',U'+HU')+\eta(U')\eta(V'))g(\phi U,V)\\&+\alpha^2(-g(U,U'+HU')+\eta(U')\eta(U))g(\phi V',V)\\&-\eta(V')g((\nabla_{U'}\phi)U,V)-\eta(U)g((\nabla_{U'}\phi)V',V),\end{aligned}$$

and after a straight forward computation, we get

$$\begin{aligned}&T_s(V',U,U',V)+T_s(U,U',V',V)-T_s(U',V',U,V)+2T_s(V',V,U,U')=\\&\alpha\Big[G(U',V',U,V)+2g(\phi V',V)g(HU',U)+2g(\phi U,V)g(HU',V')\ +2g(\phi U',V)g(HY,U)+2g(\phi U',U)g(HY,V)\Big]\\&+\alpha^2\Big[2g(\phi U,U')g(V',\phi^2V)+\eta(U')\eta(V')g(\phi U,V)-\eta(U)\eta(V')g(\phi U',V)\Big],\end{aligned}$$

where

$$G(U',V',U,V)=\alpha\Big[-\eta(V')g((\nabla_U\phi)U',V)\ +\eta(V')g((\nabla_{U'}\phi)U,V)-2\eta(V)g((\nabla_{V'}\phi)U,U')\Big].$$

The anti-symmetrization of (15) in V' and V and also using the first Bianchi identity, we have

$$\begin{aligned}&3R(\phi U',U,V',V)+3R(U',\phi U,V',V)+3R(U',U,\phi V',V)+3R(U',U,V',\phi V)\\&+\alpha\Big[4g(\phi V',V)g(HU',U)+2g(\phi U,V)g(HU',V')-2g(\phi U,V')g(HU',V)\\&+4g(\phi U',U)g(HY,V)+2g(\phi U',V)g(HY,U)-2g(\phi U',V')g(HV,U)\Big]=0,\end{aligned}$$

which implies equation (9) if one assumes $H=0$. Now we will show that $H=0$. For $U'=\xi$, $(H\xi=\phi\xi=0)$, we get

$$R(\xi,\phi U,V',V)+R(\xi,U,\phi V',V)+R(\xi,U,V',\phi V)=0, \tag{16}$$

and

$$-R(\xi,U,\phi V',V)-R(\xi,\phi U,V',V)+R(\xi,\phi U,\phi V',\phi V)+\eta(V')R(\xi,\phi U,\xi,V)=0. \tag{17}$$

Hence we have

$$R(\xi,U,V',\phi V)+R(\xi,\phi U,\phi V',\phi V)+\eta(V')R(\xi,\phi U,\xi,V)=0, \tag{18}$$

and

$$\begin{aligned}&-R(\xi,\phi U,V',V)+R(\xi,U,\phi V',V)+\eta(V)R(\xi,U,\xi,\phi V')\\&-\eta(V)R(\xi,\phi U,\xi,V')+\eta(V')R(\xi,\phi U,\xi,V)=0.\end{aligned} \tag{19}$$

From (16) and (19), we have

$$2R(\xi,\phi U,V',V)+R(\xi,U,V',\phi V)=\eta(V')R(\xi,\phi U,\xi,V)+\eta(V)\Big[R(\xi,U,\xi,\phi V')-R(\xi,\phi U,\xi,V')\Big]. \tag{20}$$

and changing U by ϕU and V by ϕV in (20), we get

$$-2R(\xi, U, V', \phi V) - R(\xi, \phi U, V', V) = -\eta(V')R(\xi, U, \xi, \phi V) + \eta(V)R(\xi, \phi U, \xi, V'). \tag{21}$$

Taking the sum of the last two equations above, we obtain

$$R(\xi, \phi U, V', V) - R(\xi, U, V', \phi V) = \eta(V')R(\xi, \phi U, \xi, V) + \eta(V)R(\xi, U, \xi, \phi V') - R(\xi, U, \xi, \phi V). \tag{22}$$

From the equations (19) and (22), we get (16) as

$$3R(\xi, \phi U, V', V) = \eta(V')\left[2R(\xi, \phi U, \xi, V) - R(\xi, U, \xi, \phi V)\right] + \eta(V)\left[2R(\xi, U, \xi, \phi V') - R(\xi, \phi U, \xi, V')\right],$$

$$R(\xi, U, \phi V', \phi V) = 0.$$

Applying $\nabla\xi = -\alpha\phi^2 + H$, we have

$$R(V', U, \xi, U') = -g((\nabla_{V'}H)U', U) + g((\nabla_U H)U', V') - g((\nabla^2_{V'}\phi)U', U) + g((\nabla^2_U\phi)U', V').$$

Using the first Bianchi identity by applying the cyclic sum on U', V', U, we obtain

$$g((\nabla_{U'}H)V', U) - g((\nabla_{V'}H)U', U) + g((\nabla_U H)U', V') = 0,$$

in this way we have

$$\begin{aligned} R(V', U, \xi, U') = &- g((\nabla_{V'}\phi^2)U', U) + g((\nabla_U\phi^2)U', V') - g((\nabla_{U'}H)V', U) \\ &+ \alpha\left[-2\eta(U')g(U, HY) + \eta(V')g(U', HU) - \eta(U)g(U', HY)\right] \\ &+ \alpha^2\left[\eta(V')g(U', U) - \eta(U)g(U', V')\right] - g((\nabla_{U'}H)V', U), \end{aligned} \tag{23}$$

$$\begin{aligned} 0 = R(\xi, U', \phi V', \phi U) &= -2\alpha\eta(U')g(H\phi V', \phi U) - g((\nabla_{U'}H)\phi V', \phi U) \\ &= \alpha\left[-2\eta(U')g(HY, U)\right] - g((\nabla_{U'}H)\phi V', \phi U). \end{aligned} \tag{24}$$

If we take V' , a unit eigenvector field on M such that $\eta(V') = 0$ and $H^2V' = \lambda V'$; in this way, note that $H^2\phi V' = \lambda\phi V'$, as $\phi H = -H\phi$. Then

$$\begin{aligned} 0 = R(\xi, U', \phi V', \phi HY) &= -2\alpha\lambda\eta(U') - g((\nabla_{U'}H)\phi V', \phi HY) - \frac{1}{2}g((\nabla_{U'}H^2)\phi V', \phi V') \\ &= 2\alpha\lambda\eta(U') - \frac{1}{2}d\lambda(U') = 0, \end{aligned} \tag{25}$$

so that $d\lambda = -4\alpha\lambda\eta$, where U' is arbitrary vector field on M.

As a result, $\lambda = 0$ or $d\eta = 0$, means $H = 0$. Then from (23), we obtain (10).

Stating the left hand side of (9) by R_*, we will prove (11).

Then, if we applying this regulation in (10), we have

$$\begin{aligned} 0 &= R_*(U', \phi V', U, V) - R_*(U', V', \phi U, V) - R_*(U', V', U, \phi V) + R_*(\phi U', V', U, V) \\ &= -2R(U', V', \phi U, \phi V) + 2R(\phi U', \phi V', U, V). \end{aligned}$$

Now, it is immediate to see (12). □

Proposition 4.2. *For a nearly $\alpha-$cosymplectic manifold (M, ϕ, ξ, η, g), following relation holds;*

$$-2\alpha g(\phi U', V')\xi + (\nabla_{U'}\phi)V' + \alpha\eta(V')\phi U' + (\nabla_{\phi U'}\phi)\phi V' = 0.$$

Proof. By $\phi^2 = -Id + \eta \otimes \xi$, we have

$$g((\nabla_{U'}\phi)\phi V', U) = \alpha\left[\eta(V')g(U', U) + \eta(U)g(U', V') - 2\eta(U')\eta(V')\eta(U)\right] + \eta(V')g(HU', U) + \eta(U)g(HU', V') + g((\nabla_{U'}\phi)V', \phi U),$$

taking into account (5), we get

$$g((\nabla_{\phi U'}\phi)V', U) = \alpha\left[2\eta(V')g(U', U) - \eta(U)g(U', V') - \eta(U')\eta(V')\eta(U)\right] + \eta(U')g(HU, V') + \eta(U)g(HU', V') + g((\nabla_{U'}\phi)V', \phi U). \tag{26}$$

From the equations above, the expression we are trying to show is obtained. □

Proposition 4.3. *For a nearly $\alpha-$cosymplectic manifold (M, ϕ, ξ, η, g), following curvature relations are hold;*

$$Ric(U', \xi) = \alpha^2\left[-2n\eta(U')\right], \tag{27}$$

$$Ric(\phi V', \phi U) = \alpha^2\left[2n\eta(V')\eta(U)\right] + Ric(V', U), \tag{28}$$

$$Ric(U, \phi V') + Ric(\phi U, V') = 0, \tag{29}$$

where Ric is the Ricci tensor of M.

Proof. In for dimension M is $2n+1$ and $(E_0 = \xi, E_1, ..., E_n, E_{n+1}, ..., E_{2n})$, orthonormal ϕ-frame satisfies $\phi E_i = E_{i+n}$, $\phi E_{i+n} = -E_i$, $i = 1, ..., n$. If we evaluate the ϕ-basis with (10), we can give the *Ricci tensor* $Ric(U', \xi)$ by (27).

Then from the equation (12) we get;

$$\begin{aligned} Ric(U', V') &= \sum_{i=1}^{n}(R(E_i, U', V', E_i) + R(E_{i+n}, U', V', E_{i+n})) + R(\xi, U', V', \xi) \\ &= Ric(\phi U', \phi V') + \eta(U')Ric(\xi, V') - R(\xi, \phi U', \phi V', \xi) + R(\xi, U', V', \xi) \\ &= Ric(\phi U', \phi V') + \eta(U')Ric(\xi, V') = Ric(\phi U', \phi V') - 2\alpha^2 n\eta(U')\eta(V'), \end{aligned} \tag{30}$$

in which we applied (27). From the direct consequence of (28), we obtain (29). □

Proposition 4.4. *The fundamental form of a nearly $\alpha-$cosymplectic manifold (M, ϕ, ξ, η, g) satisfies;*

$$3d\Phi(U', V', U) = \alpha\Big[-2\eta(U')g(\phi V', U) - \eta(V')g(\phi U', U) + \eta(U)g(\phi U', V')\Big] - 3g((\nabla_{U'}\phi)V', U). \tag{31}$$

$$d\Phi(U', V', U) = 2\alpha(\eta \wedge \Phi)(U', V', U) + \frac{1}{4}g([\phi, \phi](U', V'), \phi U). \tag{32}$$

Proof. From the well known following identities

$$3d\Phi(U', V', U) = (\nabla_{U'}\Phi)(V', U) + (\nabla_{V'}\Phi)(U, U') + (\nabla_U\Phi)(U', V')$$

and

$$[\phi, \phi](U', V') = -\phi(\nabla_{U'}\phi)V' + \phi(\nabla_{V'}\phi)U' + (\nabla_{\phi U'}\phi)V' - (\nabla_{\phi V'}\phi)U',$$

we have

$$\begin{aligned} 3d\Phi(U', V', U) &= -g((\nabla_{U'}\phi)V', U) + g((\nabla_{V'}\phi)U', U) - g((\nabla_U\phi)U', V') \\ &= \alpha\Big[-2\eta(U')g(\phi V', U) + \eta(V')g(\phi U, U') - \eta(U)g(\phi V', U')\Big] - 3g((\nabla_{U'}\phi)V', U), \end{aligned} \tag{33}$$

$$\frac{1}{2}[\phi,\phi](U',V') = \alpha\left[-\eta(U')V' + \eta(V')U'\right] - \phi(\nabla_{U'}\phi)V' + \phi(\nabla_{V'}\phi)U'. \tag{34}$$

Hence

$$\begin{aligned} 6d\Phi(U',V',U) &= \alpha\left[-\eta(U')g(\phi V',U) + \eta(V')g(\phi U',U) + 2\eta(U)g(\phi U',V')\right] - 3g((\nabla_{U'}\phi)V' - (\nabla_{V'}\phi)U',U) \\ &= 4\alpha\left[\eta(U')g(V',\phi U) + \eta(V')g(U,\phi U') + \eta(U)g(U',\phi V')\right] + \frac{3}{2}g([\phi,\phi](U',V'),\phi U) \\ &= 12\alpha(\eta\wedge\Phi)(U',V',U) + \frac{3}{2}g([\phi,\phi](U',V'),\phi U). \end{aligned}$$

□

Theorem 4.5. *Every normal nearly $\alpha-$cosymplectic manifold (M,ϕ,ξ,η,g) is cosymplectic manifold.*

Proof. We know that $d\eta = 0$ and if and only if $N = 0$, the structure is normal. According to Proposition 4.4, in the case of $N = 0$ we have

$$3d\Phi(U',V',Z) = 2\alpha(\eta\wedge\Phi)(U',V',Z)$$

and

$$d\Phi = 2\alpha\eta\wedge\Phi.$$

That is to say, M is almost $\alpha-$cosymplectic. Namely, we can see that a normal almost $\alpha-$cosymplectic manifold is $\alpha-$cosymplectic. □

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The Solution of Linear Volterra Integral Equation of the First Kind with ZZ-Transform

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Abstract. In this paper, we apply ZZ-transform to solve linear Volterra integral equation of the first kind. The several examples solve by ZZ- Transform. This means that ZZ- transform is a powerful tool for solving linear Volterra integral equations of the first kind. The Convolution theorem for the ZZ- transform has been proved. ZZ- transform for the solution of linear Volterra integral equation of the first kind submitted in application section of this paper, some applications are given to demonstrate the effectiveness of proposed scheme.

1. Introduction

Integral transformations are encountered in many fields of engineering and science such as electrical networks, heat transfer, mixing problems, springs, signal processing, bending of beams, Newton's second law of motion, carbon dating problems, decay and exponential growth problems. In later times, many the scientist are related in solving the problems of engineering and science by introducing new integral transforms. The ZZ-Transform is integral transform. There are many integral transforms in the literature. Some of these transformations are Laplace transform, Fourier transform, Elzaki transform, Sumudu transform, Aboodh transform, Kamal transform [1, 9]. These transformations are used to solve for differential equations and integral equations. The ZZ-Transform was first presented by Zain UI Abadin Zafar in 2016 [10]. The linear Volterra integral equation of the first kind is given by $f(t) = \int\limits_0^x K(x,t)u(t)dt, u(x)$ is the unknown function and occurs only inside the integral sign. The function $f(x)$and the kernel $K(x,t)$ are real-valued functions [11]. The $ZZ-$transform of the function $f(t)$for $t \geq 0$is defined as;

$$Z(u,s) = Z\{f(t)\} = s\int_0^\infty f(ut)e^{-st}dt \tag{1}$$

or

$$Z(u,s) = Z\{f(t)\} = \frac{s}{u}\int_0^\infty f(t)e^{-\frac{s}{u}t}dt \tag{2}$$

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Where $Z(u,s)$ is transform operator. Assuming that the integral on the right side in (2) exists. The unique function $f(t)$ in (2) is called the inverse transform of $Z(u,s)$is indicated by

$$f(t) = Z^{-1}\{Z(u,s)\} \tag{3}$$

If $F(t)$ is piecewise continuous and of exponential order, the $ZZ-$transform of the function $F(t)$ for $t \geq 0$exist. These conditions are only sufficent conditions for the existence of $ZZ-$transform of the function $F(t)$.

1.1. Linearity Property of ZZ−Transform:

If $Z\{F(t)\} = A(u,s)$ and $Z\{G(t)\} = B(u,s)$then $Z\{aF(t) + bG(t)\} = aZ\{F(t)\} + bZ\{G(t)\} = aA(u,s) + bB(u,s)$,where $a, b$are arbitrary constants.

1.2. ZZ− Transform of Some Elementary Functions:

Table 1: ZZ− Transform of Some Elementary Functions:

No	$f(t)$	$Zf(t)$
1	1	1
2	t	$\frac{u}{s}$
3	e^{at}	$\frac{u}{s-au}$
4	$\sin at$	$\frac{aus}{s^2+a^2u^2}$
5	$\cos at$	$\frac{s^2}{s^2+a^2u^2}$
6	t^n	$n!\frac{u^n}{s^n}$
7	$e^{at}\sin bt$	$\frac{b\frac{s}{u}}{\left(\frac{s}{u}-a\right)^2+b^2}$
8	$e^{at}\cos bt$	$\frac{\frac{s^2}{u^2}-\frac{as}{u}}{\left(\frac{s}{u}-a\right)^2+b^2}$
9	$t\cos at$	$\frac{\frac{s}{u}\left(\frac{s^2}{u^2}-a^2\right)}{\left(\frac{s^2}{u^2}+a^2\right)^2}$
10	$t\sin at$	$\frac{2a\frac{s^2}{u^2}}{\left(\frac{s^2}{u^2}+a^2\right)^2}$

1.3. Existence of ZZ−Transform

Theorem 1.1. *If $f(t)$ is piecewise continuous in interval $0 \leq t \leq K$and of exponential order γ for $t > K$, then its ZZ− transform $Z(u,s)$ exists for all $s > \gamma$,$u > \gamma$.*

Proof. We have for every positive number K,

$$\frac{s}{u}\int_0^\infty f(t)e^{-\frac{s}{u}t}dt = \frac{s}{u}\int_0^K f(t)e^{-\frac{s}{u}t}dt + \frac{s}{u}\int_K^\infty f(t)e^{-\frac{s}{u}t}dt$$

Since $f(t)$ is piecewise continuous in every finite interval $0 \leq t \leq K$, the first integral on the right side exists. Also the second integral on the right side exists. So $f(t)$ is of exponential order γ for $t > K$. To see this we have only to observe that in such case:

$$\left[\frac{s}{u}\int_{K}^{\infty} f(t)e^{-\frac{s}{u}t}dt\right] \leq \frac{s}{u}\left[f(t)e^{-\frac{s}{u}t}\right]dt$$

$$\leq \frac{s}{u}\int_{0}^{\infty} e^{-\frac{s}{u}t}(f(t))dt \leq \frac{s}{u}\int_{0}^{\infty} e^{-\frac{s}{u}t}Me^{\gamma t}dt$$

$$\leq \frac{sM}{u}\int_{0}^{\infty} e^{-\frac{s}{u}t}e^{\gamma t}dt \leq \frac{sM}{u}\int_{0}^{\infty} e^{-(\frac{s}{u}-\gamma)t}dt$$

$$= \frac{sM}{u}\frac{e^{-(\frac{s}{u}-\gamma)t}}{\left(-\frac{s}{u}-\gamma\right)}\Big|_{0}^{\infty} = \frac{sM}{s-\gamma u}$$

□

1.4. Convolution of two Functions:

Convolution of $F(t)$ and $G(t)$ functions is defined by

$$F(t)\otimes G(t) = F\otimes G = \int_{0}^{t} F(x)G(t-x)dx = \int_{0}^{t} F(t-x)G(x)dx.$$

$$\frac{u}{s}\frac{s^2}{s^2+u^2}Z\left\{x(t)\right\} = 2\frac{s^2}{u^2}\frac{u^4}{\left(s^2+u^2\right)^2}$$

1.5. Convolution Theorem for ZZ−Transforms:

Theorem 1.2. *If* $Z\{F(t)\} = A(u,s)$ *and* $Z\{G(t)\} = B(u,s)$ *then*

$$Z(f\otimes g) = \frac{u}{s}Z(f)Z(g)$$

$$Z\left\{F(t)\otimes G(t)\right\} = \frac{u}{s}Z\left\{F(t)\right\}Z\left\{G(t)\right\} = \frac{u}{s}A(u,s)B(u,s)$$

Proof.

$$Z(f)Z(g) = \frac{s}{u}\int_{0}^{\infty} f(\tau)e^{-\frac{s}{u}\tau}d\tau\,\frac{s}{u}\int_{0}^{\infty} g(\vartheta)e^{-\frac{s}{u}\vartheta}d\vartheta$$

$$Z(f)Z(g) = \frac{s^2}{u^2}\int_{0}^{\infty} f(\tau)e^{-\frac{s}{u}\tau}d\tau\int_{0}^{\infty} g(\vartheta)e^{-\frac{s}{u}\vartheta}d\vartheta \tag{4}$$

$t = \vartheta + \tau$ and $\vartheta = t - \tau$

$$Z(g) = \int_{\tau}^{\infty} g(t-\tau)e^{-\frac{s}{u}(t-\tau)}dt$$

$$= \int_{0}^{\infty} g(t-\tau)e^{-\frac{s}{u}t}e^{\frac{s}{u}\tau}dt$$

$$= \int_{\tau}^{\infty} g(t-\tau)e^{-\frac{s}{u}t}e^{\frac{s}{u}\tau}dt$$

$$= e^{\frac{s}{u}\tau}\int_{\tau}^{\infty} g(t-\tau)e^{-\frac{s}{u}t}dt$$

Thus

$$\begin{aligned}
&= \frac{s^2}{u^2}\int_{0}^{\infty} f(\tau)e^{-\frac{s}{u}\tau}d\tau e^{\frac{s}{u}\tau}\int_{\tau}^{\infty} e^{-\frac{s}{u}t}g(t-\tau)dt \\
&= \frac{s^2}{u^2}\int_{0}^{\infty} f(\tau)\int_{\tau}^{\infty} e^{-\frac{s}{u}t}g(t-\tau)dtd\tau \\
&= \frac{s^2}{u^2}\int_{0}^{\infty} e^{-\frac{s}{u}t}\int_{0}^{t} f(t)g(t-\tau)d\tau dt \\
&= \frac{s^2}{u^2}\int_{0}^{\infty} e^{-\frac{s}{u}t}(f\otimes g)(t)dt \\
&= \frac{s}{u}Z(f\otimes g) \\
Z(f\otimes g) &= \frac{u}{s}Z(f)Z(g)
\end{aligned}$$

□

1.6. Inverse of ZZ−Transforms:

If $Z\{F(t)\} = Z\{u,s\}$then $F(t)$ is called the inverse $ZZ-$transform of $Z\{u,s\}$and it is defined as $F(t) = Z^{-1}\{Z(u.s)\}$, where Z^{-1} is the inverse $ZZ-$transform operator.

1.7. Applications:

In this chapter, some applications are given to show the effectiveness of $ZZ-$transform for solving of linear Volterra integral equation of the first kind.

Example 1.3. *Consider linear Volterra integral equation of the first kind:*

$$x = \int_{0}^{x} u(t)dt \tag{5}$$

Applying the ZZ− transform to both sides of (5), we have:

$$Z\{x\} = Z\left\{\int_{0}^{x} u(t)dt\right\} \tag{6}$$

Using convolution theorem of ZZ−transform on (6), we have:

$$Z\{x\} = \frac{u}{s}Z\{1\}Z\{u(x)\}$$

$$\frac{u}{s} = \frac{u}{s}.1.Z\{u(x)\}$$

$$Z\{u(x)\} = 1 \tag{7}$$

Operating inverse ZZ−transform on both sides of (7), we have:

$$Z^{-1}\{Z\{u(x)\}\} = Z^{-1}(1)$$

$$u(x) = 1.$$

This is the exact solution of equation (5).

Example 1.4. *Consider linear Volterra integral equation of the first kind:*

$$x^2 = \frac{1}{2}\int_0^x (x-t)u(t)dt \tag{8}$$

Applying the ZZ−transform to both sides of (8), we have:

$$Z\left\{x^2\right\} = Z\left\{\frac{1}{2}\int_0^x (x-t)u(t)dt\right\} \tag{9}$$

Using convolution theorem of ZZ−transform on (9), we have:

$$2\frac{u^2}{s^2} = \frac{1}{2}\frac{u}{s}Z\{x\}Z\{u(x)\}$$

$$2\frac{u^2}{s^2} = \frac{1}{2}\frac{u}{s}\frac{u}{s}Z\{u(x)\}$$

$$Z\{u(x)\} = 4 \tag{10}$$

operating inverse ZZ−transform on both sides of (10), we have:

$$Z^{-1}\{Z\{u(x)\}\} = Z^{-1}\{4\}$$

$$u(x) = 4$$

This is the exact solution of equation (8).

Example 1.5. *Consider linear Volterra integral equation of the first kind:*

$$y(t) = t^2 + \int_0^t y(u)\sin(t-u)du \tag{11}$$

$$Z\{y(t)\} = Z\left\{t^2 + \int_0^t y(u)\sin(t-u)du\right\}$$

From the linearity property of the inverse ZZ−transform

$$Z\{y(t)\} = Z\left\{t^2\right\} + Z\left\{\int_0^t y(u)\sin(t-u)du\right\} \tag{12}$$

Using convolution theorem of transform on (12), we have:

$$Z\{y(t)\} = Z\left\{t^2\right\} + \frac{u}{s} Z\{y(t)\} Z\{\sin t\}$$

$$Z\{y(t)\} = 2\frac{u^2}{s^2} + \frac{u}{s} Z\{y(t)\} \frac{us}{s^2+u^2}$$

$$Z\{y(t)\} - Z\{y(t)\}\left(\frac{su^2}{s^3+su^2}\right) = 2\frac{u^2}{s^2}$$

$$Z\{y(t)\}\left(1 - \frac{su^2}{s^3+su^2}\right) = 2\frac{u^2}{s^2}$$

$$Z\{y(t)\}\left(\frac{s^3}{s^3+su^2}\right) = 2\frac{u^2}{s^2}$$

$$Z\{y(t)\}\left(\frac{s^2}{s^2+u^2}\right) = 2\frac{u^2}{s^2}$$

$$Z\{y(t)\} = 2\frac{u^2}{s^2}.\frac{s^2+u^2}{s^2}.$$

$$Z\{y(t)\} = 2\frac{u^2}{s^2} + 2\frac{u^4}{s^4} \tag{13}$$

operating inverse ZZ−transform on both sides of (13), we have:

$$Z^{-1}\{Z\{y(t)\}\} = Z^{-1}\left\{2\frac{u^2}{s^2} + 2\frac{u^4}{s^4}\right\}$$

From the linearity property of the inverse ZZ−transform

$$y(t) = Z^{-1}\left\{2\frac{u^2}{s^2}\right\} + Z^{-1}\left\{2\frac{u^4}{s^4}\right\}$$

$$y(t) = t^2 + \frac{t^4}{12}$$

This is the exact solution of equation (11).

Example 1.6. *Consider linear Volterra integral equation of the first kind:*

$$\int_0^t \cos(t-s)x(s)ds = t\sin t \tag{14}$$

Applying the ZZ−transform to both sides of (14), we have:

$$Z\left\{\int_0^t \cos(t-s)x(s)ds\right\} = Z\{t\sin t\} \tag{15}$$

Using convolution theorem of ZZ−transform on (15), we have:

$$\frac{u}{s} Z\{\cos t\} Z\{x(t)\} = \frac{2\frac{s^2}{u^2}}{\left(\frac{s^2}{u^2}+1\right)^2}$$

$$\frac{u}{s}\frac{s^2}{s^2+u^2}Z\{x(t)\} = 2\frac{s^2}{u^2}\frac{u^4}{(s^2+u^2)^2}$$

$$Z\{x(t)\} = 2\frac{us}{s^2+u^2} \tag{16}$$

operating inverse ZZ−transform on both sides of (16), we have:

$$Z^{-1}\{Z\{x(t)\}\} = Z^{-1}\left\{2\frac{us}{s^2+u^2}\right\}$$

$$x(t) = \sin t$$

This is the exact solution of equation (14).

2. Conclusion

In this study, we have discussed the $ZZ-$transform for the solution of linear volterra integral equation of the first kind. The given examples show that the exact solution have been obtained spending a very little time and using very less computational work.

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The Representation and Finite Sums of the Padovan-p Jacobsthal Numbers

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Abstract. In this paper, we regard the Padovan-p Jacobsthal sequence and then we discuss the connection of the Padovan-p Jacobsthal numbers and Jacobsthal numbers. Furthermore, we give the permanental, determinantal, combinatorial, and exponential representations, and the sums of the Padovan-p Jacobsthal numbers by the aid of the generating function and generating matrix of this sequence.

1. Introduction

The well-known Jacobsthal sequence $\{J_n\}$ is defined by the following recurrence relation:

$$J_n = J_{n-1} + 2J_{n-2}$$

for $n \geq 2$ in which $J_0 = 0$ and $J_1 = 1$. It is easy to see that the characteristic polynomial of the Jacobsthal sequence is $j(x) = x^2 - x - 2$.

In [2], Aküzüm defined the Padovan-p Jacobsthal sequence $\left\{J_n^p\right\}$ by the following homogeneous linear recurrence relation for any given $p\,(3,4,5,\ldots)$ and $n \geq 0$

$$J_{n+p+4}^p = J_{n+p+3}^p + 3J_{n+p+2}^p - J_{n+p+1}^p - 2J_{n+p}^p + J_{n+2}^p - J_{n+1}^p - 2J_n^p$$

in which $J_0^p = \cdots = J_{p+2}^p = 0$ and $J_{p+3}^p = 1$.

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Also in [2], she gave the generating matrix of the Padovan-p Jacobsthal sequence $\{J_n^p\}$ as follows:

$$PJ_p = \begin{bmatrix} 1 & 3 & -1 & -2 & 0 & \cdots & 0 & 1 & -1 & -2 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix}_{(p+4)\times(p+4)}.$$

The matrix PJ_p is entitled a Padovan-p Jacobsthal matrix. By an inductive argument, she obtained that

$$\left(PJ_p\right)^n = \begin{bmatrix} J_{n+p+3}^p & J_{n+p+4}^p - J_{n+p+3}^p & Pap\,(n+p+3) - J_{n+p+3}^p & Pap\,(n+p+4) - J_{n+p+4}^p - J_{n+p+3}^p & \\ J_{n+p+2}^p & J_{n+p+3}^p - J_{n+p+2}^p & Pap\,(n+p+2) - J_{n+p+2}^p & Pap\,(n+p+3) - J_{n+p+3}^p - J_{n+p+2}^p & \\ J_{n+p+1}^p & J_{n+p+2}^p - J_{n+p+1}^p & Pap\,(n+p+1) - J_{n+p+1}^p & Pap\,(n+p+2) - J_{n+p+2}^p - J_{n+p+1}^p & PJ_p^* \\ \vdots & \vdots & \vdots & \vdots & \\ J_{n+1}^p & J_{n+2}^p - J_{n+1}^p & Pap\,(n+1) - J_{n+1}^p & Pap\,(n+2) - J_{n+2}^p - J_{n+1}^p & \\ J_n^p & J_{n+1}^p - J_n^p & Pap\,(n) - J_n^p & Pap\,(n+1) - J_{n+1}^p - J_n^p & \end{bmatrix},$$

where PJ_p^* is a $(p+4)\times(p)$ matrix as follows:

$$PJ_p^* = \begin{bmatrix} Pap\,(n+3) & Pap\,(n+4) & \cdots & Pap\,(n+p) & -J_{n+p+2}^p - 2J_{n+p+1}^p & -2J_{n+p+2}^p \\ Pap\,(n+2) & Pap\,(n+3) & \cdots & Pap\,(n+p-1) & -J_{n+p+1}^p - 2J_{n+p}^p & -2J_{n+p+1}^p \\ Pap\,(n+1) & Pap\,(n+2) & \cdots & Pap\,(n+p-2) & -J_{n+p}^p - 2J_{n+p-1}^p & -2J_{n+p}^p \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ Pap\,(n-p+1) & Pap\,(n-p+2) & \cdots & Pap\,(n-2) & -J_n^p - 2J_{n-1}^p & -2J_n^p \\ Pap\,(n-p) & Pap\,(n-p+1) & \cdots & Pap\,(n-3) & -J_{n-1}^p - 2J_{n-2}^p & -2J_{n-1}^p \end{bmatrix}$$

for $n \geq p$.

In the literature, many authors studied number theoretic properties such as these obtained from homogeneous linear recurrence relations relevant to this paper; see for example, [5, 7, 8, 14, 15]. In [1, 3, 4, 10–13, 16–20, 23], the authors defined some linear recurrence sequences and gave their various properties by matrix methods. In this paper, we investigate the Padovan-p Jacobsthal sequence. Firstly, we discuss connections between the Jacobsthal and Padovan-p Jacobsthal numbers. Furthermore, we derive the permanental and determinantal representations of the Padovan-p Jacobsthal numbers by using certain matrices which are obtained from the generating matrix of this sequence. Finally, we acquire the combinatorial and exponential representations and the sums of the Padovan-p Jacobsthal numbers by the aid of the generating function and the generating matrix of this sequence.

2. Main Results

First, we derive a relationship between the above-described Padovan-p Jacobsthal sequence and Jacobsthal sequence.

Theorem 2.1. *Let $J(n)$ and J_n^p be the nth the Jacobsthal number and Padovan-p Jacobsthal numbers, respectively. Then,*

$$J(n) = J_{n+p+2}^p - J_{n+p}^p - J_n^p$$

for $n \geq 0$ and $p \geq 3$.

Proof. The assertion may be proved by induction method on n. It is clear that $J(0) = J_{p+2}^{p} - J_{p}^{p} - J_{0}^{p} = 0$. Assume that the equation holds for $n \geq 1$. Then we must show that the equation holds for $n+1$. Since the characteristic polynomial of the Jacobsthal sequence $\{J(n)\}$, is

$$j(x) = x^2 - x - 2$$

we obtain the following relations:

$$J(n+p+4) = J(n+p+3) + 3J(n+p+2) - J(n+p+1) - 2J(n+p) + J(n+2) - J(n+1) - 2J(n)$$

for $n \geq 1$. Hence, by a simple calculation, we have the conclusion. □

Now we take into account the relationship between the Padovan-p Jacobsthal numbers and the permanents of a certain matrix which is obtained using the Padovan-p Jacobsthal matrix $\left(PJ_p\right)^n$.

Definition 2.2. *A $u \times v$ real matrix $M = \left[m_{i,j}\right]$ is called a contractible matrix in the k^{th} column (resp. row.) if the k^{th} column (resp. row.) contains exactly two non-zero entries.*

Suppose that $x_1, x_2, \ldots, x_u$ are row vectors of the matrix M. If M is contractible in the k^{th} column such that $m_{i,k} \neq 0, m_{j,k} \neq 0$ and $i \neq j$, then the $(u-1) \times (v-1)$ matrix $M_{ij:k}$ obtained from M by replacing the i^{th} row with $m_{i,k}x_j + m_{j,k}x_i$ and deleting the j^{th} row. The k^{th} column is called the contraction in the k^{th} column relative to the i^{th} row and the j^{th} row.

In [6], Brualdi and Gibson obtained that $per(M) = per(N)$ if M is a real matrix of order $\alpha > 1$ and N is a contraction of M.

Now we concentrate on finding relationships among the Padovan-p Jacobsthal numbers and the permanents of certain matrices which are obtained by using the generating matrix of this sequence. Let $F_{m,p}^{Pa,J} = \left[f_{i,j}^{(p)}\right]$ be the $m \times m$ super-diagonal matrix, defined by

$$f_{i,j}^{(p)} = \begin{cases} 3 & \text{if } i=\tau \text{ and } j=\tau+1 \text{ for } 1 \leq \tau \leq m-1, \\ 1 & \begin{array}{c} \text{if } i=\tau \text{ and } j=\tau \text{ for } 1 \leq \tau \leq m, \\ i=\tau \text{ and } j=\tau+p+1 \text{ for } 1 \leq \tau \leq m-p-1 \\ \text{and} \\ i=\tau+1 \text{ and } j=\tau \text{ for } 1 \leq \tau \leq m-1, \end{array} \\ -1 & \begin{array}{c} \text{if } i=\tau \text{ and } j=\tau+2 \text{ for } 1 \leq \tau \leq m-2 \\ \text{and} \\ i=\tau \text{ and } j=\tau+p+2 \text{ for } 1 \leq \tau \leq m-p-2, \end{array} \\ -2 & \begin{array}{c} \text{if } i=\tau \text{ and } j=\tau+3 \text{ for } 1 \leq \tau \leq m-3 \\ \text{and} \\ i=\tau \text{ and } j=\tau+p+3 \text{ for } 1 \leq \tau \leq m-p-3, \end{array} \\ 0 & \text{otherwise.} \end{cases},$$

for $m \geq p+4$. Then we have the following Theorem.

Theorem 2.3. *For $m \geq p+4$,*

$$perF_{m,p}^{Pa,J} = J_{m+p+3}^{p}.$$

Proof. Let us keep in view matrix $F_{m,p}^{Pa,J}$ and let the equation be hold for $m \geq p+4$. Then we show that the equation holds for $m+1$. If we expand the $perF_{m,p}^{Pa,J}$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$perF_{m+1,p}^{Pa,J} = perF_{m,p}^{Pa,J} + 3perF_{m-1,p}^{Pa,J} - perF_{m-2,p}^{Pa,J} - 2perF_{m-3,p}^{Pa,J} + perF_{m-p-1,p}^{Pa,J} - perF_{m-p-2,p}^{Pa,J} - 2perF_{m-p-3,p}^{Pa,J}.$$

Since

$$perF^{Pa,J}_{m,p} = J^{p}_{m+p+3},$$

$$perF^{Pa,J}_{m-1,p} = J^{p}_{m+p+2},$$

$$perF^{Pa,J}_{m-2,p} = J^{p}_{m+p+1},$$

$$perF^{Pa,J}_{m-3,p} = J^{p}_{m+p},$$

$$perF^{Pa,J}_{m-p-1,p} = J^{p}_{m+2},$$

$$perF^{Pa,J}_{m-p-2,p} = J^{p}_{m+1}$$

and

$$perF^{Pa,J}_{m-p-3,p} = J^{p}_{m},$$

we easily obtain that $perF^{Pa,J}_{m+1,p} = J^{p}_{m+p+4}$. So the proof is complete. □

Let $G^{Pa,J}_{m,p} = \left[g^{(p)}_{i,j}\right]$ be the $m \times m$ matrix, defined by

$$g^{(p)}_{i,j} = \begin{cases} 3 & \text{if } i=\tau \text{ and } j=\tau+1 \text{ for } 1 \le \tau \le m-2, \\ 1 & \begin{array}{c} \text{if } i=\tau \text{ and } j=\tau \text{ for } 1 \le \tau \le m, \\ i=\tau \text{ and } j=\tau+p+1 \text{ for } 1 \le \tau \le m-p-2 \\ \text{and} \\ i=\tau+1 \text{ and } j=\tau \text{ for } 1 \le \tau \le m-2, \end{array} \\ -1 & \begin{array}{c} \text{if } i=\tau \text{ and } j=\tau+2 \text{ for } 1 \le \tau \le m-3 \\ \text{and} \\ i=\tau \text{ and } j=\tau+p+2 \text{ for } 1 \le \tau \le m-p-3, \end{array} \\ -2 & \begin{array}{c} \text{if } i=\tau \text{ and } j=\tau+3 \text{ for } 1 \le \tau \le m-4 \\ \text{and} \\ i=\tau \text{ and } j=\tau+p+3 \text{ for } 1 \le \tau \le m-p-3, \end{array} \\ 0 & \text{otherwise.} \end{cases},$$

for $m \ge p+4$. Then we have the following Theorem.

Theorem 2.4. *For $m \ge p+4$,*

$$perG^{Pa,J}_{m,p} = J^{p}_{m+p+2}.$$

Proof. Let us keep in view matrix $G^{Pa,J}_{m,p}$ and let the equation be hold for $m \ge p+4$. Then we show that the equation holds for $m+1$. If we expand the $perG^{Pa,J}_{m,p}$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$perG^{Pa,J}_{m+1,p} = perG^{Pa,J}_{m,p} + 3perG^{Pa,J}_{m-1,p} - perG^{Pa,J}_{m-2,p} - 2perG^{Pa,J}_{m-3,p} + perG^{Pa,J}_{m-p-1,p} - perG^{Pa,J}_{m-p-2,p} - 2perG^{Pa,J}_{m-p-3,p}.$$

Since

$$perG^{Pa,J}_{m,p} = J^{p}_{m+p+2},$$

$$perG^{Pa,J}_{m-1,p} = J^{p}_{m+p+1},$$

$$perG^{Pa,J}_{m-2,p} = J^{p}_{m+p},$$

$$perG^{Pa,J}_{m-3,p} = J^{p}_{m+p-1},$$

$$perG^{Pa,J}_{m-p-1,p} = J^{p}_{m+1},$$

$$perG^{Pa,J}_{m-p-2,p} = J^{p}_{m}$$

and

$$perG^{Pa,J}_{m-p-3,p} = J^{p}_{m-1},$$

we easily obtain that $perG^{Pa,J}_{m+1,p} = J^{p}_{m+p+3}$. So the proof is complete. □

Suppose that $H^{Pa,J}_{m,p} = \left[h^{(p)}_{i,j}\right]$ be the $m \times m$ matrix, defined by

$$H^{Pa,J}_{m,p} = \left[\begin{array}{cccc} & & (m-1)\,\text{th} & \\ & & \downarrow & \\ 1 & \cdots & 1 & 0 \\ 1 & & & \\ 0 & & G^{Pa,J}_{m-1,p} & \\ \vdots & & & \\ 0 & & & \end{array}\right],$$

for $m > p + 4$, then we have the following results:

Theorem 2.5. *For $m > p + 4$,*

$$perH^{Pa,J}_{m,p} = \sum_{i=0}^{m+p+1} J^{p}_{i}.$$

Proof. If we extend $per\, H^{Pa,J}_{m,p}$ with respect to the first row, we write

$$perH^{Pa,J}_{m,p} = perH^{Pa,J}_{m-1,p} + perG^{Pa,J}_{m-1,p}.$$

Thence, by the results and an inductive argument, the proof is easily seen. □

A matrix M is called convertible if there is an $n \times n$ $(1, -1)$-matrix K such that $perM = \det(M \circ K)$, where $M \circ K$ denotes the Hadamard product of M and K.

Now we give relationships among the Padovan-p Jacobsthal numbers and the determinants of certain matrices which are obtained by using the matrices $F^{Pa,J}_{m,p}$, $G^{Pa,J}_{m,p}$ and $H^{Pa,J}_{m,p}$. Let $m > p + 4$ and let R be the $m \times m$ Hadamard matrix, defined by

$$R = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}.$$

Corollary 2.6. *For $m > p + 4$,*

$$\det\left(F^{Pa,J}_{m,p} \circ R\right) = J^{p}_{m+p+3},$$

$$\det\left(G^{Pa,J}_{m,p} \circ R\right) = J^{p}_{m+p+2}$$

and

$$\det\left(H^{Pa,J}_{m,p} \circ R\right) = \sum_{i=0}^{m+p+1} J^{p}_{i}.$$

Proof. Since $perF_{m,p}^{Pa,J} = \det\left(F_{m,p}^{Pa,J} \circ R\right)$, $perG_{m,p}^{Pa,J} = \det\left(G_{m,p}^{Pa,J} \circ R\right)$ and $perH_{m,p}^{Pa,J} = \det\left(H_{m,p}^{Pa,J} \circ R\right)$ for $m > p + 4$, by Theorem 2.3, Theorem 2.4 and Theorem 2.5, we have the conclusion. □

Let $K(k_1, k_2, \ldots, k_v)$ be a $v \times v$ companion matrix as follows:

$$K(k_1, k_2, \ldots, k_v) = \begin{bmatrix} k_1 & k_2 & \cdots & k_v \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

For more details on the companion type matrices, see [21, 22].

Theorem 2.7. *(Chen and Louck [9]) The (i, j) entry $k_{i,j}^{(n)}(k_1, k_2, \ldots, k_v)$ in the matrix $K^n(k_1, k_2, \ldots, k_v)$ is given by the following formula:*

$$k_{i,j}^{(n)}(k_1, k_2, \ldots, k_v) = \sum_{(t_1, t_2, \ldots, t_v)} \frac{t_j + t_{j+1} + \cdots + t_v}{t_1 + t_2 + \cdots + t_v} \times \binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} k_1^{t_1} \cdots k_v^{t_v} \tag{1}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + vt_v = n - i + j$, $\binom{t_1+\cdots+t_v}{t_1,\ldots,t_v} = \frac{(t_1+\cdots+t_v)!}{t_1!\cdots t_v!}$ is a multinomial coefficient, and the coefficients in (1) are defined to be 1 if $n = i - j$.

Then we can give combinatorial representations for the Padovan-p Jacobsthal numbers by the following Corollary.

Corollary 2.8. *Let J_n^p be the nth the Padovan-p Jacobsthal number for $n \geq p$. Then*

i.

$$J_n^p = \sum_{(t_1, t_2, \ldots, t_{p+4})} \binom{t_1 + t_2 + \cdots + t_{p+4}}{t_1, t_2, \cdots, t_{p+4}} 3^{t_2} (-1)^{t_3 + t_{p+3}} (-2)^{t_4 + t_{p+4}}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (p+4)\, t_{p+4} = n - p - 3$.

ii.

$$F_n^{Pa,p} = -\frac{1}{2} \sum_{(t_1, t_2, \ldots, t_4)} \frac{t_{p+4}}{t_1 + t_2 + \cdots + t_{p+4}} \times \binom{t_1 + t_2 + \cdots + t_{p+4}}{t_1, t_2, \cdots, t_{p+4}} 3^{t_2} (-1)^{t_3 + t_{p+3}} (-2)^{t_4 + t_{p+4}}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (p+4)\, t_{p+4} = n + 1$.

Proof. If we take $i = p + 4$, $j = 1$ for the case i. and $i = p + 3$, $j = p + 4$ for the case ii. in Theorem 2.7, then we can directly see the conclusions from $\left(PJ_p\right)^n$. □

The generating function of the Padovan-p Jacobsthal sequence $\left\{J_n^p\right\}$ is obtained as follows:

$$g(x) = \frac{x^{p+3}}{1 - x - 3x^2 + x^3 + 2x^4 - x^{p+2} + x^{p+3} + 2x^{p+4}},$$

where $p \geq 3$.

Then, with the following theorem, we can deliver an exponential representation for the Padovan-p Jacobsthal numbers by the aid of the generating function.

Theorem 2.9. *Let $g(x)$ be generating function of the Padovan-p Jacobsthal numbers. The following exponential representation for the Padovan-p Jacobsthal numbers as follows::*

$$g(x) = x^{p+3} \exp\left(\sum_{i=1}^{\infty} \frac{(x)^i}{i} \left(1 + 3x - x^2 - 2x^3 + x^{p+1} - x^{p+2} - 2x^{p+3}\right)^i\right),$$

where $p \geq 3$.

Proof. Since

$$\ln g(x) = \ln x^{p+3} - \ln\left(1 - x - 3x^2 + x^3 + 2x^4 - x^{p+2} + x^{p+3} + 2x^{p+4}\right)$$

and

$$\begin{aligned} -\ln\left(1 - x - 3x^2 + x^3 + 2x^4 - x^{p+2} + x^{p+3} + 2x^{p+4}\right) &= -[-x\left(1 + 3x - x^2 - 2x^3 + x^{p+1} - x^{p+2} - 2x^{p+3}\right) - \\ &\quad \frac{1}{2}x^2\left(1 + 3x - x^2 - 2x^3 + x^{p+1} - x^{p+2} - 2x^{p+3}\right)^2 - \cdots \\ &\quad -\frac{1}{i}x^i\left(1 + 3x - x^2 - 2x^3 + x^{p+1} - x^{p+2} - 2x^{p+3}\right)^i - \cdots] \end{aligned}$$

it is clear that

$$g(x) = x^{p+3}\exp\left(\sum_{i=1}^{\infty}\frac{(x)^i}{i}\left(1 + 3x - x^2 - 2x^3 + x^{p+1} - x^{p+2} - 2x^{p+3}\right)^i\right)$$

by a simple calculation, we obtain the conclusion. □

Now we consider the sums of the Padovan-p Jacobsthal numbers. Let

$$T_n = \sum_{i=0}^{n} J_i^p$$

for $n \geq p$ and $p \geq 3$, and let $K_p^{Pa,J}$ and $\left(K_p^{Pa,J}\right)^n$ be the $(p+5)\times(p+5)$ matrix such that

$$K_p^{Pa,J} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & & & & & \\ 0 & & & & & \\ \vdots & & & PJ_p & & \\ 0 & & & & & \\ 0 & & & & & \end{bmatrix}.$$

If we use induction on n, then we obtain

$$\left(K_p^{Pa,J}\right)^{\alpha} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ T_{n+p+2} & & & & & \\ T_{n+p+1} & & & & & \\ \vdots & & & PJ_p & & \\ T_n & & & & & \\ T_{n-1} & & & & & \end{bmatrix}.$$

3. Conclusion

We considered a sequence called the Padovan-p Jacobsthal sequence, which is obtained using polynomials characteristic of the Padovan p-sequence and the Jacobsthal sequence. Furthermore, using the generating matrix of the Padovan-p Jacobsthal sequence, we obtained some new structural properties of the Padovan-p Jacobsthal numbers such as the generating functions, the permanental, combinatorial, determinantal, and exponential representations and the finite sums.

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The Complex-type Pell p-Numbers in Finite Groups

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Abstract. In this study, we study the complex-type Pell p-numbers modulo m and further we get the periods and the ranks of the complex-type Pell p-numbers modulo m. Additionally, we give some results on the periods and the ranks of the complex-type Pell p-numbers modulo m. Then, we consider the multiplicative orders of the complex-type Pell p-matrix when read modulo m. Also, we redefine the complex-type Pell p-numbers by means of the elements of groups. Finally, we produce the periods of the complex-type Pell 2-numbers in the semidihedral group SD_{2^m}, $(m \geq 4)$.

1. Introduction

The complex-type Pell p-numbers for any given p $(p = 2, 3, \ldots)$ is defined [2] by the following recurrence equation:

$$P_p^*(n+p+1) = 2i^{p+1} \cdot P_p^*(n+p) + i \cdot P_p^*(n) \tag{1}$$

for $n \geq 1$, where $P_p^*(1) = \cdots = P_p^*(p) = 0$, $P_p^*(p+1) = 1$ and $\sqrt{-1} = i$.

In [2], the complex-type Pell p-matrix K_p had been given as:

$$K_p = \begin{bmatrix} 2i^{p+1} & 0 & \cdots & 0 & i \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix}_{(p+1)\times(p+1)}.$$

Then, for $n \geq p$, they found that

$$\left(K_p\right)^n = \begin{bmatrix} P_p^*(n+p+1) & iP_p^*(n+1) & iP_p^*(n+2) & \cdots & iP_p^*(n+p) \\ P_p^*(n+p) & iP_p^*(n) & iP_p^*(n+1) & \cdots & iP_p^*(n+p-1) \\ \vdots & \vdots & \vdots & & \vdots \\ P_p^*(n+2) & iP_p^*(n-p+2) & iP_p^*(n-p+3) & \cdots & iP_p^*(n+1) \\ P_p^*(n+1) & iP_p^*(n-p+1) & iP_p^*(n-p+2) & \cdots & iP_p^*(n) \end{bmatrix}, \tag{2}$$

in addition, the determinant of the K_p matrix is $(-1)^p i$.

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Definition 1.1. *A sequence is well known to be periodic if after a certain point it consists only of repeats of a fixed subsequence. A sequence is simply periodic with period k if the first k elements in the sequence form a repeating subsequence.*

For a finitely generated group $G = \langle A \rangle$, where $A = \{a_1, a_2, \ldots, a_n\}$, the sequence $x_u = a_{u+1}$, $0 \le u \le n-1$, $x_{n+u} = \prod_{v=1}^{n} x_{u+v-1}$, $u \ge 0$ is called the Fibonacci orbit of G with respect to the generating set A, denoted as $F_A(G)$ in [6].

A k-nacci (k-step Fibonacci) sequence in a finite group is a sequence of group elements $x_0, x_1, x_2, \ldots, x_n, \ldots$ for which, given an initial (seed) set $x_0, x_1, x_2, \ldots, x_{j-1}$, each element is defined by

$$x_n = \begin{cases} x_0 x_1 \cdots x_{n-1} & \text{for } j \le n < k, \\ x_{n-k} x_{n-k+1} \cdots x_{n-1} & \text{for } n \ge k. \end{cases}$$

The k-nacci sequence of a group G generated by $x_0, x_1, x_2, \ldots, x_{j-1}$ is indicated by $F_k\left(G; x_0, x_1, x_2, \ldots, x_{j-1}\right)$ in [15].

In [9], Deveci and Shannon showed that the following conditions apply for every elements x, y of the group G:

Definition 1.2. (i) *Suppose that $z = a + ib$ such that a and b are integers and suppose that e is the identity of G, then*
$* \; x^z \equiv x^{a(mod|x|)+ib(mod|x|)} = x^{a(mod|x|)}x^{ib(mod|x|)} = x^{ib(mod|x|)}x^{a(mod|x|)} = x^{ib(mod|x|)+a(mod|x|)}$,
$* \; x^{ia} = \left(x^i\right)^a = (x^a)^i$,
$* \; e^u = e$,
$* \; x^{0+i0} = e$.
(ii) *Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ such that a_1, b_1, a_2 and b_2 are integers , then* $\left(x^{z_1}y^{z_2}\right)^{-1} = y^{-z_2}x^{-z_1}$.
(iii) *If $xy \ne yx$, then $x^i y^i \ne y^i x^i$.*
(iv) $(xy)^i = y^i x^i$ *and* $\left(x^i y^i\right)^i = x^{-1}y^{-1}$.
(v) $xy^i = y^i x$ *and so* $\left(xy^i\right)^i = x^i y^{-1}$ *and* $\left(x^i y\right)^i = x^{-1}y^i$.

In [1, 3, 4, 8, 11, 16], the authors have produced the cyclic groups and the semigroups through some special matrices and then, they have studied the orders of these algebraic structures. The study of the recurrence sequences in groups began with the earlier work of Wall [21]. Also, the theory extended to some special linear recurrence sequences by several authors; see for example, [5, 7, 10, 12–15, 17–20, 22]. In this study, we study the complex-type Pell p-numbers modulo m and then we get the periods and the ranks of the complex-type Pell p-numbers modulo m. Then, we consider the multiplicative orders of the complex-type Pell p-matrix when read modulo m. Also, we redefine the complex-type Pell p-numbers with the elements of groups and then we give the periods of the complex-type Pell 2-numbers in the semidihedral group.

2. The Complex-type Pell p-Numbers in Finite Groups

Reducing the complex-type Pell p-numbers by a modulus m, we obtain a repeating sequence, indicated by

$$\left\{P^*_{p,m}(n)\right\} = \left\{P^*_{p,m}(1),\; P^*_{p,m}(2),\; \ldots,\; P^*_{p,m}(j),\; \ldots\right\}$$

where $P^*_{p,m}(n) = P^*_p(n)\,(modm)$. This relation has the same recurrence relation as in (1)

Theorem 2.1. *For $p \ge 2$, the sequence $\left\{P^*_{p,m}(n)\right\}$ is simply periodic sequence.*

Proof. Consider the set

$$W = \{(w_1, w_2, \dots, w_{p+1}) \mid w_v\text{'s are complex numbers } a_v + ib_v \text{ where } a_v \text{ and } b_v \text{ are integers such that } 0 \le a_v, b_v \le m-1 \text{ and } 1 \le v \le p+1\}.$$

Suppose that the notation $|W|$ is the order of the set W. Since the set W is finite, there are $|W|$ distinct $p+1$-tuples of the complex-type Pell p-numbers modulo m. So, at least one of the $p+1$-tuples appears twice in the sequence $\{P^*_{p,m}(n)\}$. Then, the subsequence following this $p+1$-tuple repeats; that is, $\{P^*_{p,m}(n)\}$ is a periodic sequence. Let $P^*_{p,m}(k) \equiv P^*_{p,m}(l)$, $P^*_{p,m}(k+1) \equiv P^*_{p,m}(l+1)$, $\dots$, $P^*_{p,m}(k+p+1) \equiv P^*_{p,m}(l+p+1)$ and $k \ge l$, then $k \equiv l\,(mod p+1)$. It is obvious that

$$P^*_p(n) = (-i) \cdot P^*_p(n+p+1) + 2i^{p+2} \cdot P^*_p(n+p).$$

So we get $P^*_{p,m}(k-1) \equiv P^*_{p,m}(l-1)$, $P^*_{p,m}(k-2) \equiv P^*_{p,m}(l-2)$, $\dots$, $P^*_{p,m}(1) \equiv P^*_{p,m}(k-l+1)$, which indicates that $\{P^*_{p,m}(n)\}$ is a simply periodic. □

We indicate the period of the sequence $\{P^*_{p,m}(n)\}$ by $t_p(m)$.

For given a matrix $B = [b_{ij}]$ with b_{ij}'s being integers, $B\,(modm)$ means that each element of B are reduced modulo m, that is, $B\,(modm) = (b_{ij}\,(modm))$. If $(\det B, m) = 1$, then the set $\langle B\rangle_m$ is a cyclic group; if $(\det B, m) \ne 1$, then the set $\langle B\rangle_m$ is a semigroup. Let the notation $|\langle B\rangle_m|$ indicates the order of the set $\langle B\rangle_m$.

Since $\det K_p = (-1)^p\, i$, the set $\langle K_p\rangle_m$ is a cyclic group for every positive integer $m \ge 2$. It is easy to see from (2) that it is $t_p(m) = \left|\langle K_p\rangle_m\right|$.

Theorem 2.2. *Let v be a prime. If r is the smallest positive integer such that $t_p(v^{r+1}) \ne t_p(v^r)$, then $t_p(v^{r+1}) = vt_p(v^r)$ for every integer $p \ge 2$*

Proof. Suppose that r is the smallest positive integer such that $t_p(v^{r+1}) \ne t_p(v^r)$ and suppose that z is a positive integer. If $(K_p)^{t_p(v^{z+1})} \equiv I\,(modv^{z+1})$, then $(K_p)^{t_p(v^{z+1})} \equiv I\,(modv^z)$. Thus we obtain that $t_p(v^z)$ divides $t_p(v^{z+1})$. Also, writing $(K_p)^{t_p(v^z)} = I + (m^{(z)}_{i,j} \cdot v^z)$, by the binomial theorem, we obtain

$$(K_p)^{vt_p(v^z)} = \left(I + (m^{(z)}_{i,j} \cdot v^z)\right)^v = \sum_{i=0}^{v} \binom{v}{i} \left(m^{(z)}_{i,j} \cdot v^z\right)^i \equiv I\left(\text{mod } v^{z+1}\right).$$

and so it appears that $t_p(v^{z+1})$ divides $v\,t_p(v^z)$. Therefore, $t_p(v^{z+1}) = t_p(v^z)$ or $t_p(v^{z+1}) = v\,t_p(v^z)$, and the latter holds if and only if there is a $m^{(z)}_{i,j}$ which is not divisible by v. Since we assume that r is the smallest positive integer such that $t_p(v^{r+1}) \ne t_p(v^r)$, there is an $m^{(z)}_{i,j}$ that is not divisible by v. This shows that $_p(v^{r+1}) = vt_p(v^r)$. So, the proof is complete. □

Definition 2.3. *The rank of the sequence $\{P^*_{p,m}(n)\}$ is the least positive integer α such that $P^*_{p,m}(\alpha) \equiv P^*_{p,m}(\alpha+1) \equiv \cdots \equiv P^*_{p,m}(\alpha+p-1) \equiv 0\,(modm)$, and we indicate the rank of $\{P^*_{p,m}(n)\}$ by $r_p(m)$.*

If $P^*_{p,m}(\alpha+p-1) \equiv 0\,(modm)$, then the terms of the sequence $\{P^*_{p,m}(n)\}$ starting with index $r_p(m)$, namely $\underbrace{0, 0, \dots, 0}_{p}, \theta, \theta, \dots$, are exactly the initial terms of $\{P^*_{p,m}(n)\}$ multiplied by a factor θ.

The exponents ϖ for which $\left(K_p\right)^{\varpi} \equiv I\,(modm)$ form a simple aritmetic progression. So we give

$$\left(K_p\right)^{\varpi} \equiv I\,(modm) \Longleftrightarrow t_p\,(m) \mid \varpi.$$

Similarly, the exponents ϖ for which $\left(K_p\right)^{\varpi} \equiv \theta I\,(modm)$ for some $\theta \in \mathbb{C}$ form a simple aritmetic progression, and so

$$\left(K_p\right)^{\varpi} \equiv \theta I\,(modm) \Longleftrightarrow r_p\,(m) \mid \varpi.$$

Thus, it is simple to show that $r_p\,(m)$ divides $t_p\,(m)$.

The order of the sequence $\left\{P^*_{p,m}\,(n)\right\}$ is defined by $\frac{t_p(m)}{r_p(m)}$ and we indicate it by $Q_p\,(m)$. Let $\left(K_p\right)^{r_p(m)} \equiv \theta I\,(modm)$, then $ord_m\,(\theta)$ is the least positive value of δ such that $\left(K_p\right)^{\delta r_p(m)} \equiv I\,(modm)$. So it is confirm that $ord_m\,(\theta)$ is the least positive integer δ with $t_p\,(m) \mid \delta r_p\,(m)$. Thus, we obtain $ord_m\,(\theta) = \delta$. As a result, we may easily conclude that $Q_p\,(m)$ is always a positve integer, and that $Q_p\,(m) = ord_m\left(P^*_p\left(r_p\,(m) + p\right)\right)$, the multiplicative order of $P^*_{p,m}\left(r_p\,(m) + p\right)$.

Example 2.4. *Since*

$$\left\{P^*_{5,2}\,(n)\right\} = \{0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, i, 0, 0, 0, 0, 0, 1, 0, \ldots, \},$$

we have $t_5\,(2) = 12$, $r_5\,(2) = 6$ *and* $Q_5\,(2) = 2$.

Theorem 2.5. *suppose that* m_1 *and* m_2 *are positive integers with* $m_1, m_2 \geq 2$, *then* $r_p\,(lcm\,[m_1, m_2]) = lcm\left[r_p\,(m_1)\,, r_p\,(m_2)\right]$. *In the same way,* $t_p\,(lcm\,[m_1, m_2]) = lcm\left[t_p\,(m_1)\,, t_p\,(m_2)\right]$.

Proof. Let $lcm\,[m_1, m_2] = m$. Then

$$P^*_p\left(r_p\,(m)\right) \equiv P^*_p\left(r_p\,(m) + 1\right) \equiv \cdots \equiv P^*_p\left(r_p\,(m) + p - 1\right) \equiv 0\,(modm)$$

and

$$P^*_p\left(r_p\,(m_w)\right) \equiv P^*_p\left(r_p\,(m_w) + 1\right) \equiv \cdots \equiv P^*_p\left(r_p\,(m_w) + p - 1\right) \equiv 0\,(modm)$$

for $w = 1, 2$. Using the least common multiple operation implies that $P^*_p\left(r_p\,(m)\right) \equiv P^*_p\left(r_p\,(m) + 1\right) \equiv \cdots \equiv P^*_p\left(r_p\,(m) + p - 1\right) \equiv 0 modm_w$ for $w = 1, 2$. Hence we get $r_p\,(m_1) \mid r_p\,(m)$ and $r_p\,(m_2) \mid r_p\,(m)$, which signifies that $lcm\left[r_p\,(m_1)\,, r_p\,(m_2)\right]$ divides $r_p\,(lcm\,[m_1, m_2])$. We also know that

$$P^*_p\left(lcm\left[r_p\,(m_1)\,, r_p\,(m_2)\right]\right) \equiv P^*_p\left(lcm\left[r_p\,(m_1)\,, r_p\,(m_2)\right] + 1\right) \equiv \cdots \equiv P^*_p\left(lcm\left[r_p\,(m_1)\,, r_p\,(m_2)\right] + p - 1\right) \equiv 0\,(modm_w)$$

for $w = 1, 2$. Then we can write

$$P^*_p\left(lcm\left[r_p\,(m_1)\,, r_p\,(m_2)\right]\right) \equiv P^*_p\left(lcm\left[r_p\,(m_1)\,, r_p\,(m_2)\right] + 1\right) \equiv \cdots \equiv P^*_p\left(lcm\left[r_p\,(m_1)\,, r_p\,(m_2)\right] + p - 1\right) \equiv 0\,(modm)\,,$$

and it follows that $r_p\,(lcm\,[m_1, m_2])$ divides $lcm\left[r_p\,(m_1)\,, r_p\,(m_2)\right]$. Thus, the proof is complete.

The period $t_p\,(m)$ is proved with a similar proof method. □

Now we take into account the complex-type Pell p-numbers in groups.

Suppose that G be a finite j-generator group and let $X = \{(x_1, x_2, \ldots, x_j) \in \underbrace{G \times G \times \cdots \times G}_{j} \mid \;<\left\{x_1, x_2, \ldots, x_j\right\}>= G\}$. We call $\left(x_1, x_2, \ldots, x_j\right)$ a generating j-tuple for G.

Definition 2.6. *Suppose that G is a j-generator group and suppose that* $\left(x_1, x_2, \ldots, x_j\right)$ *is a generating j-tuple for G. So we define the complex-type Pell p-orbit* $P_p^*\left(G; x_1, x_2, \ldots, x_j\right) = \left\{a_p(n)\right\}$ *as shown:*

$$a_p(n+p) = a_p(n-1)^i \, a_p(n+p-1)^{2i^{p+1}} \quad (n > 1)$$

where

$$\begin{cases} a_p(1) = x_1,\ a_p(2) = x_2, \ldots,\ a_p(j) = x_j, a_p(j+1) = e, \ldots, a_p(p+1) = e & \text{if } j < p+1, \\ a_p(1) = x_1,\ a_p(2) = x_2, \ldots, a_p(p+1) = x_{p+1} & \text{if } j = p+1. \end{cases}$$

Theorem 2.7. *Suppose that G is a j-generator group. If G is finite, then the complex-type Pell p-orbit of G is periodic.*

Proof. We think of the set

$$\begin{aligned} H \;=\; & \Big\{\Big((h_1)^{a_1(mod|h_1|)+ib_1(mod|h_1|)}, \\ & (h_2)^{a_2(mod|h_2|)+ib_2(mod|h_2|)}, \ldots, \\ & \left(h_j\right)^{a_j\left(mod\left|h_j\right|\right)+ib_j\left(mod\left|h_j\right|\right)}\Big) : \\ & h_1, h_2, \ldots, h_j \in G \text{ and } a_n, b_n \in Z \text{ such that } 1 \le n \le j\Big\}. \end{aligned}$$

If G is finite, the H is a finite set. For any $c \geq 0$, there exists $k \geq c + j$ such that $a_p(c+1) = a_p(k+1)$, $a_p(c+2) = a_p(k+2), \ldots, a_p(c+j) = a_p(k+j)$. Due to repeating, for all generating j-tuples, the sequence $P_p^*\left(G; x_1, x_2, \ldots, x_j\right)$ is periodic. □

We indicate the length of the period of the complex-type Pell p-orbit $P_p^*\left(G; x_1, x_2, \ldots, x_j\right)$ by $hP_p^*\left(G; x_1, x_2, \ldots, x_j\right)$. Now we give the lengths of the periods of the complex-type Pell 2-orbit of the semidihedral group SD_{2^m}. The semidihedral group SD_{2^m} of order 2^m is defined by the presentation

$$SD_{2^m} = \langle x, y \mid x^{2^{m-1}} = y^2 = e,\ y^{-1}xy = x^{-1+2^{m-2}} \rangle$$

for every $m \geq 4$. Note that the orders x and y are 2^{m-1} and 2, respectively.

Theorem 2.8. *For generating pairs* (x, y)*, the length of the period of the complex-type Pell 2-orbit in the semidihedral group* SD_{2^m} *is* $2^{m-3} \cdot t_2(2)$.

Proof. For the complex-type Pell 2-orbit, we consider $t_2(2) = 6$. The orbit $P_2^*(SD_{2^m}; x, y)$ is

$$\begin{aligned} & x, y, e, x^i, y^i x^2, x^{-4i}, x^{-9}, yx^{20i}, x^{44}, \\ & x^{-97i}, y^i x^{42}, x^{-40i}, x^{17}, yx^{8i}, x^{56}, \ldots, \end{aligned}$$

and so the orbit becomes:

$$\begin{aligned} a_2(1) &= x,\ a_2(2) = y, a_2(3) = e, \ldots \\ a_2(2 \cdot t_2(2)\,\alpha + 1) &= x^{8\alpha\lambda_1+1}, a_2(2 \cdot t_2(2)\,\alpha + 2) = yx^{4\alpha\lambda_2 \cdot i}, a_2(2 \cdot t_2(2)\,\alpha + 3) = x^{4\alpha\lambda 3}, \ldots. \end{aligned}$$

where λ_1, λ_2 and λ_3 are positive integers such that $gcd(\lambda_1, \lambda_2, \lambda_3) = 1$. Thus, for $\beta \in \mathbb{N}$, we need the smallest integer α such that $8\alpha = 2^{m-1} \cdot \beta$. If we choose $\alpha = 2^{m-4}$, we get

$$a_2\left(2^{m-3} \cdot t_2(2) + 1\right) = x,\ a_2\left(2^{m-3} \cdot t_2(2) + 2\right) = y,\ a_2\left(2^{m-3} \cdot t_2(2) + 3\right) = e \ldots.$$

Since the elements succeeding $a_2\left(2^{m-3} \cdot t_2(2) + 1\right)$, $a_2\left(2^{m-3} \cdot t_2(2) + 2\right)$ and $a_2\left(2^{m-3} \cdot t_2(2) + 3\right)$ depend on x, y, e for their values, the cycle begins again with the $a_2\left(2^{m-3} \cdot t_2(2) + 1\right)$ nd element. Thus it is verified that the length of the period of the complex-type Pell 2-orbit in SD_{2^m} is $2^{m-3} \cdot t_2(2)$. □

Example 2.9. *The sequence $P_2^*(SD_{64}; x, y)$ is*

$$x, y, e, x^i, y^i x^2, x^{-4i}, x^{-9}, yx^{20i}, x^{12}, x^{-i}, y^i x^{10},$$
$$x^{-8i}, x^{17}, yx^{8i}, x^{24}, x^i, yx^{26}, x^{4i}, x^7, yx^{12i}, x^{20},$$
$$x^{-i}, y^i x^{18}, x^{16i}, x, yx^{16i}, x^{16}, x^i, y^i x^{18}, x^{12i}, x^{23},$$
$$yx^{4i}, x^{28}, x^{-i}, y^i x^{26}, x^{8i}, x^{17}, yx^{24i}, x^8, x^i, y^i x^{10},$$
$$x^{20i}, x^7, yx^{28i}, x^4, x^{31i}, y^i x^2, e, x, y, e, \ldots.$$

which implies that $hP_2^*(SD_{32}; x, y) = 48$.

3. Conclusion

In this study, we have considered the complex-type Pell p-numbers modulo m and then we have obtained the periods and the ranks of the complex-type Pell p-numbers modulo m. Also, we have studied the multiplicative orders of the complex-type Pell p-matrix when read modulo m. Finally, we have redefined the complex-type Pell p-numbers with the elements of groups and then we have obtained the periods of the complex-type Pell 2-numbers in the semidihedral group SD_{2^m}, $(m \geq 4)$.

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Fourier Method for Higher Order Quasi-Linear Parabolic Equation Subject with Periodic Boundary Conditions

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Abstract. In this paper, higher order inverse quasi-linear parabolic problem was investigated. It demonstrated the solution by the Fourier approximation. It proved the existence, uniqueness of the solution by Fourier and iteration method.

1. Introduction

In this study we present a high order scheme for determining unknown control parameter and unknown solution of two-dimensional parabolic inverse problem. Two- dimensional inverse parabolic problems are used especially in chemical diffusion applications, heat transfer processes have been used a lot such as population, medical area, electrochemistry, engineering, chemical area, plasma physics .This kind of problems with nonlocal boundary conditions are not easy to study. There are many papers on finding analytical and numerical solutions of inverse coefficient problems with nonlocal boundary conditions in one dimension [2, 5]. In these papers, Finite Difference Method, Boundary Element Method, Finite Element Method, etc. are examined to approximate numerical solutions.Finding of the unknown function in a nonlinear parabolic equation is used frequently by many engineers and scientists [1–5].

In this study, Fourier method is used for the for the solution of this problem.

Here $\Gamma := \{0 < x < \pi,\ 0 < y < \pi,\ 0 < t < T\}$, $\varphi(x, y)$, $f(x, y, t, u)$ are given functions.

$$\frac{\partial u}{\partial t} = b(t)\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x, y, t, u), (x, y, t) \in \Gamma \tag{1}$$

$$u(x, y, 0) = \varphi(x, y),\ x \in [0, \pi], y \in [0, \pi] \tag{2}$$

$$\begin{aligned} u(0, y, t) &= u(\pi, y, t),\ y \in [0, \pi], t \in [0, T] \\ u(x, 0, t) &= u(x, \pi, t),\ x \in [0, \pi], t \in [0, T] \end{aligned} \tag{3}$$

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$$\begin{aligned} u_x(0,y,t) &= u_x(\pi,y,t),\ y\in[0,\pi],t\in[0,T] \\ u_y(x,0,t) &= u_y(x,\pi,t),\ x\in[0,\pi],t\in[0,T] \end{aligned} \tag{4}$$

$$k(t)=\int_0^\pi\int_0^\pi xyu(x,y,t)dxdy,t\in[0,T] \tag{5}$$

where, in heat diffusion in a thin rod in which the law of variation $k(t)$ of the total quantity of heat in the bar is given. [6]

2. Solution of (1)-(4) Model

As known, in Fourier Method, the solution of problem (1)–(4) is considered in the following form :

$$\begin{aligned} u(x,y,t) &= \frac{u_0(t)}{4} \\ &+ \sum_{m,n=1}^{\infty}\left(u_{cmn}(t)\cos(2mx)\ \cos(2ny)+u_{csmn}(t)\cos(2mx)\ \sin(2ny)\right) \\ &+ \sum_{m,n=1}^{\infty}\left(u_{scmn}(t)\sin(2mx)\cos(2ny)+u_{smn}(t)\sin(2mx)\sin(2ny)\right). \end{aligned}$$

We have Fourier coefficients by applying the standart procedure of the Fourier method, as follows:

$$\begin{aligned} u_0(t) &= u_0(0)+\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi f(x,y,\tau,u)\,dxdyd\tau \\ u_{cmn}(t) &= u_{cmn}(0)e^{-\int_0^t\left[b(s)(2m)^2+(2n)^2\right]ds}+\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds} f(x,y,\tau,u)\cos(2mx)\ \cos(2ny)\,dxdyd\tau \\ u_{csmn}(t) &= u_{csmn}(0)e^{-\int_0^t\left[b(s)(2m)^2+(2n)^2\right]ds}+\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds} f(x,y,\tau,u)\cos(2mx)\ \sin(2ny)\,dxdyd\tau \\ u_{scmn}(t) &= u_{scmn}(0)e^{-\int_0^t\left[b(s)(2m)^2+(2n)^2\right]ds}+\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds} f(x,y,\tau,u)\sin(2mx)\ \cos(2ny)\,dxdyd\tau \\ u_{smn}(t) &= u_{smn}(0)e^{-\int_0^t\left[b(s)(2m)^2+(2n)^2\right]ds}+\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds} f(x,y,\tau,u)\sin(2mx)\ \sin(2ny)\,dxdyd\tau \end{aligned}$$

Then we obtain the solution:

$$
\begin{aligned}
u(x,y,t) &= \frac{1}{4}\left(\varphi_0 + \frac{4}{\pi^2}\int_0^t f_0(\tau,u)\,d\tau\right)\\
&+ \sum_{m,n=1}^{\infty}\left(\varphi_{cmn} + \frac{4}{\pi^2}\int_0^t e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds} f_{cmn}(\tau,u)\,d\tau\right)\cos(2mx)\cos(2ny)\\
&+ \sum_{m,n=1}^{\infty}\left(\varphi_{csmn} + \frac{4}{\pi^2}\int_0^t e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds} f_{csmn}(\tau,u)d\tau\right)\cos(2mx)\ \sin(2ny) \qquad (6)\\
&+ \sum_{m,n=1}^{\infty}\left(\varphi_{scmn} + \frac{4}{\pi^2}\int_0^t e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds} f_{scmn}(\tau,u)d\tau\right)\sin(2mx)\cos(2ny)\\
&+ \sum_{m,n=1}^{\infty}\left(\varphi_{smn} + \frac{4}{\pi^2}\int_0^t e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds} f_{smn}(\tau,u)d\tau\right)\sin(2mx)\sin(2ny)
\end{aligned}
$$

where $\varphi_0 = u_0(0), \varphi_{cmn} = u_{cmn}(0)e^{-\int_0^t\left[b(s)(2m)^2+(2n)^2\right]ds}, \varphi_{csmn} = u_{csmn}(0)e^{-\int_0^t\left[b(s)(2m)^2+(2n)^2\right]ds},$
$\varphi_{scmn} = u_{scmn}(0)e^{-\int_0^t\left[b(s)(2m)^2+(2n)^2\right]ds}, \varphi_{smn} = u_{smn}(0)e^{-\int_0^t\left[b(s)(2m)^2+(2n)^2\right]ds}.$

We have the following constraints for functions of the problem:

(C1) $k(t) \in C^1[0,T]$

(C2) $\varphi(x,y)\epsilon C^{1,1}([0,\pi]\times[0,\pi]), \varphi(0,y) = \varphi(\pi,y), \varphi_x(0,y) = \varphi_x(\pi,y), \varphi(x,0) = \varphi(x,\pi), \varphi_y(x,0) = \varphi_y(x,\pi)$ and

$\int_0^\pi\int_0^\pi xy\varphi(x,y)dxdy = k(0),$

(C3) $f(x,y,t,u)$ is provided following conditions:

(1)$\left|\frac{\partial f(x,y,t,u)}{\partial x} - \frac{\partial f(x,y,t,\widetilde{u})}{\partial x}\right| \le l(x,y,t)\left|u-\widetilde{u}\right|$,

$\left|\frac{\partial f(x,y,t,u)}{\partial y} - \frac{\partial f(x,y,t,\widetilde{u})}{\partial y}\right| \le l(x,y,t)\left|u-\widetilde{u}\right|$,

$\left|\frac{\partial^2 f(x,y,t,u)}{\partial x\partial y} - \frac{\partial^2 f(x,y,t,\widetilde{u})}{\partial x\partial y}\right| \le l(x,y,t)\left|u-\widetilde{u}\right|$ where $l(x,y,t) \in L_2(\Gamma), l(x,y,t) \ge 0,$

(2) $f(x,y,t,u) \in C^{2,2,0}[0,\pi], t \in [0,T],$

(3) $f(x,y,t,u)\big|_{x=0} = f(x,y,t,u)\big|_{x=\pi}, f_x(x,y,t,u)\big|_{x=0} = f_x(x,y,t,u)\big|_{x=\pi}, f_y(x,y,t,u)\big|_{y=0} = f_y(x,y,t,u)\big|_{y=\pi},$
$f_{xy}(x,y,t,u)\big|_{x=0} = f_{xy}(x,y,t,u)\big|_{x=\pi}, f_{xy}(x,y,t,u)\big|_{y=0} = f_{xy}(x,y,t,u)\big|_{y=\pi}$

(5) can be diffrentiated under the assumptions (C1)-(C3),

$$\int_0^\pi\int_0^\pi xyu_t(x,t)dxdy = k'(t),\ 0 \le t \le T. \qquad (7)$$

then the unknown coefficient is obtained in this form

$$b(t) = \frac{k'(t) - \int_0^\pi\int_0^\pi xyf(x,y,t,u)dxdy - \frac{\pi^3}{2}u_y(\pi,t)}{\frac{\pi^3}{2}u_x(\pi,t)}. \qquad (8)$$

Definition 2.1. *Show the set* $\{u(t)\} = \{u_0(t), u_{cmn}(t), u_{csmn}(t), u_{scmn}(t), u_{smn}(t), m,n = 1,...\}$ *of continuous functions on* $[0,T]$ *which satisfy the condition*

$\max\limits_{0\le t\le T}\frac{|u_0(t)|}{4}+\sum\limits_{m,n=1}^{\infty}\left(\max\limits_{0\le t\le T}|u_{cmn}(t)|+\max\limits_{0\le t\le T}|u_{csmn}(t)|+\max\limits_{0\le t\le T}|u_{scmn}(t)|+\max\limits_{0\le t\le T}|u_{smn}(t)|\right)<\infty$.

$\|u(t)\|=\max\limits_{0\le t\le T}\frac{|u_0(t)|}{4}+\sum\limits_{m,n=1}^{\infty}\left(\max\limits_{0\le t\le T}|u_{cmn}(t)|+\max\limits_{0\le t\le T}|u_{csmn}(t)|+\max\limits_{0\le t\le T}|u_{scmn}(t)|+\max\limits_{0\le t\le T}|u_{smn}(t)|\right)$ is the norm in B. (B is the Banach spaces).

Theorem 2.2. *If the conditions (C1)-(C3) be implemented. Then it has a unique solution.*

Proof. If we apply an iteration to equation (6), the following functions are obtained:

$$\begin{aligned}
u_0^{(N+1)}(t) &= \varphi_0+\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi f\left(x,y,\tau,u^{(N)}\right)dxdyd\tau,\\
u_{cmn}^{(N+1)}(t) &= \varphi_{cmn}+\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds}\cos(2mx)\cos(2ny)\ f\left(x,y,\tau,u^{(N)}\right)dxdyd\tau,\\
u_{csmn}^{(N+1)}(t) &= \varphi_{csmn}+\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds}\cos(2mx)\sin(2ny)\,f\left(x,y,\tau,u^{(N)}\right)dxdyd\tau,\\
u_{scmn}^{(N+1)}(t) &= \varphi_{scmn}+\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds}\sin(2mx)\cos(2ny)\ f\left(x,y,\tau,u^{(N)}\right)dxdyd\tau,\\
u_{smn}^{(N+1)}(t) &= \varphi_{smn}+\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi e^{-\int_\tau^t\left[b(s)(2m)^2+(2n)^2\right]ds}\sin(2mx)\sin(2ny)\ f\left(x,y,\tau,u^{(N)}\right)dxdyd\tau.
\end{aligned}$$

According to the assumptions , we get $u^{(0)}(t)\in\mathbf{B}$, $t\in[0,T]$. Using Cauchy ,Hölder, Bessel inequalities and Lipschitzs condition, finally we get:

$$\begin{aligned}
\left\|u^{(1)}(t)\right\|_{\mathbf{B}} &= \max_{0\le t\le T}\frac{\left|u_0^{(1)}(t)\right|}{4}+\sum_{m,n=1}^{\infty}\left(\max_{0\le t\le T}\left|u_{cmn}^{(1)}(t)\right|+\max_{0\le t\le T}\left|u_{csmn}^{(1)}(t)\right|+\max_{0\le t\le T}\left|u_{scmn}^{(1)}(t)\right|+\max_{0\le t\le T}\left|u_{smn}^{(1)}(t)\right|\right)\\
&\le \frac{|\varphi_0|}{2}+\sum_{m,n=1}^{\infty}\left(|\varphi_{cmn}|+|\varphi_{csmn}|+|\varphi_{scmn}|+|\varphi_{smn}|\right)\\
&\quad+\sqrt{T}\left(\frac{3\sqrt{\pi}+16}{3\pi}\right)\|l(x,y,t)\|_{L_2(\Gamma)}\left\|u^{(0)}(t)\right\|_B\\
&\quad+\sqrt{T}\left(\frac{3\sqrt{\pi}+16}{3\pi}\right)M.
\end{aligned}$$

According to the assumptions of the theorem, we have $u^{(1)}(t)\in\mathbf{B}$. The same operations for the step N,

$$\begin{aligned}
\left\|u^{(N+1)}(t)\right\|_B &= \max_{0\le t\le T}\frac{\left|u_0^{(N)}(t)\right|}{4}+\sum_{m,n=1}^{\infty}\left(\max_{0\le t\le T}\left|u_{cmn}^{(N)}(t)\right|+\max_{0\le t\le T}\left|u_{csmn}^{(N)}(t)\right|+\max_{0\le t\le T}\left|u_{scmn}^{(N)}(t)\right|+\max_{0\le t\le T}\left|u_{smn}^{(N)}(t)\right|\right)\\
&\le \frac{\left|\varphi_0\right|}{2}+\sum_{m,n=1}^{\infty}\left(\left|\varphi_{cmn}\right|+\left|\varphi_{csmn}\right|+\left|\varphi_{scmn}\right|+\left|\varphi_{smn}\right|\right)\\
&\quad+\sqrt{T}\left(\frac{3\sqrt{\pi}+16}{3\pi}\right)\left\|l(x,y,t)\right\|_{L_2(\Gamma)}\left\|u^{(N)}(t)\right\|_B\\
&\quad+\sqrt{T}\left(\frac{3\sqrt{\pi}+16}{3\pi}\right)M.
\end{aligned}$$

is obtained. We get $u^{(N+1)}(t)\in\mathbf{B}$ since $u^{(N)}(t)\in\mathbf{B}$,

$$\{u(t)\}=\{u_0(t),u_{cmn}(t),u_{csmn}(t),u_{scmn}(t),u_{smn}(t),m,n=1,...\}\in\mathbf{B}.$$

If we apply an iteration to equation (8), the following functions are obtained::

$$b^{(N+1)}(t)=\frac{k'(t)-\int_0^{\pi}\int_0^{\pi}xyf(x,y,t,u^{(N)})dxdy-\frac{\pi^3}{2}u_y^{(N)}(\pi,t)}{\frac{\pi^3}{2}u_x^{(N)}(\pi,t)}.$$

By using the same operations we obtain:

$$\left\|b^{(N+1)}(t)\right\|_{C[0,T]}\le\frac{\left|k'(t)\right|+\frac{\pi^4}{4}\left\|u^{(N)}(t)\right\|_B}{\frac{\pi^3}{2}\left\|u^{(N)}(t)\right\|_B}$$

$$\left\|b^{(N+1)}(t)\right\|_{C[0,T]}\le\frac{\pi}{2}+\frac{2\left|k'(t)\right|}{\left\|u^{(N)}(t)\right\|_B}$$

We get $b^{(N+1)}(t)\in C[0,T]$ since $u^{(N)}(t)\in B$.

Let us show that, $u^{(N+1)}(t)$, $b^{(N+1)}$ are converged for $N\rightarrow\infty$.

$$\begin{aligned}
u^{(1)}(t)-u^{(0)}(t) &= \frac{(u_0^{(1)}(t)-u_0^{(0)}(t))}{4} \\
&+[(u_{cmn}^{(1)}(t)-u_{cmn}^{(0)}(t))+(u_{csmn}^{(1)}(t)-u_{csmn}^{(0)}(t))+(u_{scmn}^{(1)}(t)-u_{scmn}^{(0)}(t))+(u_{smn}^{(1)}(t)-u_{smn}^{(0)}(t))] \\
&= \frac{1}{4}\left(\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi\left[f_{\alpha\beta}(x,y,\tau,u^{(0)})-f_{\alpha\beta}(x,y,\tau,0)\right]dxdyd\tau\right) \\
&+\sum_{m,n=1}^{\infty}\frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi\left[f_{xy}(x,y,\tau,u^{(0)})-f_{xy}(x,y,\tau,0)\right]e^{-\int_\tau^t[b(s)(2m)^2+(2n)^2]ds}\cos(2mx)\cos(2ny)\,dxdyd\tau \\
&+\sum_{m,n=1}^{\infty}\frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi e^{-\int_\tau^t[b(s)(2m)^2+(2n)^2]ds}\cos(2mx)\sin(2ny)\,dxdyd\tau \\
&+\sum_{m,n=1}^{\infty}\frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi\left[f_{xy}(x,y,\tau,u^{(0)})-f_{xy}(x,y,\tau,0)\right]e^{-\int_\tau^t[b(s)(2m)^2+(2n)^2]ds}\sin(2mx)\cos(2ny)\,dxdyd\tau \\
&+\sum_{m,n=1}^{\infty}\frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi\left[f_{xy}(x,y,\tau,u^{(0)})-f_{xy}(x,y,\tau,0)\right]e^{-\int_\tau^t[b(s)(2m)^2+(2n)^2]ds}\sin(2mx)\sin(2ny)\,dxdyd\tau \\
&+\frac{1}{4}\left(\frac{4}{\pi^2}\int_0^t\int_0^\pi\int_0^\pi f_{xy}(x,y,\tau,0)\,dxdyd\tau\right) \\
&+\sum_{m,n=1}^{\infty}\frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi f_{xy}(x,y,\tau,0)e^{-\int_\tau^t[b(s)(2m)^2+(2n)^2]ds}\cos(2mx)\cos(2ny)\ dxdyd\tau \\
&+\sum_{m,n=1}^{\infty}\frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi f_{xy}(x,y,\tau,0)e^{-\int_\tau^t[b(s)(2m)^2+(2n)^2]ds}\cos(2mx)\sin(2ny)\ dxdyd\tau \\
&+\sum_{m,n=1}^{\infty}\frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi f_{xy}(x,y,\tau,0)e^{-\int_\tau^t[b(s)(2m)^2+(2n)^2]ds}\sin(2mx)\cos(2ny)\ dxdyd\tau \\
&+\sum_{m,n=1}^{\infty}\frac{4}{\pi^2 mn}\int_0^t\int_0^\pi\int_0^\pi f_{xy}(x,y,\tau,0)e^{-\int_\tau^t[b(s)(2m)^2+(2n)^2]ds}\sin(2mx)\sin(2ny)\,dxdyd\tau.
\end{aligned}$$

Let some inequalities(Bessel, Hölder, Lipschitzs) be implemented , the following estimations are obtained:

$$\left\|u^{(1)}(t)-u^{(0)}(t)\right\|_B \leq \sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})\left(\left\|l(x,y,t)\right\|_{L_2(\Gamma)}\left\|u^{(0)}(t)\right\|_B+M\right)$$

where

$$A=\sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})\left(\left\|l(x,y,t)\right\|_{L_2(\Gamma)}\left\|u^{(0)}(t)\right\|_B+M\right).$$

$$\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\|_B\leq\frac{A\left\|l(x,y,t)\right\|_{L_2(\Gamma)}^N S^N}{\sqrt{N!}} \tag{9}$$

where

$$S=\sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})\left(1+\frac{\pi M}{2\left\|u^{(N)}(t)\right\|_B\left\|u^{(N+1)}(t)\right\|_B}+\frac{\pi\left\|l(x,y,t)\right\|_{L_2(\Gamma)}}{2\left\|u^{(N)}(t)\right\|_B}\right).$$

By using the same operations we obtain:

$$\left\|b^{(1)}(t)-b^{(0)}(t)\right\|_{C[0,T]}\leq C_1\left\|u^{(1)}(t)-u^{(0)}(t)\right\|_B .$$

The same operations for the step N :

$$\left\|b^{(N+1)}(t)-b^{(N)}(t)\right\|_{C[0,T]}\leq C_N\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\|_B$$

where $C_1=\left(\frac{\pi M}{2\left\|u^{(0)}(t)\right\|_B\left\|u^{(1)}(t)\right\|_B}+\frac{\pi\left\|l(x,y,t)\right\|_{L_2(\Gamma)}}{2\left\|u^{(0)}(t)\right\|_B}\right),...,C_N=\left(\frac{\pi M}{2\left\|u^{(N)}(t)\right\|_B\left\|u^{(N+1)}(t)\right\|_B}+\frac{\pi\left\|l(x,y,t)\right\|_{L_2(\Gamma)}}{2\left\|u^{(N)}(t)\right\|_B}\right)$.The series which is consisting of the right hand side of (9) are convergent by ratio test. So, the series which is consisting of the left hand side of (9) are convergent by comparison test. Moreover, by the Weierstrass M test , the series $\sum_{N=0}^{\infty}\left|u^{(N+1)}(t)-u^{(N)}(t)\right|$ is uniformly convergent.

We obtain $u^{(N+1)}\to u^{(N)}, b^{(N+1)}\to b^{(N)}, N\to\infty$.

Therefore $u^{(N+1)}(t)$ and $b^{(N+1)}(t)$ are converged.

Now let's show that:

$$\lim_{N\to\infty}u^{(N+1)}(t)=u(t),\ \lim_{N\to\infty}b^{(N+1)}(t)=b(t).$$

By using Cauchy, Hölder, Bessel and Lipschitzs inequalities, we have

$$\begin{aligned}\left\|u(t)-u^{(N+1)}(t)\right\|_{\mathbf{B}} &\leq \sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})\left\|l(x,y,t)\right\|_{L_2(\Gamma)}\left\|u(t)-u^{(N+1)}(t)\right\|_B\\ &+\sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})\left\|l(x,y,t)\right\|_{L_2(\Gamma)}\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\|_B\\ &+\sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})M\left|T\right|\left\|u(t)-u^{(N+1)}(t)\right\|_B .\end{aligned}$$

By using the same operations we obtain:

$$\begin{aligned}\left\|b(t)-b^{(N+1)}(t)\right\|_{C[0,T]} &\leq C_N\left\|l(x,y,t)\right\|_{L_2(\Gamma)}\left\|u(t)-u^{(N+1)}(t)\right\|_B\\ &+C_N\left\|l(x,y,t)\right\|_{L_2(\Gamma)}\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\|_{B.}\end{aligned}$$

$$\begin{aligned}\left\|u(t)-u^{(N+1)}(t)\right\|_{\mathbf{B}} &\leq \sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})\left\|l(x,y,t)\right\|_{L_2(\Gamma)}\left\|u(t)-u^{(N+1)}(t)\right\|_B\\ &+\sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})\left\|l(x,y,t)\right\|_{L_2(\Gamma)}\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\|_B\\ &+\sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})M\left|T\right|\left\|b(t)-b^{(N+1)}(t)\right\|_B\end{aligned}$$

applying Gronwall's inequality to last inequality ,we have

$$\begin{aligned}\left\|u(t)-u^{(N+1)}(t)\right\|_{\mathbf{B}}^2 &\leq 2\frac{\left(A\sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})\right)^2}{\sqrt{N!}}\left(\left\|l(x,y,t)\right\|_{L_2(\Gamma)}^{N+1}\right)^2\\ &\times\exp\left(\sqrt{T}(\frac{3\sqrt{\pi}+16}{3\pi})\right)^2\left\|l(x,y,t)\right\|_{L_2(\Gamma)}^2 .\end{aligned} \tag{10}$$

The series which is consisting of the right hand side of (10) are convergent by ratio test. So, the series which is consisting of the left hand side of (10) are convergent by comparison test. Moreover, by the Weierstrass M test , the series $\sum_{N=0}^{\infty}\left|u(t)-u^{(N+1)}(t)\right|$ is uniformly convergent.

We obtain $u^{(N+1)}\to u, b^{(N+1)}\to b, N\to\infty$.

To show the uniqueness, we get two solution pairs of the problem (1)–(5) as (c,u) and (b,v)

Applying Cauchy inequality, Hölder Inequality, Lipschitzs condition and Bessel inequality to the difference $|u(t)-v(t)|$, we obtain

$$\begin{aligned}\|u(t)-v(t)\|_{\mathbf{B}} \quad &\leq \quad \sqrt{T}\left(\frac{3\sqrt{\pi}+16}{3\pi}\right)\left\|l(x,y,t)\right\|_{L_2(\Gamma)}\|u(t)-v(t)\|_B\\ &+\sqrt{T}\left(\frac{3\sqrt{\pi}+16}{3\pi}\right)M\,|T|\,\|b(t)-c(t)\|_B\end{aligned}$$

$$\|u(t)-v(t)\|_B\leq 0\times\exp\left(\sqrt{T}\left(\frac{3\sqrt{\pi}+16}{3\pi}\right)\right)^2\left\|l(x,y,t))\right\|^2_{L_2(\Gamma)}, \tag{11}$$

we get $u(t)=v(t)$ and $c(t)=b(t)$.

The proof is over. □

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The Period and Rank of the Complex-type Padovan-p Numbers Modulo m

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Abstract. In this paper, we study the complex-type Padovan-p sequence modulo m and then we give some results concerning the periods and ranks of this sequence for any p and m. Furthermore, we produce the cyclic groups using the multiplicative orders of the generating matrix of the complex-type Padovan-p sequence when read modulo m. Finally, we give the relationships between the periods of the complex-type Padovan-p sequence modulo m and the orders of the cyclic groups produced.

1. Introduction

It is well-known that the Padovan sequence $\{P(n)\}$ is defined recursively by the equation:

$$P(n) = P(n-2) + P(n-3)$$

for $n \geq 3$, where $P(0) = P(1) = P(2) = 1$.

The Padovan p-sequence $\{Pap(n)\}$ is defined [6] by initial values $Pap(1) = Pap(2) = \cdots = Pap(p) = 0$, $Pap(p+1) = 1$, $Pap(p+2) = 0$ and the following homogeneous linear recurrence relation

$$Pap(n+p+2) = Pap(n+p) + Pap(n)$$

for any given $p\,(p = 2, 3, 4, \ldots)$ and $n \geq 1$. Note that the $(2n+1)\,th$ term of the Padovan 2-sequence $\{Pa2(n)\}$, is equal to nth Fibonacci number.

The complex-type Padovan p-sequence $\left\{Pa_p^{(i)}(n)\right\}$ is defined [11] as follows:

$$Pa_p^{(i)}(n+p+2) = i^2 \cdot Pa_p^{(i)}(n+p) + i^{p+2} \cdot Pa_p^{(i)}(n) \tag{1}$$

for any given $p\,(p = 3, 5, 7, \ldots)$ and $n \geq 1$, where $Pa_p^{(i)}(1) = \cdots = Pa_p^{(i)}(p) = 0$, $Pa_p^{(i)}(p+1) = 1$, $Pa_p^{(i)}(p+2) = 0$ and $\sqrt{-1} = i$.

A sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the shortest repeating subsequence is called the period of the sequence. For example,

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the sequence $a, b, c, d, b, c, d, b, c, d, \ldots$ is periodic after the initial element a and has period 3. A sequence is simply periodic with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, d, a, b, c, d, a, b, c, d, \ldots$ is simply periodic with period 4.

The study of the behavior of the linear recurrence sequences under a modulus began with the earlier work of Wall [17] where the periods of the ordinary Fibonacci sequences modulo m were investigated. Recently, the theory extended to some special linear recurrence sequences by several authors; see, for example, [3, 4, 12, 15, 16]. In the first part of this paper, we consider the complex-type Padovan-p sequence modulo m and then we derive some interesting results concerning the periods and ranks of the complex-type Padovan-p sequence for any p and m.

The relationships between the periods of the linear recurrence sequences modulo m and the cyclic groups which are produced using the multiplicative orders of the generating matrices of these sequences when read modulo m have been studied recently by many authors; see, for example, [1, 2, 5, 7–10, 13, 14, 18]. In the second part, we derive the cyclic groups using the multiplicative orders of the generating matrix of the complex-type Padovan-p numbers when read modulo m. Then, we give the relationships between the periods of the complex-type Padovan-p sequence modulo m and the orders of the cyclic groups produced.

2. The Main Results

If we reduce the complex-type Padovan-p sequence $\left\{Pa_p^{(i)}(n)\right\}$ by a modulus m, taking least nonnegative residues, then we get the following recurrence sequence:

$$\left\{Pa_p^{(i,m)}(n)\right\} = \left\{Pa_p^{(i,m)}(0),\ Pa_p^{(i,m)}(1),\ \ldots, Pa_p^{(i,m)}(j),\ \ldots\right\}$$

where $Pa_p^{(i,m)}(j)$ is used to mean the jth element of the complex-type Padovan-p sequence when read modulo m. We note here that the recurrence relations in the sequences $\left\{Pa_p^{(i,m)}(n)\right\}$ and $\left\{Pa_p^{(i)}(n)\right\}$ are the same.

Theorem 2.1. *For any given p $(p = 3, 5, 7, \ldots)$, the sequence $\left\{Pa_p^{(i,m)}(n)\right\}$ is simply periodic.*

Proof. Consider the set

$$\begin{aligned} C \quad = \quad & \left\{\left(c_1, c_2, \ldots, c_{p+2}\right) \mid c_n\text{'s are complex numbers } a_n + ib_n \text{ where} \right. && (2)\\ & \left. a_n \text{ and } b_n \text{ are integers such that } 0 \le a_n, b_n \le m-1 \text{ and } 1 \le n \le p+2\right\}. && (3) \end{aligned}$$

Let the notation $|C|$ indicate the cardinality of the set C. Since the set C is finite, there are $|C|$ distinct $(p+2)$-tuples of the complex-type Padovan-p numbers modulo m. Thus, it is clear that at least one of these $(p+2)$-tuples appears twice in the sequence $\left\{Pa_p^{(i,m)}(n)\right\}$. Therefore, the subsequence following this $(p+2)$-tuple repeats; that is, $\left\{Pa_p^{(i,m)}(n)\right\}$ is a periodic sequence. Let us consider $Pa_p^{(i,m)}(u) \equiv Pa_p^{(i,m)}(v)$, $Pa_p^{(i,m)}(u+1) \equiv Pa_p^{(i,m)}(v+1), \ldots, Pa_p^{(i,m)}(u+p+2) \equiv Pa_p^{(i,m)}(v+p+2)$ and $v \ge u$. Then we have $v \equiv u\,(mod(p+2))$. From the recurrence relation in (1), we can write the following recursive equations:

$$Pa_p^{(i)}(u) = i^{2-p} \cdot Pa_p^{(i)}(u+p+2) + i^{3-p} \cdot Pa_p^{(i)}(u+p)$$

and

$$Pa_p^{(i)}(v) = i^{2-p} \cdot Pa_p^{(i)}(v+p+2) + i^{3-p} \cdot Pa_p^{(i)}(v+p).$$

So we get $Pa_p^{(i,m)}(u-1) \equiv Pa_p^{(i,m)}(v-1)$, $Pa_p^{(i,m)}(u-2) \equiv Pa_p^{(i,m)}(v-2)$, $\ldots$, $Pa_p^{(i,m)}(2) \equiv Pa_p^{(i,m)}(v-u+2)$, $Pa_p^{(i,m)}(1) \equiv Pa_p^{(i,m)}(v-u+1)$, which implies that the complex-type Padovan-p sequence modulo m is simply periodic. □

Let the notation $lP_p^i(m)$ denote the smallest period of the sequence $\left\{Pa_p^{(i,m)}(n)\right\}$.

Given an integer matrix $A = \left[a_{ij}\right]$, $A\,(mod m)$ means that all entries of A are modulo m, that is, $A\,(mod m) = \left(a_{ij}\,(mod m)\right)$. Let us consider the set $\langle A\rangle_m = \{(A)^n\,(mod m) \mid n \geq 0\}$. If $(\det A, m) = 1$, then the set $\langle A\rangle_m$ is a cyclic group; if $(\det A, m) \neq 1$, then the set $\langle A\rangle_m$ is a semigroup.

In [11], the generating matrix of the complex-type Padovan-p sequence had been given as:

$$D_p = \left[d_{jk}^{(p)}\right]_{(p+2)\times(p+2)} = \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 & 0 & i^{p+2} \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

The matrix D_p is said to be the complex-type Padovan-p matrix. Then they had been written the following matrix relation:

$$\begin{bmatrix} Pa_p^{(i)}(n+p+2) \\ Pa_p^{(i)}(n+p+1) \\ \vdots \\ Pa_p^{(i)}(n+2) \\ Pa_p^{(i)}(n+1) \end{bmatrix} = D_p \cdot \begin{bmatrix} Pa_p^{(i)}(n+p+1) \\ Pa_p^{(i)}(n+p) \\ \vdots \\ Pa_p^{(i)}(n+1) \\ Pa_p^{(i)}(n) \end{bmatrix}.$$

It can be readily established by mathematical induction that for $n \geq p+1$,

$$\left(D_p\right)^n = \begin{bmatrix} Pa_p^{(i)}(n+p+1) & Pa_p^{(i)}(n+p+2) & i^{p+2}\cdot Pa_p^{(i)}(n+1) & i^{p+2}\cdot Pa_p^{(i)}(n+2) & \cdots & i^{p+2}\cdot Pa_p^{(i)}(n+p) \\ Pa_p^{(i)}(n+p) & Pa_p^{(i)}(n+p+1) & i^{p+2}\cdot Pa_p^{(i)}(n) & i^{p+2}\cdot Pa_p^{(i)}(n+1) & \cdots & i^{p+2}\cdot Pa_p^{(i)}(n+p-1) \\ Pa_p^{(i)}(n+p-1) & Pa_p^{(i)}(n+p) & i^{p+2}\cdot Pa_p^{(i)}(n-1) & i^{p+2}\cdot Pa_p^{(i)}(n) & \cdots & i^{p+2}\cdot Pa_p^{(i)}(n+p-2) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ Pa_p^{(i)}(n+1) & Pa_p^{(i)}(n+2) & i^{p+2}\cdot Pa_p^{(i)}(n-p+1) & i^{p+2}\cdot Pa_p^{(i)}(n-p+2) & \cdots & i^{p+2}\cdot Pa_p^{(i)}(n) \\ Pa_p^{(i)}(n) & Pa_p^{(i)}(n+1) & i^{p+2}\cdot Pa_p^{(i)}(n-p) & i^{p+2}\cdot Pa_p^{(i)}(n-p+1) & \cdots & i^{p+2}\cdot Pa_p^{(i)}(n-1) \end{bmatrix}. \tag{4}$$

Since $\det D_p = i^{p+2}$, the set $\left\langle D_p\right\rangle_m$ is a cyclic group for every positive integer $m \geq 2$. From Theorem 2.1 and the equation (**??**), it is easy to see that $lP_p^i(m) = \left|\left\langle D_p\right\rangle_m\right|$ for any given $p\,(p = 3, 5, 7, \ldots)$.

Clearly,

$$i^{p+2} = \begin{cases} i, & p \equiv -1\,(mod 4), \\ -i, & p \equiv 1\,(mod 4). \end{cases}$$

Since also $\det D_p = i^{p+2}$ and $lP_p^i(m) = \left|\left\langle D_p\right\rangle_m\right|$,

$$\left(i^{p+2}\right)^{lP_p^i(m)} = \left(\det D_p\right)^{lP_p^i(m)} = \det D_p^{lP_p^i(m)} \equiv 1\,(mod m).$$

From this we see that $4 \left| lP_p^i(m)\right.$.

The rank of the sequence $\left\{Pa_p^{(i,m)}(n)\right\}$ is the least positive integer r such that $Pa_p^{(i,m)}(r+1) \equiv Pa_p^{(i,m)}(r+2) \equiv Pa_p^{(i,m)}(r+p) \equiv 0(mod m)$, $Pa_p^{(i,m)}(r+p+1) \equiv u(mod m)$ $(u \in \mathbb{C})$, $Pa_p^{(i,m)}(r+p+2) \equiv 0(mod m)$, and we denote the rank of $\left\{Pa_p^{(i,m)}(n)\right\}$ by $RP_p^i(m)$. If $Pa_p^{(i,m)}(r+p+1) \equiv u(mod m)$ $(u \in \mathbb{C})$, then the terms of the sequence

$\left\{Pa_p^{(i,m)}(n)\right\}$ starting with index $RP_p^i(m)$, namely $\underbrace{0,0,\ldots,0}_{p},u,0,-u,0,u,\ldots$, are exactly the initial terms of $\left\{Pa_p^{(i,m)}(n)\right\}$ multiplied by a factor u.

Let the notation I denote the identity matrix of size $(p+2)$. The exponents n for which $\left(D_p\right)^n \equiv I(mod m)$ form a simple aritmetic progression. Then we have

$$\left(D_p\right)^n \equiv I(mod m) \Longleftrightarrow lP_p^i(m) \mid n.$$

Similarly, the exponents n for which $\left(D_p\right)^n \equiv cI(mod m)$ for some $c \in \mathbb{C}$ form a simple aritmetic progression, and hence

$$\left(D_p\right)^n \equiv cI(mod m) \Longleftrightarrow RP_p^i(m) \mid n.$$

Consequently, we can see that $RP_p^i(m)$ divides $lP_p^i(m)$ for any given $p\,(p=3,5,7,\ldots)$ and $m \geq 3$.

The order of the sequence $\left\{Pa_p^{(i,m)}(n)\right\}$, $(m \geq 3)$ is defined by $\frac{lP_p^i(m)}{RP_p^i(m)}$ and we denote it by $OP_p^i(m)$. Let $\left(D_p\right)^{RP_p^i(m)} \equiv cI(mod m)$ $(c \in \mathbb{C})$, then $ord_m(c)$ is the least positive value of λ such that $\left(D_p\right)^{\lambda RP_p^i(m)} \equiv I(mod m)$. So it is confirm that $ord_m(c)$ is the least positive integer λ with $lP_p^i(m) \mid \lambda RP_p^i(m)$ for $m \geq 3$. As a direct consequence of this we see that the smallest such λ is $OP_p^i(m)$ for $m \geq 3$. Therefore, we obtain $OP_p^i(m) = ord_m(c)$, $(m \geq 3)$ when $\left(D_p\right)^{RP_p^i(m)} \equiv cI(mod m)$. As a result, we may easily deduce that $OP_p^i(m)$ is always a positive integer, and that $OP_p^i(m) = ord_m\left(Pa_p^{(i)}\left(RP_p^i(m)+p+1\right)\right)$ for $m \geq 3$, the multiplicative order of $Pa_p^{(i,m)}\left(RP_p^i(m)+p+1\right)$.

Example 2.2. *The sequence* $\left\{Pa_3^{(i,2)}(n)\right\}$ *is as follows:*

$$\left\{\begin{array}{c} 0,0,0,1,0,1,0,1,i,1,0,1,i,,0,0,0,i,1,i,1,i, \\ 0,0,1,i,0,i,0,0,1,0,0,0,0,i,0,i,0,i,1,i,0, \\ i,1,0,0,0,1,i,1,i,1,0,0,i,1,0,1,0,0,i,0, \\ 0,0,0,1,0,1,0,1,i,\ldots. \end{array}\right\}$$

Thus it is verified that $lP_3^i(2)=62$, $RP_3^i(2)=31$ *and* $OP_3^i(2)=2$.

Example 2.3. *The sequence* $\left\{Pa_3^{(i,4)}(n)\right\}$ *is as follows:*

$$\left\{\begin{array}{c} 0,0,0,1,0,3,0,1,i,3,2i,1,3i,2,0,0,i,1,i,3,3i,0,2i,3,i,2,3i,0,0,3,2i, \\ 2,2i,2,i,0,i,2,i,1,3i,2,3i,1,2i,0,0,1,i,1,3i,3,2i,0,3i,1,0,1,0,0,i,0, \\ 0,0,0,3,0,1,2i,3,3i,1,2i,3,i,2,0,0,3i,3,3i,1,i,0,2i,1,3i,2,i,0,0,1,2i, \\ 2,2i,2,3i,0,3i,2,3i,3,i,2,i,3,2i,0,0,3,3i,3,i,1,2i,0,i,3,0,3,0,0,3i,0, \\ 0,0,0,1,0,3,0,1,i,\ldots. \end{array}\right\}$$

Thus it is verified that $lP_3^i(4)=124$, $RP_3^i(4)=62$ *and* $OP_3^i(4)=2$.

Theorem 2.4. *Let ρ be a prime. Then we have the following results for any given $p\,(p=3,5,7,\ldots)$:*
i. If t is the smallest positive integer such that $lP_p^i\left(\rho^{t+1}\right) \neq lP_p^i\left(\rho^t\right)$, then $lP_p^i\left(\rho^{t+1}\right) = \rho lP_p^i\left(\rho^t\right)$.
ii. If t is the smallest positive integer such that $RP_p^i\left(\rho^{t+1}\right) \neq RP_p^i\left(\rho^t\right)$, then $RP_p^i\left(\rho^{t+1}\right) = \rho RP_p^i\left(\rho^t\right)$.

Proof. i. Let n be a positive integer such that $\left(D_p\right)^{lP_p^i\left(\rho^{n+1}\right)} \equiv I(mod \rho^{n+1})$. Then we can easily derive $\left(D_p\right)^{lP_p^i\left(\rho^{n+1}\right)} \equiv I(mod \rho^{n})$, which implies that $lP_p^i\left(\rho^{n+1}\right)$ is divided by $lP_p^i\left(\rho^n\right)$. On the other hand, we may write $\left(D_p\right)^{lP_p^i(\rho^n)} = I + \left(\left(d_{jk}^{(p)}\right)^n \cdot \rho^n\right)$. Thus, we get the following matrix equation by using binomial expansion

$$\left(D_p\right)^{\rho \cdot lP_p^i(\rho^n)} = \left(I + \left(\left(d_{jk}^{(p)}\right)^n \cdot \rho^n\right)\right)^\rho = \sum_{k=0}^{\rho} \binom{\rho}{k} \left(\left(d_{jk}^{(p)}\right)^n \cdot \rho^n\right)^k \equiv I\left(\bmod \rho^{n+1}\right),$$

which yields that $\rho \cdot lP_p^i(\rho^n)$ is divided by $lP_p^i\left(\rho^{n+1}\right)$. Hence, $lP_p^i\left(\rho^{n+1}\right) = lP_p^i(\rho^n)$ or $lP_p^i\left(\rho^{n+1}\right) = \rho \cdot lP_p^i(\rho^n)$, and the latter holds if and only if there is a $\left(d_{jk}^{(p)}\right)^n$ which is not divisible by ρ. Due to fact that we assume t is the smallest positive integer such that $lP_p^i\left(\rho^{t+1}\right) \neq lP_p^i(\rho^t)$, there is an $\left(d_{jk}^{(p)}\right)^n$ which is not divisible by ρ. This shows that $lP_p^i\left(\rho^{t+1}\right) = \rho lP_p^i(\rho^t)$.

ii. The proof is similar to the above and is omitted. □

Theorem 2.5. *Let m_1 and m_2 be positive integers with $m_1, m_2 \geq 2$, then $RP_p^i(lcm[m_1, m_2]) = lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right]$ and $lP_p^i(lcm[m_1, m_2]) = lcm\left[lP_p^i(m_1), lP_p^i(m_2)\right]$ for any given p $(p = 3, 5, 7, \ldots)$.*

Proof. Let us consider the ranks $RP_p^i(m_1)$ and $RP_p^i(m_2)$. Suppose that $lcm[m_1, m_2] = m$. Then we may write

$$Pa_p^{(i)}\left(RP_p^i(m_1) + 1\right) \equiv Pa_p^{(i)}\left(RP_p^i(m_1) + 2\right) \equiv \cdots \equiv Pa_p^{(i)}\left(RP_p^i(m_1) + p\right) \equiv 0(modm),$$

$$Pa_p^{(i)}\left(RP_p^i(m_1) + p + 1\right) \equiv u_1(modm),\ Pa_p^{(i)}\left(RP_p^i(m_1) + p + 2\right) \equiv 0(modm),$$

$$Pa_p^{(i)}\left(RP_p^i(m_2) + 1\right) \equiv Pa_p^{(i)}\left(RP_p^i(m_2) + 2\right) \equiv \cdots \equiv Pa_p^{(i)}\left(RP_p^i(m_2) + p\right) \equiv 0(modm),$$

$$Pa_p^{(i)}\left(RP_p^i(m_2) + p + 1\right) \equiv u_2(modm),\ Pa_p^{(i)}\left(RP_p^i(m_2) + p + 2\right) \equiv 0(modm)$$

and

$$Pa_p^{(i)}\left(RP_p^i(m) + 1\right) \equiv Pa_p^{(i)}\left(RP_p^i(m) + 2\right) \equiv \cdots \equiv Pa_p^{(i)}\left(RP_p^i(m) + p\right) \equiv 0(modm),$$

$$Pa_p^{(i)}\left(RP_p^i(m) + p + 1\right) \equiv u(modm),\ Pa_p^{(i)}\left(RP_p^i(m) + p + 2\right) \equiv 0(modm)$$

where u_1, u_2 and u are complex numbers. Using the least common multiple operation this implies that

$$Pa_p^{(i)}\left(RP_p^i(m) + 1\right) \equiv Pa_p^{(i)}\left(RP_p^i(m) + 2\right) \equiv \cdots \equiv Pa_p^{(i)}\left(RP_p^i(m) + p\right) \equiv 0(modm_j),$$

$$Pa_p^{(i)}\left(RP_p^i(m) + p + 1\right) \equiv u(modm_j),\ Pa_p^{(i)}\left(RP_p^i(m) + p + 2\right) \equiv 0(modm_j)$$

for $j = 1, 2$. So we get $RP_p^i(m_1) \mid RP_p^i(m)$ and $RP_p^i(m_2) \mid RP_p^i(m)$, which means that $RP_p^i(lcm[m_1, m_2])$ is divided by $lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right]$. We also know that

$$Pa_p^{(i)}\left(lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right] + 1\right) \equiv Pa_p^{(i)}\left(lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right] + 2\right) \equiv \cdots \equiv Pa_p^{(i)}\left(lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right] + p\right) \equiv 0(modm_j),$$

$$Pa_p^{(i)}\left(lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right] + p + 1\right) \equiv u_j(modm_j),\ Pa_p^{(i)}\left(lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right] + p + 2\right) \equiv 0(modm_j)$$

for $j = 1, 2$. Then we can write

$$Pa_p^{(i)}\left(lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right] + 1\right) \equiv Pa_p^{(i)}\left(lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right] + 2\right) \equiv \cdots \equiv Pa_p^{(i)}\left(lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right] + p\right) \equiv 0(modm),$$

$$Pa_p^{(i)}\left(lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right] + p + 1\right) \equiv u(modm),\ Pa_p^{(i)}\left(lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right] + p + 2\right) \equiv 0(modm),$$

which yields that $lcm\left[RP_p^i(m_1), RP_p^i(m_2)\right]$ is divided by $RP_p^i(lcm[m_1, m_2])$. So we have the conclusion.

There is a similar proof for the periods $lP_p^i(m_1)$ and $lP_p^i(m_2)$. □

3. Conclusion

We have examined the complex-type Padovan-p sequence modulo m and then we give some results concerning the periods and ranks of this sequence for any p and m. In addition, we have considered the complex-type Padovan-p matrix and we obtained cyclic groups by taking the multiplicative order of this matrix according to m. Finally, we have reached that the periods of the complex-type Padovan-p sequence according to modulo m are equal to the order the cyclic groups obtained.

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Mathematical Modeling of the Shape of Cavity Created with Laser Using Melting, Boiling Temperature

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Abstract. In this study, a groove was formed on the Ti-6Al-4V plate with Nd:YAG laser. The structure of the resulting grooves was examined. Mathematical modeling of the heat dissipation was made by making use of the melting and boiling temperatures observed on the plate. Later, to prove the validity of the obtained mathematical model, grooves with different geometries were obtained with different laser energies. The results obtained with the proposed mathematical model are quite compatible with the experimental results.

1. Introduction

When it is desired to modify the mechanical properties of materials such as friction and adhesion, one of the most used methods is the surface texturing process. For different materials, different texturing methods can be applied according to the material properties and the intended pattern properties. Surface texturing processes can be divided into three main groups as chemical, mechanical and thermal.

Ti-6Al-4V alloy is widely used in industry and especially in the healthcare industry due to its low density and high toughness. Titanium and titanium alloys are used in the production of many parts, especially in aviation, health and space technology, because they are more durable than steel but much lighter [1]. Ti-6Al-4V is a titanium alloy with high specific strength and excellent corrosion resistance. It is one of the most commonly used titanium alloys and is applied in a wide range of applications where low density and excellent corrosion resistance are necessary such as e.g. biomechanical applications (implants and prostheses) [2]. Additive Manufacturing [3], racing and aerospace industry [4], marine applications and chemical industry [5], etc..

Although Ti-6Al-4V alloy has superior properties and is preferred in many applications, tribological performance is inadequate. The tribologic properties can be improved by surface texturing. The process of creating regular patterns on the material surface by various methods can be called surface texturing. The sizes, shapes and proportions of these patterns on the surface greatly affect the adhesion and friction properties of the surface. Due to the different physical and chemical properties of materials, it can be processed with different methods for different materials. In addition to the many advantages of these methods, they also have some disadvantages such as environmental pollution, increased burrs from the material and wear of the parts.

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Lasers have many advantages in material processing. Lasers are preferred in many areas, especially thanks to their superior qualities such as the absence of wear on parts, precise processing and the preservation of this sensitivity in almost all products. Although many metals can be processed very easily by laser, there are difficulties in laser processing of Al and its alloys due to their high reflectivity.

Many studies have been carried out to control the dimensions and geometries of the patterns created on the material with the laser and to determine the appropriate laser parameters [6, 7]. Many parameters, including the properties of the laser used in laser material processing and the ambient conditions, affect the properties of the patterns obtained on the material. Numerous experimental studies have been carried out to obtain suitable parameters [8–10]. Since there are many parameters affecting the result in laser material processing, classical experimental methods take a lot of time and have high costs. For this, successful results are obtained with mathematical modeling as well as optimization studies [11–13].

In this study, a groove was created on the Ti-6Al-4V plate and mathematical modeling of the heat distribution was made with the measurements taken from the geometries of these grooves and the data obtained. In the mathematical modeling using the Fourier method, Melting, Boiling and melting Temperature were used as boundary conditions.

The effects of the laser beam energy on the groove width of Ti-6Al-4V plate were investigated in the mathematical model. Physical properties of Ti-6Al-4V and laser parameters were used to conduct model.

The heat distribution equation on surface can be written as below;

$$\frac{\partial T(x,t)}{\partial t} = \alpha^2 \frac{\partial^2 T(x,t)}{\partial x^2} \tag{1}$$

where, T is the temperature as a function of time "t" and distance "x", α is the thermal diffusivity of the material that can be obtained as below;

$\alpha^2 = \frac{\lambda}{c\rho}$

where, λ is the thermal conductivity, c specific heat and ρ density of material.

Let $t_p > 0$ be a fixed number and denote by $D = \{(x.t) : 0 < x < l, 0 < t < t_p\}$

where x is the investigated length that varies between zero and l. t_p is the pulse duration that means laser beam start at "0" and laser is beam is cut of at t_p.

Therefore one of the initial condition can be written as;

$T(x,0) = T_0, 0 < x < l$

where T_0 is the initial temperature of the material. It was assumed that all the energy absorbed by the surface was transmitted to the material. Thus, in the absence of heat loss, the boundary condition ($x = 0$) on the surface can be written as follows:

$(\partial T(0,t))/\partial t = 0, (\partial T(l,t))/\partial t = 0 (t > 0)$

This is a parabolic problem. Classical solution of the problem (1)-(3) is $T(x,t) \in C^{2,1}(D) \cap C^{1,0}(D)$. The heat source problem has been investigated with parabolic equation in many studies.

By applying the standard procedure of the Fourier method, we obtain the following representation for the solution of (1)-(3).

$T(x,t) = Z(x)T(t)$

$(X''(x))/(X(x)) = (T'(t))/(\alpha^2 T(t)) = -\lambda^2$

where λ is fix number.

The eigen values are

$\lambda_k = (2\pi k/l)^2, k = 1, ..., \infty$

The eigien functions are

$X_1(x) = cos\frac{2\pi k}{l}x, X_2 = sin\frac{2\pi k}{l}x,$

$X(x) = C_1 cos\frac{2\pi k}{l}x + C_2 sin\frac{2\pi k}{l}x.$

$T(t) = C_3 e^{-(\frac{2\pi\alpha k}{l})^2 t}.$

Then the following solution is obtained using Fourier method.

$T(x,t) = \sum_{k=1}^{\infty} \left(T_{ck}\cos\frac{2\pi\alpha k}{l} + T_{sk}\sin\frac{2\pi\alpha k}{l}\right) e^{-(\frac{2\pi\alpha k}{l})^2 t}.$

The laser intensity within the material can be found using the Beer-Lambert's Law:
$\frac{dI(x)}{dx} = -al$
Where $I(x)$ is the laser intensity as a function of distance from laser spot and a is the absorption coefficient of the material respectively. Although absorption coefficient is changed within the material but it was taken as constant in our study. Laser intensity as a function of distance within material can be written as;
$I = I_0 e^{-\int_b^z a dx}$.
Actually most of the beam intensities have Gaussian distribution. We made one more assumption that our laser beam is top-hat beam that means intensity is homogeneously distributed in spot area.
The heat generation from the laser beam absorbed by the material is defined as,
$S = \frac{-dI}{dx}$
Using Leibniz rule yields, the heat source can be written as;
$S = I_0 e^{-\int_b^z a dx}$
The temperature distribution as a function was obtained as given below;

$$\begin{aligned} T(x,t) &= \sum_{k=1}^{\infty}\left(\varphi_{ck} e^{-(\frac{2\pi\alpha k}{l})^2 t} + \int_0^t \int_0^\pi S(x,t)\cos\frac{2\pi\alpha k}{l} x e^{-(\frac{2\pi\alpha k}{l})^2 (t-\tau)} dx d\tau\right)\cos\frac{2\pi\alpha k}{l} x \\ &+ \sum_{k=1}^{\infty}\left(\varphi_{sk} e^{-(\frac{2\pi\alpha k}{l})^2 t} + \int_0^t \int_0^\pi S(x,t)\sin\frac{2\pi\alpha k}{l} x e^{-(\frac{2\pi\alpha k}{l})^2 (t-\tau)} dx d\tau\right)\sin\frac{2\pi\alpha k}{l} x - -xH/l\lambda. \end{aligned} \tag{2}$$

2. MATERIAL AND EXPERIMENTAL SETUP

The Ti-6Al-4V plates with 2.5 cm x 2.5 cm having an area of 3 mm thick were used for to surface machining process. Some physical and thermal properties of Ti-6Al-4V which were used in mathematical modeling have been listed in Table 1. In the ablation process commercial Nd:YAG laser was used with different energy at constant scan speed. The laser beams were focused 1 mm above the surface, the spot diameters were obtained as 580 μm.

Table 1 Some physical and thermal properties of Ti-6Al-4V

Properties	Value	Unit
Density	4410	kg/m3
Specific Heat Capacity	5263	kJ / kg.K
Melting point	1650	K
Boiling Temperature	3133	K
Thermal Conductivity	6.7	W/mK

3. RESULTS AND DISCUSSION

In this study, mathematical model has been proposed for the groove width on Ti-6Al-4V plate with 3 J of energy and 2 mm/s scan speed. An optical microscope was used to take the images of ablated surfaces of Ti-6Al-4V plate. Groove widths were measured from these images.

The Boiling and molten zone boundary distances were measured as 1310 μm and 1120 μm respectively. Temperatures at Boiling and molten zone boundary are 3133 K and 1650 K respectively. These temperatures are used in obtained mathematical model obtain the Fourier coefficients. These coefficients depend on the material properties. The coefficients in the temperature distribution equation (2) were calculated as φ_{ck} (=701,68) and φ_{sk} (-112.48). Then, in order to verify the validity of mathematical model, new grooves were created obtained using 2.5, 3.5,4, and 4.5 Joules of laser energies. The coefficients obtained with first experiment (conducted with 3 Joule of laser energy) were used to calculate temperature distribution for the Ti-6Al-4V plate and different laser beam energies.

Table 2 Laser Energies and groove widths measured from images.

Laser Energy Joule	Boiled Zone width	Melt Zone width
26	1252	1434
39	1319	1513
52	1367	1568
65	1404	1611
78	1434	1646

The calculated temperatures for boundaries are given in Table 3.

Table 3. Melting and Boiling Temperatures calculated with mathematical model, real values and percent error between them.

Energy Joule		T(x,t) (K)	T(x,t) (K) (calculated)	% error
2.5	Melting	1650	1640	0.61
2.5	Boiling	3133	3121	0.38
3	Melting	1650	(ref)	
3	Boiling	3133	(ref)	
3.5	Melting	1650	1662	0.72
3.5	Boiling	3133	3178	1.44
4	Melting	1650	1674	1.45
4	Boiling	3133	3191	1.85
4.5	Melting	1650	1692	2.55
4.5	Boiling	3133	3209	2.43

4. CONCLUSION

Micro-scale patterns created on metal surfaces change the mechanical properties of the surfaces. In addition to the many advantages of laser surface treatment, it is very difficult to accurately predict the properties of the surface to be obtained due to the complexity of the laser-material interaction. Thanks to the mathematical modeling of the heat distribution of the surface to be obtained with the laser texture, the properties of the product to be obtained can be known in advance. In mathematical modeling, as in parameter optimizations, both time and material can be saved in experimental studies.

In this study, firstly, grooves were created on the Ti-6Al-4V plate with a 3 J laser. Measurements were made on the obtained through and the constants to be used in the temperature distribution equation were calculated. Then, grooves were obtained with 2.5, 3.5, 4 and 4.5 Joules energies to prove the validity of the mathematical model obtained. The measurements made on these grooves and the results obtained with the mathematical model were compared. The error rates of the results obtained vary between 0.38 and 2.55 %. The fact that the error rates are so low indicates that the proposed model is an acceptable one.

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Partial Sums of The Miller-Ross Function

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Abstract. This article deals with the ratio of normalized Miller-Ross function $\mathbb{E}_{\nu,c}(z)$ and its sequence of partial sums $(\mathbb{E}_{\nu,c})_m(z)$. Several examples which illustrate the validity of our results are also given.

1. Introduction

Let $\mathcal{A}$ be the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.
Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ which consists of univalent functions in $\mathcal{U}$. Consider the function $E_{\nu,c}(z)$ defined by

$$E_{\nu,c}(z) = z^{\nu} \sum_{n=0}^{\infty} \frac{(cz)^n}{\Gamma(\nu+n+1)} \tag{2}$$

where Γ stands for the Euler gamma function and $\nu > -1$, $c \in \mathbb{C}$ and $z \in \mathcal{U}$. This function was introduced by Miller and Ross in 1993 [9] and is therefore known as the Miller-Ross function.
The function defined by (2) does not belong to the class $\mathcal{A}$. Therefore, we consider the following normalization of the Miller-Ross function $E_{\nu,c}(z)$: for $z \in \mathcal{U}$,

$$\mathbb{E}_{\nu,c}(z) = \Gamma(\nu+1)\, z^{1-\nu} E_{\nu,c}(z) = \sum_{n=0}^{\infty} \frac{c^n \Gamma(\nu+1)}{\Gamma(\nu+n+1)} z^{n+1} \tag{3}$$

where $\nu > -1$ and $c \in \mathbb{C}$.
Note that some special cases of $\mathbb{E}_{\nu,c}(z)$ are:

$$\begin{cases} \mathbb{E}_{0,1}(z) = e^z z \\ \mathbb{E}_{1,1}(z) = e^z - 1 \\ \mathbb{E}_{3,1}(z) = \frac{3\left(2e^z - z^2 - 2z - 2\right)}{z^2} \\ \mathbb{E}_{\frac{1}{2},\frac{1}{2}}(z) = e^{\frac{z}{2}} \sqrt{\frac{\pi}{2}} \sqrt{z}\,\mathrm{Erf} \sqrt{\frac{z}{2}}, \end{cases} \tag{4}$$

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where Erf $\sqrt{z}$ is the error function.
For various interesting developments concerning partial sums of analytic univalent functions, the reader may be (for examples) refered to the works of Brickman et al. [1], Kazımoğlu et al. [7], Çağlar and Orhan [2], Lin and Owa [8], Deniz and Orhan [4, 5], Owa et al. [11], Sheil-Small [14], Silverman [15] and Silvia [16]. Recently, some researchers have studied on partial sums of special functions (see [3, 7, 10, 13, 17]).
By using the Pochhammer (or Appell) symbol, defined in terms of Euler's gamma functions, by $(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \lambda(\lambda+1)\cdots(\lambda+n+1)$, we obtain the following series representation for the ratio of normalized Miller-Ross function $\mathbb{E}_{\nu,c}(z)$ given by (3):

$$\begin{cases} (\mathbb{E}_{\nu,c})_0(z) = z \\ (\mathbb{E}_{\nu,c})_m(z) = z + \sum\limits_{n=1}^{m} A_n z^{n+1},\ m \in \mathbb{N} = \{1,2,3,\ldots\}, \end{cases} \tag{5}$$

where

$$A_n = \frac{c^n \Gamma(\nu+1)}{\Gamma(\nu+n+1)} = \frac{c^n}{(\nu+1)_n},\ \nu > -1 \text{ and } c \in \mathbb{C}.$$

We obtain lower bounds on ratios like

$$\Re\left\{\frac{\mathbb{E}_{\nu,c}(z)}{(\mathbb{E}_{\nu,c})_m(z)}\right\},\ \Re\left\{\frac{(\mathbb{E}_{\nu,c})_m(z)}{\mathbb{E}_{\nu,c}(z)}\right\},\ \Re\left\{\frac{\mathbb{E}'_{\nu,c}(z)}{(\mathbb{E}_{\nu,c})'_m(z)}\right\},\ \Re\left\{\frac{(\mathbb{E}_{\nu,c})'_m(z)}{\mathbb{E}'_{\nu,c}(z)}\right\}.$$

Several examples will be also given.
Results concerning partial sums of analytic functions may be found in [6, 12] etc.

2. MAIN RESULTS

In order to obtain our results we need the following lemma.

Lemma 2.1. *Let $\nu > -1$, $c \in \mathbb{C}$ and $|c| < \nu + 1$. Then the function $\mathbb{E}_{\nu,c}(z)$ satisfies the next two inequalities:*

$$\left|\mathbb{E}_{\nu,c}(z)\right| \leq \frac{\nu+1}{\nu-|c|+1} \quad (z \in \mathcal{U}) \tag{6}$$

$$\left|\mathbb{E}'_{\nu,c}(z)\right| \leq 1 + \frac{2\nu|c| + 2|c| - |c|^2}{(\nu-|c|+1)^2} \quad (z \in \mathcal{U}). \tag{7}$$

Proof. By using the well-known triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

and the inequality

$$(\nu+1)_n \geq (\nu+1)^n,\ n \in \mathbb{N}, \tag{8}$$

we have

$$\begin{aligned} \left|\mathbb{E}_{\nu,c}(z)\right| &= \left|z + \sum_{n=1}^{\infty} \frac{c^n \Gamma(\nu+1)}{\Gamma(\nu+n+1)} z^{n+1}\right| \leq 1 + \sum_{n=1}^{\infty} \frac{|c|^n \Gamma(\nu+1)}{\Gamma(\nu+n+1)} \\ &= 1 + \sum_{n=1}^{\infty} \frac{|c|^n}{(\nu+1)_n} \leq 1 + \sum_{n=1}^{\infty} \left(\frac{|c|}{\nu+1}\right)^n = \frac{\nu+1}{\nu-|c|+1}, \quad (|c| < \nu+1) \end{aligned}$$

and thus, inequality (6) is proved.
To prove (7), using again (8) and the triangle inequality, for $z \in \mathcal{U}$, we obtain

$$\begin{aligned}\left|\mathbb{E}'_{\nu,c}(z)\right| &= \left|1+\sum_{n=1}^{\infty}\frac{(n+1)\,c^n\Gamma(\nu+1)}{\Gamma(\nu+n+1)}z^n\right| \leq 1+\sum_{n=1}^{\infty}\frac{(n+1)\,|c|^n\Gamma(\nu+1)}{\Gamma(\nu+n+1)}\\ &= 1+\sum_{n=1}^{\infty}\frac{(n+1)\,|c|^n}{(\nu+1)_n} \leq 1+\sum_{n=1}^{\infty}(n+1)\left(\frac{|c|}{\nu+1}\right)^n = 1+\frac{2\nu\,|c|+2\,|c|-|c|^2}{(\nu-|c|+1)^2},\quad (|c|<\nu+1)\end{aligned}$$

and thus, inequality (7) is proved. □

Let $w(z)$ be an analytic function in $\mathcal{U}$. In the sequel, we will frequently use the following well-known result:

$$\Re\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0,\ z\in\mathcal{U} \text{ if and only if } |w(z)|<1,\ z\in\mathcal{U}.$$

Theorem 2.2. *Let* $\nu > -1$ *and* $0 < 2\,|c| \leq \nu+1$. *Then*

$$\Re\left\{\frac{\mathbb{E}_{\nu,c}(z)}{(\mathbb{E}_{\nu,c})_m(z)}\right\} \geq \frac{\nu-2\,|c|+1}{\nu-|c|+1},\ z\in\mathcal{U} \tag{9}$$

and

$$\Re\left\{\frac{(\mathbb{E}_{\nu,c})_m(z)}{\mathbb{E}_{\nu,c}(z)}\right\} \geq \frac{\nu-|c|+1}{\nu+1}. \tag{10}$$

Proof. From inequality (6) we get

$$1+\sum_{n=1}^{\infty}A_n \leq \frac{\nu+1}{\nu-|c|+1}, \text{where } A_n = \frac{c^n\Gamma(\nu+1)}{\Gamma(\nu+n+1)},\ \nu>-1,\ c\in\mathbb{C} \text{ and } n\in\mathbb{N}.$$

The last inequality is equivalent to

$$\left(\frac{\nu-|c|+1}{|c|}\right)\sum_{n_1}^{\infty}A_n \leq 1.$$

In order to prove the inequality (9), we consider the function $w(z)$ defined by

$$\frac{1+w(z)}{1-w(z)} = \left(\frac{\nu-|c|+1}{|c|}\right)\frac{\mathbb{E}_{\nu,c}(z)}{(\mathbb{E}_{\nu,c})_m(z)} - \left(\frac{\nu-2\,|c|+1}{|c|}\right)$$

or

$$\frac{1+w(z)}{1-w(z)} = \frac{1+\sum_{n=1}^{m}A_nz^n+\left(\frac{\nu-|c|+1}{|c|}\right)\sum_{n=m+1}^{\infty}A_nz^n}{1+\sum_{n=1}^{m}A_nz^n}. \tag{11}$$

From (11), we obtain

$$w(z) = \frac{\left(\frac{\nu-|c|+1}{|c|}\right)\sum_{n=m+1}^{\infty}A_nz^n}{2+2\sum_{n=1}^{m}A_nz^n+\left(\frac{\nu-|c|+1}{|c|}\right)\sum_{n=m+1}^{\infty}A_nz^n}$$

and

$$|w(z)| \leq \frac{\left(\frac{\nu-|c|+1}{|c|}\right)\sum_{n=m+1}^{\infty}A_n}{2-2\sum_{n=1}^{m}A_n-\left(\frac{\nu-|c|+1}{|c|}\right)\sum_{n=m+1}^{\infty}A_n}.$$

Now, $|w(z)| \leq 1$ if and only if

$$2\left(\frac{\nu-|c|+1}{|c|}\right)\sum_{n=m+1}^{\infty}A_n \leq 2-2\sum_{n=1}^{m}A_n$$

which is equivalent to

$$\sum_{n=1}^{m} A_n + \left(\frac{\nu - |c| + 1}{|c|}\right) \sum_{n=m+1}^{\infty} A_n \leq 1. \tag{12}$$

To prove (12), it suffices to show that its left-hand side is bounded above by

$$\left(\frac{\nu - |c| + 1}{|c|}\right) \sum_{n=1}^{\infty} A_n$$

which is equivalent to

$$\left(\frac{\nu - 2|c| + 1}{|c|}\right) \sum_{n=1}^{m} A_n \geq 0.$$

The last inequality holds true for $0 < 2|c| \leq \nu + 1$.
We use the same method to prove the inequality (10). Consider the function $w(z)$ given by

$$\begin{aligned} \frac{1 + w(z)}{1 - w(z)} &= \left(\frac{\nu + 1}{|c|}\right) \frac{\mathbb{E}_{\nu,c}(z)}{(\mathbb{E}_{\nu,c})_m(z)} - \left(\frac{\nu - |c| + 1}{|c|}\right) \\ &= \frac{1 + \sum_{n=1}^{m} A_n z^n - \left(\frac{\nu - |c| + 1}{|c|}\right) \sum_{n=m+1}^{\infty} A_n z^n}{1 + \sum_{n=1}^{m} A_n z^n}. \end{aligned}$$

From the last equality we get

$$w(z) = \frac{-\left(\frac{\nu + 1}{|c|}\right) \sum_{n=m+1}^{\infty} A_n z^n}{2 + 2\sum_{n=1}^{m} A_n z^n - \left(\frac{\nu - 2|c| + 1}{|c|}\right) \sum_{n=m+1}^{\infty} A_n z^n}$$

and

$$|w(z)| \leq \frac{\left(\frac{\nu + 1}{|c|}\right) \sum_{n=m+1}^{\infty} A_n}{2 - 2\sum_{n=1}^{m} A_n - \left(\frac{\nu - 2|c| + 1}{|c|}\right) \sum_{n=m+1}^{\infty} A_n}.$$

Then, $|w(z)| \leq 1$ if and only if

$$\sum_{n=1}^{m} A_n + \left(\frac{\nu - |c| + 1}{|c|}\right) \sum_{n=m+1}^{\infty} A_n \leq 1. \tag{13}$$

Since the left-hand side of (13) is bounded above by

$$\left(\frac{\nu - |c| + 1}{|c|}\right) \sum_{n=1}^{\infty} A_n,$$

we have that the inequality (10) holds true. Now, the proof of our theorem is completed. □

Theorem 2.3. *Let* $\nu > -1$ *and* $0 < 2\nu|c| + 2|c| - |c|^2 \leq \frac{(\nu+1)^2}{2}$. *Then*

$$\Re \left\{ \frac{\mathbb{E}'_{\nu,c}(z)}{(\mathbb{E}_{\nu,c})'_m(z)} \right\} \geq 1 - \frac{2\nu|c| + 2|c| - |c|^2}{(\nu - |c| + 1)^2}, \ z \in \mathcal{U} \tag{14}$$

and

$$\Re \left\{ \frac{(\mathbb{E}_{\nu,c})'_m(z)}{\mathbb{E}'_{\nu,c}(z)} \right\} \geq \frac{(\nu - |c| + 1)^2}{(\nu - |c| + 1)^2 + 2\nu|c| + 2|c| - |c|^2}, \ z \in \mathcal{U}. \tag{15}$$

Proof. From (7) we have

$$1+\sum_{n=1}^{\infty}(n+1)A_n \leq 1+\frac{2\nu|c|+2|c|-|c|^2}{(\nu-|c|+1)^2},$$

where $A_n=\frac{c^n\Gamma(\nu+1)}{\Gamma(\nu+n+1)}$, $\nu>-1$, $c\in\mathbb{C}$ and $n\in\mathbb{N}$. The above inequality is equivalent to

$$\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\sum_{n=1}^{\infty}(n+1)A_n \leq 1.$$

To prove (14), define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)}=\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\frac{\mathbb{E}'_{\nu,c}(z)}{(\mathbb{E}_{\nu,c})'_m(z)}-\left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}-1\right)$$

which gives

$$w(z)=\frac{\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\sum_{n=m+1}^{\infty}(n+1)A_nz^n}{2+2\sum_{n=1}^{m}(n+1)A_nz^n+\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\sum_{n=m+1}^{\infty}(n+1)A_nz^n}$$

and

$$|w(z)|\leq\frac{\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\sum_{n=m+1}^{\infty}(n+1)A_n}{2-2\sum_{n=1}^{m}(n+1)A_n-\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\sum_{n=m+1}^{\infty}(n+1)A_n}.$$

The condition $|w(z)|\leq 1$ holds true if and only if

$$\sum_{n=1}^{m}(n+1)A_n+\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\sum_{n=m+1}^{\infty}(n+1)A_n\leq 1. \tag{16}$$

The left-hand side of (16) is bounded above by

$$\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\sum_{n=1}^{\infty}(n+1)A_n$$

which is equivalent to

$$\left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}-1\right)\sum_{n=1}^{m}(n+1)A_n\geq 0$$

which holds true for $0<2\nu|c|+2|c|-|c|^2\leq\frac{(\nu+1)^2}{2}$.
The proof of (15) follows the same pattern. Consider the function $w(z)$ given by

$$\begin{aligned}\frac{1+w(z)}{1-w(z)} &= \left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}+1\right)\frac{\mathbb{E}'_{\nu,c}(z)}{(\mathbb{E}_{\nu,c})'_m(z)}-\left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\right)\\ &= \frac{1+\sum_{n=1}^{m}(n+1)A_nz^n-\left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\right)\sum_{n=m+1}^{\infty}(n+1)A_nz^n}{1+\sum_{n=1}^{\infty}(n+1)A_nz^n}.\end{aligned}$$

Consequently, we have that

$$w(z)=\frac{-\left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}+1\right)\sum_{n=m+1}^{\infty}(n+1)A_nz^n}{2+2\sum_{n=1}^{m}(n+1)A_nz^n-\left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}-1\right)\sum_{n=m+1}^{\infty}(n+1)A_nz^n}$$

and

$$|w(z)| \leq \frac{\left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}+1\right)\sum_{n=m+1}^{\infty}(n+1)A_n}{2-2\sum_{n=1}^{m}(n+1)A_n-\left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}-1\right)\sum_{n=m+1}^{\infty}(n+1)A_n}.$$

The last inequality implies that $|w(z)| \leq 1$ if and only if

$$\left(\frac{2(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\right)\sum_{n=m+1}^{\infty}(n+1)A_n \leq 2-2\sum_{n=1}^{m}(n+1)A_n$$

or equivalently

$$\sum_{n=1}^{m}(n+1)A_n+\left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\right)\sum_{n=m+1}^{\infty}(n+1)A_n \leq 1. \tag{17}$$

It remains to show that the left-hand side of (17) is bounded above by

$$\left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}\right)\sum_{n=1}^{\infty}(n+1)A_n.$$

This is equivalent to

$$\left(\frac{(\nu-|c|+1)^2}{2\nu|c|+2|c|-|c|^2}-1\right)\sum_{n=1}^{m}(n+1)A_n \geq 0,$$

which holds true for $0 < 2\nu|c|+2|c|-|c|^2 \leq \frac{(\nu+1)^2}{2}$. Now, the proof of our theorem is completed. □

3. Examples

In this section, we give several examples which illustrate our main theorems in Sections 2. In Theorem 2.2 and Theorem 2.3, we obtain the following corollaries for special cases of ν and c.

Corollary 3.1. *If we take $\nu = 3$ and $c = 1$, we have*

$$\mathbb{E}_{3,1}(z) = \frac{3\left(2e^z-z^2-2z-2\right)}{z^2}, \quad \mathbb{E}'_{3,1}(z) = \frac{6\left(e^z(z-2)+z+2\right)}{z^3}$$

and for $m = 0$ we get

$$\left(\mathbb{E}_{3,1}(z)\right)_0(z) = z, \quad \left(\mathbb{E}'_{3,1}(z)\right)_0(z) = 1,$$

so,

$$\begin{aligned}
\Re\left\{\frac{\left(2e^z-z^2-2z-2\right)}{z^3}\right\} &\geq \frac{2}{9} \approx 0.222, \quad z \in \mathcal{U},\\
\Re\left\{\frac{z^3}{\left(2e^z-z^2-2z-2\right)}\right\} &\geq \frac{9}{4} \approx 2.25, \quad z \in \mathcal{U},\\
\Re\left\{\frac{\left(e^z(z-2)+z+2\right)}{z^3}\right\} &\geq \frac{1}{27} \approx 0.037, \quad z \in \mathcal{U},\\
\Re\left\{\frac{z^3}{\left(e^z(z-2)+z+2\right)}\right\} &\geq \frac{27}{8} \approx 3.375, \quad z \in \mathcal{U}.
\end{aligned}$$

Setting $m = 0$, $\nu = \frac{3}{2}$ and $c = \frac{1}{2}$ in Theorem 2.2 and Theorem 2.3 respectively, we obtain the next result involving the function $\mathbb{E}_{\frac{1}{2},\frac{1}{2}}(z)$, defined by (4), and its derivative.

Corollary 3.2. *The following inequıalities hold true:*

$$\Re\left\{\frac{e^{\frac{z}{2}}\sqrt{\frac{\pi}{2}}Erf\sqrt{\frac{z}{2}}-\sqrt{z}}{z\sqrt{z}}\right\} \geq \frac{1}{4}\approx 0.25,\ z\in\mathcal{U},$$

$$\Re\left\{\frac{z\sqrt{z}}{e^{\frac{z}{2}}\sqrt{\frac{\pi}{2}}Erf\sqrt{\frac{z}{2}}-\sqrt{z}}\right\} \geq \frac{12}{5}\approx 2.4,\ z\in\mathcal{U},$$

$$\Re\left\{\frac{e^{\frac{z}{2}}\sqrt{2\pi}\,(z-1)\,Erf\sqrt{\frac{z}{2}}+2\sqrt{z}}{z\sqrt{z}}\right\} \geq \frac{7}{12}\approx 0.583,\ z\in\mathcal{U},$$

$$\Re\left\{\frac{z\sqrt{z}}{e^{\frac{z}{2}}\sqrt{2\pi}\,(z-1)\,Erf\sqrt{\frac{z}{2}}+2\sqrt{z}}\right\} \geq \frac{12}{25}\approx 0.48,\ z\in\mathcal{U}.$$

Example 3.3. *The image domains of* $f_1(z)=\frac{e^{\frac{z}{2}}\sqrt{\frac{\pi}{2}}Erf\sqrt{\frac{z}{2}}-\sqrt{z}}{z\sqrt{z}}$, $f_2(z)=\frac{z\sqrt{z}}{e^{\frac{z}{2}}\sqrt{\frac{\pi}{2}}Erf\sqrt{\frac{z}{2}}-\sqrt{z}}$, $f_3(z)=\frac{e^{\frac{z}{2}}\sqrt{2\pi}(z-1)Erf\sqrt{\frac{z}{2}}+2\sqrt{z}}{z\sqrt{z}}$ *and* $f_4(z)=\frac{z\sqrt{z}}{e^{\frac{z}{2}}\sqrt{2\pi}(z-1)Erf\sqrt{\frac{z}{2}}+2\sqrt{z}}$ *are shown in Figure 1.*

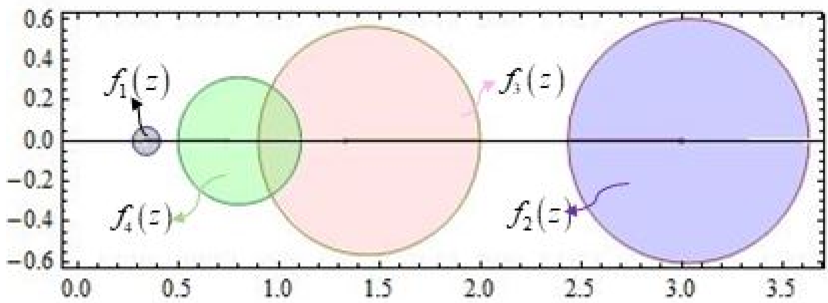

Figure1.

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