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B– Theoretical Sciences

ESKİŞEHİR TEKNİK ÜNİVERSİTESİ BİLİM VE TEKNOLOJİ DERGİSİ
B– Teorik Bilimler

Volume/Cilt 12 Number/Sayı 2 – August / Ağustos 2024

ESKİŞEHİR TEKNİK ÜNİVERSİTESİ BİLİM VE TEKNOLOJİ DERGİSİ
B- TEORİK BİLİMLER

Eskişehir Technical University Journal of Science and Technology B- Theoretical Sciences

Volume: 12 / Number: 2 – August / 2024

Eskişehir Technical University Journal of Science and Technology B - Theoretical Sciences (formerly Anadolu University Journal of Science and Technology B – Theoretical Sciences) is an **peer-reviewed** and **refereed international journal** by Eskişehir Technical University. Since 2010, it has been regularly published and distributed biannually and it has been published biannually and **electronically only since 2016**.

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Volume: 12 / Number: 2 – August / 2024

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ABOUT

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Books
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Kennedy J and Eberhart R. Swarm Intelligence. San Diego, CA, USA: Academic Press, 2001.

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B – Teorik Bilimler

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RESEARCH ARTICLE

GUANIDINE DELIVERY BY Si-DOPED C_{60} AND SWCNT: A DFT APPROACH

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Abstract

The interactions between guanidine and silicon decorated fullerene or single walled carbon nanotube were examined for insight into the drug delivery approach. The calculations show that the chemical reactivity and interaction energies are strongly dependent on the interaction site of the guanidine molecule. Depending on the purpose, by determining the interaction sites, it is possible to use Si decorated fullerenes and single walled carbon nanotubes as selective drug delivery vehicles. The results will contribute to further searches on improving drug delivery platforms.

Keywords

Guanidine,
Fullerene,
Carbon nanotubes,
DFT

Time Scale of Article

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1. INTRODUCTION

Guanidines are known as nitrogen rich small organic compounds having found large scale application areas. Guanidines and their derivatives are part of different biological molecules including arginine and agmatine [1, 2]. Moreover, guanidine containing drug molecules are center to many clinical assessments due to their versatile pharmaceutical properties [3, 4]. It is also known that guanidine groups show very good hydrophilicity and anti-algal growth characteristics and further, the amino site in the guanidine molecule plays an important role as an active uranium adsorption site [5, 6].

Nanotechnological materials have been lately grasping considerable attention among the scientific community. Especially carbon-based materials such as fullerenes carbon nanotubes (CNTs) have found many scientific application possibilities including drug delivery and sensor applications due to their inherent physical and chemical properties [7-10]. The main drawback for possible medicinal applications of fullerenes and CNT based systems appears as their solubility problems. However, this problem might be overcome by adding impurity atoms while keeping the system structurally stable [11].

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In the scope of this work, based on density functional theory (DFT) calculations, possible interaction sites, electronic and some important vibrational properties of silicon decorated fullerene (SiC_{59}) and single walled carbon nanotube (Si@SWCNT) towards the guanidine molecule have been investigated. The reactivity and sensing assessments of SiC_{59} and Si@SWCNT against the guanidine molecules have been also examined. All the obtained results were discussed in brief.

2. CALCULATIONS

For the optimization process of the examined structures, the following procedure was followed: first, guanidine molecule, C_{60} and SWCNT structures were optimized. Then one carbon atom on the surface of C_{60} and SWCNT was replaced with one silicon atom and the resultant structures were re-optimized. Based on the charge distribution on the surface of the guanidine molecule (See supplementary file) interaction sites were determined. Finally, the optimization process of the interacted systems was carried out. To reach a real global energy minimum, at the end of each optimization process, vibrational frequency calculations were also performed to see if no negative frequencies were observed at the end of each calculation. In any case where there appear any negative frequencies, the optimization process was repeated to reach a result that is free of all possible negative vibrational frequencies. The computations were carried out in both the gas phase and in the water media to see the solvation effect. The effect of solvent media was taken into account using the polarizable continuum model [12].

The binding energy (E_b) between guanidine and SiC_{59} or Si@SWCNT was calculated using the following equation:

$$E_b = E_{Guanidine\&SiC59}-(E_{Guanidine}+E_{SiC59})$$
$$E_b = E_{Guanidine\&Si@SWCNT}-(E_{Guanidine}+E_{Si@SWCNT})$$

where all the structures given above are the optimized energies of the related structures. In the given equations, the structures with more negative E_b values are referred to as the most stable structures. E_g energies are taken as the magnitudes of differences between the highest occupied molecular orbitals (HOMO) and the lowest unoccupied molecular orbitals (LUMO). E_b values were recalculated to get rid of possible basis sets superposition errors (BSSE) by including the counterpoise correction method [13]. For all calculations, the B3LYP/6-31G(d) level of theory was preferred due to their acceptable results [14, 15]. It can be used to investigate the binding energy and reactivity properties for such interactions of the compounds compared to the highly demanding cc-pvdz basis set [16]. Gaussian and GaussView programs were used for DFT computations, molecular design of the examined systems and visualization [17, 18].

3. RESULTS AND DISCUSSIONS

3.1. Analysis of SiC_{59} Interacted Guanidine System

The optimized structures of the guanidine interacted SiC_{59} system are given in Figure 1. Si…NH_2 and Si…NH inter-atomic distances were calculated as 1.983 & 1.841 Å in the gas phase and 1.940 & 1.799 Å in the water media, respectively. It was observed that Si…N inter-atomic distances are shorter in the water media leading to more stable structures. This fact can also be verified from the analysis of E_b energies which were calculated as -39.87 & -50.70 kcal/mol for gas and water media calculations for Si…NH interacted system. As for the Si…NH_2 interacted system, E_b energies were found as -15.78 & -21.76 kcal/mol correspondingly. It is seen that systems are more stable in water media and the strongest interaction occurs at the NH site of the guanidine molecule.

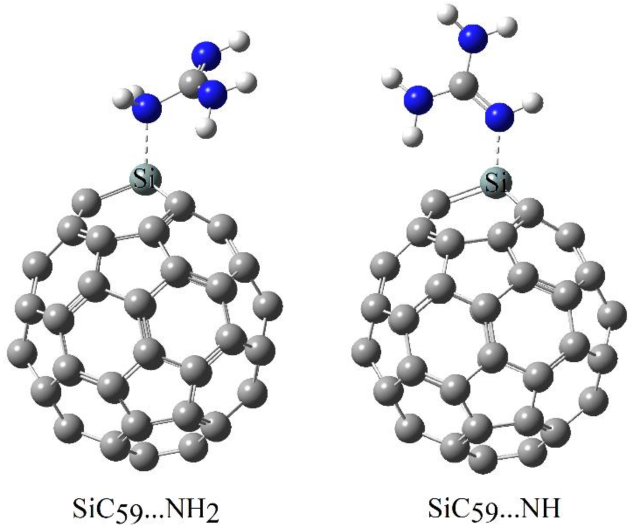

Figure 1. The optimized structures of the guanidine interacted SiC_{59} system in the gas phase.

NH or NH_2 stretching vibrations can easily be identified in the IR spectrum. Therefore, at this point, it is better to analyze the alterations of NH or NH_2 stretching vibrations before and after the interaction of the examined systems. NH_2-asymmetric stretching (as), NH_2-symmetric stretching (ss) and NH-s vibrations (scaled by 0.9614 [19]) for the isolated guanidine were calculated (gas & water media) as 3499 & 3501, 3396 & 3405 and 3332 & 3332 cm^{-1}, respectively. Following the NH_2 site interaction of the guanidine molecule with SiC_{59}, NH_2-as and NH_2-ss vibrations (gas & water media) were shifted to 3350 & 3339 and 3274 & 3275 cm^{-1}, respectively. The results suggest a red-shift in the IR spectra upon interaction. Further, as for the NH site interaction of the guanidine molecule with SiC_{59}, NH-s vibrations (gas & water media) shifted to 3425 & 3424 cm^{-1}. The blue shifts are more dominant when with NH site interacts fullerene system since as indicated the interactions are stronger with the NH site of the guanidine molecule when compared to the NH_2 site. Therefore, the obtained results support each other.

Single isolated SiC_{59} produced E_g values of 2.169 and 2.170 eV in the gas and water phases. Upon interaction from NH_2 and NH site of guanidine, these values (gas & water media) decreased to 2.092 & 1.998 eV and 2.143 & 1.968 eV, respectively. The decrease in E_g values proposes that no matter which site or which phase the interaction occurs, SiC_{59} is sensitive to the presence of guanidine molecule.

3.2. Analysis of Si@SWCNT Interacted Guanidine System

The optimized structures of guanidine interacted Si@SWCNT system are given in Figure 2. Si…NH_2 and Si…NH inter-atomic distances were computed as 2.036 & 1.869 Å in the gas phase and 1.981 & 1.819 Å in the water media, respectively. E_b energies were found as -9.09 & -13.02 kcal/mol for gas and water media calculations for Si…NH_2 interacted system. As for Si…NH interacted system, E_b values were calculated as -28.29 and -38.07 kcal/mol, respectively. As it happens for silicon decorated fullerene systems, here water as solvent stabilizes the examined silicon decorated SWCNT system and yields more negative E_b energies with respect to gas phase calculations. The strongest interaction also occurs with the NH site of the guanidine molecule.

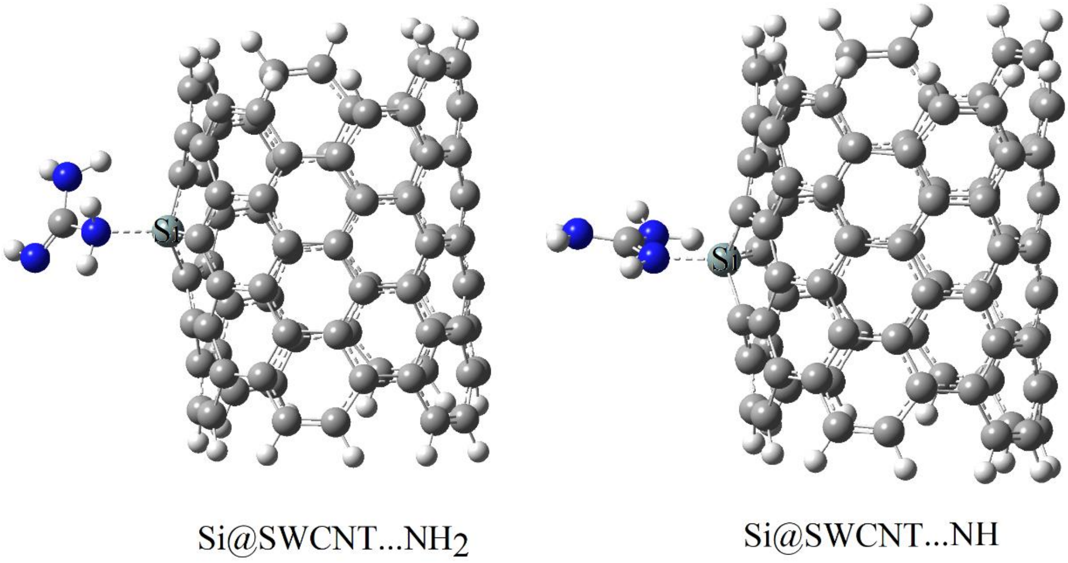

Figure 2. The optimized structures of the guanidine interacted Si@SWCNT system.

Following the NH_2 site interaction of guanidine molecule with Si@SWCNT, NH_2-as, NH_2-ss vibrations (gas & water media) were shifted to 3368 & 3357 and 3287 & 3270 cm^{-1}, respectively. When compared to the isolated guanidine molecule, a red-shift was observed for the interacted system. Moreover, as for the NH site interaction of the guanidine molecule with Si@SWCNT, NH-s vibrations (gas & water media) shifted to 3415 & 3418 cm^{-1}.

Single, isolated Si@SWCNT yielded E_g values of 1.719 and 1.706 eV in the gas and water phases. After the interaction from NH_2 and NH site of guanidine, the related values (gas & water media) altered to 1.756 & 1.729 eV and 1.740 & 1.689 eV, respectively. The obtained results here are quite different from the results obtained with fullerene interacted systems. The only decrease in E_g values for Si@SWCNT interacted guanidine system was observed for NH site interaction in the water phase. Therefore, it can be concluded that, if the interaction between Si@SWCNT and guanidine molecule occurs at the NH edge in water media then the Si@SWCNT system is sensitive to the presence of guanidine molecule otherwise it is not.

4. CONCLUSIONS

In this study, the interactions between guanidine molecule and silicon decorated fullerene C_{60} and single walled carbon nanotube were examined by DFT calculations. It was observed that the strongest interactions occurred at the NH site of the guanidine molecule and the strength of the interactions is far stronger with the fullerene system compared to the SWCNT system. It was also observed that water as a solvent and the site of interaction have impacts on the sensitivity of the fullerene and SWCNT systems towards the guanidine molecule. By defining the interaction site of the guanidine molecule, the electronic properties of the interacted systems can be manipulated in a preferred manner.

CONFLICT OF INTEREST

The authors stated that there are no conflicts of interest regarding the publication of this article.

CRediT AUTHOR STATEMENT

Shohrat Ovezov: Investigation, Software, Writing - original draft. **Cemal Parlak:** Supervision, Writing – Review & Editing, Software, Visualization. **Özgür Alver:** Conceptualization, Formal analysis, Methodology.

REFERENCES

[1] Kim SH, Semenya D, Castagnolo D. Antimicrobial drugs bearing guanidine moieties: A review. Eur. J. Med. Chem. 2021; 216: 113293.

[2] Raczyńska ED, Cyrański MK, Gutowski M, Rak J, Gal JF, Maria PC, Darowska M, Duczmal K. Consequences of proton transfer in guanidine. J. Phys. Org. Chem. 2003; 16: 91-106.

[3] Berlinck RGS, Bernardi DI, Fill T, Fernandes AAG, Jurberg ID. The chemistry and biology of guanidine secondary metabolites. Nat. Prod. Rep. 2021; 38(3): 586-667.

[4] Shaw JW, Grayson DH, Rozas I. Synthesis of guanidines and some of their biological applications. Top. Heterocycl. Chem. 2015; 50: 1-51.

[5] Yang H, Guo X, Chen R, Liu Q, Liu J, Yu J, Lin C, Wang J, Zhang M. A hybrid sponge with guanidine and phytic acid enriched surface for integration of antibiofouling and uranium uptake from seawater. Appl. Surf. Sci. 2020; 525: 146611.

[6] Jin D, Wang Y, Song D, Zhu J, Yu J, Liu Q, Liu J, Li R, Liu P, Wang J. "Bridging hydrogen bonds" from guanidine and amidoxime groups on natural bamboo strips as a superantibacterial and knitted adsorbent to efficiently adsorb uranium from simulated seawater. Sep. Purif. Technol. 2024; 330: 125186.

[7] Saadh MJ, Alsaedi II, Abbood MA, Yadav A, Alsailawi HA, Mudhafar M, Al-Athari AJH, Elmasry Y, Alawadi AH. Therapeutic potential of CX (X =48, 60, and 70) fullerenes as drug delivery carriers for ifosfamide anti-cancer drug. Diam. Relat. Mater. 2023; 140: 110494.

[8] Salem-Bekhit MM, Al Zahrani S, Alhabib NA, Maaliw III RR, Da'I M, Mirzaei M. Metal-doped fullerenes as promising drug carriers of hydroxycarbamide anticancer: Insights from density functional theory. Chem. Phys. Impact 2023; 7: 100347.

[9] Ha T, Park S, Shin M, Lee JY, Choi JH, Choi JW. Biosensing system for drug evaluation of amyotrophic lateral sclerosis based on muscle bundle and nano-biohybrid hydrogel composed of multiple motor neuron spheroids and carbon nanotubes. Chem. Eng. J. 2023; 463: 142284.

[10] Singh K, Sharma S, Singh B, Gupta M, Tripathi CC. Fabrication of graphene, graphite and multi wall carbon nano tube based thin films and their potential application as strain sensor. Thin Solid Films 2022; 761: 139540.

[11] Clemmer DE, Hunter JM, Shelimov KB, Jarrold MF. Physical and chemical evidence for metallofullerenes with metal atoms as part of the cage. Nature 1994; 372: 248–250.

[12] Tomasi J, Mennucci B, Cammi R. Quantum mechanical continuum solvation models. Chem. Rev. 2005; 105: 2999–3094.

[13] Gutowski M, Chalasinski G. Critical evaluation of some computational approaches to the problem of basis set superposition error. J. Chem. Phys. 1993; 98: 5540–5554.

[14] Becke AD. Density-functional thermochemistry. III. The role of exact exchange. J. Chem. Phys. 1993; 98: 5648-5652.

[15] Lee C, Yang W, Parr RG. Development of the Colle-Salvetti correlation-energy formula into a functional of the electron density. Phys. Rev. B 1988; 37: 785-789.

[16] Parlak C, Alver Ö. A density functional theory investigation on amantadine drug interaction with pristine and B, Al, Si, Ga, Ge doped C60 fullerenes. Chem. Phys. Lett. 2017; 678: 85–90.

[17] Frisch MJ, Trucks GW, Schlegel HB et al., Gaussian 09, Revision A.1, Gaussian Inc., Wallingford, CT, 2009.

[18] Dennington RD, Keith TA, Millam JM. GaussView 5.0.8, Gaussian Inc., 2008.

[19] Scott AP, Radom L. Harmonic vibrational frequencies: An evaluation of Hartree-Fock, Møller-Plesset, quadratic configuration interaction, density functional theory, and semiempirical scale factors. J. Phys. Chem. 1996; 100: 16502–16513.

ESKİŞEHİR TEKNİK ÜNİVERSİTESİ BİLİM VE TEKNOLOJİ DERGİSİ
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RESEARCH ARTICLE

SOFT INTERSECTION ALMOST QUASI-INTERIOR IDEALS OF SEMIGROUPS

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Abstract

Similar to how the quasi-interior ideal generalizes the ideal and interior ideal of a semigroup, the concept of soft intersection quasi-interior ideal generalizes the idea of soft intersection ideal and soft intersection interior ideal of a semigroup. In this study, we provide the notion of soft intersection almost quasi-interior ideal as well as the soft intersection weakly almost quasi-interior ideal in a semigroup. We show that any nonnull soft intersection quasi-interior ideal is a soft intersection almost quasi-interior ideal; and soft intersection almost quasi-interior ideal is a soft intersection weakly almost quasi-interior ideal, but the converses are not true. We further demonstrate that any idempotent soft intersection almost quasi-interior ideal is a soft intersection almost subsemigroup. With the established theorem that states that if a nonempty set A is almost quasi-interior ideal, then its soft characteristic function is a soft intersection almost quasi-interior ideal, and vice versa, we are also able to derive several intriguing relationships concerning minimality, primeness, semiprimeness, and strongly primeness between almost quasi-interior ideals, and soft intersection almost quasi-interior ideals.

Keywords

Soft set,
Semigroup,
(Almost) quasi-interior ideals,
Soft intersection (almost) quasi-interior ideals

Time Scale of Article

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1. INTRODUCTION

As semigroups provide the abstract algebraic basis for "memoryless" systems, which restart on each iteration, semigroups are essential in many disciplines of mathematics. The formal study of semigroups began in the early 1900s. In practical mathematics, semigroups are essential models for linear time-invariant systems. Since finite semigroups are inherently connected to finite automata, studying them is essential to theoretical computer science. Furthermore, in probability theory, semigroups, and Markov processes are related.

Ideals are necessary to understand algebraic structures and their applications. The earliest ideals to help with the study of algebraic numbers were offered by Dedekind. Noether generalized them further by adding associative rings. Bi-ideals for semigroups were first introduced by Good and Hughes in 1952 [1]. Steinfeld [2] first established the idea of quasi-ideals for semigroups and later expanded it to rings. For many mathematicians, generalizing ideals in algebraic structures has been a key field of research.

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The concept of almost left, right, and two-sided ideals of semigroups was first developed by Grosek and Satko [3] in 1980. Later, in 1981, Bogdanovic [4] extended the concept of bi-ideals to almost bi-ideals in semigroups. By combining the concepts of almost ideals and quasi-ideals of semigroups, Wattanatripop et al. [5] proposed almost quasi-ideals in 2018. In 2020, Kaopusek et al. [6] introduced almost interior ideals and weakly almost interior ideals of semigroups, extending and analyzing the notions of almost ideals and interior ideals of semigroups. Iampan [7] in 2022, Gaketem [9] in 2022, Chinram and Nakkhasen [8] in 2022, and Gaketem and Chinram [10] in 2023 introduced almost subsemigroups, almost bi-quasi-interior ideals, almost bi-interior ideals, and almost bi-quasi-ideals of semigroups, respectively. Furthermore, in [5, 7–12], several almost fuzzy semigroup ideal types were studied.

The concept of soft sets as a means of modeling uncertainty was initially proposed by Molodtsov [13] in 1999, and it has since attracted interest from several of disciplines. The basic operations of soft sets were studied in [14–29]. Çağman and Enginoğlu [30] modified the concept of soft sets and soft set operations and in [31] Çağman et al. introduced soft intersection groups by which research on other soft algebraic systems was inspired. Soft sets were also conveyed to semigroup theory through the notions of soft intersection semigroup, left, right, and two-sided, quasi-ideals, interior ideals, and (generalized) bi-ideals, which were extensively explored in [32–33]. Sezgin and Orbay [34] studied soft intersection substructures to classify various semigroups. A variety of soft algebraic structures became the subject of additional investigation in [35–44].

Bi-interior ideals, bi-quasi-interior ideals, bi-quasi-ideals, quasi-interior ideals, and weak-interior ideals are some of the new semigroup types that Rao [45–48] has proposed. These ideals are expansions of existing ideals. Moreover, Baupradist et al. [49] proposed the notion of essential ideals in semigroups.

While the quasi-interior ideal of semigroups was introduced by Rao [47] as a generalization of ideal and interior ideal of a semigroup; soft intersection quasi-interior ideal of a semigroup was proposed in [50] as a generalization of the soft intersection ideal and interior ideal of a semigroup. In this study, as a further generalization of the nonnull soft intersection quasi-interior ideal, the concept of soft intersection almost quasi-interior and its generalization soft intersection weakly almost quasi-interior ideals are introduced. Moreover, we demonstrate that an idempotent soft intersection almost quasi-interior ideal is a soft intersection almost subsemigroup. We observe that under the binary operation of soft union, a semigroup may be constructed by soft intersection almost quasi-interior ideals; however, this is not the case under the soft intersection operation. We also establish the relationship between the soft intersection almost quasi-interior ideal and almost quasi-interior ideal of a semigroup in terms of minimality, primeness, semiprimeness, and strongly primeness. This is achieved by deriving that if a nonempty set A is an almost quasi-interior ideal, then its soft characteristic function is also a soft intersection almost quasi-interior ideal, and vice versa. This paper is organized in the following manner. Section 2 reviews the fundamental principles of soft set theory, including semigroup ideals. Section 3 looks at the definition and thorough study of soft intersection almost quasi-interior ideals. In the conclusion section, we highlight the significance of the study's findings and their possible impact on the field.

2. PRELIMINARIES

In this section, we review several fundamental notions related to semigroups and soft sets.

Definition 2.1. Let U be the universal set, E be the parameter set, and $P(U)$ be the power set of U and $K \subseteq E$. A soft set f_K over U is a set-valued function such that $f_K: E \to P(U)$ such that for all $x \notin K$, $f_K(x) = \emptyset$. A soft set over U can be represented by the set of ordered pairs

$$f_K = \{(x, f_K(x)): x \in E, f_K(x) \in P(U)\}$$

[13, 30]. Throughout this paper, the set of all the soft sets over U is designated by $S_E(U)$.

Definition 2.2. Let $f_A \in S_E(U)$. If $f_A(\text{x}) = \emptyset$ for all $x \in E$, then f_A is called a null soft set and denoted by $\emptyset_E$. If $f_A(x) = U$ for all $x \in E$, then f_A is called an absolute soft set and denoted by U_E [30].

Definition 2.3. Let f_A, $f_B \in S_E(U)$. If for all $x \in E$, $f_A(\mathrm{x}) \subseteq f_B(\mathrm{x})$, then f_A is a soft subset of f_B and denoted by $f_A \tilde{\subseteq} f_B$. If $f_A(x) = f_B(x)$ for all $x \in E$, then f_A is called soft equal to f_B and denoted by $f_A = f_B$ [30].

Definition 2.4. Let $f_A, f_B \in S_E(U)$. The union of f_A and f_B is the soft set $f_A \tilde{\cup} f_B$, where $(f_A \tilde{\cup} f_B)(x) = f_A(x) \cup f_B(x)$ for all $x \in E$**.** The intersection of f_A and f_B is the soft set $f_A \tilde{\cap} f_B$, where $(f_A \tilde{\cap} f_B)(x) = f_A(x) \cap f_B(x)$ for all $x \in E$ [30].

Definition 2.5. For a soft set f_A, the support of f_A is defined by

$$supp(f_A) = \{x \in A: f_A(x) \neq \emptyset\}[18]$$

Thus, a null soft set in is indeed a soft set with an empty support, and we say that a soft set f_A is nonnull if $supp(f_A) \neq \emptyset$ [18].

Note 2.6. If $f_A \tilde{\subseteq} f_B$, then $supp(f_A) \subseteq supp(f_B)$ [51].

A semigroup S is a nonempty set with an associative binary operation and throughout this paper, S stands for a semigroup and all the soft sets are the elements of $S_S(U)$ unless otherwise specified. A nonempty subset A of S is called a left quasi-interior ideal of S if $SASA \subseteq A$; and is called a right quasi-interior ideal of S if $ASAS \subseteq A$; and is called a quasi-interior ideal of S if A is both a left quasi-interior ideal and a right quasi-interior ideal of S [47].

Definition 2.7. A nonempty subset A of S is called an almost left quasi-interior ideal of S if for all $x, y \in S$; $xAyA \cap A \neq \emptyset$, and is called an almost right quasi-interior ideal of S if for all $x, y \in S$, $AxAy \cap A \neq \emptyset$; and is called an almost quasi-interior ideal of S when A is both an almost left quasi-interior ideal of S and an almost right quasi-interior ideal of S.

Example 2.8. Let $S = \mathbb{Z}$ and $\emptyset \neq 2\mathbb{Z} \subseteq \mathbb{Z}$. Since $x(2\mathbb{Z})y(2\mathbb{Z}) \cap 2\mathbb{Z} \neq \emptyset$ and $(2\mathbb{Z})x(2\mathbb{Z})y \cap 2\mathbb{Z} \neq \emptyset$ for all $x, y \in \mathbb{Z}$, $2\mathbb{Z}$ is an almost quasi-interior ideal of S.

An almost (left/right) quasi-interior ideal A of S is called a minimal almost (left/right) quasi-interior ideal of S if for any almost (left/right) quasi-interior ideal B of S if whenever $B \subseteq A$, then $A = B$.

An almost (left/right) quasi-interior ideal P of S is called a prime almost (left/right) quasi-interior ideal if for any almost (left/right) quasi-interior ideals A and B of S such that $AB \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$. An almost (left/right) quasi-interior ideal P of S is called a semiprime almost (left/right) quasi-interior ideal if for any almost (left/right) quasi-interior ideal A of S such that $AA \subseteq P$ implies that $A \subseteq P$. An almost (left/right) quasi-interior ideal P of S is called a strongly prime almost (left/right) quasi-interior ideal if for any almost (left/right) quasi-interior ideals A and B of S such that $AB \cap BA \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$.

Definition 2.9. Let f_S and g_S be soft sets over the common universe U. Then, soft intersection product $f_S \circ g_S$ is defined by [32]

$$(f_S \circ g_S)(x) = \begin{cases} \bigcup_{x=yz} \{f_S(y) \cap g_S(z)\}, & if \ \exists y, z \in S \ such \ that \ x = yz \\ \emptyset, & otherwise \end{cases}$$

Theorem 2.10. Let $f_S, g_S, h_S \in S_S(U)$. Then,

i) $(f_S \circ g_S) \circ h_S = f_S \circ (g_S \circ h_S)$.

ii) $f_S \circ g_S \neq g_S \circ f_S$ *, generally.*

iii) $f_S \circ (g_S \tilde{\cup} h_S) = (f_S \circ g_S) \tilde{\cup} (f_S \circ h_S)$ and $(f_S \tilde{\cup} g_S) \circ h_S = (f_S \circ h_S) \tilde{\cup} (g_S \circ h_S)$.

iv) $f_S \circ (g_S \tilde{\cap} h_S) = (f_S \circ g_S) \tilde{\cap} (f_S \circ h_S)$ and $(f_S \tilde{\cap} g_S) \circ h_S = (f_S \circ h_S) \tilde{\cap} (g_S \circ h_S)$.

v) If $f_S \tilde{\subseteq} g_S$, then $f_S \circ h_S \tilde{\subseteq} g_S \circ h_S$ and $h_S \circ f_S \tilde{\subseteq} h_S \circ g_S$.

vi) If $t_S, k_S \in S_S(U)$ such that $t_S \tilde{\subseteq} f_S$ and $k_S \tilde{\subseteq} g_S$, then $t_S \circ k_S \tilde{\subseteq} f_S \circ g_S$ [32].

Lemma 2.11. Let f_S and g_S be soft sets over U. Then, $f_S \circ g_S = \emptyset_S$ if and only if $f_S = \emptyset_S$ or $g_S = \emptyset_S$.

Definition 2.12. Let A be a subset of S. We denote by S_A the soft characteristic function of A and define as

$$S_A(x) = \begin{cases} U, & if\ x \in A \\ \emptyset, & if\ x \in S \backslash A \end{cases}$$

The soft characteristic function of A is a soft set over U, that is, $S_A: S \longrightarrow P(U)$ [32].

Corollary 2.13. $supp(S_A) = A$ [51].

Theorem 2.14. Let X and Y be nonempty subsets of S. Then, the following properties hold [32,51]:

i) $X \subseteq Y$ if and only if $S_X \tilde{\subseteq} S_Y$

ii) $S_X \tilde{\cap} S_Y = S_{X \cap Y}$ and $S_X \tilde{\cup} S_Y = S_{X \cup Y}$

iii) $S_X \circ S_Y = S_{XY}$

Proof: In [32], (i) is given as if $X \subseteq Y$, then if $S_X \tilde{\subseteq} S_Y$. In [51], it was shown that if $S_X \tilde{\subseteq} S_Y$, then $X \subseteq Y$. Let $S_X \tilde{\subseteq} S_Y$ and $x \in X$. Then, $S_X(x) = U$, and this implies that $S_Y(x) = U$ since $S_X \tilde{\subseteq} S_Y$, Hence, $x \in Y$, and so $X \subseteq Y$. Now let $x \notin Y$. Then, $S_Y(x) = \emptyset$, and this implies that $S_X(x) = \emptyset$ since $S_X \tilde{\subseteq} S_Y$. Hence, $x \notin X$, and so $Y' \subseteq X'$, implying that $X \subseteq Y$.

Definition 2.15. Let x be an element in S. We denote by S_x the soft characteristic function of x and define as

$$S_x(y) = \begin{cases} U, & if\ \ y = x \\ \emptyset, & if\ \ y \neq x \end{cases}$$

The soft characteristic function of x is a soft set over U, that is, $S_x: S \longrightarrow P(U)$ [52].

Corollary 2.16. Let $x, y \in S$, f_S and S_x be soft sets over U. Then, $S_x \circ f_S \circ S_y \circ f_S = \emptyset_S$ if and only if $f_S = \emptyset_S$ and $f_S \circ S_x \circ f_S \circ S_y = \emptyset_S$ if and only if $f_S = \emptyset_S$.

Proof: By Lemma 2.11, $S_x \circ f_S \circ S_y \circ f_S = \emptyset_S$ if and only if $f_S = \emptyset_S$ or $S_x = \emptyset_S$ or $S_y = \emptyset_S$. Since $S_x \neq \emptyset_S$ and $S_y \neq \emptyset_S$ for all $x, y \in S$ by Definition 2.15, hence $f_S = \emptyset_S$. The rest of the proof is obvious by Lemma 2.11.

Definition 2.17. A soft set f_S over U is called a soft intersection left (resp. right) quasi-interior ideal of S over U if $f_S(xyzt) \supseteq f_S(y) \cap f_S(t)$ $(f_S(xyzt) \supseteq f_S(x) \cap f_S(z))$ for all $x, y, z, t \in S$. A soft set f_S over U is called a soft intersection quasi-interior ideal of S if it is both a soft intersection left quasi-interior ideal and a soft intersection right quasi-interior ideal of S over U [50].

It is easy to see that if $f_S(x) = U$ for all $x \in S$, then f_S is a soft intersection (left/right) quasi-interior ideal of S. We denote such a kind of (left/right) quasi-interior ideal by $\widetilde{\mathbb{S}}$. It is obvious that $\widetilde{\mathbb{S}} = S_S$, that is, $\widetilde{\mathbb{S}}(x) = U$ for all $x \in S$ [50].

Theorem 2.18. Let f_S be a soft set over U. Then, f_S is a soft intersection left (resp. right) quasi-interior ideal of S if and only if $\widetilde{\mathbb{S}} \circ f_S \circ \widetilde{\mathbb{S}} \circ f_S \,\widetilde{\subseteq}\, f_S$ $\left(f_S \circ \widetilde{\mathbb{S}} \circ f_S \circ \widetilde{\mathbb{S}} \,\widetilde{\subseteq}\, f_S\right)$. f_S is a soft intersection quasi-interior ideal of S if and only if $\widetilde{\mathbb{S}} \circ f_S \circ \widetilde{\mathbb{S}} \circ f_S \,\widetilde{\subseteq}\, f_S$ and $f_S \circ \widetilde{\mathbb{S}} \circ f_S \circ \widetilde{\mathbb{S}} \,\widetilde{\subseteq}\, f_S$ [50].

From now on, soft intersection left (right) quasi-interior ideal of S is denoted by SI-left (right) QI-ideal.

Definition 2.19. Let f_S be a soft set over U. Then, f_S is called a soft intersection almost subsemigroup of S if $(f_S \circ f_S) \,\widetilde{\cap}\, f_S \neq \emptyset_S$ [51].

We refer to [53] for the implications of network analysis and graph applications with respect to soft sets, which are determined by the divisibility of determinants and to [54-56] for more about soft set operations.

3. SOFT INTERSECTION ALMOST QUASI-INTERIOR IDEALS OF SEMIGROUPS

Definition 3.1. Let f_S be a soft set over U.

1) f_S is called a soft intersection almost left (resp. right) quasi-interior ideal of S if for all $x, y \in S$,
$$\left(S_x \circ f_S \circ S_y \circ f_S\right) \widetilde{\cap}\, f_S \neq \emptyset_S \ \left(\left(f_S \circ S_x \circ f_S \circ S_y\right) \widetilde{\cap}\, f_S \neq \emptyset_S\right).$$

f_S is called a soft intersection almost quasi-interior ideal of S if f_S is both a soft intersection almost left quasi-interior ideal of S and a soft intersection almost right quasi-interior ideal of S.

2) f_S is called a soft intersection weakly almost left (resp. right) quasi-interior ideal of S if for all $x \in S$,
$$\left(S_x \circ f_S \circ S_x \circ f_S\right) \widetilde{\cap}\, f_S \neq \emptyset_S \ \left(\left(f_S \circ S_x \circ f_S \circ S_x\right) \widetilde{\cap}\, f_S \neq \emptyset_S\right).$$

f_S is called a soft intersection weakly almost quasi-interior ideal of S if f_S is both a soft intersection weakly almost left quasi-interior ideal of S and a soft intersection weakly almost right quasi-interior ideal of S.

Hereafter, for brevity, soft intersection is abbreviated as SI, left (right) quasi-interior is as left (right) QI; so soft intersection (weakly) almost left (right) quasi-interior ideal of S is denoted by SI-(weakly) almost left (right) QI-ideal.

Example 3.2. Let $S = \{z, k\}$ be the semigroup with the following Cayley Table.

	z	k
z	z	z
k	z	k

Let f_S, h_S, and g_S be soft sets over $U = \mathbb{Z}^-$ as follows:

$$f_S = \{(z, \{-3, -2\}), (k, \{-5\})\}$$

$$h_S = \{(z, \{-9, -8\}), (k, \{-1\})\}$$

$$g_S = \{(z, \emptyset), (k, \{-7, -4\})\}$$

Here, f_S and h_S are SI-almost QI-ideals. Let's first show that f_S is an SI-weakly almost QI-ideal:

$[(S_k \circ f_S \circ S_k \circ f_S) \tilde{\cap} f_S](z) = [(S_k \circ f_S) \circ (S_k \circ f_S)](z) \cap f_S(z) = \Big[\big((S_k \circ f_S)(z) \cap (S_k \circ f_S)(z)\big) \cup \big((S_k \circ f_S)(z) \cap (S_k \circ f_S)(k)\big) \cup \big((S_k \circ f_S)(k) \cap (S_k \circ f_S)(z)\big)\Big] \cap f_S(z) = \Big[\Big[\Big(\big(S_k(z) \cap f_S(z)\big) \cup \big(S_k(z) \cap f_S(k)\big) \cup \big(S_k(k) \cap f_S(z)\big)\Big) \cap \Big(\big(S_k(z) \cap f_S(z)\big) \cup \big(S_k(z) \cap f_S(k)\big) \cup \big(S_k(k) \cap f_S(z)\big)\Big)\Big] \cup \Big[\Big(\big(S_k(z) \cap f_S(z)\big) \cup \big(S_k(z) \cap f_S(k)\big) \cup \big(S_k(k) \cap f_S(z)\big)\Big) \cap \Big(\big(S_k(k) \cap f_S(k)\big)\Big)\Big] \cup \Big[\Big(\big(S_k(k) \cap f_S(k)\big)\Big) \cap \Big(\big(S_k(z) \cap f_S(z)\big) \cup \big(S_k(z) \cap f_S(k)\big) \cup \big(S_k(k) \cap f_S(z)\big)\Big)\Big]\Big] \cap f_S(z) = \Big[\big(f_S(z) \cap f_S(z)\big) \cup \big(f_S(z) \cap f_S(k)\big) \cup \big(f_S(k) \cap f_S(z)\big)\Big] \cap f_S(z) = f_S(z) = \{-3, -2\}$

$[(S_k \circ f_S \circ S_k \circ f_S) \tilde{\cap} f_S](k) = [(S_k \circ f_S) \circ (S_k \circ f_S)](k) \cap f_S(k) = [(S_k \circ f_S)(k) \cap (S_k \circ f_S)(k)] \cap f_S(k) = \big[\big(S_k(k) \cap f_S(k)\big) \cap \big(S_k(k) \cap f_S(k)\big)\big] \cap f_S(k) = f_S(k) \cap f_S(k) \cap f_S(k) = f_S(k) = \{-5\}$

Consequently,

$$(S_k \circ f_S \circ S_k \circ f_S) \tilde{\cap} f_S = \{(z, \{-3, -2\}), (k, \{-5\})\} \neq \emptyset_S$$

Similarly,

$$(S_z \circ f_S \circ S_z \circ f_S) \tilde{\cap} f_S = \{(z, \{-3, -2\}), (k, \emptyset)\} \neq \emptyset_S$$

Therefore, for all $x \in S$, $(S_x \circ f_S \circ S_x \circ f_S) \tilde{\cap} f_S \neq \emptyset_S$, so f_S is an SI-weakly almost left QI-ideal. And also,

$$(f_S \circ S_k \circ f_S \circ S_k) \tilde{\cap} f_S = \{(z, \{-3, -2\}), (k, \{-5\})\} \neq \emptyset_S$$

$$(f_S \circ S_z \circ f_S \circ S_z) \tilde{\cap} f_S = \{(z, \{-3, -2\}), (k, \emptyset)\} \neq \emptyset_S$$

Therefore, for all $x \in S$, $(f_S \circ S_x \circ f_S \circ S_x) \tilde{\cap} f_S \neq \emptyset_S$, so f_S is an SI-weakly almost right QI-ideal. Thus, f_S is an SI-weakly almost QI-ideal.

With similar calculations, we have

$$(S_k \circ f_S \circ S_z \circ f_S) \tilde{\cap} f_S = \{(z, \{-3, -2\}), (k, \emptyset)\} \neq \emptyset_S$$

$$(S_z \circ f_S \circ S_k \circ f_S) \tilde{\cap} f_S = \{(z, \{-3, -2\}), (k, \emptyset)\} \neq \emptyset_S$$

Therefore, we have shown that for all $x, y \in S$, $\big(S_x \circ f_S \circ S_y \circ f_S\big) \tilde{\cap} f_S \neq \emptyset_S$, so f_S is an SI-almost left QI-ideal. Now let's show that f_S is an SI-almost right QI-ideal. As usual calculations, we have

$$(f_S \circ S_k \circ f_S \circ S_z) \tilde{\cap} f_S = \{(z, \{-3, -2\}), (k, \emptyset)\} \neq \emptyset_S$$

$$(f_S \circ S_z \circ f_S \circ S_k) \tilde{\cap} f_S = \{(z, \{-3, -2\}), (k, \emptyset)\} \neq \emptyset_S$$

Therefore, we have shown that for all $x, y \in S$, $\big(f_S \circ S_x \circ f_S \circ S_y\big) \tilde{\cap} f_S \neq \emptyset_S$, so f_S is an SI-almost right QI-ideal. Thus f_S is an SI-almost QI-ideal. Similarly, h_S is an SI-weakly almost QI-ideal since

$$(S_k \circ h_S \circ S_k \circ h_S) \tilde{\cap} h_S = \{(z, \{-9, -8\}), (k, \{-1\})\} \neq \emptyset_S$$

$$(S_z \circ h_S \circ S_z \circ h_S) \tilde{\cap} h_S = \{(z, \{-9, -8\}), (k, \emptyset)\} \neq \emptyset_S$$

$$(h_S \circ S_k \circ h_S \circ S_k) \tilde{\cap} h_S = \{(z, \{-9, -8\}), (k, \{-1\})\} \neq \emptyset_S$$

$$(h_S \circ S_z \circ h_S \circ S_z) \tilde{\cap} h_S = \{(z, \{-9, -8\}), (k, \emptyset)\} \neq \emptyset_S$$

and also since

$$(S_k \circ h_S \circ S_z \circ h_S) \tilde{\cap} h_S = \{(z, \{-9, -8\}), (k, \emptyset)\} \neq \emptyset_S$$

$$(S_z \circ h_S \circ S_k \circ h_S) \tilde{\cap} h_S = \{(z, \{-9, -8\}), (k, \emptyset)\} \neq \emptyset_S$$

$$(h_S \circ S_k \circ h_S \circ S_z) \tilde{\cap} h_S = \{(z, \{-9, -8\}), (k, \emptyset)\} \neq \emptyset_S$$

$$(h_S \circ S_z \circ h_S \circ S_k) \tilde{\cap} h_S = \{(z, \{-9, -8\}), (k, \emptyset)\} \neq \emptyset_S$$

h_S is an SI-almost QI-ideal. One can also show that g_S is not an SI-almost QI-ideal. In fact;

$[(S_z \circ g_S \circ S_z \circ g_S) \tilde{\cap} g_S](z) = [(S_z \circ g_S) \circ (S_z \circ g_S)](z) \cap g_S(z) = \Big[\big((S_z \circ g_S)(z) \cap (S_z \circ g_S)(z)\big) \cup \big((S_z \circ g_S)(z) \cap (S_z \circ g_S)(k)\big) \cup \big((S_z \circ g_S)(k) \cap (S_z \circ g_S)(z)\big)\Big] \cap g_S(z) = \Big[\Big[\Big(\big(S_z(z) \cap g_S(z)\big) \cup \big(S_z(z) \cap g_S(k)\big) \cup \big(S_z(k) \cap g_S(z)\big)\Big) \cap \Big(\big(S_z(z) \cap g_S(z)\big) \cup \big(S_z(z) \cap g_S(k)\big) \cup \big(S_z(k) \cap g_S(z)\big)\Big)\Big] \cup \Big[\Big(\big(S_z(z) \cap g_S(z)\big) \cup \big(S_z(z) \cap g_S(k)\big) \cup \big(S_z(k) \cap g_S(z)\big)\Big) \cap \Big(\big(S_z(k) \cap g_S(k)\big)\Big)\Big] \cup \Big[\Big(\big(S_z(k) \cap g_S(k)\big)\Big) \cap \Big(\big(S_z(z) \cap g_S(z)\big) \cup \big(S_z(z) \cap g_S(k)\big) \cup \big(S_z(k) \cap g_S(z)\big)\Big)\Big]\Big] \cap g_S(z) = \Big[\big[\big(g_S(z) \cup g_S(k)\big) \cap \big(g_S(z) \cup g_S(k)\big)\big] \cup \big[\big(g_S(z) \cup g_S(k)\big) \cap \emptyset\big] \cup \big[\emptyset \cap \big(g_S(z) \cup g_S(k)\big)\big]\Big] \cap g_S(z) = \big[\big(g_S(z) \cup g_S(k)\big) \cup \emptyset \cup \emptyset\big] \cap g_S(z) = g_S(z) = \emptyset$

$[(S_z \circ g_S \circ S_z \circ g_S) \tilde{\cap} g_S](k) = [(S_z \circ g_S) \circ (S_z \circ g_S)](k) \cap g_S(k) = [(S_z \circ g_S)(k) \cap (S_z \circ g_S)(k)] \cap g_S(k) = \big[\big(S_z(k) \cap g_S(k)\big) \cap \big(S_z(k) \cap g_S(k)\big)\big] \cap g_S(k) = \emptyset \cap \emptyset \cap g_S(k) = \emptyset$

Thus, for $z \in S$;

$$(S_z \circ g_S \circ S_z \circ g_S) \tilde{\cap} g_S = \{(z, \emptyset), (k, \emptyset)\} = \emptyset_S$$

Hence, g_S is not an SI-weakly almost QI-ideal, thus it is not an SI-almost QI-ideal.

Proposition 3.3. Every SI-almost QI-ideal is an SI-weakly almost QI-ideal.

Proof: Let f_S be an SI-almost QI-ideal of S. Then, for all $x, y \in S$, $\big(S_x \circ f_S \circ S_y \circ f_S\big) \tilde{\cap} f_S \neq \emptyset_S$ and $\big(f_S \circ S_x \circ f_S \circ S_y\big) \tilde{\cap} f_S \neq \emptyset_S$. Hence, for all $x \in S$, $(S_x \circ f_S \circ S_x \circ f_S) \tilde{\cap} f_S \neq \emptyset_S$ and $(f_S \circ S_x \circ f_S \circ S_x) \tilde{\cap} f_S \neq \emptyset_S$. So, f_S is an SI-weakly almost QI-ideal.

Since SI-weakly almost QI-ideal is a generalization of SI-almost QI-ideal, from now on, all the theorems and proofs are given for SI-almost QI-ideals instead of SI-weakly almost QI-ideals.

The converse of Proposition 3.3 is not true in general as shown in the following example:

Example 3.4. Let $S = \{a, r\}$ be the semigroup with the following Cayley Table.

	a	r
a	r	a
r	a	r

f_s be soft sets over $U = \mathbb{Z}$ as follows:

$$f_S = \{(a, \emptyset), (r, \{-6,3,6\})\}$$

Let's first show that f_S is an SI-weakly almost QI-ideal, that is for all $x \in S$,

$$(S_x \circ f_S \circ S_x \circ f_S) \tilde{\cap} f_S \neq \emptyset_S$$

Let's start with S_a, S_a:

$[(S_a \circ f_S \circ S_a \circ f_S) \tilde{\cap} f_S](a) = [(S_a \circ f_S) \circ (S_a \circ f_S)](a) \cap f_S(a) = \big[\big((S_a \circ f_S)(a) \cap (S_a \circ f_S)(r)\big) \cup \big((S_a \circ f_S)(r) \cap (S_a \circ f_S)(a)\big)\big] \cap f_S(a) = \Big[\Big[\Big(\big(S_a(a) \cap f_S(r)\big) \cup \big(S_a(r) \cap f_S(a)\big)\Big) \cap \Big(\big(S_a(a) \cap f_S(a)\big) \cup \big(S_a(r) \cap f_S(r)\big)\Big)\Big] \cup \Big[\Big(\big(S_a(a) \cap f_S(a)\big) \cup \big(S_a(r) \cap f_S(r)\big)\Big) \cap \Big(\big(S_a(a) \cap f_S(r)\big) \cup \big(S_a(r) \cap f_S(a)\big)\Big)\Big]\Big] \cap f_S(a) = \big[\big(f_S(r) \cap f_S(a)\big) \cup \big(f_S(a) \cap f_S(r)\big)\big] \cap f_S(a) = \big(f_S(a) \cap f_S(r)\big) \cap f_S(a) = f_S(a) \cap f_S(r) = \emptyset$

$[(S_a \circ f_S \circ S_a \circ f_S) \tilde{\cap} f_S](r) = [(S_a \circ f_S) \circ (S_a \circ f_S)](r) \cap f_S(r) = \big[\big((S_a \circ f_S)(a) \cap (S_a \circ f_S)(a)\big) \cup \big((S_a \circ f_S)(r) \cap (S_a \circ f_S)(r)\big)\big] \cap f_S(r) = \Big[\Big[\Big(\big(S_a(a) \cap f_S(r)\big) \cup \big(S_a(r) \cap f_S(a)\big)\Big) \cap \Big(\big(S_a(a) \cap f_S(r)\big) \cup \big(S_a(r) \cap f_S(a)\big)\Big)\Big] \cup \Big[\Big(\big(S_a(a) \cap f_S(a)\big) \cup \big(S_a(r) \cap f_S(r)\big)\Big) \cap \Big(\big(S_a(a) \cap f_S(a)\big) \cup \big(S_a(r) \cap f_S(r)\big)\Big)\Big]\Big] \cap f_S(r) = \big[\big(f_S(r) \cap f_S(r)\big) \cup \big(f_S(a) \cap f_S(a)\big)\big] \cap f_S(r) = \big(f_S(r) \cup f_S(a)\big) \cap f_S(r) = f_S(r) = \{-6,3,6\}$

Hence,

$$(S_a \circ f_S \circ S_a \circ f_S) \tilde{\cap} f_S = \{(a, \emptyset), (r, \{-6,3,6\})\} \neq \emptyset_S$$

And also,

$$(S_r \circ f_S \circ S_r \circ f_S) \tilde{\cap} f_S = \{(a, \emptyset), (r, \{-6,3,6\})\} \neq \emptyset_S$$

Therefore, for all $x \in S$, $(S_x \circ f_S \circ S_x \circ f_S) \tilde{\cap} f_S \neq \emptyset_S$, so f_S is an SI-weakly almost left QI-ideal. Similarly,

$$(f_S \circ S_a \circ f_S \circ S_a) \tilde{\cap} f_S = \{(a, \emptyset), (r, \{-6,3,6\})\} \neq \emptyset_S$$

$$(f_S \circ S_r \circ f_S \circ S_r) \tilde{\cap} f_S = \{(a, \emptyset), (r, \{-6,3,6\})\} \neq \emptyset_S$$

Hence, for all $x \in S$, $(f_S \circ S_x \circ f_S \circ S_x) \tilde{\cap} f_S \neq \emptyset_S$, so f_S is an SI-weakly almost right QI-ideal. Thus f_S is an SI-weakly almost QI-ideal.

However, here not that f_S is not an SI-almost QI-ideal. In deed;

$[(S_a \circ f_S \circ S_r \circ f_S) \tilde{\cap} f_S](a) = [(S_a \circ f_S) \circ (S_r \circ f_S)](a) \cap f_S(a) = \big[\big((S_a \circ f_S)(a) \cap (S_r \circ f_S)(r)\big) \cup \big((S_a \circ f_S)(r) \cap (S_r \circ f_S)(a)\big)\big] \cap f_S(a) = \Big[\Big[\Big(\big(S_a(a) \cap f_S(r)\big) \cup \big(S_a(r) \cap f_S(a)\big)\Big) \cap \Big(\big(S_r(a) \cap f_S(a)\big) \cup \big(S_r(r) \cap f_S(r)\big)\Big)\Big] \cup \Big[\Big(\big(S_a(a) \cap f_S(a)\big) \cup \big(S_a(r) \cap f_S(r)\big)\Big) \cap \Big(\big(S_r(a) \cap f_S(r)\big) \cup \big(S_r(r) \cap f_S(a)\big)\Big)\Big]\Big] \cap f_S(a) = \big[\big(f_S(r) \cap f_S(r)\big) \cup \big(f_S(a) \cap f_S(a)\big)\big] \cap f_S(a) = \big(f_S(r) \cup f_S(a)\big) \cap f_S(a) = f_S(a) = \emptyset$

$[(S_a \circ f_S \circ S_r \circ f_S) \tilde{\cap} f_S](r) = [(S_a \circ f_S)\circ(S_r \circ f_S)](r) \cap f_S(r) = \big[\big((S_a \circ f_S)(a) \cap (S_r \circ f_S)(a)\big) \cup \big((S_a \circ f_S)(r) \cap (S_r \circ f_S)(r)\big)\big] \cap f_S(r) = \Big[\Big[\Big(\big(S_a(a) \cap f_S(r)\big) \cup \big(S_a(r) \cap f_S(a)\big)\Big) \cap \Big(\big(S_r(a) \cap f_S(r)\big) \cup \big(S_r(r) \cap f_S(a)\big)\Big)\Big] \cup \Big[\Big(\big(S_a(a) \cap f_S(a)\big) \cup \big(S_a(r) \cap f_S(r)\big)\Big) \cap \Big(\big(S_r(a) \cap f_S(a)\big) \cup \big(S_r(r) \cap f_S(r)\big)\Big)\Big]\Big] \cap f_S(r) = \big[\big(f_S(r) \cap f_S(a)\big) \cup \big(f_S(a) \cap f_S(r)\big)\big] \cap f_S(r) = \big(f_S(a) \cap f_S(r)\big) \cap f_S(r) = f_S(a) \cap f_S(r) = \emptyset$

Consequently,

$$(S_a \circ f_S \circ S_r \circ f_S) \tilde{\cap} f_S = \{(a,\emptyset),(r,\emptyset)\} = \emptyset_S$$

Thus, f_S is not an SI-almost QI-ideal.

Proposition 3.5. If f_S is an SI-left (resp. right) QI-ideal such that $f_S \neq \emptyset_S$, then f_S is an SI-almost left (resp. right) QI-ideal.

Proof: Let $f_S \neq \emptyset_S$ be an SI-left QI-ideal, then $\tilde{\mathbb{S}} \circ f_S \circ \tilde{\mathbb{S}} \circ f_S \tilde{\subseteq} f_S$. Since $f_S \neq \emptyset_S$, by Corollary 2.16, it follows that $S_x \circ f_S \circ S_y \circ f_S \neq \emptyset_S$. We need to show that for all $x, y \in S$,

$$\big(S_x \circ f_S \circ S_y \circ f_S\big) \tilde{\cap} f_S \neq \emptyset_S.$$

Since $S_x \circ f_S \circ S_y \circ f_S \tilde{\subseteq} \tilde{\mathbb{S}} \circ f_S \circ \tilde{\mathbb{S}} \circ f_S \tilde{\subseteq} f_S$, it follows that $S_x \circ f_S \circ S_y \circ f_S \tilde{\subseteq} f_S$. Thus,

$$\big(S_x \circ f_S \circ S_y \circ f_S\big) \tilde{\cap} f_S = S_x \circ f_S \circ S_y \circ f_S \neq \emptyset_S$$

implying that f_S is an SI-almost left QI-ideal.

Here it is obvious that $\emptyset_S$ is an SI-left QI-ideal, as $\tilde{\mathbb{S}} \circ \emptyset_S \circ \tilde{\mathbb{S}} \circ \emptyset_S \tilde{\subseteq} \emptyset_S$; but it is not an SI-almost QI-ideal, since $\big(S_x \circ \emptyset_S \circ S_y \circ \emptyset_S\big) \tilde{\cap} \emptyset_S = \emptyset_S \tilde{\cap} \emptyset_S = \emptyset_S$ for all $x, y \in S$.

Here note that if f_S is an SI-almost left (resp. right) QI-ideal, then f_S needs not to be an SI-left (resp. right) QI-ideal as shown in the following example:

Example 3.6. In Example 3.2, it is shown that f_S and h_S are SI-almost QI-ideals; however f_S and h_S are not SI-QI-ideals. In fact;

$(\tilde{\mathbb{S}} \circ f_S \circ \tilde{\mathbb{S}} \circ f_S)(z) = \big[\big(\tilde{\mathbb{S}} \circ f_S\big)(z) \cap \big(\tilde{\mathbb{S}} \circ f_S\big)(z)\big] \cup \big[\big(\tilde{\mathbb{S}} \circ f_S\big)(z) \cap \big(\tilde{\mathbb{S}} \circ f_S\big)(k)\big] \cup \big[\big(\tilde{\mathbb{S}} \circ f_S\big)(k) \cap \big(\tilde{\mathbb{S}} \circ f_S\big)(z)\big] = \Big[\Big[\Big(\tilde{\mathbb{S}}(z) \cap f_S(z)\Big) \cup \Big(\tilde{\mathbb{S}}(z) \cap f_S(k)\Big) \cup \Big(\tilde{\mathbb{S}}(k) \cap f_S(z)\Big)\Big] \cap \Big[\Big(\tilde{\mathbb{S}}(z) \cap f_S(z)\Big) \cup \Big(\tilde{\mathbb{S}}(z) \cap f_S(k)\Big) \cup \Big(\tilde{\mathbb{S}}(k) \cap f_S(z)\Big)\Big]\Big] \cup \Big[\Big[\Big(\tilde{\mathbb{S}}(z) \cap f_S(z)\Big) \cup \Big(\tilde{\mathbb{S}}(z) \cap f_S(k)\Big) \cup \Big(\tilde{\mathbb{S}}(k) \cap f_S(z)\Big)\Big] \cap \Big[\Big(\tilde{\mathbb{S}}(k) \cap f_S(k)\Big)\Big]\Big] \cup \Big[\Big[\Big(\tilde{\mathbb{S}}(k) \cap f_S(k)\Big)\Big] \cap \Big[\Big(\tilde{\mathbb{S}}(z) \cap f_S(z)\Big) \cup \Big(\tilde{\mathbb{S}}(z) \cap f_S(k)\Big) \cup \Big(\tilde{\mathbb{S}}(k) \cap f_S(z)\Big)\Big]\Big] = \big[\big(f_S(z) \cup f_S(k) \cup f_S(z)\big) \cap \big(f_S(z) \cup f_S(k) \cup f_S(z)\big)\big] \cup \big[\big(f_S(z) \cup f_S(k) \cup f_S(z)\big) \cap f_S(k)\big] \cup \big[\big(f_s(k) \cap \big(f_S(z) \cup f_S(k) \cup f_S(z)\big)\big] = \big(f_S(z) \cup f_S(k)\big) \cup f_s(k) \cup f_s(k) = \big(f_S(z) \cup f_S(k)\big) = \{-5,-3,-2\} \not\subseteq f_S(z) = \{-3,-2\}$

Or similarly,

$(f_S \circ \tilde{\mathbb{S}} \circ f_S \circ \tilde{\mathbb{S}})(z) = \big[(f_S \circ \tilde{\mathbb{S}})(z) \cap (f_S \circ \tilde{\mathbb{S}})(z)\big] \cup \big[(f_S \circ \tilde{\mathbb{S}})(z) \cap (f_S \circ \tilde{\mathbb{S}})(k)\big] \cup \big[(f_S \circ \tilde{\mathbb{S}})(k) \cap (f_S \circ \tilde{\mathbb{S}})(z)\big] = \Big[\Big[\Big(f_S(z) \cap \tilde{\mathbb{S}}(z)\Big) \cup \Big(f_S(z) \cap \tilde{\mathbb{S}}(k)\Big) \cup \Big(f_S(k) \cap \tilde{\mathbb{S}}(z)\Big)\Big] \cap \Big[\Big(f_S(z) \cap \tilde{\mathbb{S}}(z)\Big) \cup \Big(f_S(z) \cap \tilde{\mathbb{S}}(k)\Big) \cup \Big(f_S(k) \cap \tilde{\mathbb{S}}(z)\Big)\Big]\Big] \cup \Big[\Big[\Big(f_S(z) \cap \tilde{\mathbb{S}}(z)\Big) \cup \Big(f_S(z) \cap \tilde{\mathbb{S}}(k)\Big) \cup \Big(f_S(k) \cap \tilde{\mathbb{S}}(z)\Big)\Big] \cap \Big[\Big(f_S(k) \cap$

$\widetilde{\mathbb{S}}(k))]] \cup [[(f_S(k) \cap \widetilde{\mathbb{S}}(k))] \cap [(f_S(z) \cap \widetilde{\mathbb{S}}(z)) \cup (f_S(z) \cap \widetilde{\mathbb{S}}(k)) \cup (f_S(k) \cap \widetilde{\mathbb{S}}(z))]] = [(f_S(z) \cup f_S(z) \cup f_S(k)) \cap (f_S(z) \cup f_S(z) \cup f_S(k))] \cup [(f_S(z) \cup f_S(z) \cup f_S(k)) \cap f_S(k)] \cup [(f_S(k) \cap (f_S(z) \cup f_S(z) \cup f_S(k))] = (f_S(z) \cup f_S(k)) \cup f_S(k) \cup f_S(k) = (f_S(z) \cup f_S(k)) = \{-5,-3,-2\} \nsubseteq f_S(z) = \{-3,-2\}$

Thus, f_S is not an SI-QI-ideal. Similarly,

$$(\widetilde{\mathbb{S}} \circ h_S \circ \widetilde{\mathbb{S}} \circ h_S)(z) = (h_S(z) \cup h_S(k)) = \{-9,-8,-1\} \nsubseteq h_S(z) = \{-9,-8\} \text{ or}$$

$$(h_S \circ \widetilde{\mathbb{S}} \circ h_S \circ \widetilde{\mathbb{S}})(z) = (h_S(z) \cup h_S(k)) = \{-9,-8,-1\} \nsubseteq h_S(z) = \{-9,-8\}$$

Hence, h_S is not an SI-QI-ideal.

Proposition 3.7. Let f_S be an idempotent SI-almost left (right) QI-ideal. Then, f_S is an SI-almost subsemigroup.

Proof: Assume that f_S is an idempotent SI-almost left QI-ideal. Then, $f_S \circ f_S = f_S$ and for all $x, y \in S$, $(f_S \circ S_x \circ f_S \circ S_y) \widetilde{\cap} f_S \neq \emptyset_S$. We need to show that for all $x \in S$,

$$(f_S \circ f_S) \widetilde{\cap} f_S \neq \emptyset_S$$

Since,

$$\emptyset_S \neq (f_S \circ S_x \circ f_S \circ S_y) \widetilde{\cap} f_S = [(f_S \circ S_x \circ f_S \circ S_y) \widetilde{\cap} f_S] \widetilde{\cap} f_S = [(f_S \circ S_x \circ f_S \circ S_y) \widetilde{\cap} f_S] \widetilde{\cap} (f_S \circ f_S) \widetilde{\subseteq} (f_S \circ f_S) \widetilde{\cap} f_S$$

hence f_S is an SI-almost subsemigroup.

Theorem 3.7. Let $f_S \widetilde{\subseteq} h_S$ such that f_S is an SI-almost left (resp. right) QI-ideal, then h_S is an SI-almost left (resp. right) QI-ideal.

Proof: Assume that f_S is an SI-almost left QI-ideal. Hence, for all $x, y \in S$, $(S_x \circ f_S \circ S_y \circ f_S) \widetilde{\cap} f_S \neq \emptyset_S$. We need to show that $(S_x \circ h_S \circ S_y \circ h_S) \widetilde{\cap} h_S \neq \emptyset_S$. In fact,

$$(S_x \circ f_S \circ S_y \circ f_S) \widetilde{\cap} f_S \widetilde{\subseteq} (S_x \circ h_S \circ S_y \circ h_S) \widetilde{\cap} h_S$$

Since $(S_x \circ f_S \circ S_y \circ f_S) \widetilde{\cap} f_S \neq \emptyset_S$, it is obvious that $(S_x \circ h_S \circ S_y \circ h_S) \widetilde{\cap} h_S \neq \emptyset_S$. This completes the proof.

Theorem 3.8. Let f_S and h_S be SI-almost left (resp. right) QI-ideals. Then, $f_S \widetilde{\cup} h_S$ is an SI-almost left (resp. right) QI-ideal.

Proof: Since f_S is an SI-almost left QI-ideal by assumption and $f_S \widetilde{\subseteq} f_S \widetilde{\cup} h_S$, $f_S \widetilde{\cup} h_S$ is an SI-almost left QI-ideal by Theorem 3.7.

Here note that if f_S and h_S are SI-almost left (resp. right) QI-ideals, then $f_S \widetilde{\cap} h_S$ needs not to be an SI-almost left (resp. right) QI-ideal.

Example 3.9. Consider the SI-almost QI-ideals f_S and h_S in Example 3.2. Since,

$$f_S \widetilde{\cap} h_S = \{(z,\emptyset),(k,\emptyset)\} = \emptyset_S$$

$f_S \widetilde{\cap} h_S$ is not an SI-almost QI-ideals.

Now, we give the relationship between almost QI-ideal and SI-almost QI-ideal. But first of all, we remind the following lemma to use it in Theorem 3.11.

Lemma 3.10. Let $x \in S$ and Y be nonempty subset of S. Then, $S_x \circ S_Y = S_{xY}$. If X is a nonempty subset of S and $y \in S$, then it is obvious that $S_X \circ S_y = S_{Xy}$ [52].

Theorem 3.11. Let A be a nonempty subset of S. Then, A is an almost left (resp. right) QI-ideal if and only if S_A, the soft characteristic function of A, is an SI-almost left (resp. right) QI-ideal.

Proof: Assume that $\emptyset \neq A$ is an almost left QI-ideal. Then, $xAyA \cap A \neq \emptyset$ for all $x, y \in S$, and so there exist $t \in S$ such that $t \in xAyA \cap A$. Since,

$$\left((S_x \circ S_A \circ S_y \circ S_A) \tilde{\cap} S_A\right)(t) = (S_{xAyA} \tilde{\cap} S_A)(t) = (S_{xAyA \cap A})(t) = U \neq \emptyset$$

it follows that $(S_x \circ S_A \circ S_y \circ S_A) \tilde{\cap} S_A \neq \emptyset_S$. Thus, S_A is an SI-almost left QI-ideal.

Conversely assume that S_A is an SI-almost left QI-ideal. Hence, we have $(S_x \circ S_A \circ S_y \circ S_A) \tilde{\cap} S_A \neq \emptyset_S$ for all $x, y \in S$. To show that A is an almost left QI-ideal of S, we should show that $A \neq \emptyset$ and $xAyA \cap A \neq \emptyset$ for all $x, y \in S$. $A \neq \emptyset$ is obvious from assumption. Now,

$$\begin{aligned}
\emptyset_S \neq (S_x \circ S_A \circ S_y \circ S_A) \tilde{\cap} S_A &\Rightarrow \exists n \in S \,;\, \left((S_x \circ S_A \circ S_y \circ S_A) \tilde{\cap} S_A\right)(n) \neq \emptyset \\
&\Rightarrow \exists n \in S \,;\, (S_{xAyA} \tilde{\cap} S_A)(n) \neq \emptyset \\
&\Rightarrow \exists n \in S \,;\, (S_{xAyA \cap A})(n) \neq \emptyset \\
&\Rightarrow \exists n \in S \,;\, (S_{xAyA \cap A})(n) = U \\
&\Rightarrow n \in xAyA \cap A
\end{aligned}$$

Hence, $xAyA \cap A \neq \emptyset$ for all $x, y \in S$. Consequently, A is an almost left QI-ideal.

Lemma 3.12. Let f_S be a soft set over U. Then, $f_S \tilde{\subseteq} S_{supp(f_S)}$ [51].

Theorem 3.13. If f_S is an SI-almost left (resp. right) QI-ideal, then $supp(f_S)$ is an almost left (resp. right) QI-ideal.

Proof: Assume that f_S is an SI-almost left QI-ideal. Thus, $f_S \neq \emptyset_S$, $supp(f_S) \neq \emptyset$ and $(S_x \circ f_S \circ S_y \circ f_S) \tilde{\cap} f_S \neq \emptyset_S$ for all $x, y \in S$. To show that $supp(f_S)$ is an almost left QI-ideal, by Theorem 3.11, it is enough to show that $S_{supp(f_S)}$ is an SI-almost left QI-ideal. By Lemma 3.12,

$$(S_x \circ f_S \circ S_y \circ f_S) \tilde{\cap} f_S \tilde{\subseteq} (S_x \circ S_{supp(f_S)} \circ S_y \circ S_{supp(f_S)}) \tilde{\cap} S_{supp(f_S)}$$

and since $(S_x \circ f_S \circ S_y \circ f_S) \tilde{\cap} f_S \neq \emptyset_S$, it implies that $(S_x \circ S_{supp(f_S)} \circ S_y \circ S_{supp(f_S)}) \tilde{\cap} S_{supp(f_S)} \neq \emptyset_S$. Consequently, $S_{supp(f_S)}$ is an SI-almost left QI-ideal of S and by Theorem 3.11, $supp(f_S)$ is an almost left QI-ideal.

Here note that the converse of Theorem 3.13 is not true in general as shown in the following example.

Example 3.14. Let $S = \{h, i, k\}$ be the semigroup with the following Cayley Table.

	h	i	k
h	k	h	h
i	h	k	k
k	h	k	k

f_S be soft sets over $U = \mathbb{Z}$ as follows:

$$f_S = \{(h, \{1,9\}), (i, \{0,5\}), (k, \emptyset)\}$$

Let's first show that f_S is not an SI-almost QI-ideal:

$[(f_S \circ S_i \circ f_S \circ S_i) \widetilde{\cap} f_S](h) = [(f_S \circ S_i) \circ (f_S \circ S_i)](h) \cap f_S(h) = \Big[\big((f_S \circ S_i)(h) \cap (f_S \circ S_i)(i)\big) \cup \big((f_S \circ S_i)(h) \cap (f_S \circ S_i)(k)\big) \cup \big((f_S \circ S_i)(i) \cap (f_S \circ S_i)(h)\big) \cup \big((f_S \circ S_i)(k) \cap (f_S \circ S_i)(h)\big)\Big] \cap f_S(h) = \Bigg[\bigg[\Big(\big(f_S(h) \cap S_i(i)\big) \cup \big(f_S(h) \cap S_i(k)\big) \cup \big(f_S(i) \cap S_i(h)\big) \cup \big(f_S(k) \cap S_i(h)\big)\Big) \cap \emptyset\bigg] \cup \bigg[\Big(\big(f_S(h) \cap S_i(i)\big) \cup \big(f_S(h) \cap S_i(k)\big) \cup \big(f_S(i) \cap S_i(h)\big) \cup \big(f_S(k) \cap S_i(h)\big)\Big) \cap \Big(\big(f_S(h) \cap S_i(h)\big) \cup \big(f_S(i) \cap S_i(i)\big) \cup \big(f_S(i) \cap S_i(k)\big) \cup \big(f_S(k) \cap S_i(i)\big) \cup \big(f_S(k) \cap S_i(k)\big)\Big)\bigg] \cup \bigg[\emptyset \cap \Big(\big(f_S(h) \cap S_i(i)\big) \cup \big(f_S(h) \cap S_i(k)\big) \cup \big(f_S(i) \cap S_i(h)\big) \cup \big(f_S(k) \cap S_i(h)\big)\Big)\bigg] \cup \bigg[\Big(\big(f_S(h) \cap S_i(h)\big) \cup \big(f_S(i) \cap S_i(i)\big) \cup \big(f_S(i) \cap S_i(k)\big) \cup \big(f_S(k) \cap S_i(i)\big) \cup \big(f_S(k) \cap S_i(k)\big)\Big) \cap \Big(\big(f_S(h) \cap S_i(i)\big) \cup \big(f_S(h) \cap S_i(k)\big) \cup \big(f_S(i) \cap S_i(h)\big) \cup \big(f_S(k) \cap S_i(h)\big)\Big)\bigg]\Bigg] \cap f_S(h) = \Big[\emptyset \cup \big[f_S(h) \cap \big(f_S(i) \cup f_S(k)\big)\big] \cup \emptyset \cup \big[\big(f_S(i) \cup f_S(k)\big) \cap f_S(h)\big]\Big] \cap f_S(h) = \big[\big(f_S(i) \cup f_S(k)\big) \cap f_S(h)\big] \cap f_S(h) = \emptyset$.

Similarly, $[(f_S \circ S_i \circ f_S \circ S_i) \widetilde{\cap} f_S](i) = \emptyset$ and $[(f_S \circ S_i \circ f_S \circ S_i) \widetilde{\cap} f_S](k) = \emptyset$. Thus,

$$(f_S \circ S_i \circ f_S \circ S_i) \widetilde{\cap} f_S = \{(h, \emptyset), (i, \emptyset), (k, \emptyset)\} = \emptyset_S$$

Thus f_s is not an SI-almost QI-ideal. Let's show that $supp(f_S) = \{h, i\}$ is an almost QI-ideal. In deed

$$[\{h\}supp(f_S)\{h\}supp(f_S)] \cap supp(f_S) = [\{h\}\{h,i\}\{h\}\{h,i\}] \cap \{h,i\} = \{h\} \neq \emptyset$$

$$[\{h\}supp(f_S)\{i\}supp(f_S)] \cap supp(f_S) = [\{h\}\{h,i\}\{i\}\{h,i\}] \cap \{h,i\} = \{h\} \neq \emptyset$$

$$[\{h\}supp(f_S)\{k\}supp(f_S)] \cap supp(f_S) = [\{h\}\{h,i\}\{k\}\{h,i\}] \cap \{h,i\} = \{h\} \neq \emptyset$$

$$[\{i\}supp(f_S)\{h\}supp(f_S)] \cap supp(f_S) = [\{i\}\{h,i\}\{h\}\{h,i\}] \cap \{h,i\} = \{h\} \neq \emptyset$$

$$[\{i\}supp(f_S)\{i\}supp(f_S)] \cap supp(f_S) = [\{i\}\{h,i\}\{i\}\{h,i\}] \cap \{h,i\} = \{h\} \neq \emptyset$$

$$[\{i\}supp(f_S)\{k\}supp(f_S)] \cap supp(f_S) = [\{i\}\{h,i\}\{k\}\{h,i\}] \cap \{h,i\} = \{h\} \neq \emptyset$$

$$[\{k\}supp(f_S)\{h\}supp(f_S)] \cap supp(f_S) = [\{k\}\{h,i\}\{h\}\{h,i\}] \cap \{h,i\} = \{h\} \neq \emptyset$$

$$[\{k\}supp(f_S)\{i\}supp(f_S)] \cap supp(f_S) = [\{k\}\{h,i\}\{i\}\{h,i\}] \cap \{h,i\} = \{h\} \neq \emptyset$$

$$[\{k\}supp(f_S)\{k\}supp(f_S)] \cap supp(f_S) = [\{k\}\{h,i\}\{k\}\{h,i\}] \cap \{h,i\} = \{h\} \neq \emptyset$$

it is seen that $[\{x\}supp(f_S)\{y\}supp(f_S)] \cap supp(f_S) \neq \emptyset$ for all $x, y \in S$. That is to say, $supp(f_S)$ is an almost left QI-ideal of S. Similarly,

$$[supp(f_S)\{h\}supp(f_S)\{h\}] \cap supp(f_S) = [\{h,i\}\{h\}\{h,i\}\{h\}] \cap \{h,i\} = \{h\} \neq \emptyset$$

$$[supp(f_S)\{h\}supp(f_S)\{i\}] \cap supp(f_S) = [\{h,i\}\{h\}\{h,i\}\{i\}] \cap \{h,i\} = \{h\} \neq \emptyset$$

$$[supp(f_S)\{h\}supp(f_S)\{k\}] \cap supp(f_S) = [\{h,i\}\{h\}\{h,i\}\{k\}] \cap \{h,i\} = \{h\} \neq \emptyset$$

$$[supp(f_S)\{i\}supp(f_S)\{h\}] \cap supp(f_S) = [\{h,i\}\{i\}\{h,i\}] \cap \{h,i\} = \{h\} \neq \emptyset$$

$$[supp(f_S)\{i\}supp(f_S)\{i\}] \cap supp(f_S) = [\{h,i\}\{i\}\{h,i\}\{i\}] \cap \{h,i\} = \{h\} \neq \emptyset$$

$$[supp(f_S)\{i\}supp(f_S)\{k\}] \cap supp(f_S) = [\{h,i\}\{i\}\{h,i\}\{k\}] \cap \{h,i\} = \{h\} \neq \emptyset$$

$$[supp(f_S)\{k\}supp(f_S)\{h\}] \cap supp(f_S) = [\{h,i\}\{k\}\{h,i\}\{h\}] \cap \{h,i\} = \{h\} \neq \emptyset$$

$$[supp(f_S)\{k\}supp(f_S)\{i\}] \cap supp(f_S) = [\{h,i\}\{k\}\{h,i\}] \cap \{h,i\} = \{h\} \neq \emptyset$$

$$[supp(f_S)\{k\}supp(f_S)\{k\}] \cap supp(f_S) = [\{h,i\}\{k\}\{h,i\}\{k\}] \cap \{h,i\} = \{h\} \neq \emptyset$$

It is seen that $[supp(f_S)\{x\}supp(f_S)\{y\}] \cap supp(f_S) \neq \emptyset$ for all $x, y \in S$. That is to say, $supp(f_S)$ is an almost right QI-ideal of S. Consequently, $supp(f_S)$ is an almost QI-ideal of S; however f_S is not an SI-almost QI-ideal.

Definition 3.15. An SI-almost left (resp. right) QI-ideal f_S is called minimal if any SI-almost left (resp. right) QI-ideal h_S if whenever $h_S \tilde{\subseteq} f_S$, then $supp(h_S) = supp(f_S)$.

Theorem 3.16. A is a minimal almost left (resp. right) QI-ideal if and only if S_A, the soft characteristic function of A, is a minimal SI-almost left (resp. right) QI-ideal, where $\emptyset \neq A \subseteq S$.

Proof: Assume that A is a minimal almost left QI-ideal. Thus, A is an almost left QI-ideal of S, and so S_A is an SI-almost left QI-ideal by Theorem 3.11. Let f_S be an SI-almost left QI-ideal such that $f_S \tilde{\subseteq} S_A$. By Theorem 3.13, $supp(f_S)$ is an almost left QI-ideal and by Note 2.6 and Corollary 2.13,

$$supp(f_S) \subseteq supp(S_A) = A.$$

Since A is a minimal almost left QI-ideal, $supp(f_S) = supp(S_A) = A$. Thus, S_A is a minimal SI-almost left QI-ideal by Definition 3.15.

Conversely, let S_A be a minimal SI-almost left QI-ideal. Thus, S_A is an SI-almost left QI-ideal of S and A is an almost left QI-ideal by Theorem 3.13. Let B be an almost left QI-ideal such that $B \subseteq A$. By Theorem 3.11, S_B is an SI-almost left QI-ideal, and by Theorem 2.14 (i), $S_B \tilde{\subseteq} S_A$. Since S_A is a minimal SI-almost left QI-ideal,

$$B = supp(S_B) = supp(S_A) = A$$

by Corollary 2.13. Thus, A is a minimal almost left QI-ideal.

Definition 3.17. Let f_S, g_S and h_S be any SI-almost left (resp. right) QI-ideals. If $h_S \circ g_S \tilde{\subseteq} f_S$ implies that $h_S \tilde{\subseteq} f_S$ or $g_S \tilde{\subseteq} f_S$, then f_S is called an SI-prime almost left (resp. right) QI-ideal.

Definition 3.18. Let f_S and h_S be any SI-almost left (resp. right) QI-ideals. If $h_S \circ h_S \tilde{\subseteq} f_S$ implies that $h_S \tilde{\subseteq} f_S$, then f_S is called an SI-semiprime almost left (resp. right) QI-ideal.

Definition 3.19. Let f_S, g_S and h_S be any SI-almost left (resp. right) QI-ideals. If $(h_S \circ g_S) \tilde{\cap} (g_S \circ h_S) \tilde{\subseteq} f_S$ implies that $h_S \tilde{\subseteq} f_S$ or $g_S \tilde{\subseteq} f_S$, then f_S is called an SI-strongly prime almost left (resp. right) QI-ideal.

It is obvious that every SI-strongly prime almost left (resp. right) QI-ideal of S is an SI-prime almost left (resp. right) QI-ideal and every SI-prime almost (left/right) QI-ideal of S is an SI-semiprime almost left (resp. right) QI-ideal.

Theorem 3.20. If S_P, the soft characteristic function of P, is an SI-prime almost left (resp. right) QI-ideal, then P is a prime almost left (resp. right) QI-ideal, where $\emptyset \neq P \subseteq S$.

Proof: Assume that S_P is an SI-prime almost left QI-ideal. Thus, S_P is an SI-almost left QI-ideal of S and thus, P is an almost left QI-ideal by Theorem 3.11. Let A and B be almost left QI-ideals such that $AB \subseteq P$. Thus, by Theorem 3.11, S_A and S_B are SI-almost left QI-ideals and by Theorem 2.14 (i) and (iii),

$$S_A \circ S_B = S_{AB} \tilde{\subseteq} S_P.$$

Since S_P is an SI-prime almost left QI-ideal and $S_A \circ S_B \tilde{\subseteq} S_P$, it follows that $S_A \tilde{\subseteq} S_P$ or $S_B \tilde{\subseteq} S_P$. Therefore, by Theorem 2.14 (i), $A \subseteq P$ or $B \subseteq P$. Consequently, P is a prime almost left QI-ideal.

Theorem 3.21. If S_P, the soft characteristic function of P, is an SI-semiprime left (resp. right) almost QI-ideal of S, then P is a semiprime almost left (resp. right) QI-ideal, where $\emptyset \neq P \subseteq S$.

Proof: Assume that S_P is an SI-semiprime almost left QI-ideal. Thus, S_P is an SI-almost left QI-ideal and thus, P is an almost left QI-ideal of S by Theorem 3.11. Let A be an almost left QI-ideal such that $AA \subseteq P$. Thus, by Theorem 3.11, S_A is an SI-almost left QI-ideal and by Theorem 2.14 (i) and (iii),

$$S_A \circ S_A = S_{AA} \tilde{\subseteq} S_P$$

Since S_P is an SI-semiprime almost left QI-ideal of S, and $S_A \circ S_A \tilde{\subseteq} S_P$, it follows that $S_A \tilde{\subseteq} S_P$. Therefore, by Theorem 2.14 (i), $A \subseteq P$. Consequently, P is a semiprime almost left QI-ideal.

Theorem 3.22. If S_P, the soft characteristic function of P, is an SI-strongly prime almost left (resp. right) QI-ideal, then P is a strongly prime almost left (resp. right) QI-ideal of S, where $\emptyset \neq P \subseteq S$.

Proof: Assume that S_P is an SI-strongly prime almost left QI-ideal. Thus, S_P is an SI-almost left QI-ideal of S and thus, P is an almost left QI-ideal by Theorem 3.11. Let A and B be almost left QI-ideals such that $AB \cap BA \subseteq P$. Thus, by Theorem 3.11, S_A and S_B are SI-almost left QI-ideals and by Theorem 2.14,

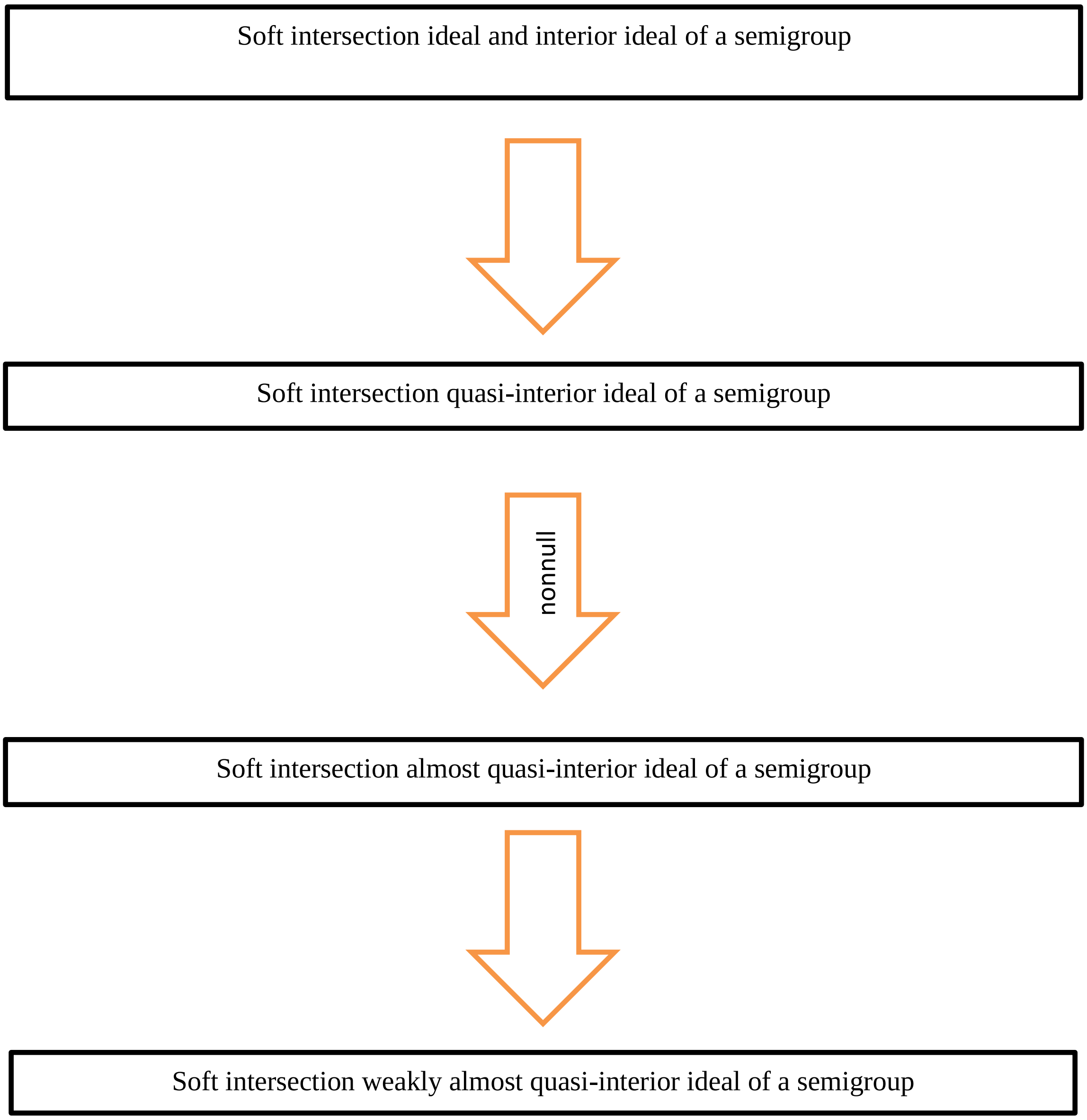
$$(S_A \circ S_B) \tilde{\cap} (S_B \circ S_A) = S_{AB} \tilde{\cap} S_{BA} = S_{AB \cap BA} \tilde{\subseteq} S_P$$

Since S_P is an SI-strongly prime almost left QI-ideal and $(S_A \circ S_B) \tilde{\cap} (S_B \circ S_A) \tilde{\subseteq} S_P$, it follows that $S_A \tilde{\subseteq} S_P$ or $S_B \tilde{\subseteq} S_P$. Thus, by Theorem 2.14 (i), $A \subseteq P$ or $B \subseteq P$. Therefore, P is a strongly prime almost left QI-ideal

4. CONCLUSION

In this study, we defined two notions of semigroups: "soft intersection almost quasi-interior ideal" and "soft intersection weakly almost quasi-interior ideal". We showed that while every nonnull soft intersection quasi-interior ideal is a soft intersection almost quasi-interior, and every soft intersection almost quasi-interior is a soft intersection weakly almost quasi-interior ideal; the converses are not true for counterexamples. Furthermore, it was demonstrated that an idempotent soft intersection almost quasi-interior ideal is a soft intersection almost subsemigroup. With the obtained theorem that if a nonempty set A is almost quasi-interior ideal, then its soft characteristic function is soft intersection almost quasi-interior ideal and vice versa, we obtained the relation among soft intersection almost quasi-interior ideal of a semigroup and almost quasi-interior ideal of a semigroup according with minimality, primeness, semiprimeness, and strongly primeness. Furthermore, we derived that the binary operation of soft union, in contrast to soft intersection operation, constructs a semigroup with the collection of soft intersection almost quasi-interior ideals. In future studies, many types of soft intersection almost ideals, including quasi-ideal, interior ideal, bi-ideal, bi-interior ideal, bi-quasi ideal, and bi-quasi-interior ideal of semigroups and their interrelations can be examined.

The relation between several soft intersection ideals and their generalized ideals is depicted in the following figure, where A➡B denotes that A is B but B may not always be A.



CONFLICT OF INTEREST

The authors stated that there are no conflicts of interest regarding the publication of this article.

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REFERENCES

[1] Good RA, Hughes DR. Associated groups for a semigroup. Bull Amer Math Soc 1952; 58: 624–625.

[2] Steinfeld O. Uher die quasi ideals. Von halbgruppend Publ Math Debrecen 1956; 4: 262–275.

[3] Grosek O, Satko L. A new notion in the theory of semigroup. Semigroup Forum 1980; 20: 233–240.

[4] Bogdanovic S. Semigroups in which some bi-ideal is a group. Univ u Novom Sadu Zb Rad Prirod Mat Fak Ser Mat 1981; 11: 261–266.

[5] Wattanatripop K, Chinram R, Changphas T. Quasi-A-ideals and fuzzy A-ideals in semigroups. J Discrete Math Sci Cryptogr 2018; 21: 1131–1138.

[6] Kaopusek N, Kaewnoi T, Chinram R. On almost interior ideals and weakly almost interior ideals of semigroups. J Discrete Math Sci Cryptogr 2020; 23: 773–778.

[7] Iampan A, Chinram R, Petchkaew P. A note on almost subsemigroups of semigroups. Int J Math Comput Sci 2021; 16(4): 1623–1629.

[8] Chinram R, Nakkhasen W. Almost bi-quasi-interior ideals and fuzzy almost bi-quasi-interior ideals of semigroups. J Math Comput Sci 2022; 26: 128–136.

[9] Gaketem T. Almost bi-interior ideal in semigroups and their fuzzifications. Eur J Pure Appl Math 2022; 15(1): 281–289.

[10] Gaketem T, Chinram R. Almost bi-quasi ideals and their fuzzifcations in semigroups. Ann Univ Craiova Math Comput Sci Ser 2023; 50(2): 42–352.

[11] Wattanatripop K, Chinram R, Changphas T. Fuzzy almost bi-ideals in semigroups. Int J Math Comput Sci 2018; 13: 51–58.

[12] Krailoet W, Simuen A, Chinram R, Petchkaew P. A note on fuzzy almost interior ideals in semigroups. Int J Math Comput Sci 2021; 16: 803–808.

[13] Molodtsov D. Soft set theory-first results. Comput Math Appl 1999; 37(1): 19–31.

[14] Maji PK, Biswas R, Roy AR. Soft set theory. Comput Math Appl 2003; 45(1): 555–562.

[15] Pei D, Miao D. From soft sets to information systems. In: Proceedings of Granular Computing. IEEE 2005; 2: 617–621.

[16] Ali MI, Feng F, Liu X, Min WK, Shabir M. On some new operations in soft set theory. Comput Math Appl 2009; 57(9): 1547–1553.

[17] Sezgin A, Atagün AO. On operations of soft sets. Comput Math Appl 2011; 61(5):1457–1467.

[18] Feng F, Jun YB, Zhao X. Soft semirings. Comput Math Appl 2008; 56(10): 2621–2628.

[19] Ali MI, Shabir M, Naz M. Algebraic structures of soft sets associated with new operations. Comput Math Appl 2011; 61: 2647–2654.

[20] Sezgin A, Shahzad A, Mehmood A. New operation on soft sets: Extended difference of soft sets. J New Theory 2019; 27: 33–42.

[21] Stojanovic NS. A new operation on soft sets: Extended symmetric difference of soft sets. Military Technical Courier 2021; 69(4): 779–791.

[22] Sezgin A, Atagün AO. New soft set operation: Complementary soft binary piecewise plus operation. Matrix Science Mathematic 2023; 7(2): 125–142.

[23] Sezgin A, Aybek FN. New soft set operation: Complementary soft binary piecewise gamma operation. Matrix Science Mathematic 2023; (7)1: 27–45.

[24] Sezgin A, Aybek FN, Atagün AO. New soft set operation: Complementary soft binary piecewise intersection operation. BSJ Eng Sci 2023; 6(4): 330–346.

[25] Sezgin A, Aybek FN, Güngör NB. New soft set operation: Complementary soft binary piecewise union operation. Acta Informatica Malaysia 2023; 7(1): 38–53.

[26] Sezgin A, Demirci AM. New soft set operation: Complementary soft binary piecewise star operation. Ikonion Journal of Mathematics 2023; 5(2): 24–52.

[27] Sezgin A, Yavuz E. New soft set operation: Complementary Soft Binary Piecewise Lambda Operation. Sinop University Journal of Natural Sciences 2023; 8(2): 101–133.

[28] Sezgin A, Yavuz E. A new soft set operation: Soft binary piecewise symmetric difference operation. Necmettin Erbakan University Journal of Science and Engineering 2023; 5(2): 189–208.

[29] Sezgin A, Çağman N. New soft set operation: Complementary soft binary piecewise difference operation. Osmaniye Korkut Ata Üniv Fen Biliml Derg 2024; 7(1): 58–94.

[30] Çağman N, Enginoğlu S. Soft set theory and uni-int decision making. Eur J Oper Res 2010; 20, 7(2): 848–855.

[31] Çağman N, Çitak F, Aktaş H. Soft int-group and its applications to group theory. Neural Comput Appl 2012; 2: 151–158.

[32] Sezer AS, Çağman N, Atagün AO, Ali MI, Türkmen E. Soft intersection semigroups, ideals and bi-ideals; a new application on semigroup theory I. Filomat 2015; 29(5): 917–946.

[33] Sezer AS, Çağman N, Atagün AO. Soft intersection interior ideals, quasi-ideals and generalized bi-ideals; a new approach to semigroup theory II. J Mult-Valued Log Soft Comput 2014; 23(1-2): 161–207.

[34] Sezgin A, Orbay M. Analysis of semigroups with soft intersection ideals. Acta Univ Sapientiae Math 2022; 14(1): 166–210.

[35] Mahmood T, Rehman ZU, Sezgin A. Lattice ordered soft near rings. Korean J Math 2018; 26(3): 503–517.

[36] Jana C, Pal M, Karaaslan F, Sezgin A. (α, β)-soft intersectional rings and ideals with their applications. New Math Nat Comput 2019; 15(2): 333–350.

[37] Muştuoğlu E, Sezgin A, Türk ZK. Some characterizations on soft uni-groups and normal soft uni-groups. Int J Comput Appl 2016; 155(10): 1–8.

[38] Sezer AS, Çağman N, Atagün AO. Uni-soft substructures of groups. Ann Fuzzy Math Inform 2015; 9(2): 235–246.

[39] Sezer AS. Certain Characterizations of LA-semigroups by soft sets. J Intel Fuzzy Syst 2014; 27(2): 1035–1046.

[40] Özlü Ş, Sezgin A. Soft covered ideals in semigroups. Acta Univ Sapientiae Math 2020; 12(2): 317–346.

[41] Atagün AO, Sezgin A. Soft subnear-rings, soft ideals and soft n-subgroups of near-rings. Math Sci Letters 2018; 7(1): 37–42.

[42] Sezgin A. A new view on AG-groupoid theory via soft sets for uncertainty modeling. Filomat 2018; 32(8): 2995–3030.

[43] Sezgin A, Çağman N, Atagün AO. A completely new view to soft intersection rings via soft uni-int product. Appl Soft Comput 2017; 54: 366–392.

[44] Sezgin A, Atagün AO, Çağman N, Demir H. On near-rings with soft union ideals and applications. New Math Nat Comput 2022; 18(2): 495–511.

[45] Rao MMK. Bi-interior ideals of semigroups. Discuss Math Gen Algebra Appl 2018; 38: 69–78.

[46] Rao MMK. A study of a generalization of bi-ideal, quasi ideal and interior ideal of semigroup. Mathematica Moravica 2018; 22: 103–115.

[47] Rao MMK. Left bi-quasi ideals of semigroups. Southeast Asian Bull Math 2020; 44: 369–376.

[48] Rao MMK. Quasi-interior ideals and weak-interior ideals. Asia Pac Journal Mat 2020; 7(21): 1–20.

[49] Baupradist S, Chemat B, Palanivel K, Chinram R. Essential ideals and essential fuzzy ideals in semigroups. J Discrete Math Sci Cryptogr 2021; 24(1): 223–233.

[50] Sezgin A, Kocakaya FZ. Soft intersection quasi-interior ideals of semigroups. JuTISI 2024; (in press).

[51] Sezgin A, İlgin A. Soft intersection almost subsemigroups of semigroups. Int J Math Phys 2024; 15(1): 13-20.

[52] Sezgin A, İlgin A. Soft intersection almost ideals of semigroups. J Innovative Eng Nat Sci 2024; 4(2): 466-481.

[53] Pant S, Dagtoros K, Kholil MI, Vivas A. Matrices: Peculiar determinant property. OPS Journal 2024, 1: 1–7.

[54] Sezgin A, Çalışıcı H. A comprehensive study on soft binary piecewise difference operation. Eskişehir Teknik Univ Bilim Teknol Derg Teor Bilim 2024; 12(1): 32-54.

[55] Sezgin A, Dagtoros K. Complementary soft binary piecewise symmetric difference operation: A novel soft set operation. Scientific Journal of Mehmet Akif Ersoy University 2023; 6(2): 31-45.

[56] Sezgin A, Sarıalioğlu M. New soft set operation: Complementary soft binary piecewise theta operation. Journal of Kadirli Faculty of Applied Sciences 2023; 4(2): 1-33.

ESKİŞEHİR TEKNİK ÜNİVERSİTESİ BİLİM VE TEKNOLOJİ DERGİSİ
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RESEARCH ARTICLE

ON SEMI-EXHAUSTIVENESS, SEMI-UNIFORM CONVERGENCE AND KOROVKIN-TYPE THEOREMS

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Abstract

In this study, we scrutinize the Korovkin-type theorems based on various forms of convergence, such as almost uniform convergence, semi-uniform convergence, and the concept of semi-exhaustiveness. Since it is known that the convergence types mentioned above are between point-wise and uniform convergence, it will be noticed that the circumstances can be mitigated in the Korovkin theorem.

Keywords

Function sequences,
Convergence,
Exhaustiveness,
Korovkin theorem

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1. INTRODUCTION

Let (X, d) and (Y, ρ) be metric spaces, (f_n) be a sequence of functions from X to Y and f be a function from X to Y. In this study, there are two main types of convergence of sequences of functions: "exhaustiveness" and "almost uniform convergence". In 2008, V. Gregoriades and N. Papanastassiou introduced the notion of exhaustiveness which is closely connected to the notion of equicontinuity as follows:

Definition 1.1. [16] The sequence (f_n) is called *exhaustive* at $x_0 \in X$, if for every $\varepsilon > 0$ there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $x \in B_d(x_0, \delta)$ and all $n \geq n_0$ we have that $\rho(f_n(x), f_n(x_0)) < \varepsilon$, where $B_d(x_0, \delta)$ is the ball with radius δ centered at x_0 according to the metric d.

The concept of exhaustiveness allows us to understand the convergence of a sequence of functions based on properties of the sequence itself, rather than properties of individual functions within the sequence [16]. In the following years, many generalizations of this concept were carried out. Z.H. Toyganozu and S. Pehlivan introduced the concept of exhaustiveness in the context of asymmetric metric spaces and examined several of its properties [21]. A. Caserta and Lj.D.R. Kočinac defined statistical versions of notions, exhaustiveness and weak exhaustiveness. Moreover, they presented several findings regarding the continuity of the statistical pointwise limit of a sequence of functions and elucidated the relationships between st-exhaustiveness and other forms of st-convergence [7]. E. Athanassiadou et al. introduced

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and examined the fundamental properties of I-exhaustiveness and I-convergence in sequences of real-valued functions, providing certain characterizations [4]. Subsequently, H. Albayrak and S. Pehlivan [1] introduced the concepts of $\mathcal{F}$-exhaustiveness, where $\mathcal{F}$ represents a filter on $\mathbb{N}$. See also [15].

The concept "almost uniform convergence" was defined by J. Ewert in 1993 [13]:

Definition 1.2. The sequence (f_n) is called *almost uniformly convergent* at x_0 to a function f and denoted by "$f_n \overset{a.u.}{\to} f$ at x_0" if for every $\varepsilon > 0$ there exists $\delta(\varepsilon, x_0) > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $x \in B_d(x_0, \delta)$ implies $\rho(f_n(x), f(x)) < \varepsilon$.
Ewert provided instances of this form of convergence that exist between the concepts of uniform convergence and quasi-uniform convergence. Ewert also proved in which cases these are equivalent concepts [13]. R. Drozdowski et al. discussed Ewert's concept of "almost uniform convergence" with the same name but with a different approach [11].

Korovkin's Theorem stands as one of the fundamental theorems in constructive approximation theory [18]. While the original theorem was given according to the concept of uniform convergence, in recent years it has been given according to many different concepts of convergence and summability methods. A seminal paper discussing Korovkin-type theorems in the context of statistical convergence can be found in [14]. In the paper by K. Demirci et al. [9], the concept of relative uniform convergence of a sequence of functions at a specific point was introduced and they utilized this new form of convergence to prove a Korovkin-type approximation theorem. Additionally, they delved into the investigation of convergence rates in their study. Numerous studies have also been conducted on Korovkin-type theorems that are closely linked to convergence connected with summability methods, statistical convergence and filter convergence ([3,5,6,10,12,14,17,19,22,23]).

As of 2020, with the work of N. Papanastassiou [20], in addition to the ones mentioned above, semi-types of many convergence types for function sequences have been defined and their relationships with each other have been examined. See for example [8].

This paper focuses on dealing the Korovkin-type theorems that are contingent upon the semi-types of "exhaustiveness" and "almost uniform convergence". Since it is known that the convergence types mentioned above are between point-wise and uniform convergence, it will be noticed that the circumstances can be mitigated in the classical Korovkin's Theorem.

2. DEFINITIONS AND AUXILIARY RESULTS

Let (X, d) and (Y, ρ) be metric spaces, (f_n) be a sequence of functions from X to Y and f be a function from X to Y. For $x_0 \in X$ and $\delta > 0$, $B_d(x_0, \delta)$ denotes the ball with radius δ centered at x_0 according to the metric d. Let us recall the definitions of exhaustiveness, semi-exhaustiveness, almost uniform convergence and semi-uniformly convergence.

Definition 2.1. [20] The sequence (f_n) is called *semi-exhaustive* at $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ $(m > n)$ such that for all $x \in B_d(x_0, \delta)$ we have that $\rho(f_m(x), f_m(x_0)) < \varepsilon$.

From the definition, an exhaustive sequence of functions is semi-exhaustive, although the reverse implication may not hold. In [20] (Remark 4.2 (2) (Example 3.3)), the given example is accidentally overlooked. For correction, one can use the following example:

Example 2.2. Let $f_n: (-1,1) \to \mathbb{R}$, $f_n(x) = \begin{cases} nx, & n \text{ is odd} \\ x/n, & n \text{ is even} \end{cases}$. It is evident that while (f_n) is semi-exhaustive at $x = 0$ but lacks exhaustiveness at the same point.

Proposition 2.3. Let $x_0 \in X$. The sequence (f_n) is semi-exhaustive at x_0 iff there exists a strictly increasing sequence of positive integers (n_k) such that (f_{n_k}) is exhaustive at x_0.

Proof.
Let $\varepsilon > 0$ and $x_0 \in \mathbb{R}$ are given. Let's assume we have a sequence of positive integers, denoted by (n_k), which strictly increases such that (f_{n_k}) is exhaustive at x_0. From exhaustiveness there exists $\delta > 0$ and $k^* \in \mathbb{N}$ such that for all $k \geq k^*$ and for all $x \in B(x_0, \delta)$ we have $\rho(f_{n_k}(x), f_{n_k}(x_0)) < \varepsilon$. Since $n_k \geq n_{k^*} \geq k^*$ so that $n_{k^*+n} \geq k^* + n > n$ for all $k \geq k^*$, then if we choose $m = n_{k^*+n}$ for all $n \in \mathbb{N}$ then for all $x \in B(x_0, \delta)$ we have

$$\rho(f_m(x), f_m(x_0)) = \rho\big(f_{n_{k^*+n}}(x), f_{n_{k^*+n}}(x_0)\big) < \varepsilon.$$

Now, assume that the sequence (f_n) semi-exhaustive at x_0. From here we construct the desired subsequence (n_k) as follows: From the Definition 2.1., there exists $n_1 \geq 1$ such that $\rho(f_{n_1}(x), f_{n_1}(x_0)) < \varepsilon$ for all $x \in B_d(x_0, \delta)$. Similarly, there exists $n_2 \geq n_1 + 1$ such that $\rho(f_{n_2}(x), f_{n_2}(x_0)) < \varepsilon$ for all $x \in B_d(x_0, \delta)$. If it continues in this way, there exists $n_k \geq n_{k-1} + 1$ such that $\rho(f_{n_k}(x), f_{n_k}(x_0)) < \varepsilon$ for all $x \in B_d(x_0, \delta)$. Consequently, we get a strictly increasing sequence of positive integers (n_k) such that f_{n_k} is exhaustive at x_0.

■

Definition 2.4. [22] The sequence (f_n) is called *uniformly exhaustive* on X if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $x, y \in X$ that satisfy $d(x, y) < \delta$ implies $\rho(f_n(x), f_n(y)) < \varepsilon$.

Definition 2.5. The sequence (f_n) is called *semi-bounded* on X if there exists $M > 0$ such that for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ $(m > n)$ such that the sequence $\rho(f_m(x), 0) \leq M$ for all $x \in X$.

Definition 2.6. The sequence (f_n) is called *semi-boundedly exhaustive* at x_0 if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ $(m > n)$ such that

i. $\rho(f_m(x), f(x_0)_m) < \varepsilon$ for all $x \in B(x_0, \delta)$
ii. $\rho(f_m(x_0), 0) < M$

where $M > 0$ is a constant independent from ε and n.

Definition 2.7. The sequence (f_n) is called *almost uniformly bounded* on X if there exists $n_0 \in \mathbb{N}$ and $M > 0$ such that $\rho(f_n(x), 0) \leq M$ for all $n \geq n_0$ and all $x \in X$.

Remark 2.8. It is clear that the uniform boundedness of a sequence implies almost uniformly boundedness. The inverse of this assertion is not true. For example, for $f_n\colon (1, \infty) \to \mathbb{R}$, $f_n(x) = x^{2n-n^2}$, the sequence (f_n) is not uniformly bounded, but almost uniformly bounded.

Definition 2.8. The sequence (f_n) is called *locally almost uniformly bounded* on X, if for all $x \in X$, there exists $\delta > 0$ such that the sequence (f_n) is almost uniformly bounded on $B_d(x, \delta)$.

Proposition 2.9. If the sequence (f_n) is exhaustive at x_0 and $(f_n(x_0))$ is bounded then (f_n) is almost uniformly bounded in a neighborhood at x_0.

Proof.
By boundedness of the sequence $(f_n(x_0))$, there exists a number $M > 0$ such that $\rho(f_n(x_0), 0) \leq M$ for all $n \in \mathbb{N}$. From exhaustiveness of the sequence (f_n) at x_0, there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $x \in B_d(x_0, \delta)$ we have $\rho(f_n(x), f_n(x_0)) < 1$. Since $\rho(f_n(x), 0) \leq 1 + \rho(f_n(x_0), 0) \leq 1 + M$ for all $n \geq n_0$ and all $x \in B_d(x_0, \delta)$, we get the desired result.

■

Corollary 2.10. Let the sequence (f_n) is exhaustive and pointwise bounded on X then (f_n) is locally almost uniformly bounded on X.

Proposition 2.11. If $(f_n) \overset{a.u.}{\to} f$ at x_0, then $(f_{n_k}) \overset{a.u.}{\to} f$ at x_0, for any strictly increasing sequence of positive integers (n_k).

Proof. Let strictly increasing sequence of positive integers (n_k) and $x_0 \in \mathbb{R}$ are given. From almost convergency of (f_n) to f at x_0 there exists $\delta > 0$ and $n^* \in \mathbb{N}$ such that for all $n \geq n^*$ and for all $x \in B(x_0, \delta)$ we have $\rho(f_n(x), f(x)) < \varepsilon$ for all $\varepsilon > 0$. For given $\varepsilon > 0$ if we choose $\delta^* = \delta$ and $k^* = \min\{k : n_k \geq n^*\}$ then for all $k \geq k^*$ and for all $x \in B(x_0, \delta)$ we have $\rho(f_{n_k}(x), f(x)) < \varepsilon$.

■

Definition 2.12. [20] The sequence (f_n) is called *semi-uniformly convergent* to a function f at x_0 if

i. $f_n(x_0) \to f(x_0)$
ii. For every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ $(m > n)$ such that for all $x \in B(x_0, \delta)$ implies $\rho(f_m(x), f(x)) < \varepsilon$.

The notation "$f_n \overset{semi-un.}{\to} f$ at x_0" will be used for semi-uniformly convergence of the sequence (f_n) to f at x_0.

Remark 2.13. Obviously, if a sequence of functions converges almost uniformly to a function at a specific point, then it can be inferred that the sequence converges semi-uniformly to the same function at the same point. Nevertheless, it should be noted that the converse statement does not hold true. For instance, considering the sequence provided in Example 2.2. even though it converges semi-uniformly to the function $f = 0$ at the point $x = 0$, it is not characterized by almost uniform convergence.

Proposition 2.14. Let $x_0 \in X$. The sequence (f_n) semi-uniformly converges to f at x_0 iff

i. $f_n(x_0) \to f(x_0)$
ii. There exists a strictly increasing sequence of positive integers (n_k) such that (f_{n_k}) is almost uniformly convergent to f at x_0.

Proof. Let $\varepsilon > 0$ and $x_0 \in \mathbb{R}$ are given. Assume that $(f_n(x_0))$ converges to $f(x_0)$ and there exists a strictly increasing sequence of positive integers (n_k) such that (f_{n_k}) is almost uniformly convergent to f at x_0. From almost uniform convergency there exists $\delta > 0$ and $k^* \in \mathbb{N}$ such that for all $k \geq k^*$ and for all $x \in B(x_0, \delta)$ we have $\rho(f_{n_k}(x), f(x)) < \varepsilon$. Since $n_k \geq n_{k^*} \geq k^*$ so that $n_{k^*+n} \geq k^* + n > n$ for all $k \geq k^*$, then if we choose $m = n_{k^*+n}$ for all $n \in \mathbb{N}$ then for all $x \in B(x_0, \delta)$ we have $\rho(f_m(x), f(x)) = \rho\left(f_{n_{k^*+n}}(x), f(x)\right) < \varepsilon$.

Now, assume that the sequence (f_n) semi-uniformly converges to f at x_0. From here we construct the desired subsequence (n_k) as follows: From the second condition of Definition 2.12. there exists $n_1 \geq 1$ such that $\rho(f_{n_1}(x), f(x)) < \varepsilon$ for all $x \in B_d(x_0, \delta)$. Similarly, there exists $n_2 \geq n_1 + 1$ such that $\rho(f_{n_2}(x), f(x)) < \varepsilon$ for all $x \in B_d(x_0, \delta)$. If it continues in this way, there exists $n_k \geq n_{k-1} + 1$ such that $\rho(f_{n_k}(x), f(x)) < \varepsilon$ for all $x \in B_d(x_0, \delta)$. Consequently, we get a strictly increasing sequence of positive integers (n_k) such that $f_{n_k} \overset{semi-un.}{\to} f$.

■

With the proposition mentioned earlier serving as motivation, a specific type of semi-exhaustiveness can be defined by incorporating the concept of natural density. However, before delving into this definition, let's review the definition of natural density. For $A \subseteq \mathbb{N}$, we denote the natural density of A by

$$d(A)=\lim_{n\to\infty}\frac{|\{k\in A:k\leq n\}|}{n}$$

if the limit exists, where $|A|$ denotes of the cardinality of the finite set A. Let $A\subset$. Then A is called

- a statistically thin set is $d(A)=0$
- a statistically thick set is $d(A)\neq 0$
- a statistically dense set if $d(A)=1$.

It is well known that if $d(A_1)=d(A_2)=1$ for $A_1,A_2\subset\mathbb{N}$ then $d(A_1\cap A_2)=1$ [8].

Definition 2.15. It is called that the sequence (f_n) is *densely semi-exhaustive* at $x_0\in X$ if there exists a strictly increasing sequence of positive integers (n_k), with $d(\{n_k\})=1$, such that (f_{n_k}) is exhaustive at x_0.

It's clear that if a function sequence is densely semi-exhaustive then it is semi-exhaustive. Reverse implication could not be true. For example, the sequence (f_n) defined by $f_n:(-1,1)\to\mathbb{R}$,

$$f_n(x)=\begin{cases} nx, & n \text{ is prime}\\ x/n, & n \text{ is non-prime}\end{cases}$$

is semi-exhaustive at $x_0=0$, however it is not densely semi-exhaustive at $x_0=0$.

Definition 2.16.

It is called that the sequence (f_n) is *densely semi-uniformly converges* to f at $x_0\in X$ if there exists a strictly increasing sequence of positive integers (n_k), with $d(\{n_k\})=1$, such that (f_{n_k}) almost converges to f at x_0.

It's clear that if a function sequence is densely semi-uniformly convergent then it is semi-uniformly convergent. Reverse implication could not be true. For instance, consider the sequence (f_n) defined by: $f_n:(-1,1)\to\mathbb{R}$,

$$f_n(x)=\begin{cases} \frac{x}{n}, & n \text{ is prime}\\ nx, & n \text{ is not prime}\end{cases}$$

is semi-uniformly convergent at $x_0=0$, it is not densely semi-uniformly convergent at $x_0=0$.

Let $C(X)$ denote the space of real valued continuous functions and $B(X)$ denote the space of real valued bounded functions on the metric space (X,ρ). We will deal with the positive and linear operators defined on these spaces. The positivity of an L operator defined on these spaces will be understood as the fact that the $L(f)$ function is also positive for every positive function f. Let be $e_k(x)=x^k$ for $k\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$ and $x\in\mathbb{R}$. For $X=[a,b]$, let us give Korovkin's Theorem to deal with an approximation property of the sequences of positive and linear operators on $C(X)$:

Theorem 2.17. [18] Let (L_n) be a sequence of positive linear operators on $C[a,b]$. If the sequence $L_n(e_k)$ converges uniformly to e_k on $[a,b]$, for $k=0,1,2$ then the sequence $L_n(f)$ converges uniformly to f on $[a,b]$ for all $f\in C[a,b]$.

In the next section, we deal with Korovkin-type theorems depending upon the kind of convergences such as almost uniform convergence, semi-uniformly convergence and the notion of semi-exhaustiveness.

3. MAIN RESULTS

Let (X,ρ) be a metric space for a bounded set $X \subset \mathbb{R}$ and $C_b(X)$ be the space of real valued, bounded and continuous functions on the metric space (X,ρ). For every $x \in X$ denote by $B(x;\delta)$, the set $\{y \in X: \rho(y,x) < \delta\}$ and by ρ_x the function $\rho_x(y) = \rho(x,y)$, $(y \in X)$. It is clear that $\rho_x \in C_b(X)$. In [2], Altomare's contribution involved extending Korovkin's Theorem to include metric spaces, thus presenting a broader and more encompassing version of the theorem.

Theorem 3.1. [2] Let $(L_n)_{n\geq 1}$ be a sequence of positive linear operators on $C(X)$ and assume that for some compact subset K of X the following properties hold true:

- $\lim_{n\to\infty} L_n(e_0) = e_0$ uniformly on K.
- $\lim_{n\to\infty} L_n(\rho_x) = 0$ uniformly on K.

Then for every function $f \in C(K)$, $lim_{n\to\infty} L_n(f) = f$ uniformly on K.
Using similar method in [2], we give the Korovkin-type theorems based on the concept of semi-exhaustiveness, almost uniform convergence and semi-uniformly convergence.

Theorem 3.2. Let (L_n) be a sequence of positive linear operators on $C(X)$ and $x_0 \in X$. If $L_n(e_0) \overset{a.u.}{\to} e_0$ and $L_n(\rho_{x_0}^r)$ is almost uniformly converges to 0 at x_0 for some $r > 0$, then $L_n(f) \overset{a.u.}{\to} f$ at x_0 for all $f \in C_b(X)$.

Proof. Let $f \in C_b(X)$ and $x_0 \in X$. By the continuity of f at x_0, there exists $\delta > 0$ such that
$$|f(t) - f(x_0)| < \varepsilon$$
holds for all $t \in X$ that satisfies $\rho(x_0,t) < \delta$. On the other hand, in the case $\rho(x_0,t) \geq \delta$, we have
$$|f(t) - f(x_0)| \leq 2\sup_{x\in X}|f(x)| \leq \frac{2M}{\delta}\rho(x_0,t)$$
where $M := \sup_{x\in X}|f(x)|$. Let $r > 0$. From the discussion above, the inequality pertaining to the set X can be written as follows:
$$|f(t) - f(x_0)| \leq \varepsilon e_0 + \frac{2M}{\delta^r}\rho_{x_0}^r.$$
By almost uniformly convergence of $(L_n(e_0))$ at x_0, there exists $\delta_1 > 0$ and $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ and for all $x \in B(x_0,\delta_1)$ we have
$$|L_n(e_0;x) - e_0(x)| < 1.$$
Also, by almost uniformly convergence of $\left(L_n(\rho_{x_0}^r)\right)$ at x_0, there exists $\delta_2 > 0$ and $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$ and for all $x \in B(x_0,\delta_2)$ we have
$$L_n(\rho_{x_0}^r;x) < \frac{\varepsilon\delta^2}{6M}.$$
By utilizing the well-known properties of positive and linear operators, we can establish the following:
$$\begin{aligned}|L_n(f;x) - f(x)| &\leq L_n(|f - f(x)|;x)\\ &\leq \varepsilon L_n(e_0;x) + \frac{2M}{\delta^r}L_n(\rho_{x_0}^r;x)\\ &\leq \frac{\varepsilon}{3}|L_n(e_0;x) - e_0(x)| + \frac{2M}{\delta^r}L_n(\rho_{x_0}^r;x) + \frac{\varepsilon}{3}e_0(x)\\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$
for all $n \geq n_0$ and for all $x \in B(x_0,\delta)$ where $n_0 = \max\{n_1,n_2\}$ and $\delta_0 = \min\{\delta_1,\delta_2,\delta\}$. Consequently, we obtain the almost uniform convergence of the sequence $(L_n(f))$ to f at x_0.

■

Corollary 3.3. Let (L_n) be a sequence of positive linear operators on $C(X)$. If $L_n(e_0) \overset{a.u.}{\to} e_0$ and $L_n(\rho_{x_0}^r)$ is almost uniformly converges to 0 on X for some $r > 0$, then $L_n(f) \overset{a.u.}{\to} f$ on X for all $f \in C_b(X)$.

Example 3.4. For $X = (0,2)$, consider the operators L_n on $C_b(X)$

$$L_n(f;x) = \begin{cases} f(1) + nf(x), & x \leq 1/2^n \\ f(x), & x > 1/2^n. \end{cases}$$

It is evident that the operators L_n possess both linearity and positivity. While the sequence (L_n) does not meet the conditions of Theorem 3.1, it does satisfy the conditions outlined in Theorem 3.2.

Remark 3.5. Korovkin's Theorem is not true for the concept of semi-uniformly convergence. However, as we can see in the next theorem, it can be written for densely semi-uniformly convergence. An example is given after the next theorem.

Theorem 3.6. Let (L_n) be a sequence of positive linear operators on $C_b(X)$ and $x_0 \in X$. If the sequence $(L_n(e_0))$ densely semi-uniformly convergent to e_0 and the sequence $L_n(\rho_{x_0}^r)$ densely semi-uniformly convergent to 0, for some $r > 0$, at x_0, then $L_n(f)$ densely semi-uniformly convergent to f at x_0 for all $f \in C_b(X)$.

Proof. Let $f \in C_b(X)$ and $\varepsilon > 0$ be given. Since the sequence $(L_n(e_0))$ has densely semi-uniformly convergent to e_0 at x_0, then there exists a strictly increasing sequence of positive integers $(n_k^{(1)})$, with $d(\{n_k^{(1)}\}) = 1$, such that $(L_{n_k^{(1)}}(e_0))$ is almost uniformly convergent to e_0 at x_0. Similarly, since the sequence $(L_n(\rho_{x_0}^r))$ has densely semi-uniformly convergent to 0 at x_0 then there exists a strictly increasing sequence of positive integers $(n_k^{(2)})$, with $d(\{n_k^{(2)}\}) = 1$, such that $(L_{n_k^{(2)}}(\rho_{x_0}^r))$ is almost uniformly convergent to 0 at x_0. Because of the densely semi-uniformly convergence implies the semi-uniformly convergence , if we take the strictly increasing sequence of positive integers 0 in the set $\{n_k^{(1)}\} \cap \{n_k^{(2)}\}$ which has natural density 1, we obtain that $L_{n_k}(e_0) \overset{a.u}{\to} e_0$ and $L_{n_k}(\rho_x^r)$ almost uniformly converges to 0 at x_0 by using Proposition 2.11. Now, the desired result follows from Theorem 3.2.

■

Example 3.7. Let the linear positive operators L_n on $C[0,1]$ defined by

$$L_n(f;x) = \begin{cases} f(\frac{1}{2}), & x = \frac{1}{2} \\ \int_0^1 f(t)K_n(t,x)\,dt, & x \neq \frac{1}{2} \end{cases}$$

where $K_n(t,x) = (m+1)x^m + \frac{1}{n}\left|x - \frac{1}{2}\right|$ with $n \equiv m(\text{mod}3)$ for $n \in \mathbb{N}$. It's obvious that $L_n(e_i) \overset{\text{semi}-u}{\to} e_i$ at $\frac{1}{2}$ for $i = 0,1,2$ but $L_n(f)$ does not semi-uniformly converge to f at $\frac{1}{2}$ for $f(x) = x^3$.

In the next theorem, let $X \subset \mathbb{R}$ be any set, bounded or unbounded.

Theorem 3.8. Let (L_n) be a sequence of positive linear operators on $C(X)$. If $(L_n(e_0))$ is semi-exhaustive and bounded at $x_0 \in X$, then $(L_n(f))$ is semi-exhaustive at x_0 for all $f \in C(X)$.

Proof.
Let $f \in C(X)$, $x_0 \in X$ and $\varepsilon > 0$ be given. By semi-exhaustiveness of $(L_n(e_0))$ at x_0, there exists $\delta_0 > 0$ and for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that for all $x \in X$ satisfying $\rho(x, x_0) < \delta_0$, we have

$$|L_m(e_0;x) - L_m(e_0;x_0)| < \frac{\varepsilon}{3(|f(x_0)|+1)} := A_1(\varepsilon).$$

By boundedness of the sequence $(L_n(e_0;x_0))$, there exists $M > 0$ such that $L_n(e_0;x_0) \leq M$. By the continuity of f at x_0, there exists $\delta_1 > 0$ such that for all $x \in X$ that satisfies $\rho(x, x_0) < \delta_1$, we get

$$|f(x) - f(x_0)| < \frac{\varepsilon}{3(A_1(\varepsilon)+M)} := A_2(\varepsilon).$$

From properties of positive linear operators, we have

$$L_n(|f - f(x_0)|;x) < A_2(\varepsilon)|L_n(e_0;x) - L_n(e_0;x_0)| + A_2(\varepsilon)|L_n(e_0;x_0)|.$$

Now, if we choose $\delta = \min\{\delta_0, \delta_1\}$ and for all $n \in \mathbb{N}$, $m^* = m$ $(m^* > n)$ then for all $x \in B(x_0; \delta)$, we have

$$\begin{aligned} |L_m(f;x) - L_m(f;x_0)| &\leq |L_m(f;x) - L_m(f(x_0);x)| + |L_m(f(x_0);x) - L_m(f(x_0);x_0)| \\ &\quad + |L_m(f(x_0);x_0) - L_m(f;x_0)| \\ &\leq L_m(|f - f(x_0)|;x) + |f(x_0)||L_m(e_0;x) - L_m(e_0;x_0)| \\ &\quad + L_m(|f - f(x_0)|;x_0) \\ &\leq 2A_2(\varepsilon)(A_1(\varepsilon) + M) + |f(x_0)|A_1(\varepsilon) \\ &< \varepsilon. \end{aligned}$$

Hence $(L_m(f))$ is semi-exhaustive at x_0. ■

Theorem 3.8 can also be expressed as follows:

Theorem 3.9. Let (L_n) be a sequence of positive linear operators on $C(X)$. If $(L_n(e_0))$ is semi-boundedly exhaustive at $x_0 \in X$, then $(L_n(f))$ is semi-exhaustive at x_0 for all $f \in C(X)$.

Theorem 3.10. Let (L_n) be positive linear operators on $C(X)$. If $(L_n(e_0))$ is semi-exhaustive and pointwise bounded on X then $(L_n(f))$ is semi-exhaustive on X for all $f \in C(X)$.

Proof.

Let $f \in C(X)$, $x_0 \in X$ and $\varepsilon > 0$ be given. From Proposition 2.3. there exists an increasing sequence of positive integers (n_k) such that (L_{n_k}) is exhaustive at x_0. Then by exhaustiveness of $\left(L_{n_k}(e_0)\right)$ at x_0, there exists $\delta_0 > 0$ and $k_0 \in \mathbb{N}$ such that for all $x \in X$ and for all $k \geq k_0$ that satisfy $\rho(x, x_0) < \delta_0$, we have

$$\rho\left(L_{n_k}(e_0;x), L_{n_k}(e_0;x_0)\right) < \frac{\varepsilon}{3(|f(x_0)|+1)} = A_1(\varepsilon).$$

Exhaustiveness and pointwise boundedness of $(L_n(e_0))$ on X implies locally almost uniformly boundedness from Corollary 2.10. Consequently, there is a positive real number that exists $M > 0$, $\delta_1 > 0$ and $k_1 \in \mathbb{N}$ such that for all $x \in X$ that satisfy $\rho(x, x_0) < \delta_1$ and for all $k \geq k_1$, we have $|L_{n_k}(e_0;x)| \leq M$. By the continuity of f at x_0, there exists $\delta_2 > 0$ such that for all $x \in X$ that satisfies $\rho(x, x_0) < \delta_2$, we have

$$|f(x) - f(x_0)| < \frac{\varepsilon}{3M}.$$

From properties of positive linear operators, we have

$$L_{n_k}(|f - f(x_0)|;x) < \frac{\varepsilon}{3M}|L_{n_k}(e_0;x)|.$$

Now, if we choose $\delta = \min\{\delta_0, \delta_1, \delta_2\}$ and for all $n \in \mathbb{N}$, $m = n_k(m > n_k > n)$ then for all $x \in B(x_0, \delta)$, we have

$$\begin{aligned} |L_m(f;x) - L_m(f;x_0)| &\leq |L_m(f;x) - L_m(f(x_0);x)| + |L_m(f(x_0);x) - L_m(f(x_0);x_0)| \\ &\quad + |L_m(f(x_0);x_0) - L_m(f;x_0)| \\ &\leq L_m(|f - f(x_0)|;x) + |f(x_0)||L_m(e_0;x) - L_m(e_0;x_0)| \\ &\quad + L_m(|f - f(x_0)|;x_0) \\ &\leq 2\frac{\varepsilon}{3M}|L_m(e_0;x)| + |f(x_0)|A_1(\varepsilon) \\ &< \varepsilon. \end{aligned}$$

Hence $(L_n(f))$ is semi-exhaustive at x_0. Thus $(L_n(f))$ is semi-exhaustive on X.

■

Example 3.11. Scrutinize the linear positive operators L_n on $C[-1,1]$ defined by

$$L_n(f;x) = \begin{cases} f(x)/n, & x \leq 0 \text{ and n is prime} \\ f(x)/2n, & x > 0 \text{ and n is prime} \\ f(0), & \text{n is not prime.} \end{cases}$$

It is clear that $(L_n(e_0))$ is semi-exhaustive at $x = 0$ and bounded on $[-1,1]$, so for every $f \in C[-1,1]$, $(L_n(f))$ is semi-exhaustive at $x = 0$. Indeed, for every $\varepsilon > 0$, we choose $\delta < 1/2$ and for every $n \in \mathbb{N}$

we choose m to be the least prime integer that greater than n, then $|L_m(f;x) - L_m(f;0)| < \varepsilon$ hold for all $x \in B(0,\delta)$.

Example 3.12. Consider the linear positive operators L_n on $C(0,1)$ defined by

$$L_n(f;x) = \begin{cases} f(x) + nf(x_0), & n \text{ is odd} \\ nf(x), & n \text{ is even} \end{cases}$$

and $x_0 \in (0,1)$ be fixed. For a function $f \in C(0,1)$ with $f(x_0) \neq 0$. the sequence $(L_n(f))$ does not converge uniformly on $(0,1)$, but it is semi-exhaustive on $(0,1)$.

Remark 3.13. The condition about boundedness cannot remove from Theorem 3.8.

Example 3.14. Consider the linear positive operators L_n on $C[0,1]$ defined by $L_n(f;x;) = n^2 f(x)$. It's clear that $(L_n(e_0))$ is not bounded. Although $(L_n(e_0))$ is semi-exhaustive, the sequence $(L_n(f))$ is not semi-exhaustive on $[0,1]$ for every $f \in C[0,1]$ which is not constant.

CONFLICT OF INTEREST

The authors stated that there are no conflicts of interest regarding the publication of this article.

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Alper Erdem: Formal analysis, Conceptualization, Investigation, Writing - original draft.
Tuncay Tunç: Supervision, Investigation, Conceptualization.

REFERENCES

[1] Albayrak H, Pehlivan S. Filter exhaustiveness and F−α-convergence of function sequences. Filomat, 2013; 27 (8), 1373−1383.

[2] Altomare F. Korovkin-type theorems and local approximation problems. Expositiones Mathematicae, 2022; 40 (4), 1229−1243.

[3] Anastassiou GA, Duman O. Towards intelligent modeling: Statistical approximation theory. Springer, Berlin, 2011.

[4] Athanassiadou E, Boccuto A, Dimitriou X, Papanastassiou N. Ascoli-type theorems and ideal (α)-convergence. Filomat, 2012; 26 (2), 397−405.

[5] Bardaro C, Boccuto A, Demirci K, Mantellini I, Orhan S. Triangular A-Statistical approximation by double sequences of positive linear operators. Results in Mathematics, 2015; 68, 271–291.

[6] Boccuto A, Demirci K and Yildiz S. Abstract korovkin-type theorems in the filter setting with respect to relative uniform convergence. Turkish J. of Mathematics, 2020; 44 (4), 1238–1249.

[7] Caserta A, Kočinac LD. On statistical exhaustiveness. Applied Mathematics Letters, 2012; 25 (10), 1447–1451.

[8] Das S, Ghosh A. A study on statistical versions of convergence of sequences of functions. Mathematica Slovaca, 2022; 72 (2), 443–458.

[9] Demirci K, Boccuto A, Yıldız S, Dirik F. Relative uniform convergence of a sequence of functions at a point and korovkin-type approximation theorems. Positivity, 2020; 24, 1–11.

[10] Demirci K, Orhan S. Statistically relatively uniform convergence of positive linear operators. Results in Mathematics, 2016; 69, 359–367.

[11] Drozdowski R, Jedrzejewski J, Sochaczewska A. On the almost uniform convergence. Scientific Issues of Jan Dlugosz University in Czestochowa Mathematics, 2013; 18, 11–17.

[12] Duman O, Özarslan MA, Erkuş-Duman E. Rates of ideal convergence for approximation operators. Mediterranean. Journal of Mathematics, 2010; 7, 111–121.

[13] Ewert J. Almost uniform convergence. Periodica Mathematica Hungarica, 1993; 26 (1), 77–84.

[14] Gadjiev AD, Orhan C. Some approximation theorems via statistical convergence. The Rocky Mountain Journal of Mathematics, 2002; 32, 129–138.

[15] Ghosh A. $I^*-\alpha$ convergence and I^*-exhaustiveness of sequences of metric functions. Matematicki Vesnik, 2022; 74 (2), 110–118.

[16] Gregoriades V, Papanastassiou N. The notion of exhaustiveness and Ascoli-type theorems. Topology and its Applications, 2008; 155 (10), 1111–1128.

[17] Karakuş S, Demirci K, Duman O. Statistical approximation by positive linear operators on modular spaces. Positivity, 2010; 14, 321–334.

[18] Korovkin PP. Linear operators and approximation theory, Hindustan Publising Corp., 1960.

[19] Kosar C, Kosar NP. Simultaneous statistical approximation of analytic functions in annulus by k-positive linear operators. Journal of Mathematical Analysis, 2019; 10 (2), 46–57.

[20] Papanastassiou N. A note on convergence of sequences of functions. Topology and its Applications, 2020; 275, 107017.

[21] Toyganözü ZH, Pehlivan S. Some results on exhaustiveness in asymmetric metric spaces. Filomat, 2015; 29 (1), 183–192.

[22] Tunc T, Erdem A. Korovkin-type theorems via some modes of convergence. Filomat, 2024; 38 (2), 523−530.

[23] Zeren Y, Ismailov M, Karacam C. Korovkin-type theorems and their statistical versions in grand Lebesgue spaces. Turkish Journal of Mathematics, 2020; 44 (3), 1027–1041.