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Research Article

A Study on the *k*-Mersenne and *k*-Mersenne-Lucas Sequences

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Abstract

In this study, we examine an application of $k-$Mersenne and $k-$Mersenne-Lucas sequences. We present the Catalan transforms of these sequences and give the properties of these Catalan transforms. Catalan transforms of $k-$Mersenne and $k-$Mersenne-Lucas sequences have terms according to different values of k, and some of them are associated with the sequences in OEIS. We obtain the generating functions of the Catalan transforms of $k-$Mersenne and $k-$Mersenne-Lucas sequences. Moreover, we apply the Hankel transform to the Catalan transforms of these sequences. Finally, the determinants of the created matrices are calculated by applying the Hankel transform to the terms of the Catalan transforms of these sequences.

1. Introduction

Number sequences constitute an important subject area in mathematics. These sequences usually consist of consecutive numbers that follow a certain rule. The Fibonacci sequence is one of the most well-known and frequently encountered number sequences. These sequences have intrigued scientists for a long time. Fibonacci and Lucas numbers and many of their applications have been studied and many properties of these sequences have been investigated [1]. New sequences are obtained by changing the recurrence relation and initial conditions of the generalized Fibonacci sequence.

In [2], Falcon defined $k-$Fibonacci sequence. This paper is an extension of the work of Falcon [3]. In [3], Falcon gave an application of the Catalan transform to the k-Fibonacci sequences.

Mersenne and Mersenne-Lucas numbers are a generalization of these numbers, similar to the Fibonacci sequence. Many researchers have studied Mersenne number sequences and their different applications have been investigated [4–17].

Definition 1.1. *The Mersenne numbers M_n are defined by the following recurrence relation for $n \in \mathbb{N}$,*

$$M_{n+2} = 3M_{n+1} - 2M_n$$

with the initial conditions $M_0 = 0$, $M_1 = 1$ [16].

Definition 1.2. *The Mersenne-Lucas numbers m_n are defined by the following recurrence relation for $n \in \mathbb{N}$*

$$m_{n+2} = 3m_{n+1} - 2m_n,$$

with the initial conditions $m_0 = 2$, $m_1 = 3$, [17].

Definition 1.3. *[16] The Binet formula for Mersenne numbers is defined by*

$$M_n = 2^n - 1.$$

Definition 1.4. *[17] The Binet formula for Mersenne-Lucas numbers is defined by*

$$m_n = 2^n + 1.$$

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Catalan and Hankel transforms of some number sequences have been investigated by some authors and their properties have been examined [18–24].

In this paper, the Catalan transforms of these sequences are examined in detail. We give the generating functions for the Catalan transformations of the sequences. Finally, the Hankel transforms of the newly obtained sequences and the determinants of these transforms are discussed.

Definition 1.5. *Let k be any positive number, then the $k-$Mersenne sequence is recursively given for $n \in \mathbb{N}$ as*

$$M_{k,n+2} = 3kM_{k,n+1} - 2M_{k,n},$$

where $M_{k,0} = 0$, $M_{k,1} = 1$. From now on we will show the sequence as $M_{k,n}$. When $k = 1$, Mersenne sequence is obtained [16]. The characteristic equation of the sequence is as follows

$$x^2 - 3kx + 2 = 0.$$

Its characteristic roots are

$$x_1 = \frac{3k + \sqrt{9k^2 - 8}}{2} \text{and}$$

$$x_2 = \frac{3k - \sqrt{9k^2 - 8}}{2}.$$

Characteristic roots prove the following properties to be true.

$$x_1 + x_2 = 3k,$$
$$x_1 x_2 = 2,$$
$$x_1 - x_2 = \sqrt{9k^2 - 8}.$$

Binet's formula for $M_{k,n}$ is

$$M_{k,n} = \frac{{x_1}^n - {x_2}^n}{x_1 - x_2}.$$

Definition 1.6. *Let k be any positive number, the $k-$Mersenne-Lucas number sequence is recursively given for $n \in \mathbb{N}$ as*

$$m_{k,n+2} = 3km_{k,n+1} - 2m_{k,n},$$

where $m_{k,0} = 2, m_{k,1} = 3k$ [9].From now on we will show the sequence as $m_{k,n}$. When $k = 1$, Mersenne-Lucas sequence is obtained. The characteristic equation of the sequence is as follows

$$x^2 - 3kx + 2 = 0.$$

Its characteristic roots are

$$x_1 = \frac{3k + \sqrt{9k^2 - 8}}{2}$$

and

$$x_2 = \frac{3k - \sqrt{9k^2 - 8}}{2}.$$

Binet's formula for $m_{k,n}$ is

$$m_{k,n} = {x_1}^n + {x_2}^n.$$

2. Catalan Transforms for the $k-$Mersenne and $k-$Mersenne Lucas Sequences

In this section, we present the Catalan transform of these sequences and give the properties of these Catalan transforms. Catalan transforms of k-Mersenne and k-Mersenne-Lucas sequences have terms according to different values of k, and some of them are associated with the sequences in OEIS.

Definition 2.1. *The n^{th} Catalan number is introduced by Barry [8]. For $n \geq 0$, it is*

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

or

$$C_n = \frac{2n!}{(n+1)!n!}.$$

The Catalan number generating function is

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

Some Catalan numbers are in the order $1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, \ldots$

Using the definition of Catalan numbers, the Catalan transform of these sequences can be defined as follows.

Definition 2.2. *Let $k > 0, n > 0$ and $CM_{k,0} = 0$;*

$$CM_{k,n} = \sum_{i=0}^{n} \frac{i}{2n-i}\binom{2n-i}{n-i} M_{k,i}$$

equation is called Catalan transforms in $k-$Mersenne sequences. Based on this definition, the terms of the Catalan transforms of the $k-$Mersenne sequence are as follows.

$$CM_{k,1} = \sum_{i=0}^{1} \frac{i}{2-i}\binom{2-i}{1-i} M_{k,i} = \binom{1}{0} M_{k,1} = 1,$$
$$CM_{k,2} = \sum_{i=0}^{2} \frac{i}{4-i}\binom{4-i}{2-i} M_{k,i} = \frac{1}{3}\binom{3}{1} M_{k,1} + \binom{2}{0} M_{k,2} = 3k+1,$$
$$CM_{k,3} = \sum_{i=0}^{3} \frac{i}{6-i}\binom{6-i}{3-i} M_{k,i} = 9k^2+6k,$$
$$CM_{k,4} = \sum_{i=0}^{4} \frac{i}{8-i}\binom{8-i}{4-i} M_{k,i} = 27k^3+27k^2+3k-1,$$
$$CM_{k,5} = \sum_{i=0}^{5} \frac{i}{10-i}\binom{10-i}{5-i} M_{k,i} = 81k^4+108k^3+27k^2-6k,$$
$$CM_{k,6} = \sum_{i=0}^{6} \frac{i}{12-i}\binom{12-i}{6-i} M_{k,i} = 243k^5+405k^4+162k^3-18k^2-6k+6,$$
$$CM_{k,7} = \sum_{i=0}^{7} \frac{i}{14-i}\binom{14-i}{7-i} M_{k,i} = 729k^6+1458k^5+2025k^4-864k^2+36k+84.$$

We can write the found $CM_{k,n}$ sequences as the product of the $nx1$-dimensional $M_{k,n}$ matrix and the lower triangular matrix C.

$$\begin{bmatrix} CM_{k,1} \\ CM_{k,2} \\ CM_{k,3} \\ CM_{k,4} \\ CM_{k,5} \\ CM_{k,6} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 2 & 1 & & & \\ 5 & 5 & 3 & 1 & & \\ 14 & 14 & 9 & 4 & 1 & \\ 42 & 42 & 28 & 14 & 5 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} M_{k,1} \\ M_{k,2} \\ M_{k,3} \\ M_{k,4} \\ M_{k,5} \\ M_{k,6} \\ \vdots \end{bmatrix}$$

In this case, the 1^{st} and 2^{nd} columns of the C matrix are filled with Catalan numbers, and the elements of this matrix validate a recurrence relation. The Table 2.1 shows the coefficients of the terms of the Catalan $k-$Mersenne sequence, which we call the Catalan triangle.

$CM_{k,1}$	1					
$CM_{k,2}$	3	1				
$CM_{k,3}$	9	6				
$CM_{k,4}$	27	27	3	-1		
$CM_{k,5}$	81	108	27	-6		
$CM_{k,6}$	243	405	162	-18	-6	6
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 2.1: Catalan triangle of $k-$Mersenne sequence

The results of some Catalan transforms are presented below.

$CM_1 = \{0, 1, 4, 15, 56, 210, 792, \ldots\}$ It is indexed as A001791 in OEIS.

$CM_2 = \{0, 1, 7, 48, 329, 2256, 15474, \ldots\}$

$CM_3 = \{0, 1, 10, 99, 980, 9702, 96054, \ldots\}$

$CM_4 = \{0, 1, 13, 168, 2171, 28056, 362524, \ldots\}$

$CM_5 = \{0, 1, 16, 255, 4064, 64770, 1032276, \ldots\}$

$CM_6 = \{0, 1, 19, 360, 6821, 129240, 2464882, \ldots\}$

Definition 2.3. *Let $k>0, n>0$ and $Cm_{k,0}=0$;*

$$Cm_{k,n} = \sum_{i=0}^{n} \frac{i}{2n-i} \binom{2n-i}{n-i} m_{k,i}$$

equation is called Catalan transform in k-Mersenne-Lucas sequences. Based on this definition, the Catalan transform of the k-Mersenne-Lucas sequence are found below.

$$\begin{aligned}
Cm_{k,1} &= \sum_{i=0}^{1} \frac{i}{2-i}\binom{2-i}{1-i} m_{k,i} = \binom{1}{0} m_{k,1} = 3k,\\
Cm_{k,2} &= \sum_{i=0}^{2} \frac{i}{4-i}\binom{4-i}{2-i} m_{k,i} = \frac{1}{3}\binom{3}{1} m_{k,1} + \binom{2}{0} m_{k,2} = 9k^2+3k-4,\\
Cm_{k,3} &= \sum_{i=0}^{3} \frac{i}{6-i}\binom{6-i}{3-i} m_{k,i} = 27k^3+18k^2-12k-8,\\
Cm_{k,4} &= \sum_{i=0}^{4} \frac{i}{8-i}\binom{8-i}{4-i} m_{k,i} = 81k^4+81k^3-27k^2-39k-12,\\
Cm_{k,5} &= \sum_{i=0}^{5} \frac{i}{10-i}\binom{10-i}{5-i} m_{k,i} = 243k^5+324k^4-27k^3-162k^2-60k-24,\\
Cm_{k,6} &= \sum_{i=0}^{6} \frac{i}{12-i}\binom{12-i}{6-i} m_{k,i} = 729k^6+1215k^5+162k^4-594k^3-306k^2-78k-72,\\
Cm_{k,7} &= \sum_{i=0}^{7} \frac{i}{14-i}\binom{14-i}{7-i} m_{k,i} = 2187k^7+4374k^6+1458k^5-1944k^4-1458k^3-324k^2-192k-240.
\end{aligned}$$

We can write the found $Cm_{k,n}$ sequences as the product of the $nx1$-dimensional $m_{k,n}$ matrix and the lower triangular matrix C.

$$\begin{bmatrix} Cm_{k,1}\\ Cm_{k,2}\\ Cm_{k,3}\\ Cm_{k,4}\\ Cm_{k,5}\\ Cm_{k,6}\\ \vdots \end{bmatrix} = \begin{bmatrix} 1 &&&&& \\ 1 & 1 &&&& \\ 2 & 2 & 1 &&& \\ 5 & 5 & 3 & 1 && \\ 14 & 14 & 9 & 4 & 1 & \\ 42 & 42 & 28 & 14 & 5 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} m_{k,1}\\ m_{k,2}\\ m_{k,3}\\ m_{k,4}\\ m_{k,5}\\ m_{k,6}\\ \vdots \end{bmatrix}$$

In this case, the 1^{st} and 2^{nd} columns of the C matrix are filled with Catalan numbers, and the elements of this matrix validate a recurrence relation. The Table 2.2 shows the coefficients of the terms of the Catalan transform of the k-Mersenne sequence, which we call the Catalan triangle.

$Cm_{k,1}$	3						
$Cm_{k,2}$	9	3	-4				
$Cm_{k,3}$	27	18	-12	-8			
$Cm_{k,4}$	81	81	-27	-39	-12		
$Cm_{k,5}$	243	324	-27	-162	-60	-24	
$Cm_{k,6}$	729	1215	162	-594	-306	-78	-72
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 2.2: Catalan triangle of $k-$Mersenne-Lucas sequence

3. The Generating Functions of the Catalan Transform of the $k-$ Mersenne and $k-$Mersenne-Lucas Sequences

In this section, we obtain the generating functions of the Catalan transforms of k-Mersenne and k-Mersenne-Lucas sequences.

Theorem 3.1. *The generating function of the k-Mersenne sequence is*

$$M(x) = \frac{x}{2x^2-3kx+1}$$

[9].

Theorem 3.2. *The generating function of the Catalan transform of the k-Mersenne sequences is*

$$T\left(x\right)=\frac{1-\sqrt{1-4x}}{4-4x-3k+\left(3k-2\right)\sqrt{1-4x}}.$$

Proof. The generating function of k-Mersenne and Catalan sequences are as follows, respectively,

$$M\left(x\right)=\frac{x}{2x^2-3kx+1}$$

and

$$C\left(x\right)=\frac{1-\sqrt{1-4x}}{2x}$$

If this information and the properties of the composition process in the functions are used, the desired result is achieved. Thus, the following equations for the generating function of the Catalan transform of the k-Mersenne sequence are obtained.

$$\begin{aligned} T\left(x\right)=M\left(x\right)*C= T\left(x*C\left(x\right)\right) &= \frac{\frac{1-\sqrt{1-4x}}{2}}{2\left(\frac{1-\sqrt{1-4x}}{2}\right)^2-\frac{3k\left(1-\sqrt{1-4x}\right)}{2}+1} \\ &= \frac{\frac{1-\sqrt{1-4x}}{2}}{\frac{1-2\sqrt{1-4x}+1-4x}{2}-\frac{3k-3k\sqrt{1-4x}}{2}+1} \\ &= \frac{\frac{1-\sqrt{1-4x}}{2}}{\frac{1-2\sqrt{1-4x}+1-4x-3k+3k\sqrt{1-4x}+2}{2}} \\ &= \frac{1-\sqrt{1-4x}}{4-4x-3k+\left(3k-2\right)\sqrt{1-4x}}. \end{aligned}$$

□

Theorem 3.3. *The generating function of the k−Mersenne-Lucas sequence is*

$$m\left(x\right)=\frac{2-3kx}{2x^2-3kx+1}$$

Theorem 3.4. *The generating function of the Catalan transform of the k-Mersenne-Lucas number is*

$$L\left(x\right)=\frac{4-3k+3k\sqrt{1-4x}}{4-4x-3k+\left(3k-2\right)\sqrt{1-4x}}.$$

Proof. The generating function of k-Mersenne-Lucas and Catalan sequences are as follows respectively,

$$m\left(x\right)=\frac{2-3kx}{2x^2-3kx+1}$$

and

$$C\left(x\right)=\frac{1-\sqrt{1-4x}}{2x}.$$

Thus, the following equations for the generating function of the Catalan transform of the k-Mersenne-Lucas sequence is obtained.

$$\begin{aligned} L\left(x\right)=m\left(x\right)*C=m\left(x*C\left(x\right)\right) &= \frac{2-3k\left(\frac{1-\sqrt{1-4x}}{2}\right)}{2\left(\frac{1-\sqrt{1-4x}}{2}\right)^2-\frac{3k\left(1-\sqrt{1-4x}\right)}{2}+1} \\ &= \frac{2-3k\left(\frac{1-\sqrt{1-4x}}{2}\right)}{\frac{1-2\sqrt{1-4x}+1-4x}{2}-\frac{3k-3k\sqrt{1-4x}}{2}+1} \\ &= \frac{\frac{4-3k+3k\sqrt{1-4x}}{2}}{\frac{1-2\sqrt{1-4x}+1-4x-3k+3k\sqrt{1-4x}+2}{2}} \\ &= \frac{4-3k+3k\sqrt{1-4x}}{4-4x-3k+\left(3k-2\right)\sqrt{1-4x}}. \end{aligned}$$

□

4. Hankel Transform

In this section, we apply the Hankel transform to the Catalan transforms of these sequences. In addition, the determinants of the created matrices are calculated by applying the Hankel transform to the terms of the Catalan transforms of these sequences.

Definition 4.1. *Let $R = (r_1, r_2, r_3, r_4, \ldots)$ be a sequence of real numbers. The Hankel transform of R is the sequence of determinants* $H_n = \det\left[r_{i+j-2}\right]$. *That is,*

$$H_n = \begin{vmatrix} r_0 & r_1 & r_2 & r_3 & \cdots \\ r_1 & r_2 & r_3 & r_4 & \cdots \\ r_2 & r_3 & r_4 & r_5 & \cdots \\ r_3 & r_4 & r_5 & r_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

the n^{th} order Hankel determinant of R is the upper left nxn subdeterminant of H.

Based on this definition, we apply the Hankel transform to the Catalan transforms of $k-$Mersenne and $k-$ Mersenne-Lucas sequences, respectively, as follows.

$$HCM_{k,1} = |1| = 1,$$

$$HCM_{k,2} = \begin{vmatrix} 1 & 3k + 1 \\ 3k + 1 & 9k^2 + 6k \end{vmatrix} = -1,$$

$$HCM_{k,3} = -1,$$

and

$$HCm_{k,1} = det\,|3k| = 3k,$$

$$HCm_{k,2} = \begin{vmatrix} 3k & 9k^2+3k-4 \\ 9k^2+3k-4 & 27k^3+18k^2-12k-8 \end{vmatrix} = 27k^2-16,$$

$$HCm_{k,3} = 270k^4+297k^3-304k^2-240k-128.$$

5. Conclusion

In this study, we presented the Catalan transform of $k-$Mersenne and $k-$Mersenne-Lucas sequences and obtained the generating functions of the Catalan transform of these sequences. Catalan transform of $k-$Mersenne and $k-$ Mersenne-Lucas sequences have terms according to different values of k, and some of them are associated with the sequences in OEIS. Moreover, we applied the Hankel transform to the Catalan transforms of $k-$Mersenne and $k-$Mersenne-Lucas sequences. Finally, the determinants of the created matrices are calculated by applying the Hankel transform to the terms of the Catalan transform of the $k-$Mersenne and $k-$Mersenne-Lucas sequences. If this study is examined, the same results can be found for other sequences. For example, the Catalan and Hankel transforms of Perrin and Padovan sequences.

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References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, 2019.
[2] S. Falcon, A. Plaza, *On the Fibonacci k-Numbers*, Chaos Solutions Fractals, **32**(5) (2007), 1615-1624.
[3] S. Falcon, *Catalan transform of the k-Fibonacci sequence*, Communications of the Korean Mathematical Society, **28**(4) (2013), 827–832.
[4] Y. Soykan, *On generalized p-Mersenne numbers*, Earthline Journal of Mathematical Sciences, **8**(1) (2022), 83-120.
[5] R. Frontczak, T. P. Goy, *Mersenne-Horadam identities using generating functions*, Carpathian Math. Publ., **12**(1) (2020), 34-45.
[6] M. Uysal, M. Kumari, B. Kuloğlu, K. Prasad, E. Özkan, *On the hyperbolic k-Mersenne and k-Mersenne-Lucas octonions*, Kragujevac J. Math., **49**(5) (2025), 765-779.
[7] D. Tasci, *On Gaussian Mersenne numbers*, J. Sci. Arts, **21**(4) (2021), 1021-1028.
[8] P. Barry, *A Catalan transform and related transformations on integer sequences*, J. Integer Seq. J., **8**(4) (2005), 1-24.
[9] M. Chelgham, A. Boussayoud, *On the k-Mersenne-Lucas numbers*, Notes Number Theory Discrete Math., **27**(1) (2021), 7-13.
[10] M. Kumari, K, Prasad, J, Tanti, *On the generalization of Mersenne and Gaussian Mersenne polynomials*, J. Anal., **32**(3) (2024), 931-947.
[11] Y. Soykan, *A study on generalized Mersenne numbers*, Journal of Progressive Research in Mathematics, **18**(3) (2021), 90-112.
[12] S. Demiriz, S. Erdem, *Mersenne matrix operator and its application in p-summable sequence space*, Commun. Adv. Math. Sci., **7**(1)(2024), 42-55.

[13] E. Özkan, M. Uysal, *Mersenne-Lucas hybrid numbers*, Math. Montisnigri, **52**(2021), 17-29.
[14] M. Kumari, K. Prasad, R. Mohanta, *Algebra of quaternions and octonions involving higher order Mersenne numbers*, Proc. Indian Nat. Sci. Acad., (2024), 1-10.
[15] M. Kumari, J. Tanti, K. Prasad, *On some new families of k-Mersenne and generalized k-Gaussian Mersenne numbers and their polynomials*, (2021), arXiv preprint arXiv:2111.09592.
[16] P. Catarino, H. Campos, H., P. Vasco, *On the Mersenne sequence*, Ann. Math. Inform., **46** (2016), 37-53.
[17] N. Saba, A. Boussayoud, K. V. V. Kanuri, *Mersenne Lucas numbers and complete homogeneous symmetric functions*, J. Math. Comput. Sci., **24**(2) (2021), 127- 139.
[18] A. Patra, M. K. Kaabar, *Catalan transform of k-Balancing sequences*, Int. J. Math. Math. Sci., **2021**(1) 2021, 9987314.
[19] S. Kapoor, P. Kumar, *Catalan Transformation of (s; t) Padovan Sequences*, Asian Journal of Pure and Applied Mathematics, **5**(1) (2023), 170-178.
[20] P. J. Larcombe, P. D. Wilson, *On the generating function of the Catalan sequence: A historical perspective*, Congr. Numer., **149** (2001), 97-108.
[21] J. W. Layman, *The Hankel transform and some of its properties*, J. Integer Seq., **4**(1) 2001, 1-11.
[22] P. M. Rajković, M. D. Petković, P. Barry, *The Hankel transform of the sum of consecutive generalized Catalan numbers*, Integral Transforms Spec. Funct., **18**(4) (2007), 285-296.
[23] K. Parmar, V. R. Gorty, *Quaternion Hankel Transform and its Generalization*, Sahand Commun. Math. Anal., **21**(1) (2024), 67-81.
[24] E. Özkan, M. Uysal, B. Kuloğlu, *Catalan transform of the incomplete Jacobsthal numbers and incomplete generalized Jacobsthal polynomials*, Asian-Eur. J. Math., **15**(6) (2022), 2250119.

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Research Article

Some Novel Proximal Point Results and Applications

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Abstract

The current study provides significant findings on coincidence the best proximity points for proximal-contractions in the context of extended b-metric spaces. To substantiate our assertions, we present illustrative examples across several circumstances. The conclusions presented in this study provide an expanded and more nuanced viewpoint, extending and generalizing multiple previous findings in optimal proximity theory. These findings present a new method for comprehending proximal coincidence locations in multi-valued mappings, representing a substantial advancement in the existing research landscape. This work gives practical benchmarks for implementing optimal proximity results, provided that existence and uniqueness constraints are met.

1. Introduction

The theory of metric spaces has been a fundamental aspect of mathematical analysis since its inception. One of the earliest and most significant contributions to this area was made by Fréchet (1906) [1], who introduced the foundational concepts of metric spaces in his seminal work on functional calculus. This laid the groundwork for the subsequent exploration of generalized metric spaces, which have been extensively studied and applied in various branches of mathematics and science. Over time, the classical metric space structure has been extended to encompass a range of generalizations, including b-metric spaces and their variants, as explored by Kamran et al. (2017) [2], who established fixed point results in extended b-metric spaces.
The concept of the best proximity points emerged as an important generalization of fixed point theory, addressing situations where mappings lack fixed points but exhibit minimal distances. This line of inquiry gained prominence with the pioneering work of Basha and Veeramani (1997) [3], who introduced the concept of best approximations, providing a framework for analyzing such mappings. Subsequently, their work was extended in 2000 [4], where they proposed theorems for best proximity pairs in multifunctions. These studies established a foundation for connecting approximation theory with optimization problems, marking an important milestone in the development of best proximity point theory; see, for example, [5].
The integration of contraction principles into the study of best proximity points has been another significant advancement. For instance, Suzuki et al. (2009) [6] investigated the existence of best proximity points in metric spaces satisfying the UC property, introducing new methods to address proximity problems. These foundational results were further enhanced by Asadi and Afshar (2022) [7], who introduced fixed point theorems for rational C-class functions in b-metric spaces and demonstrated their application to integral equations. Similarly, Alqahtani et al. (2018) [8] explored extended b-metric spaces to establish common fixed point results, offering a deeper understanding of the relationships between different types of metric spaces.
Further advancements in fixed point theory have been made through the study of generalized contraction mappings. For example, Chandok and Karapinar (2012) [9] examined generalized rational-type contractions in partially ordered metric spaces. This approach provided new perspectives on the structural properties of metric spaces and their implications for fixed point existence. The study of fuzzy metric spaces also gained traction with contributions like those by Jabeen et al. (2020) [10], who applied weakly compatible and quasi-contraction results to fuzzy cone metric spaces.
The application of these theories has extended beyond pure mathematics to practical and computational domains. For instance, McConnell et al. (1991) [11] applied metric concepts to dynamic time warping techniques in geosciences, demonstrating the utility of these mathematical structures in real-world problems. More recently, Younis et al. (2021) [12] utilized graphical structures in b-metric spaces to study the

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transverse oscillations of homogeneous bars, while Younis et al. (2023) [13] explored applications in elastic beam deformations. For further synthesis in this direction, we refer to [14, 15].
The evolution of the best proximity point theory has also been marked by innovative applications in optimization and computational modelling. The work of Savanovic et al. (2022) [16] on multi-valued mappings in b-metric spaces illustrates the growing adaptability of proximity point results to diverse mathematical contexts. Moreover, the recent study by Younis et al. (2024) [17] connected best proximity points to the equations of motion, highlighting the increasing relevance of these concepts in applied mathematics and physics. The chronological development of metric space theories and best proximity point research demonstrates a profound progression from foundational concepts to modern applications. These contributions, spanning more than a century, underscore the importance of metric spaces and their generalizations in addressing both theoretical and practical challenges in mathematics and related fields.
This research is inspired by the current literature regarding rational-type contractive conditions in diverse metric spaces. This work presents a series of coincidence best proximity point theorems that are specifically designed for extended b-metric spaces. The results for coincidence points in this paper are new because they include contraction requirements for both multivalued and single-value mappings in extended b-metric spaces. The main goal is to come up with coincidence best proximity point theorems for generalized and modified proximal-contractions in extended b-metric spaces. This will help to find the best solutions for the equation $\nabla(x) = x$. Secondly, it utilizes these findings to establish adequate conditions for the existence of solutions to nonlinear differential and integral equations, demonstrating the practical significance of the proposed theory. The uniqueness of these conclusions is emphasized by instances that validate their generalization as noteworthy outcomes in the current state of the art.

2. Preliminaries

Before delving into the main content of the article, we review some fundamental concepts and notations that will be utilized throughout this study.

Definition 2.1. *[2] Let $\mathbb{J}$ be a nonempty set, and let $\pi : \mathbb{J} \times \mathbb{J} \to [1, \infty)$ be a mapping. A function $\partial : \mathbb{J} \times \mathbb{J} \to [0, \infty)$ is said to be an* extended b-metric *if it satisfies the following conditions:*

1. $\partial(\tau_1, \tau_2) = 0$ if and only if $\tau_1 = \tau_2$,
2. $\partial(\tau_1, \tau_2) = \partial(\tau_2, \tau_1)$ for all $\tau_1, \tau_2 \in \mathbb{J}$,
3. $\partial(\tau_1, \tau_2) \leq \pi(\tau_1, \tau_2)[\partial(\tau_1, \tau_3) + \partial(\tau_3, \tau_2)]$ for all $\tau_1, \tau_2, \tau_3 \in \mathbb{J}$.

The pair $(\mathbb{J}, \partial)$ is then called an *extended b-metric space*.

Remark 2.2. *If $\pi(\tau_1, \tau_2) = s$ for $s \geq 1$, the extended b-metric space $(\mathbb{J}, \partial)$ reduces to a b-metric space.*

Let $(\mathbb{J}, \partial)$ be an extended b-metric space, and let $\complement$ and $\backepsilon$ be two nonempty subsets of $\mathbb{J}$.

$$\complement_0 = \{\tau \in \complement : \partial(\tau, \tau^*) = \partial(\complement, \backepsilon) \text{ for some } \tau^* \in \backepsilon\},$$
$$\backepsilon_0 = \{\tau^* \in \backepsilon : \partial(\tau, \tau^*) = \partial(\complement, \backepsilon) \text{ for some } \tau \in \complement\},$$

where

$$\partial(\complement, \backepsilon) = \inf\{\partial(\tau, \tau^*) : \tau \in \complement, \tau^* \in \backepsilon\}.$$

Definition 2.3. *[2] Let $(\mathbb{J}, \partial)$ be an extended b-metric space. A sequence $\{\tau_n\}$ in $\mathbb{J}$ is said to* converge *to $\tau \in \mathbb{J}$ if for every $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that $\partial(\tau_n, \tau) < \varepsilon$ for all $n \geq N_\varepsilon$. This is denoted as*

$$\lim_{n\to\infty} \tau_n = \tau.$$

Definition 2.4. *[2] A sequence $\{\tau_n\}$ in an extended b-metric space $(\mathbb{J}, \partial)$ is called a* Cauchy sequence *if for every $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that $\partial(\tau_n, \tau_m) < \varepsilon$ for all $m, n \geq N_\varepsilon$.*

Definition 2.5. *[2] An extended b-metric space $(\mathbb{J}, \partial)$ is said to be* complete *if every Cauchy sequence in $\mathbb{J}$ converges to a point in $\mathbb{J}$.*

Lemma 2.6. *[18] Let $(\mathbb{J}, \partial)$ be a complete extended b-metric space. If ∂ is continuous, then every convergent sequence in $\mathbb{J}$ has a unique limit.*

Definition 2.7. *[18] Let $(\complement, \backepsilon)$ be a pair of nonempty subsets of a metric space such that $\complement_0$ is nonempty. The pair $(\complement, \backepsilon)$ is said to have the* P-property *if*

$$\partial(\tau_1, \tau_1^*) = \partial(\complement, \backepsilon) \text{ and } \partial(\tau_2, \tau_2^*) = \partial(\complement, \backepsilon) \implies \partial(\tau_1, \tau_2) = \partial(\tau_1^*, \tau_2^*),$$

where $\tau_1, \tau_2 \in \complement$ and $\tau_1^*, \tau_2^* \in \backepsilon$.

Definition 2.8. *[19] Let $\mathscr{CB}(\mathbb{J})$ denote the set of all closed and bounded subsets of $\mathbb{J}$. The Pompeiu–Hausdorff metric $\mathscr{H}$ induced by ∂ is defined as*

$$\mathscr{H}(\complement, \backepsilon) = \max\{\sup_{a\in\complement} \wp(a, \backepsilon), \sup_{b\in\backepsilon} \wp(b, \complement)\},$$

for $\complement, \backepsilon \in \mathscr{CB}(\mathbb{J})$, where

$$\wp(a, \backepsilon) = \inf\{\partial(\sigma_1, \sigma_2) : b \in \backepsilon\}.$$

Additionally, we define

$$\wp^*(\sigma_1, \sigma_2) = \wp(\sigma_1, \sigma_2) - \partial(\complement, \backepsilon), \quad \forall a \in \complement, b \in \backepsilon.$$

From this point onward, let $\complement$ and $\ni$ be nonempty subsets of a complete extended b-metric space $(\mathbb{J},\partial)$ define $\pi_*:\mathbb{J}^2\to[1,\infty)$ as

$$\pi_*(\tau,\ni)=\inf\{\pi(\tau,\tau^*):\tau^*\in\ni\}$$

and

$$\pi_*(\complement,\ni)=\inf\{\pi(\tau,\tau^*):\tau\in\complement,\tau^*\in\ni\},$$

where $\pi:\mathbb{J}\times\mathbb{J}\to[1,\infty)$.

3. Coincidence Proximal Point Results for Multi-Valued Mappings

In this section, we will discuss some of the best coincidence proximity point theorems using multi-valued concepts on extended b-metric space.

Definition 3.1. *Given $\nabla:\complement\to\mathscr{CB}(\ni)$ and $\mho:\complement\to\complement$ be mappings. The pair $(\mho,\nabla)$ is said to be a $\varphi_{max}-$proximal-contraction if there exists a real number $\varphi\in[0,1)$ such that*

$$\left.\begin{array}{l}\wp(\mho u,\nabla\tau)=\partial(\complement,\ni)\\ \wp(\mho v,\nabla\tau^*)=\partial(\complement,\ni)\end{array}\right\}\text{ implies }\mathscr{H}(\nabla\tau,\nabla\tau^*)\leq\varphi\mathbb{M}(u,v,\tau,\tau^*),$$

where

$$\mathbb{M}(u,v,\tau,\tau^*)=\max\left\{\ \partial(\mho u,\mho v),\frac{\wp(\mho\tau,\nabla v)-\pi_*(\mho\tau,\nabla v)\wp(\mho u,\nabla v)}{\pi_*(\mho\tau,\nabla v)},\wp^*(\mho v,\nabla u)\ \right\}$$

for all u,v,τ,τ^* in $\complement$.

Example 3.2. *Let $\mathbb{J}=\{1,2,3,4\}$. Consider the function ∂ given as $\partial(\tau,\tau^*)=\partial(\tau^*,\tau)$ and $\partial(\tau,\tau)=0$, where*

∂	1	2	3	4
1	0	3	5	7
2	3	0	4	6
3	5	4	0	2
4	7	6	2	0

.

Take $\pi_*:\mathbb{J}\times\mathbb{J}\to[1,\infty)$ to be symmetric, defined as $\pi(\tau,\tau^*)=10\tau+5\tau^*$. It is easy to see that $(\mathbb{J},\partial)$ is an extended $b-$metric space. Let $\complement=\{1,2\}$ and $\ni=\{3,4\}$ be two non-empty subsets of the extended $b-$metric space $(\mathbb{J},\partial)$. After routine calculations, we get $\partial(\complement,\ni)=4$, $\complement_0=\complement$ and $\ni_0=\ni$. Since $\nabla:\complement\to\mathscr{CB}(\ni)$, we define

$$\nabla\tau=\left\{\ \{3,4\},\quad \text{if }\tau\in\{1,2\}\ \right\},$$

and for $\mho:\complement\to\complement$, we have

$$\mho\tau=\left\{\begin{array}{ll}1, & \text{if }\tau=2,\\ 2, & \text{if }\tau=1.\end{array}\right.$$

We show that the pair $(\mho,\nabla)$ satisfies φ_M-proximal-contraction

$$\mathscr{H}(\nabla\tau,\nabla\tau^*)\leq\varphi\mathbb{M}(u,v,\tau,\tau^*),$$

for all $u,v,\tau,\tau^*\in\complement$, where $\mathbb{M}(u,v,\tau,\tau^*)$ is defined in Definition 3.1.
Now,

$$\left.\begin{array}{l}\wp(\mho 2,\nabla 1)=\partial(\complement,\ni),\\ \wp(\mho 1,\nabla 2)=\partial(\complement,\ni),\end{array}\right\}\text{ implies }\mathscr{H}(\nabla 1,\nabla 2)\leq\varphi\mathbb{M}(u,v,\tau,\tau^*).$$

Hence, $\mathscr{H}(\nabla 1,\nabla 2)=0$, for every $\varphi\in[0,1)$, and Definition 3.1 is satisfied.

Definition 3.3. *Given $\nabla:\complement\to\mathscr{CB}(\ni)$ and $\mho:\complement\to\complement$ be mappings. The pair $(\mho,\nabla)$ is said to be a $\varphi_\nabla-$proximal-contraction if there exists a real number $\varphi\in[0,1)$ such that*

$$\left.\begin{array}{l}\wp(u,\nabla\tau)=\partial(\complement,\ni)\\ \wp(v,\nabla\tau^*)=\partial(\complement,\ni)\end{array}\right\}\text{ implies }\mathscr{H}(\nabla\tau,\nabla\tau^*)\leq\varphi\mathbb{M}(u,v,\tau,\tau^*),$$

where

$$\mathbb{M}(u,v,\tau,\tau^*)=\max\left\{\ \partial(u,v),\frac{\wp(\tau,\nabla v)-\pi_*(\tau,\nabla v)\wp(u,\nabla v)}{\pi_*(\tau,\nabla v)},\wp^*(v,\nabla u)\ \right\}$$

for all u,v,τ,τ^* in $\complement$.
Note that, if we take $\mho=I_{\complement}$ ($\mho$ as an identity mapping on $\complement$), then every φ_M-proximal-contraction will reduce to a $\varphi_\nabla-$proximal-contraction.

Definition 3.4. *A mapping $\nabla : \mathbb{J} \to \mathscr{CB}(\mathbb{J})$ is continuous in an extended b-metric space $(\mathbb{J}, \partial)$ at $\tau \in \mathbb{J}$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\nabla(K(\tau, \delta)) \subseteq K(\nabla\tau, \varepsilon),$$

where $K(\tau, \varepsilon)$ is given as

$$K(\tau, \varepsilon) = \{\tau^* \in \mathbb{J}, \partial(\tau, \tau^*) < \varepsilon\}.$$

Clearly, if ∇ is continuous at τ, then $\tau_n \to \tau$ implies that $\nabla\tau_n \to \nabla\tau$ as $n \to \infty$.

The following theorem is based on Definition 3.1, which is more general than the results discussed in the introduction within the context of extended $b-$metric spaces. Furthermore, Example 3.7 illustrates our fact.

Theorem 3.5. *Let $\nabla : \mathsf{C} \to \mathscr{CB}(\Game)$, $\mho : \mathsf{C} \to \mathsf{C}$ and $\pi_* : \mathbb{J}^2 \to [1, \infty)$ be mappings, where C is closed subset of an extended $b-$metric space $(\mathbb{J}, \partial)$. Let the pair $(\mathsf{C}, \Game)$ satisfies the P-property with $\nabla(\mathsf{C}_0) \subseteq \Game_0$ and $\mathsf{C}_0 \subseteq \mho(\mathsf{C}_0)$. Assume that the pair of continuous mappings $(\mho, \nabla)$, where $\mho$ is one-to-one, satisfies φ_M-proximal-contraction the following statement holds*

$$\lim_{n,m\to\infty} \pi(\mho\tau_n, \mho\tau_m) < \frac{1}{k}, \quad where \quad k \in (0, 1).$$

Then, there exists a unique coincidence best proximity point of the pair $(\mho, \nabla)$.

Proof. Let τ_0 be an arbitrary element in C_0. Since $\nabla(\mathsf{C}_0)$ is contained in $\Game_0$ and C_0 is contained in $\mho(\mathsf{C}_0)$, there exists an element τ_1 in C_0 such that

$$\wp(\mho\tau_1, \nabla\tau_0) = \partial(\mathsf{C}, \Game).$$

Again, since $\nabla\tau_1$ is an element of $\nabla(\mathsf{C}_0)$ which is contained in $\Game_0$, and C_0 is contained in $\mho(\mathsf{C}_0)$, it follows that there is an element τ_2 in C_0 such that

$$\wp(\mho\tau_2, \nabla\tau_1) = \partial(\mathsf{C}, \Game).$$

Making use of the property P, we acquire

$$\partial(\mho\tau_1, \mho\tau_2) = \mathscr{H}(\nabla\tau_0, \nabla\tau_1).$$

Since the pair of mappings $(\mho, \nabla)$ satisfies φ_M-proximal-contraction, we obtain

$$\partial(\mho\tau_1, \mho\tau_2) \leq \varphi\mathbb{M}(\tau_1, \tau_2, \tau_0, \tau_1),$$

where

$$\begin{aligned} M_\nabla(\tau_1, \tau_2, \tau_0, \tau_1) &\leq \max\left\{ \partial(\tau_1, \tau_2), \frac{\wp(\tau_0, \nabla\tau_2) - \pi_*(\tau_0, \nabla\tau_2)\wp(\tau_1, \nabla\tau_2)}{\pi_*(\tau_0, \nabla\tau_2)}, \wp^*(\tau_2, \nabla\tau_1) \right\} \\ &\leq \max\left\{ \begin{array}{c} \partial(\tau_1, \tau_2), \frac{\pi_*(\tau_0, \nabla\tau_2)[\partial(\tau_0, \tau_1) + \wp(\tau_1, \nabla\tau_2)] - \pi_*(\tau_0, \nabla\tau_2)\wp(\tau_1, \nabla\tau_2)}{\pi_*(\tau_0, \nabla\tau_2)}, \\ \wp(\tau_2, \nabla\tau_1) - \partial(\mathsf{C}, \Game) \end{array} \right\} \\ &\leq \max\left\{ \partial(\tau_1, \tau_2), \frac{\pi_*(\tau_0, \nabla\tau_2)\partial(\tau_0, \tau_1)}{\pi_*(\tau_0, \nabla\tau_2)}, 0 \right\} \\ &\leq \max\{\partial(\tau_1, \tau_2), \partial(\tau_0, \tau_1), 0\}. \end{aligned}$$

Hence, we have

$$M_\nabla(\tau_1, \tau_2, \tau_0, \tau_1) \leq \max\{\partial(\mho\tau_1, \mho\tau_2), \partial(\mho\tau_0, \mho\tau_1)\}.$$

If $\max\{\partial(\mho\tau_1, \mho\tau_2), \partial(\mho\tau_0, \mho\tau_1)\} = \partial(\mho\tau_1, \mho\tau_2)$, then the above inequality implies

$$\partial(\mho\tau_1, \mho\tau_2) < \varphi\partial(\mho\tau_1, \mho\tau_2),$$

which is a contradiction. Therefore, we conclude that

$$\max\{\partial(\mho\tau_1, \mho\tau_2), \partial(\mho\tau_0, \mho\tau_1)\} = \partial(\mho\tau_0, \mho\tau_1).$$

Further, by the fact that $\nabla\tau_2$ is a member of $\nabla(\mathsf{C}_0)$, which is contained in $\Game_0$, and C_0 is contained in $\mho(\mathsf{C}_0)$, there exists $\tau_3 \in \mathsf{C}_0$ such that

$$\wp(\mho\tau_2, \nabla\tau_1) = \partial(\mathsf{C}, \Game),$$

$$\wp(\mho\tau_3, \nabla\tau_2) = \partial(\mathsf{C}, \Game).$$

Again utilizing the P-property, we get

$$\partial(\mho\tau_2, \mho\tau_3) = \mathscr{H}(\nabla\tau_1, \nabla\tau_2).$$

Also, the pair of mappings $(\mho, \nabla)$ satisfies φ_M-proximal-contraction, we obtain

$$\partial(\mho\tau_2, \mho\tau_3) \leq \varphi\mathbb{M}(\tau_2, \tau_3, \tau_1, \tau_2),$$

where

$$\begin{aligned}
\mathbb{M}(\tau_2,\tau_3,\tau_1,\tau_2) &\leq \max\left\{ \partial(\mho\tau_2,\mho\tau_3), \frac{\wp(\mho\tau_1,\nabla\tau_3)-\pi_*(\mho\tau_1,\nabla\tau_3)\wp(\mho\tau_2,\nabla\tau_3)}{\pi_*(\mho\tau_1,\nabla\tau_3)}, \wp^*(\mho\tau_3,\nabla\tau_2) \ ) \right\} \\
&\leq \max\left\{ \begin{array}{c} \partial(\mho\tau_2,\mho\tau_3), \frac{\pi_*(\mho\tau_1,\nabla\tau_3)[\partial(\mho\tau_1,\mho\tau_2)+\wp(\mho\tau_2,\nabla\tau_3)]-\pi_*(\mho\tau_1,\nabla\tau_3)\wp(\mho\tau_2,\nabla\tau_3)}{\pi_*(\mho\tau_1,\nabla\tau_3)}, \\ \wp(\mho\tau_3,\nabla\tau_2)-\partial(\mathsf{C},\Game) \end{array} \right\} \\
&\leq \max\left\{ \partial(\mho\tau_2,\mho\tau_3), \frac{\pi_*(\mho\tau_1,\nabla\tau_3)\partial(\mho\tau_1,\mho\tau_2)}{\pi_*(\mho\tau_1,\nabla\tau_3)}, 0 \right\} \\
&\leq \max\left\{\partial(\mho\tau_2,\mho\tau_3), \partial(\mho\tau_1,\mho\tau_2), 0\right\}.
\end{aligned}$$

That is

$$\mathbb{M}(\tau_2,\tau_3,\tau_1,\tau_2) \leq \max\left\{\partial(\mho\tau_2,\mho\tau_3), \partial(\mho\tau_1,\mho\tau_2)\right\}.$$

If $\max\left\{\partial(\mho\tau_2,\mho\tau_3), \partial(\mho\tau_1,\mho\tau_2)\right\} = \partial(\mho\tau_2,\mho\tau_3)$, then the above inequality implies

$$\partial(\mho\tau_2,\mho\tau_3) < \varphi\partial(\mho\tau_2,\mho\tau_3),$$

which is a contradiction. Thus we conclude that

$$\max\left\{\partial(\mho\tau_2,\mho\tau_3), \partial(\mho\tau_1,\mho\tau_2)\right\} = \partial(\mho\tau_1,\mho\tau_2).$$

This process could be continued further. Having chosen $\tau_n \in \mathsf{C}_0$, it is clear that there exists an element $\tau_{n+1} \in \mathsf{C}_0$. Since, τ_{n+1} is a member of $\nabla(\mathsf{C}_0)$ which is contained in $\Game_0$ and C_0 is contained in $\mho(\mathsf{C}_0)$ such that

$$\wp(\mho\tau_n,\nabla\tau_{n-1}) = \partial(\mathsf{C},\Game),$$

$$\wp(\mho\tau_{n+1},\nabla\tau_n) = \partial(\mathsf{C},\Game).$$

Making use of the property P, we obtain the following

$$\partial(\mho\tau_n,\mho\tau_{n+1}) = \mathscr{H}(\nabla\tau_n,\nabla\tau_{n-1})$$

and

$$\partial(\mho\tau_n,\mho\tau_{n+1}) \leq \varphi\mathbb{M}(\tau_n,\tau_{n+1},\tau_{n-1},\tau_n),$$

where

$$\begin{aligned}
\mathbb{M}(\tau_n,\tau_{n+1},\tau_{n-1},\tau_n) &\leq \max\left\{ \partial(\mho\tau_n,\mho\tau_{n+1}), \frac{\wp(\mho\tau_{n-1},\nabla\tau_{n+1})-\pi_*(\mho\tau_{n-1},\nabla\tau_{n+1})\wp(\mho\tau_n,\nabla\tau_{n+1})}{\pi_*(\mho\tau_{n-1},\nabla\tau_{n+1})}, \wp^*(\mho\tau_{n+1},\nabla\tau_n) \right\} \\
&\leq \max\left\{ \begin{array}{c} \partial(\mho\tau_n,\mho\tau_{n+1}), \wp(\mho\tau_{n+1},\nabla\tau_n)-\partial(\mathsf{C},\Game), \\ \frac{\pi_*(\mho\tau_{n-1},\nabla\tau_{n+1})[\partial(\mho\tau_{n-1},\mho\tau_n)+\wp(\mho\tau_n,\nabla\tau_{n+1})]-\pi_*(\mho\tau_{n-1},\nabla\tau_{n+1})\wp(\mho\tau_n,\nabla\tau_{n+1})}{\pi_*(\mho\tau_{n-1},\nabla\tau_{n+1})} \end{array} \right\} \\
&\leq \max\left\{ \partial(\mho\tau_n,\mho\tau_{n+1}), 0, \frac{\pi_*(\mho\tau_{n-1},\nabla\tau_{n+1})\partial(\mho\tau_{n-1},\mho\tau_n)}{\pi_*(\mho\tau_{n-1},\nabla\tau_{n+1})} \right\} \\
&\leq \max\left\{\partial(\mho\tau_n,\mho\tau_{n+1}), 0, \partial(\mho\tau_{n-1},\mho\tau_n), \partial(\mho\tau_n,\mho\tau_{n+1})\right\}.
\end{aligned}$$

Hence,

$$\mathbb{M}(\tau_n,\tau_{n+1},\tau_{n-1},\tau_n) \leq \max\left\{\partial(\mho\tau_n,\mho\tau_{n+1}), \partial(\mho\tau_{n-1},\mho\tau_n)\right\}.$$

If $\max\left\{\partial(\mho\tau_n,\mho\tau_{n+1}), \partial(\mho\tau_{n-1},\mho\tau_n)\right\} = \partial(\mho\tau_n,\mho\tau_{n+1})$, then the above inequality implies

$$\partial(\mho\tau_n,\mho\tau_{n+1}) < \varphi\partial(\mho\tau_n,\mho\tau_{n+1}),$$

which is a contradiction. Hence, we have

$$\max\left\{\partial(\mho\tau_n,\mho\tau_{n+1}), \partial(\mho\tau_{n-1},\mho\tau_n)\right\} = \partial(\mho\tau_{n-1},\mho\tau_n).$$

Keeping with the same pattern, we assert that

$$\partial(\mho\tau_n,\mho\tau_{n+1}) < \varphi^n\partial(\mho\tau_0,\mho\tau_1).$$

Now, we show that $\{\mho\tau_n\}$ is a Cauchy sequence. Since $(\mathbb{J},\partial)$ is a complete extended $b-$metric space, for all $n,m\in\mathbb{N}$ with $n<m$, we have

$$\begin{aligned}\partial(\mho\tau_n,\mho\tau_m) &\leq \pi(\mho\tau_n,\mho\tau_m)\partial(\mho\tau_n,\mho\tau_{n+1})+\pi(\mho\tau_n,\mho\tau_m)\partial(\mho\tau_{n+1},\mho\tau_m)\\ &\leq \pi(\mho\tau_n,\mho\tau_m)\partial(\mho\tau_n,\mho\tau_{n+1})+\pi(\mho\tau_n,\mho\tau_m)\pi(\mho\tau_{n+1},\mho\tau_m)\partial(\mho\tau_{n+1},\mho\tau_{n+2})\\ &+\pi(\mho\tau_n,\mho\tau_m)\pi(\mho\tau_{n+1},\mho\tau_m)\partial(\mho\tau_{n+2},\mho\tau_m)\\ &\leq \pi(\mho\tau_n,\mho\tau_m)\partial(\mho\tau_n,\mho\tau_{n+1})+\pi(\mho\tau_n,\mho\tau_m)\pi(\mho\tau_{n+1},\mho\tau_m)\partial(\mho\tau_{n+1},\mho\tau_{n+2})+\ldots\\ &+\pi(\mho\tau_n,\mho\tau_m)\pi(\mho\tau_{n+1},\mho\tau_m)...\pi(\mho\tau_{m-2},\mho\tau_m)\pi(\mho\tau_{m-1},\mho\tau_m)\partial(\mho\tau_{m-1},\mho\tau_m)\\ &\leq \pi(\mho\tau_n,\mho\tau_m)\varphi^n\partial(\mho\tau_0,\mho\tau_1)+\pi(\mho\tau_n,\mho\tau_m)\pi(\mho\tau_{n+1},\mho\tau_m)\varphi^{n+1}\partial(\mho\tau_0,\mho\tau_1)+...\\ &+\pi(\mho\tau_n,\mho\tau_m)\pi(\mho\tau_{n+1},\mho\tau_m)...\pi(\mho\tau_{m-2},\mho\tau_m)\pi(\mho\tau_{m-1},\mho\tau_m)\varphi^{m-1}\partial(\mho\tau_0,\mho\tau_1)\\ &=\partial(\mho\tau_0,\mho\tau_1)\sum_{i=1}^{m-1}\varphi^i\prod_{j=1}^{i}\pi(\mho\tau_i,\mho\tau_m).\end{aligned}$$

Assume that

$$S_n=\sum_{i=1}^{n}\varphi^i\prod_{j=1}^{i}\pi(\mho\tau_i,\mho\tau_m).$$

We can write

$$\partial(\mho\tau_n,\mho\tau_m)\leq[S_{m-1}-S_n]. \tag{3.1}$$

Using the ratio test, we get

$$a_i=\varphi^i\prod_{j=1}^{i}\pi(\mho\tau_i,\mho\tau_m),\text{ where }\frac{a_{i+1}}{a_i}<\frac{1}{k}.$$

Further taking the limit as $n\to\infty$ in inequality (3.1), we infer

$$\lim_{n\to\infty}\partial(\mho\tau_n,\mho\tau_m)=0,$$

which implies that $\{\mho\tau_n\}$ is a Cauchy sequence in a complete extended $b-$metric space $(\mathbb{J},\partial)$, hence it is convergent. Suppose that $\{\mho\tau_n\}$ converges to some τ^* in $\mathtt{C}$ (as the set $\mathtt{C}$ is closed), assuring that the sequence $\{\tau_n\}\subseteq\mathtt{C}_0$, since $\{\tau_n\}\to\tau^*$. Therefore, $(\mho,\nabla)$ is a pair of continuous mappings, so one can write

$$\partial(\mho\tau^*,\nabla\tau^*)=\partial(\mathtt{C},\ni).$$

Therefore, τ^* is a coincidence best proximity point of the pair of mappings $(\mho,\nabla)$.
Uniqueness: Suppose that there are two distinct coincidence best proximity points of $(\mho,\nabla)$ with $\tau\neq\tau^*$. Then $q=\partial(\mho\tau,\mho\tau^*)>0$. Since $\wp(\mho\tau,\nabla\tau)=\wp(\mho\tau^*,\nabla\tau^*)=\partial(\mathtt{C},\ni)$, using the P-property, we conclude that $q=\mathscr{H}(\nabla\tau,\nabla\tau^*)$. Moreover the pair of mappings $(\mho,\nabla)$ satisfies φ_M-proximal-contraction, then we obtain $q\leq\varphi q$.
Hence, $\varphi\geq 1$. Since, $\varphi\leq 1$, we conclude that $\varphi=1$, which is again a contradiction. Hence the uniqueness is certified. □

Corollary 3.6. *Let $\nabla:\mathtt{C}\to\mathscr{CB}(\ni)$ and $\pi_*:\mathbb{J}\times\mathbb{J}\to[1,\infty)$ be mappings, where $\mathtt{C}$ is closed subset of an extended $b-$metric space $(\mathbb{J},\partial)$ and the pair $(\mathtt{C},\ni)$ satisfies the P-property with $\nabla(\mathtt{C}_0)\subseteq\ni_0$. Assume that the continuous mapping ∇ satisfies $\varphi_\nabla-$proximal contraction and the following expression holds*

$$\lim_{n,m\to\infty}\pi(\tau_n,\tau_m)<\frac{1}{k},\ \textit{where}\ \ k\in(0,1).$$

Then there exists a unique best proximity point of the mapping ∇.

Proof. Taking into account the identity mapping $\mho=I_{\mathtt{C}}$ ($\mho$ is the identity on $\mathtt{C}$), the proof can be completed on the similar lines as in Theorem 3.5. □

Example 3.7. *Let $\mathbb{J}=\{0,1,2,3,4\}$ and consider the function $\partial:\mathbb{J}\times\mathbb{J}\to[0,\infty)$ defined as $\partial(\tau,\tau^*)=\partial(\tau^*,\tau)$ and $\partial(\tau,\tau)=0$, where*

∂	0	1	2	3	4
0	0	$\frac{1}{7}$	$\frac{1}{9}$	$\frac{1}{11}$	$\frac{1}{13}$
1	$\frac{1}{7}$	0	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{8}$
2	$\frac{1}{9}$	$\frac{1}{5}$	0	$\frac{1}{4}$	$\frac{1}{3}$
3	$\frac{1}{11}$	$\frac{1}{6}$	$\frac{1}{4}$	0	$\frac{1}{2}$
4	$\frac{1}{13}$	$\frac{1}{8}$	$\frac{1}{3}$	$\frac{1}{2}$	0

.

Take $\pi_* : \mathbb{J} \times \mathbb{J} \to [1,\infty)$ to be symmetric and defined as $\pi_*(\tau,\tau^*) = 2\tau + 3\tau^* + 1$. It is straightforward to verify that $(\mathbb{J},\partial)$ is an extended b-metric space. Let $\complement = \{0,1\}$ and $\eth = \{2,3,4\}$ be nonempty subsets of $\mathbb{J}$. After simple calculations, we attain $\partial(\complement,\eth) = \frac{1}{7}$. It can also be shown that the pair $(\complement,\eth)$ satisfies the P-property, with $\complement_0 = \complement$ and $\eth_0 = \eth$.
Now, consider the mappings $\nabla : \complement \to \mathscr{CB}(\eth)$ and $\mho : \complement \to \complement$ defined as follows

$$\nabla(\tau) = \begin{cases} \{2,3\}, & \text{if } \tau = 0, \\ \{3,4\}, & \text{if } \tau = 1, \end{cases}$$

and

$$\mho(\tau) = \begin{cases} 1, & \text{if } \tau = 0, \\ 0, & \text{if } \tau = 1. \end{cases}$$

Clearly, $\nabla(\complement_0) \subseteq \eth_0$ and $\complement_0 \subseteq \mho(\complement_0)$. We now show that the pair $(\mho,\nabla)$ satisfies φ_M-proximal-contraction

$$\mathscr{H}(\nabla(\tau),\nabla(\tau^*)) \le \varphi\mathbb{M}(u,v,\tau,\tau^*),$$

for all $u,v,\tau,\tau^* \in \complement$, where $\mathbb{M}(u,v,\tau,\tau^*)$ is as defined in Definition 3.1. Since

$$\wp(\mho 0,\nabla 1) = \partial(\complement,\eth)$$

$$\wp(\mho 1,\nabla 0) = \partial(\complement,\eth),$$

and

$$\wp(\mho 0,\nabla 0) = \partial(\complement,\eth)$$

$$\wp(\mho 1,\nabla 1) = \partial(\complement,\eth),$$

then we analyze the following cases:

Case (i): If $\wp(\mho 0,\nabla 1) = \wp(\mho 1,\nabla 0) = \partial(\complement,\eth)$, let $u = 0, v = 1, \tau = 0, \tau^* = 1$, then

$$\mathscr{H}(\nabla(0),\nabla(1)) = \mathscr{H}(\{2,3\},\{3,4\}) = \frac{1}{4}.$$

Also,

$$\mathbb{M}(0,1,0,1) = \max\left\{\frac{1}{7}, \frac{\wp(1,\{3,4\}) - \pi_*(1,\{3,4\})\wp(0,\{3,4\})}{\pi_*(1,\{3,4\})}, \frac{1}{4}\right\} = \frac{1}{4}.$$

Thus,

$$\mathscr{H}(\nabla(0),\nabla(1)) \le \varphi\mathbb{M}(0,1,0,1),$$

for $\varphi \in [0,1)$.

Case (ii): If $\wp(\mho 0,\nabla 0) = \wp(\mho 1,\nabla 1) = \partial(\complement,\eth)$, let $u = 0, v = 1, \tau = 1, \tau^* = 0$, then

$$\mathscr{H}(\nabla(1),\nabla(0)) = \mathscr{H}(\{3,4\},\{2,3\}) = \frac{1}{4}.$$

Similarly,

$$\mathbb{M}(0,1,1,0) = \frac{1}{4}.$$

Thus,

$$\mathscr{H}(\nabla(1),\nabla(0)) \le \varphi\mathbb{M}(0,1,1,0),$$

for $\varphi \in [0,1)$. Since $\mathscr{H}(\nabla(\tau),\nabla(\tau^*)) \le \varphi\mathbb{M}(u,v,\tau,\tau^*)$ holds in all cases, it is concluded that $(\mho,\nabla)$ satisfies φ_M-proximal-contraction. Finally, $\wp(\mho 0,\nabla 0) = \partial(\complement,\eth)$, so 0 is the unique coincidence best proximity point of $(\mho,\nabla)$. Therefore, Theorem 3.5 is validated.

Remark 3.8. *It may be noted that the pair of mapping $(\mho,\nabla)$ does not satisfy the proximal conditions of the main results in [20, 21] within the realm of extended b-metric spaces. Hence the results of [20, 21] can not be applied on the pair $(\mho,\nabla)$, showing that under the context of extended b−metric spaces, our generalizations are applicable and generalizations in the true sense.*

4. Coincidence Proximal Point Results for Single-Valued Mappings

This section is devoted to discussing some of best coincidence proximity point theorems for single-valued mappings in the framework of extended $b-$metric spaces.

Definition 4.1. *Given $\nabla : \complement \to \backepsilon$ and $\mho : \complement \to \complement$ be mappings. The pair $(\mho, \nabla)$ is said to be a φ_S-proximal-contraction if there exists a real number $\varphi \in [0,1)$ such that*

$$\left.\begin{array}{l}\partial(\mho u, \nabla\tau) = \partial(\complement, \backepsilon)\\ \partial(\mho v, \nabla\tau^*) = \partial(\complement, \backepsilon)\end{array}\right\} \text{ implies } \partial(\nabla\tau, \nabla\tau^*) \leq \varphi\,\mathbb{M}(u,v,\tau,\tau^*),$$

where

$$\mathbb{M}(u,v,\tau,\tau^*) = \max\left\{ \partial(\mho u, \mho v), \frac{\partial(\mho\tau,\nabla v) - \pi(\mho\tau,\nabla v)\partial(\mho u,\nabla v)}{\pi(\mho\tau,\nabla v)}, \partial^*(\mho v, \nabla u) \right\},$$

for all u,v,τ,τ^* in $\complement$.

Definition 4.2. *Given $\nabla : \complement \to \backepsilon$ and $\mho : \complement \to \complement$ be mappings. The pair $(\mho, \nabla)$ is said to be a $\varphi_{S^*}-$proximal-contraction if there exists a real number $\varphi \in [0,1)$ such that*

$$\left.\begin{array}{l}\partial(u, \nabla\tau) = \partial(\complement, \backepsilon)\\ \partial(v, \nabla\tau^*) = \partial(\complement, \backepsilon)\end{array}\right\} \text{ implies } \partial(\nabla\tau, \nabla\tau^*) \leq \varphi\mathbb{M}(u,v,\tau,\tau^*),$$

where

$$\mathbb{M}(u,v,\tau,\tau^*) = \max\left\{\partial(u,v), \frac{\partial(\tau,\nabla v) - \pi(\tau,\nabla v)\partial(u,\nabla v)}{\pi(\tau,\nabla v)}, \partial^*(v,\nabla u)\right\},$$

for all u,v,τ,τ^* in $\complement$.
Note that, if we take $\mho = I_{\complement}$ ($\mho$ as an identity mapping on $\complement$), then every φ_S-proximal-contraction will reduce to a $\varphi_{S^*}-$proximal-contraction.

Theorem 4.3. *Let $\nabla : \complement \to \backepsilon$, $\mho : \complement \to \complement$ and $\pi : \mathbb{J} \times \mathbb{J} \to [1,\infty)$ be mappings, where $\complement$ is closed subset of an extended $b-$metric space $(\mathbb{J}, \partial)$ and the pair $(\complement, \backepsilon)$ satisfies the P-property with $\nabla(\complement_0) \subseteq \backepsilon_0$ and $\complement_0 \subseteq \mho(\complement_0)$. Assume that the pair of continuous mappings $(\mho, \nabla)$, where $\mho$ is one-to-one satisfies φ_S-proximal-contraction and the following statement holds*

$$\lim_{n,m\to\infty} \pi(\mho\tau_n, \mho\tau_m) < \frac{1}{k}, \text{ where } k \in (0,1).$$

Then, there exists a coincidence best proximity point of the pair $(\mho, \nabla)$.

Proof. Since every single valued mapping is multi-valued mapping, the remaining proof is the same as Theorem 3.5. □

Corollary 4.4. *Let $\nabla : \complement \to \backepsilon$ and $\pi : \mathbb{J} \times \mathbb{J} \to [1,\infty)$ be mappings, where $\complement$ is closed subset and the pair $(\complement, \backepsilon)$ satisfies the P-property with $\nabla(\complement_0) \subseteq \backepsilon_0$. Assume that the continuous mapping ∇ satisfies $\varphi_{S^*}-$proximal-contraction and the following statement holds*

$$\lim_{n,m\to\infty} \pi(\tau_n, \tau_m) < \frac{1}{k}, \text{ where } k \in (0,1).$$

Then, there exists a best proximity point of the mapping ∇.

Proof. If we take identity mapping $\mho = I_{\complement}$ ($\mho$ is identity on $\complement$), the remaining proof is the same as Theorem 4.3.

□

Corollary 4.5. *Let $(\mathbb{J}, \partial)$ be an extended $b-$metric space and $\nabla : \mathbb{J} \to \mathbb{J}$, $\pi : \mathbb{J} \times \mathbb{J} \to [1,\infty)$ be mappings. If $\lim_{n,m\to\infty} \pi(\tau_n, \tau_m) < \frac{1}{k}$, where $k \in (0,1)$, the mapping ∇ is continuous and there exists a real number $\varphi \in [0,1)$ such that the following $\varphi_{S^*}-$proximal-type contraction is satisfied*

$$\partial(\nabla\tau, \nabla\tau^*) \leq \varphi\,\mathbb{M}(u,v,\tau,\tau^*),$$

where

$$\mathbb{M}(u,v,\tau,\tau^*) = \max\left\{\partial(u,v), \frac{\partial(\tau,\nabla v) - \pi(\tau,\nabla v)\partial(u,\nabla v)}{\pi(\tau,\nabla v)}, \partial^*(v,\nabla u), \frac{\partial(u,\nabla\tau^*) - \pi(u,\nabla\tau^*)\partial(v,\nabla\tau^*)}{\pi(u,\nabla\tau^*)}\right\},$$

for all u,v,τ,τ^* in $\mathbb{J}$, then there exists a unique fixed point of the mapping ∇ in $(\mathbb{J}, \partial)$.

Example 4.6. *Let $\mathbb{J} = \{5,6,7,8,9,10\}$ be a complete extended b-metric space $(\mathbb{J},\partial)$, where the distance function is defined as*

$$\partial(\tau,\tau^*) = |\tau-\tau^*|^3, \text{ for all } \tau,\tau^* \in \mathbb{J}.$$

Additionally, let $\complement = \{5,7,9\}$ and $\ni = \{6,8,10\}$ be two non-empty subsets of $\mathbb{J}$. It can be verified through straightforward calculations that $\partial(\complement,\ni) = 1$, and the pair $(\complement,\ni)$ satisfies the P-property. Here $\complement_0 = \complement$, $\ni_0 = \ni$, and we define

$$\pi(\tau,\tau^*) = 12\tau^2 + 10\tau^{*2} + 5.$$

Next, we define the mappings $\nabla : \complement \to \ni$ and $\mho : \complement \to \complement$ as follows

$$\nabla(\tau) = \begin{cases} 6, & \text{if } \tau \in \{5,7\}, \\ 8, & \text{if } \tau = 9, \end{cases}$$

and

$$\mho(\tau) = \begin{cases} 5, & \text{if } \tau = 7, \\ 7, & \text{if } \tau = 9, \\ 9, & \text{if } \tau = 5. \end{cases}$$

Clearly, $\nabla(\complement_0) \subseteq \ni_0$ and $\mho(\complement_0) \subseteq \complement_0$. Now, we verify that the pair $(\mho,\nabla)$ satisfies the φ_S-proximal-contraction condition

$$\partial(\nabla\tau,\nabla\tau^*) \le \varphi\mathbb{M}(u,v,\tau,\tau^*), \text{ for all } u,v,\tau,\tau^* \in \complement.$$

Let us consider the case where

$$\partial(\mho 5,\nabla 7) = \partial(\complement,\ni), \quad \partial(\mho 7,\nabla 9) = \partial(\complement,\ni),$$

and take $u = 5$, $v = 7$, $\tau = 7$, $\tau^* = 9$. After routine calculations, we derive that

$$\partial(\nabla 7,\nabla 9) = \partial(6,8) = 8,$$

and

$$\mathbb{M}(5,7,7,9) = \max\left\{ \partial(\mho 5,\mho 7), \tfrac{\partial(\mho 7,\nabla 7)-\pi(\mho 7,\nabla 7)\partial(\mho 5,\nabla 7)}{\pi(\mho 7,\nabla 7)}, \partial^*(\mho 7,\nabla 5) \right\}.$$

Breaking this down

$$\partial(\mho 5,\mho 7) = \partial(5,9) = 64,$$

$$\frac{\partial(\mho 7,\nabla 7)-\pi(\mho 7,\nabla 7)\partial(\mho 5,\nabla 7)}{\pi(\mho 7,\nabla 7)} = \frac{\partial(7,6)-\pi(7,6)\partial(5,6)}{\pi(7,6)} = \frac{1-(615)\partial(5,6)}{615} = \frac{-3097}{615},$$

and

$$\partial^*(\mho 7,\nabla 5) = \partial(9,6) = 27.$$

Thus, we have

$$\mathbb{M}(5,7,7,9) = \max\{64, \frac{-3097}{615}, 27\} = 64.$$

Finally, we attain

$$\partial(\nabla 7,\nabla 9) = 8 \le \varphi\mathbb{M}(5,7,7,9),$$

which holds for $\varphi = \frac{1}{8}$, thereby satisfying the φ_S-proximal-contraction condition. Therefore, $\partial(\mho 5,\nabla 5) = \partial(\complement,\ni)$, it is concluded that 5 is a coincidence best proximity point of the mappings $(\mho,\nabla)$. This confirms that all the conditions of Theorem 4.3 are fulfilled.

5. Applications

Fundamental instruments in science, including basic and differential conditions, significantly influence various logical domains. These conditions illustrate relationships regarding rates of advancement or accumulation, rendering them essential for representing dynamic systems. Differential conditions highlight the aspects of characteristics that are subject to change, whereas essential conditions enhance these analyses by incorporating cumulative effects or boundary conditions. Fundamental and differential conditions are essential in mathematics for the advancement of hypotheses in applied science, control systems, and optimization. They constitute the foundation of mathematical inquiry and computational modelling. In practical contexts, their applications encompass a remarkable variety: from assessing meteorological anomalies and examining liquid substances to simulating biological structures, such as disease propagation, and engineering designs like bridges and aircraft. Furthermore, the partial plans of these scenarios simplify the demonstration of processes involving memory effects, such as the transformation or uneven distribution of materials in chemistry and physical science. Recently, necessary and differential conditions have gained prominence for their capacity to elucidate phenomena across various domains, including epidemiology, finance, and energy. In 2021 Khan et al. [22] employed partial differential equations to illustrate the transmission of Coronavirus, integrating the effects of isolation on individuals with diabetes. Abdou [23] studied nonlinear fragmentary differential conditions in symmetrical measurement spaces in 2023. He used a fixed-direct hypothesis to show that complex frameworks were possible. These models demonstrate the sufficiency of fundamental and differential conditions as quantitative tools for evaluating and resolving intricate problems in science and engineering. Fundamental and differential equations are essential numerical tools for characterizing and analyzing systems that evolve over time or space. Differential conditions show how quickly things are changing in different frameworks, while fundamental conditions show how effects add up across a domain, which is common in boundary value or inverse problems. Collectively, they present a theoretical rationale for valuing diverse physical, natural, and financial characteristics. The fixed-point hypothesis has emerged as a crucial tool for analyzing these situations, providing robust methods to ascertain the existence and uniqueness of solutions. Experts transform necessary and sufficient conditions into corresponding fixed-point problems, utilizing fixed-point theorems such as Banach's and Schauder's principles to verify the solveability of these problems. This procedure has demonstrated considerable progress in both theoretical and practical mathematics. Hamdan and Kechil [24] illustrate through numerical simulations that fractional-order models can accurately characterize the dissemination of COVID-19, providing insights into the effects of different control methods. Their research indicates that fractional-order differential equations are effective instruments for comprehending and forecasting the dynamics of infectious diseases, especially those characterized by intricate transmission patterns such as COVID-19. Abdou [23] employed fixed-direct hypotheses in symmetrical measurement spaces to address nonlinear fragmentary differential equations and manage intricate limit conditions. Cabada and Hamdi [25] examined essential limit value problems, highlighting the importance of fixed-point theory in nonlinear fractional differential equations. These commitments highlight the importance of fundamental and differential conditions in the dynamics of numerical speculations and the pursuit of viable issues. Their versatility and significance guarantee their essential role in both fundamental and applied research, propelling advancements across science, engineering, and various disciplines.

Let $\mathbb{J}=\mathscr{C}[\sigma_1,\sigma_2]$ be a set of all real valued continuous functions on $[\sigma_1,\sigma_2]$. Define the mappings $\partial:\mathbb{J}\times\mathbb{J}\to[0,+\infty)$ by

$$\partial(\tau,\tau^*)=\sup_{c\in[\sigma_1,\sigma_2]}|\tau(c)-\tau^*(c)|^p,$$

for all $\tau,\tau^*\in\mathbb{J}$ and

$$e(\tau,\tau^*)=r+\tau+\tau^*,\ \ p\geq 2,\ \ r>2.$$

Then, $(\mathbb{J},\partial)$ is a complete extended b-metric space. Consider the Fredholm integral equation given by

$$\tau(\varpi)=\varphi(\varpi)+\beth\int_{\sigma_1}^{\sigma_2}\mathscr{I}(\varpi,\kappa,\tau(\kappa))d\kappa, \tag{5.1}$$

where $t\in[\sigma_1,\sigma_2],|\beth|>0$ and $\mathscr{I}:[\sigma_1,\sigma_2]\times[\sigma_1,\sigma_2]\times\mathbb{J}\to\Re$ and $\varphi:[\sigma_1,\sigma_2]\to\Re$ are continuous functions. Let $\nabla:\mathbb{J}\to\mathbb{J}$ be an integral operator defined by

$$\nabla\tau(\varpi)=f(\varpi)+\beth\int_{\sigma_1}^{\sigma_2}\mathscr{I}(\varpi,\kappa,\tau(\kappa))d\kappa. \tag{5.2}$$

Then $\tau(\varpi)$ is a fixed point of ∇ if and only it is a solution of the Fredholm integral equation (5.1).
We now offer the following subsequent theorem to establish the existence of a solution to the Fredholm integral equation (5.1).

Theorem 5.1. *Let $\nabla:\mathbb{J}\to\mathbb{J}$ be an integral operator defined in (5.2). Suppose that the following assumptions hold*

1. *for any $\tau_0\in\mathbb{J},\lim_{n,m\to\infty}\pi(\nabla^n\tau_0,\nabla^m\tau_0)<\frac{1}{\varphi}$, where $\varphi=\frac{1}{2^p}$*
2. *for any $\tau,\tau^*\in\mathbb{J},\tau\neq\tau^*$, $\mathscr{I}:[\sigma_1,\sigma_2]\times[\sigma_1,\sigma_2]\times\mathbb{J}\to\Re$ satisfies*

$$|\mathscr{I}(\varpi,\kappa,\tau(\kappa))-\mathscr{I}(\varpi,\kappa,\tau^*(\kappa))|\leq\xi(\varpi,\kappa)\mid\tau(\kappa)-\tau^*(\kappa))\mid,$$

where $(\kappa,t)\in[\sigma_1,\sigma_2]\times[\sigma_1,\sigma_2]$ and $\xi:[\sigma_1,\sigma_2]\times[\sigma_1,\sigma_2]\to\Re$ is a continuous function satisfying

$$\sup_{t\in[\sigma_1,\sigma_2]}\int_{\sigma_1}^{\sigma_2}\xi^p(\varpi,\kappa)|\tau(\kappa)-\tau^*(\kappa)|^pd\kappa\leq\frac{1}{2^p|\beth|^p(\sigma_2-\sigma_1)^{p-1}}\mathbb{M}(\varpi,\kappa,\tau,\tau^*),$$

where $\mathbb{M}(\varpi,\kappa,\tau,\tau^*)$ is defined as in Definition 4.2.
Then, the integral operator ∇ has a unique solution in $\mathbb{J}$.

Proof. Let $\tau_0\in\mathbb{J}$ and define a sequence $\{\tau_n\}$ in $\mathbb{J}$ by $\tau_n=\nabla^n\tau_0,\ \ n\geq 1$. From (5.2), we obtain

$$\tau_{n+1}=\nabla\tau_n(\varpi)=L(\varpi)+\beth\int_{\sigma_1}^{\sigma_2}\mathscr{I}(\varpi,\kappa,\tau_n(\kappa))d\kappa.$$

Let $q>1$ be a constant with $\frac{1}{p}+\frac{1}{q}=1$. By the Holder's inequality, we speculate that

$$\begin{aligned}|\nabla\tau(\varpi)-\nabla\tau^*(\varpi)|^p&=\left|\beth\int_{\sigma_1}^{\sigma_2}\mathscr{I}(\varpi,\kappa,\tau(\kappa))d\kappa-\beth\int_{\sigma_1}^{\sigma_2}\mathscr{I}(\varpi,\kappa,\tau^*(\kappa))d\kappa\right|^p\\&\leq\left(\int_{\sigma_1}^{\sigma_2}|\beth||\mathscr{I}(\varpi,\kappa,\tau(\kappa))-\mathscr{I}(\varpi,\kappa,\tau^*(\kappa))|d\kappa\right)^p\\&\leq\left(\int_{\sigma_1}^{\sigma_2}|\beth|^qd\kappa\right)^{\frac{p}{q}}\left(\left(\int_{\sigma_1}^{\sigma_2}|\mathscr{I}(\varpi,\kappa,\tau(\kappa))-\mathscr{I}(\varpi,\kappa,\tau^*(\kappa))|^pd\kappa\right)^{\frac{1}{p}}\right)^p\\&=|\beth|^p(\sigma_2-\sigma_1)^{p-1}\left(\int_{\sigma_1}^{\sigma_2}|\mathscr{I}(\varpi,\kappa,\tau(\kappa))-\mathscr{I}(\varpi,\kappa,\tau^*(\kappa))|^pd\kappa\right)\\&\leq|\beth|^p(\sigma_2-\sigma_1)^{p-1}\int_{\sigma_1}^{\sigma_2}\xi^p(\varpi,\kappa)|\tau(\kappa)-\tau^*(\kappa)|^pd\kappa.\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}\partial(\nabla\tau,\nabla\tau^*) &= \sup_{t\in[\sigma_1,\sigma_2]}|\nabla\tau(\varpi)-\nabla\tau^*(\varpi)|^p\\ &\leq |\beth|^p(\sigma_2-\sigma_1)^{p-1}\sup_{t\in[\sigma_1,\sigma_2]}\left[\int_{\sigma_1}^{\sigma_2}\xi^p(\varpi,\kappa)|\tau(\kappa)-\tau^*(\kappa)|^p d\kappa\right]\\ &\leq \frac{1}{2^p}\mathscr{M}(\varpi,\kappa,\tau,\tau^*).\end{aligned}$$

Setting $\varphi=\frac{1}{2^p}$, we obtain

$$\partial(\nabla\tau,\nabla\tau^*)\leq\varphi\mathscr{M}(\varpi,\kappa,\tau,\tau^*).$$

Thus, all the conditions of Corollary 4.5 are satisfied and hence ∇ possesses a unique fixed point in $\mathbb{J}$, which means the integral operator ∇ has a unique solution in $\mathbb{J}$.

□

5.1. An application to the solution of a second-order differential equation

The existence of a solution for the preceding second-order boundary value problem is manifested in this section

$$\begin{cases} u''(\varpi)=W\big(\varpi,u(\varpi),\kappa(\varpi)\big), & \varpi\in[0,1];\\ u(0)=u_0, u(1)=u_1, \end{cases} \tag{5.3}$$

where $W:[0,1]\times[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.
Initially, consider the space $\mathbb{J}=C(\mathbb{Q})(\mathbb{Q}=[0,1],\mathbb{R})$ of continuous functions defined on $\mathbb{Q}$. Obviously this space with metric given by

$$\partial(u,v)=\sup_{c\in[\sigma_1,\sigma_2]}|u(c)-v(c)|^p,$$

for all $u,v\in\mathbb{J}$ and $e(u,v)=r+u+v, p\geq 2, r>2$ is a complete extended b-metric space.

Theorem 5.2. *Consider the boundary value problem given in (5.3). Suppose that for any $\tau_0\in\mathbb{J}$, $\lim_{n,m\to\infty}\pi(\nabla^n\tau_0,\nabla^m\tau_0)<\frac{1}{\varphi}$, where $\varphi=\frac{1}{2^p}$, and for suitable value of $\beth$ if $|\beth|^p\leq\frac{1}{2^p}$, then the second-order boundary value problem given in (5.3) has a unique solution.*

Proof. The boundary value problem given in (5.3) is equivalent to the second kind Fredholm integral equation

$$u(\varpi)=L(\varpi)+\beth\int_0^1\mathfrak{D}(\varpi,\kappa)u(\kappa)d\kappa,\quad \varpi\in[0,1], \tag{5.4}$$

in which $L(\varpi)=u_0+\varpi(u_1-u_0)$ and $\mathfrak{D}(\varpi,\kappa)$ is the Green's function, given by

$$\mathfrak{D}(\varpi,\kappa)=\begin{cases}\kappa(1-\kappa) & 0\leq\kappa\leq\varpi;\\ \varpi(1-\kappa) & \varpi\leq\kappa\leq 1.\end{cases} \tag{5.5}$$

Note that if $u\in C(\mathbb{Q})$ is a fixed point of ∇, then u is a solution of (5.4), consequently a solution of (5.3).
Let $u,v\in C(\mathbb{Q})$ and $q>1$ be a constant with $\frac{1}{p}+\frac{1}{q}=1$. By the Hölder's inequality, we acquire that

$$\begin{aligned}|\nabla u(\varpi)-\nabla v(\varpi)|^p &= \left|\beth\int_0^1\mathfrak{D}(\varpi,\kappa)u(\kappa)d\kappa-\beth\int_0^1\mathfrak{D}(\varpi,\kappa)v(\kappa)d\kappa\right|^p\\ &\leq\left(\int_0^1|\beth||\mathfrak{D}(\varpi,\kappa)u(\kappa)-\mathfrak{D}(\varpi,\kappa)v(\kappa)|d\kappa\right)^p\\ &\leq\left(\int_0^1|\beth|^q d\kappa\right)^{\frac{p}{q}}\left(\left(\int_0^1|\mathfrak{D}(\varpi,\kappa)u(\kappa)-\mathfrak{D}(\varpi,\kappa)v(\kappa)|^p d\kappa\right)^{\frac{1}{p}}\right)^p\\ &=|\beth|^p\left(\int_0^1|\mathfrak{D}(\varpi,\kappa)u(\kappa)-\mathfrak{D}(\varpi,\kappa)v(\kappa)|^p d\kappa\right)\\ &\leq|\beth|^p|u(\varpi)-v(\varpi)|^p\left(\int_0^1|\mathfrak{D}(\varpi,\kappa)|^p d\kappa\right).\end{aligned}$$

Then we conclude that

$$\begin{aligned}\partial(\nabla\tau,\nabla\tau^*) &= \sup_{t\in[\sigma_1,\sigma_2]}|\nabla\tau(\varpi)-\nabla\tau^*(\varpi)|^p\\ &\leq|\beth|^p\sup_{t\in[\sigma_1,\sigma_2]}|u(\varpi)-v(\varpi)|^p\left(\int_0^1|\mathfrak{D}(\varpi,\kappa)|^p d\kappa\right)\\ &\leq|\beth|^p\partial(u,v),\ \text{for any value of } p \text{ and utilizing (5.5)}\\ &\leq\frac{1}{2^p}\mathscr{M}(\varpi,\kappa,\tau,\tau^*).\end{aligned}$$

Fixing $\varphi = \frac{1}{2^p}$, we get

$$\partial(\nabla\tau, \nabla\tau^*) \leq \varphi\mathscr{M}(\varpi, \kappa, \tau, \tau^*).$$

Hence, we conclude that the proximal condition of Corollary 4.5 is satisfied, so ∇ has a unique fixed point, which is the solution of the integral equation (5.4). Consequently, the differential equation (5.3) has a solution.

□

6. Conclusion

This study elucidated the ideas of the proximal-contractions by developing some generalized proximal-contractions for both multivalued and single-valued mappings and demonstrated their applicability in extended b-metric spaces. The findings extended traditional proximity point theorems, guaranteeing the existence of unique coincidence-best proximity points under less stringent conditions. This framework streamlines current hypotheses and sets a basis for investigation in generalized metric structures, flexible mappings, and practical applications, presenting the substantial potential for advancing the theory of metric fixed points and the best proximity point results.

Some open problems for future research:

- Is it possible to apply the findings to real-world optimization problems, such as those in architecture or finance, where constraints inherently prompt a framework that incorporates proximal mappings?
- Is it possible to relax the continuity condition for $\mho$ and ∇ or substitute it with milder forms of continuity, such as lower semi-continuity or upper semi-continuity, while still obtaining comparable outcomes?
- Can the findings shown in this article be used to generate a solution to the following first-order periodic boundary value problem:

$$\begin{cases} \xi'(\tau) = f(\tau, \xi(\tau)), \\ \xi(0) = \xi(\nabla), \end{cases}$$

  where $\tau \in I = [0, \nabla]$, $\xi(\tau)$ is a real-valued function on I, and $f : I \times \mathbb{C} \to \mathbb{C}$ is a continuous function.
- Is it possible to explore effective numerical techniques to approximate optimal proximity points for large-scale data sets represented in an extended b-metric space?

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References

[1] M. M. Fréchet, *Sur quelques points du calcul fonctionnel*, Rend. Circ. Mat. Palermo, **22**(1) (1906), 1-72.
[2] T. Kamran, M. Samreen, Q. Ul. Ain, *A generalization of b-metric space and some fixed point theorems*, Math., **5**(2) (2017), 1-19.
[3] S. Basha, P. Veeramani, *Best approximations and best proximity pairs*, Acta Sci. Math, **63** (1997), 289-300.
[4] S. Basha and P. Veeramani, *Best proximity pair theorems for multifunction with open fibers*, J. Approx. Theory, **103**(1) (2000), 119-129.
[5] S. Ghezellou, M. Azhini, M. Asadi, *Best proximity point theorems by K, C and $\mathscr{MT}$ types in b-metric spaces with an application*. Int. J. Nonlinear Anal. Appl., **12**(2) (2021), 1317-1329.
[6] T. Suzuki, M. Kikkawa, C. Vetro, *The existence of best proximity points in metric spaces with the property UC*, Nonlinear Anal., **71**(7-8) (2009), 2918-2926.
[7] M. Asadi, M. Afshar, *Fixed point theorems in the generalized rational type of C-class functions in b-metric spaces with application to integral equation*. 3c Empresa: Investigación y pensamiento crítico, **11**(2) (2022), 64-74.
[8] B. Alqahtani, A. Fulg, E. Karapınar, *Common fixed point results on an extended b-metric space*, J. Inequal. Appl., **2018**(1) (2018), 1-15.
[9] S. Chandok, E. Karapinar, *Common fixed point of generalized rational type contraction mappings in partially ordered metric spaces*, Thai J. Math., **11**(2) (2012), 251-260.
[10] S. Jabeen, S. Ur Rehman, Z. Zheng, W. Wei, *Weakly compatible and quasi-contraction results in fuzzy cone metric spaces with application to the Urysohn type integral equations*, Adv. Differ. Equ., **2020**, 280. https://doi.org/10.1186/s13662-020-02743-5
[11] R. McConnell, R. Kwok, J. Curlander, W. Kober, S. Pang, *Ψ-S correlation and dynamic time warping: Two methods for tracking ice floes*, IEEE Trans. Geosci. Remote Sens., **29** (1991), 1004–1012.
[12] M. Younis, D. Singh, I. Altun, V. Chauhan, *Graphical structure of extended b-metric spaces: An application to the transverse oscillations of a homogeneous bar*, Internat. J. Nonlinear Sci. Numer. Simul., 23(7-8) (2022), 1239-1252. https://doi.org/10.1515/ijnsns-2020-0126
[13] M. Younis, H. Ahmad, L. Chen, M. Han, *Computation and convergence of fixed points in graphical spaces with an application to elastic beam deformations*, J. Geom. Phys., **192** (2023), 104955. https://doi.org/10.1016/j.geomphys.2023.104955
[14] M. Younis, D. Singh, M. Asadi, V. Joshi, *Results on contractions of Reich type in graphical b-metric spaces with applications*, Filomat, **33**(17) (2019), 5723-5735.
[15] M. E. Samani, S. M. Vaezpour, M. Asadi, New fixed point results with α_{qs^p}-admissible contractions on b-Branciari metric spaces, J. Inequal. Spec. Funct., **9**(4) (2018), 101-112.
[16] N. Savanovic, I. D. Arandelovic, Z. D. Mitrovic, *The results on coincidence and common fixed points for a new type multi-valued mappings in b-metric spaces*, Math., **10**(6) (2022), 856.
[17] M. Younis, H. Ahmad, W. Shahid, *Best proximity points for multi-valued mappings and equation of motion*, J. Appl. Anal. Comput., **14**(1) (2024), 298-316.
[18] V. S. Raj, *A best proximity point theorem for weakly contractive non-self-mappings*, Nonlinear. Anal., **74**(2011), 4804–4808.
[19] T. R. Rockafellar, R. J. V. Wets, *Variational Analysis*, Springer Berlin, Germany, (2005).
[20] M. Gabeleh, C. Vetro, *A note on best proximity point theory using proximal-contractions*, J. Fixed Point Theory Appl., **20**(4) (2018), 1-11.

[21] N. Saleem, J. Vujaković, W.U. Baloch, S. Radenović, *Coincidence point results for multi-valued Suzuki type mappings using θ-contraction in b-metric spaces*, Math., **7**(11) (2019), 1017.

[22] M.S. Khan, M. Ozair, T. Hussain, J. F. Gómez-Aguilar, *Bifurcation analysis of a discrete-time compartmental model for hypertensive or diabetic patients exposed to COVID-19*, The European Physical Journal Plus, **136**(8), (2021), 1-26.

[23] A. A. N. Abdou, *Solving a nonlinear fractional differential equation using fixed point results in orthogonal metric spaces*, Fractal Fract., **7**(11) (2023), 817. https://doi.org/10.3390/fractalfract7110817

[24] N. I. Hamdan, S. A., Kechil, *Mathematical model of COVID-19 transmission using the fractional-order differential equation. In AIP Conference Proceedings, AIP Publishing*, **2905**(1) (2024).

[25] A. Cabada, Z. Hamdi, *Nonlinear fractional differential equations with integral boundary value conditions*, Appl. Math. Comput., **228**(1) (2014), 251-257.

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Research Article

Contributions to the Fractional Hardy Integral Inequality

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Abstract

This article makes three contributions to the fractional Hardy integral inequality. First, we refine an existing result in the literature by improving the main constant and relaxing some assumptions on the parameters. We then propose a fractional-type Hardy integral inequality for an under-studied case, with a significant adaptation of the existing general proof. Finally, a version of this result is established when the integral domain is finite. The proofs are given in detail, with the exact expression of the constants involved at each step. We also mention that almost no intermediate results are used.

1. Introduction

The study of integral inequalities is a central topic in mathematics, particularly in real and functional analysis. It is used to determine the possible values of complex integrals, i.e., integrals that cannot be evaluated exactly using the usual techniques. Let us take a brief look at the most famous integral inequalities. The Hölder integral inequality connects integrals of products, the Minkowski integral inequality generalizes the triangular inequality to integrals, the Cauchy-Schwarz integral inequality, which can be seen as a special case of the Hölder integral inequality, gives a fundamental bound for the integral version of the inner product, the Jensen integral inequality applies to convex functions and integrals, the Grönwall integral inequality estimates solutions to differential inequalities, the Sobolev integral inequality relates function norms in Sobolev spaces, the Chebyshev integral inequality bounds probabilities using integrals, the Young integral inequality helps with convolution estimates, and the Hardy integral inequality gives bounds on weighted integrals and establishes key relationships in functional spaces. These results have many applications in physics, probability and optimization. The mathematical details of them can be found in the following books: [1–5].
For the purposes of this article, we will emphasize the Hardy integral inequality. A brief review of the results used for the purposes of the article is given below; the full historical facts and details can be found in [6, 7]. The classical Hardy integral inequality states that, for $p \in (1,+\infty)$ and $f : (0,+\infty) \mapsto \mathbb{R}$ such that

$$\int_0^{+\infty} |f(x)|^p dx < +\infty,$$

we have

$$\int_0^{+\infty} \left[\frac{1}{x}\int_0^x |f(t)|dt\right]^p dx \le C_p \int_0^{+\infty} |f(x)|^p dx,$$

where

$$C_p = \left(\frac{p}{p-1}\right)^p. \tag{1.1}$$

See [1, 8]. A finite integration interval version was also established, attributed to [9]. It states that, for $(a,b) \in (0,+\infty)^2 \cup \{\pm\infty\}^2$ with $a < b$, $p \in (1,+\infty)$ and $f : (a,b) \mapsto \mathbb{R}$ such that

$$\int_a^b |f(x)|^p dx < +\infty,$$

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we have

$$\int_a^b \left[\frac{1}{x}\int_a^x |f(t)|dt\right]^p dx \leq C_p \int_a^b |f(x)|^p dx,$$

where C_p is given in Equation (1.1). For more information, we refer to [6, Section 1.5].
Another famous variant of the Hardy integral inequality is the fractional Hardy integral inequality, which states that, for $p \in [1,+\infty)$, $\lambda \in (0,+\infty)\backslash\{1\}$ and $f : (0,+\infty) \mapsto \mathbb{R}$ such that

$$\int_0^{+\infty} \frac{|f(x)|^p}{x^\lambda} dx < +\infty,$$

we have

$$\int_0^{+\infty} \frac{|f(x)|^p}{x^\lambda} dx \leq D \int_0^{+\infty}\int_0^{+\infty} \frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}} dxdy,$$

where D is a certain constant (to be discussed later). We also refer to [6, Chapter 5] for the relevant historical background and developments. There are other versions of these integral inequalities that extend their applicability. They are used in harmonic analysis, partial differential equations, mathematical physics and probability theory. In addition to the classic books [6, 7], a selection of articles on the subject is given below: [10–29].
In particular, in [23], some results and proofs have attracted our attention. The fractional Hardy integral inequality is demonstrated in an original and simple way, with clear assumptions and exact constants. In particular, we emphasize two results, described below.

- The result in [23, Lemma 2] is formulated below, modulo some minor changes in the presentation. Let $p \in [1,+\infty)$, $\lambda \in (0,+\infty)\backslash\{1\}$ (note that the value $\lambda = 1$ is excluded) and $f : (0,+\infty) \mapsto \mathbb{R}$ such that

$$\int_0^{+\infty} \frac{|f(x)|^p}{x^\lambda} dx < +\infty.$$

  Then the following integral inequality holds:

$$\int_0^{+\infty} \frac{|f(x)|^p}{x^\lambda} dx \leq D_{\alpha,p,\lambda} \int_0^{+\infty}\int_0^{+\infty} \frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}} dxdy,$$

  where

$$D_{\alpha,p,\lambda} := \frac{2^p}{\alpha}\left\{\max\left[|1-\alpha|, |2\alpha-1|\right]\right\}^{1+\lambda}$$

  and $\alpha \in (0,+\infty)$ can be arbitrarily chosen such that

$$\frac{2^{p-1}}{\lambda}(2^\lambda - 1)\alpha^{\lambda-1} \leq \frac{1}{2}. \tag{1.2}$$

- In the same framework and under the same assumptions, but with a finite and $f : (0,a) \mapsto \mathbb{R}$, [23, Corollary 1] ensures that

$$\int_0^{a} \frac{|f(x)|^p}{x^\lambda} dx \leq D_{\alpha,p,\lambda} \int_0^{a}\int_0^{a} \frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}} dxdy.$$

The proofs of [23, Lemma 2] and [23, Corollary 1] are based on the triangular inequality, a thorough decomposition of the integral, the use of the Fubini-Tonelli integral theorem, integral calculus and basic arithmetic. The fact that these comprehensible developments are combined with the exact expressions of the constants involved is a real plus for a deeper understanding of these inequalities.
This article contributes to the topic in three related ways. First, we generalize the result in [23, Lemma 2]. In particular, we extend it to $p \in (0,+\infty)$, including the new case $p \in (0,1]$, and we slightly relax the assumption on α described in Equation (1.2). Second, the fractional Hardy integral inequality in [23, Lemma 2] is not valid for $\lambda = 1$. So we have no upper bound on the term

$$\int_0^{+\infty} \frac{|f(x)|^p}{x} dx,$$

which we will call "the case $\lambda = 1$" for our purposes.
This raises the following question: Can we prove a fractional-type Hardy integral inequality for this case? We provide an answer to this question by modifying the proof of [23, Lemma 2] on several crucial points. The case $p \in (0,+\infty)$ is also considered and the constants involved are given. Third, a similar question arises for the term

$$\int_0^{a} \frac{|f(x)|^p}{x} dx,$$

where a is finite. An answer is also given. Thus, new integral inequalities are established, considering understudied cases of the fractional Hardy integral inequality. The proofs are detailed for accuracy and reproducibility.
The rest of the article is divided into four sections: Section 2 is devoted to the generalization of [23, Lemma 2]. Fractional-type Hardy integral inequalities for "the case $\lambda = 1$" and for infinite and finite intervals are considered in Sections 3 and 4, respectively. A conclusion is proposed in Section 5.

2. Generalization of an Existing Result

The first proposition offers a generalized version of [23, Lemma 2]. In particular, the points below are developed.

- The condition on p, i.e., $p \in (1,+\infty)$, can be relaxed as $p \in (0,+\infty)$, with a slight modification of a constant.
- The condition on α recalled in Equation (1.2) can be slightly relaxed with little mathematical effort.

These modifications give more flexibility to the constant in the factor in the main inequality.

Proposition 2.1. *Let $p \in (0,+\infty)$, $\lambda \in (0,+\infty)\backslash\{1\}$ and $f:(0,+\infty)\mapsto \mathbb{R}$ such that*

$$\int_0^{+\infty} \frac{|f(x)|^p}{x^\lambda} dx < +\infty.$$

Then the following integral inequality holds:

$$\int_0^{+\infty} \frac{|f(x)|^p}{x^\lambda} dx \leq F_{\beta,p,\lambda} \int_0^{+\infty}\int_0^{+\infty} \frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}} dxdy,$$

where

$$F_{\beta,p,\lambda} := \left[1-\frac{\max(2^{p-1},1)}{\lambda}(2^\lambda-1)\beta^{\lambda-1}\right]^{-1} \frac{\max(2^{p-1},1)}{\beta}\left\{\max\left[|1-\beta|,|2\beta-1|\right]\right\}^{1+\lambda} \tag{2.1}$$

and $\beta \in (0,+\infty)$ can be arbitrarily chosen such that

$$\frac{\max(2^{p-1},1)}{\lambda}(2^\lambda-1)\beta^{\lambda-1} < 1.$$

Proof. The proof revisits that of [23, Lemma 2]. The details are given below. First, for any $p \in (0,+\infty)$ and $(u,v)\in\mathbb{R}^2$, we have

$$|u+v|^p \leq \max(2^{p-1},1)[|u|^p+|v|^p]. \tag{2.2}$$

See [30, Chapter 1]. This inequality applied to $u = f(y)$ and $v = f(x)-f(y)$ gives

$$|f(x)|^p = |f(y)+[f(x)-f(y)]|^p \leq \max(2^{p-1},1)|f(y)|^p + \max(2^{p-1},1)|f(x)-f(y)|^p.$$

Now, since $\beta \in (0,+\infty)$ and $x^{1+\lambda} \in (0,+\infty)$, dividing by $\beta x^{1+\lambda}$, we have

$$\frac{|f(x)|^p}{\beta x^{1+\lambda}} \leq \max(2^{p-1},1)\frac{|f(y)|^p}{\beta x^{1+\lambda}} + \max(2^{p-1},1)\frac{|f(x)-f(y)|^p}{\beta x^{1+\lambda}}.$$

Integrating with respect to $y \in (\beta x, 2\beta x)$, we get

$$\int_{\beta x}^{2\beta x} \frac{|f(x)|^p}{\beta x^{1+\lambda}} dy \leq \max(2^{p-1},1)\int_{\beta x}^{2\beta x} \frac{|f(y)|^p}{\beta x^{1+\lambda}} dy + \max(2^{p-1},1)\int_{\beta x}^{2\beta x} \frac{|f(x)-f(y)|^p}{\beta x^{1+\lambda}} dy.$$

For the left term, we have

$$\int_{\beta x}^{2\beta x} \frac{|f(x)|^p}{\beta x^{1+\lambda}} dy = \frac{|f(x)|^p}{\beta x^{1+\lambda}}\int_{\beta x}^{2\beta x} dy = \frac{|f(x)|^p}{x^\lambda}.$$

Using this and integrating with respect to $x \in (0,+\infty)$, we find that

$$\int_0^{+\infty} \frac{|f(x)|^p}{x^\lambda} dx \leq \max(2^{p-1},1)P + \max(2^{p-1},1)Q, \tag{2.3}$$

where

$$P := \int_0^{+\infty}\int_{\beta x}^{2\beta x} \frac{|f(y)|^p}{\beta x^{1+\lambda}} dydx$$

and

$$Q := \int_0^{+\infty}\int_{\beta x}^{2\beta x} \frac{|f(x)-f(y)|^p}{\beta x^{1+\lambda}} dydx.$$

Let us now bound P and Q successively.
For P, the Fubini-Tonelli integral theorem ensures the change in the order of integration, which gives

$$P = \int_0^{+\infty}\int_{y/(2\beta)}^{y/\beta} \frac{|f(y)|^p}{\beta x^{1+\lambda}} dxdy = \int_0^{+\infty} |f(y)|^p \left[\int_{y/(2\beta)}^{y/\beta} \frac{1}{\beta x^{1+\lambda}} dx\right] dy$$
$$= \int_0^{+\infty} |f(y)|^p \frac{1}{\beta\lambda}\left[\left(\frac{2\beta}{y}\right)^\lambda - \left(\frac{\beta}{y}\right)^\lambda\right] dy = \frac{1}{\lambda}(2^\lambda-1)\beta^{\lambda-1}\int_0^{+\infty} \frac{|f(y)|^p}{y^\lambda} dy. \tag{2.4}$$

On the other hand, for Q, with a suitable decomposition of the integrated term, we have

$$\begin{aligned}
Q &= \frac{1}{\beta}\int_0^{+\infty}\int_{\beta x}^{2\beta x}\frac{|x-y|^{1+\lambda}}{x^{1+\lambda}}\times\frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}dydx \\
&\leq \frac{1}{\beta}\left[\sup_{x\in(0,+\infty)}\sup_{y\in(\beta x,2\beta x)}\frac{|x-y|^{1+\lambda}}{x^{1+\lambda}}\right]\int_0^{+\infty}\int_{\beta x}^{2\beta x}\frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}dydx \\
&\leq \frac{1}{\beta}\left[\sup_{x\in(0,+\infty)}\sup_{y\in(\beta x,2\beta x)}\frac{|x-y|^{1+\lambda}}{x^{1+\lambda}}\right]\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}dydx.
\end{aligned}$$

The term in square brackets can be developed as follows:

$$\begin{aligned}
\sup_{x\in(0,+\infty)}\sup_{y\in(\beta x,2\beta x)}\frac{|x-y|^{1+\lambda}}{x^{1+\lambda}} &= \sup_{x\in(0,+\infty)}\max\left[\frac{|x-\beta x|^{1+\lambda}}{x^{1+\lambda}},\frac{|x-2\beta x|^{1+\lambda}}{x^{1+\lambda}}\right] \\
&= \{\max[|1-\beta|,|2\beta-1|]\}^{1+\lambda}.
\end{aligned}$$

We therefore obtain

$$Q \leq \frac{1}{\beta}\{\max[|1-\beta|,|2\beta-1|]\}^{1+\lambda}\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}dydx. \tag{2.5}$$

Combining Equations (2.3), (2.4) and (2.5) together, we get

$$\begin{aligned}
\int_0^{+\infty}\frac{|f(x)|^p}{x^\lambda}dx \leq{}& \frac{\max(2^{p-1},1)}{\lambda}(2^\lambda-1)\beta^{\lambda-1}\int_0^{+\infty}\frac{|f(x)|^p}{x^\lambda}dx \\
&+\frac{\max(2^{p-1},1)}{\beta}\{\max[|1-\beta|,|2\beta-1|]\}^{1+\lambda}\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}dydx.
\end{aligned}$$

This is equivalent to the following inequality:

$$\left[1-\frac{\max(2^{p-1},1)}{\lambda}(2^\lambda-1)\beta^{\lambda-1}\right]\int_0^{+\infty}\frac{|f(x)|^p}{x^\lambda}dx \leq \frac{\max(2^{p-1},1)}{\beta}\{\max[|1-\beta|,|2\beta-1|]\}^{1+\lambda}\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}dydx.$$

Since $\beta \in (0,+\infty)$ is chosen such that

$$\frac{\max(2^{p-1},1)}{\lambda}(2^\lambda-1)\beta^{\lambda-1} < 1,$$

we also have

$$\begin{aligned}
\int_0^{+\infty}\frac{|f(x)|^p}{x^\lambda}dx &\leq \left[1-\frac{\max(2^{p-1},1)}{\lambda}(2^\lambda-1)\beta^{\lambda-1}\right]^{-1}\frac{\max(2^{p-1},1)}{\beta}\{\max[|1-\beta|,|2\beta-1|]\}^{1+\lambda}\times\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}dxdy \\
&= F_{\beta,p,\lambda}\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}dxdy,
\end{aligned}$$

where $F_{\beta,p,\lambda}$ is given in Equation (2.1). The proof of Proposition 2.1 ends. □

This result shows greater flexibility than [23, Lemma 2], on the same mathematical basis. Indeed, the case $p \in (0,1]$ is now considered, and β can be chosen more flexibly than α in [23, Lemma 2]. Note that, for any $p \in (0,1]$, based on Equation (2.1), we have

$$F_{\beta,p,\lambda} = \left[1-\frac{1}{\lambda}(2^\lambda-1)\beta^{\lambda-1}\right]^{-1}\frac{1}{\beta}\{\max[|1-\beta|,|2\beta-1|]\}^{1+\lambda}.$$

The presence of p in the index of $F_{\beta,p,\lambda}$ is due to the fact that β may depend on p.
These aspects have a positive effect on the constant factor in the main inequality.

3. A New Fractional-Type Hardy Integral Inequality

The proposition below fills a gap in [23, Lemma 2] and in the literature on integral inequalities in general. It provides a valuable fractional-type inequality for "the case $\lambda = 1$", i.e., for the following integral as the left term:

$$\int_0^{+\infty}\frac{|f(x)|^p}{x}dx,$$

which was avoided in [23, Lemma 2]. As in Proposition 2.1, we assume that $p \in (0,+\infty)$ and reuse some techniques from the proof of that proposition.

Proposition 3.1. *Let* $p \in (0,+\infty)$ *and* $f:(0,+\infty) \mapsto \mathbb{R}$ *such that*

$$\int_0^{+\infty} \frac{|f(x)|^p}{x} dx < +\infty.$$

Then we have

$$\int_0^{+\infty} \frac{|f(x)|^p}{x} dx \leq G_{\theta,p,\alpha} \int_0^{+\infty} \int_0^{+\infty} \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx,$$

where

$$G_{\theta,p,\alpha} := \frac{\max(2^{p-1},1)\theta}{\alpha\left[\theta - \max(2^{p-1},1)\right]} \{\max[|1-\alpha|,|2\alpha-1|]\}^{1+1/\theta}, \tag{3.1}$$

$\theta > \max(2^{p-1},1)$ *and* $\alpha \in (0,+\infty)$ *(it is completely arbitrary).*

Proof. A significant modification of the proof of [23, Lemma 2] is necessary, where the new constant θ plays an important role. First, it follows from Equation (2.2) applied to $u = f(y)$ and $v = f(x) - f(y)$ that

$$|f(x)|^p = |f(y) + [f(x) - f(y)]|^p \leq \max(2^{p-1},1)|f(y)|^p + \max(2^{p-1},1)|f(x) - f(y)|^p.$$

We now divide the above terms by $\alpha x^{1+\theta}$, activating the parameter θ, which is positive. This gives us

$$\frac{|f(x)|^p}{\alpha x^{1+\theta}} \leq \max(2^{p-1},1)\frac{|f(y)|^p}{\alpha x^{1+\theta}} + \max(2^{p-1},1)\frac{|f(x)-f(y)|^p}{\alpha x^{1+\theta}}.$$

Integrating with respect to $y \in (\alpha x^\theta, 2\alpha x^\theta)$, we get

$$\int_{\alpha x^\theta}^{2\alpha x^\theta} \frac{|f(x)|^p}{\alpha x^{1+\theta}} dy \leq \max(2^{p-1},1)\int_{\alpha x^\theta}^{2\alpha x^\theta} \frac{|f(y)|^p}{\alpha x^{1+\theta}} dy + \max(2^{p-1},1)\int_{\alpha x^\theta}^{2\alpha x^\theta} \frac{|f(x)-f(y)|^p}{\alpha x^{1+\theta}} dy.$$

Noticing that

$$\int_{\alpha x^\theta}^{2\alpha x^\theta} \frac{|f(x)|^p}{\alpha x^{1+\theta}} dy = \frac{|f(x)|^p}{\alpha x^{1+\theta}} \int_{\alpha x^\theta}^{2\alpha x^\theta} dy = \frac{|f(x)|^p}{x},$$

and integrating with respect to $x \in (0,+\infty)$, we find that

$$\int_0^{+\infty} \frac{|f(x)|^p}{x} dx \leq \max(2^{p-1},1)U + \max(2^{p-1},1)V, \tag{3.2}$$

where

$$U := \int_0^{+\infty} \int_{\alpha x^\theta}^{2\alpha x^\theta} \frac{|f(y)|^p}{\alpha x^{1+\theta}} dydx$$

and

$$V := \int_0^{+\infty} \int_{\alpha x^\theta}^{2\alpha x^\theta} \frac{|f(x)-f(y)|^p}{\alpha x^{1+\theta}} dydx.$$

Let us now bound U and V successively.
For U, the Fubini-Tonelli integral theorem ensures the change in the order of integration, which gives

$$\begin{aligned} U &= \int_0^{+\infty} \int_{[y/(2\alpha)]^{1/\theta}}^{(y/\alpha)^{1/\theta}} \frac{|f(y)|^p}{\alpha x^{1+\theta}} dxdy = \int_0^{+\infty} |f(y)|^p \left[\int_{[y/(2\alpha)]^{1/\theta}}^{(y/\alpha)^{1/\theta}} \frac{1}{\alpha x^{1+\theta}} dx\right] dy \\ &= \int_0^{+\infty} |f(y)|^p \frac{1}{\alpha\theta} \left[\left(\frac{2\alpha}{y}\right) - \left(\frac{\alpha}{y}\right)\right] dy = \frac{1}{\theta}\int_0^{+\infty} \frac{|f(y)|^p}{y} dy. \end{aligned} \tag{3.3}$$

Note that the resulting term no longer depends on α.
On the other hand, for V, we have

$$\begin{aligned} V &= \frac{1}{\alpha}\int_0^{+\infty} \int_{\alpha x^\theta}^{2\alpha x^\theta} \frac{|x^\theta - y|^{1+1/\theta}}{x^{1+\theta}} \times \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx \\ &\leq \frac{1}{\alpha}\left[\sup_{x\in(0,+\infty)} \sup_{y\in(\alpha x^\theta, 2\alpha x^\theta)} \frac{|x^\theta - y|^{1+1/\theta}}{x^{1+\theta}}\right] \int_0^{+\infty} \int_{\alpha x^\theta}^{2\alpha x^\theta} \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx \\ &\leq \frac{1}{\alpha}\left[\sup_{x\in(0,+\infty)} \sup_{y\in(\alpha x^\theta, 2\alpha x^\theta)} \frac{|x^\theta - y|^{1+1/\theta}}{x^{1+\theta}}\right] \int_0^{+\infty} \int_0^{+\infty} \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx. \end{aligned}$$

Thanks to the introduction of the key term $|x^\theta - y|^{1+1/\theta}$, we have

$$\sup_{x\in(0,+\infty)}\sup_{y\in(\alpha x^\theta,2\alpha x^\theta)}\frac{|x^\theta-y|^{1+1/\theta}}{x^{1+\theta}} = \sup_{x\in(0,+\infty)}\max\left[\frac{|x^\theta-\alpha x^\theta|^{1+1/\theta}}{x^{1+\theta}},\frac{|x^\theta-2\alpha x^\theta|^{1+1/\theta}}{x^{1+\theta}}\right]$$
$$= \{\max[|1-\alpha|,|2\alpha-1|]\}^{1+1/\theta}.$$

We therefore obtain

$$V \leq \frac{1}{\alpha}\{\max[|1-\alpha|,|2\alpha-1|]\}^{1+1/\theta}\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|x^\theta-y|^{1+1/\theta}}dydx. \tag{3.4}$$

It follows from Equations (3.2), (3.3) and (3.4) that

$$\int_0^{+\infty}\frac{|f(x)|^p}{x}dx \leq \frac{\max(2^{p-1},1)}{\theta}\int_0^{+\infty}\frac{|f(x)|^p}{x}dx + \frac{\max(2^{p-1},1)}{\alpha}\{\max[|1-\alpha|,|2\alpha-1|]\}^{1+1/\theta}\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|x^\theta-y|^{1+1/\theta}}dydx.$$

This is equivalent to

$$\left[1-\frac{\max(2^{p-1},1)}{\theta}\right]\int_0^{+\infty}\frac{|f(x)|^p}{x}dx \leq \frac{\max(2^{p-1},1)}{\alpha}\{\max[|1-\alpha|,|2\alpha-1|]\}^{1+1/\theta}\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|x^\theta-y|^{1+1/\theta}}dydx.$$

Since $\theta > \max(2^{p-1},1)$, we get

$$\int_0^{+\infty}\frac{|f(x)|^p}{x}dx \leq \frac{\max(2^{p-1},1)\theta}{\alpha\left[\theta-\max(2^{p-1},1)\right]}\{\max[|1-\alpha|,|2\alpha-1|]\}^{1+1/\theta}\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|x^\theta-y|^{1+1/\theta}}dydx$$
$$= G_{\theta,p,\alpha}\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|x^\theta-y|^{1+1/\theta}}dydx,$$

where $G_{\theta,p,\alpha}$ is given in Equation (3.1). The proof of Proposition 3.1 ends. □

This result is thus a proposal of a fractional-type Hardy integral inequality for an under-explored case. The key point was the use of an adaptable parameter θ, which activates numerous intermediate terms, including $|x^\theta - y|^{1+1/\theta}$. We should also mention the presence of the parameter α, which can be set to any positive value.
Note that, when $p \in (0,1]$, the constant $G_{\theta,p,\alpha}$ in Equation (3.1) is reduced to

$$G_{\theta,p,\alpha} = \frac{\theta}{\alpha(\theta-1)}\{\max[|1-\alpha|,|2\alpha-1|]\}^{1+1/\theta},$$

which is independent of p.
Furthermore, for $p \in (0,+\infty)$ taking $\alpha = 1/2$, we find that

$$G_{\theta,p,\alpha} = \frac{\max(2^{p-1},1)\theta}{(1/2)\left[\theta-\max(2^{p-1},1)\right]}\left\{\max\left[\left|1-\frac{1}{2}\right|,\left|2\times\frac{1}{2}-1\right|\right]\right\}^{1+1/\theta}$$
$$= \frac{\max(2^{p-1},1)\theta 2^{-1/\theta}}{\theta-\max(2^{p-1},1)},$$

and the inequality in Proposition 3.1 gives

$$\int_0^{+\infty}\frac{|f(x)|^p}{x}dx \leq \frac{\max(2^{p-1},1)\theta 2^{-1/\theta}}{\theta-\max(2^{p-1},1)}\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|x^\theta-y|^{1+1/\theta}}dydx,$$

with $\theta > \max(2^{p-1},1)$. Especially, for $p \in (0,1]$, we have

$$\int_0^{+\infty}\frac{|f(x)|^p}{x}dx \leq \frac{\theta 2^{-1/\theta}}{\theta-1}\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|x^\theta-y|^{1+1/\theta}}dydx.$$

Besides filling a gap in the literature, this inequality has the merit of having a simple and original constant in the factor.
In the general case, we can also note that the change of variables $z = x^\theta$ gives

$$\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|x^\theta-y|^{1+1/\theta}}dydx = \frac{1}{\theta}\int_0^{+\infty}\int_0^{+\infty}\frac{|f(z^{1/\theta})-f(y)|^p}{|z-y|^{1+1/\theta}}z^{1/\theta-1}dydz.$$

Therefore, the inequality in Proposition 3.1 with the denominator term $|x-y|^{1+1/\theta}$ can be reformulated as follows:

$$\int_0^{+\infty}\frac{|f(x)|^p}{x}dx \leq \frac{1}{\theta}G_{\theta,p,\alpha}\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x^{1/\theta})-f(y)|^p}{|x-y|^{1+1/\theta}}x^{1/\theta-1}dydx.$$

From this point of view, the inequality can be seen as a special type of fractional Hardy integral inequality.

4. A New Fractional-Type Hardy Integral Inequality on a Finite Interval

The proposition below fills a gap in [23, Corollary 1] and in the literature on integral inequalities in general. It provides a valuable fractional-type inequality for "the case $\lambda = 1$" when the integration domain is finite, i.e., for the following integral as the left term:

$$\int_0^a \frac{|f(x)|^p}{x} dx,$$

where a is finite. This case was avoided in [23, Corollary 1]. We also assume that $p \in (0,+\infty)$, including the not yet considered case $p \in (0,1]$.

Proposition 4.1. *Let $a \in (0,+\infty)$, $p \in (0,+\infty)$ and $f : (0,a) \mapsto \mathbb{R}$ such that*

$$\int_0^a \frac{|f(x)|^p}{x} dx < +\infty.$$

Then we have

$$\int_0^a \frac{|f(x)|^p}{x} dx \leq H_{\theta,p,\alpha} \int_0^a \int_0^a \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx,$$

where

$$H_{\theta,p,\alpha} := \frac{\max(2^{p-1},1)\theta}{\alpha\left[\theta - \max(2^{p-1},1)\right]} \left\{\max\left[|1-\alpha|, |2\alpha-1|\right]\right\}^{1+1/\theta}, \tag{4.1}$$

$\theta > \max(2^{p-1},1)$ and $\alpha \in (0,+\infty)$ is such that

$$\alpha \leq \frac{1}{2} a^{1-\theta}. \tag{4.2}$$

Note that α may depend on a.

Proof. The proof is similar to that of Proposition 3.1, but a special treatment of the integration interval has to be done at several strategic points. We detail it to understand the assumption made on α in Equation (4.2), which depends on a. First, the inequality in Equation (2.2) gives

$$|f(x)|^p = |f(y) + [f(x)-f(y)]|^p \leq \max(2^{p-1},1)|f(y)|^p + \max(2^{p-1},1)|f(x)-f(y)|^p.$$

Dividing the above terms by $\alpha x^{1+\theta}$, which is positive, we get

$$\frac{|f(x)|^p}{\alpha x^{1+\theta}} \leq \max(2^{p-1},1)\frac{|f(y)|^p}{\alpha x^{1+\theta}} + \max(2^{p-1},1)\frac{|f(x)-f(y)|^p}{\alpha x^{1+\theta}}.$$

Integrating with respect to $y \in (\alpha x^\theta, 2\alpha x^\theta)$ (with $x \in (0,a)$), we obtain

$$\int_{\alpha x^\theta}^{2\alpha x^\theta} \frac{|f(x)|^p}{\alpha x^{1+\theta}} dy \leq \max(2^{p-1},1) \int_{\alpha x^\theta}^{2\alpha x^\theta} \frac{|f(y)|^p}{\alpha x^{1+\theta}} dy + \max(2^{p-1},1) \int_{\alpha x^\theta}^{2\alpha x^\theta} \frac{|f(x)-f(y)|^p}{\alpha x^{1+\theta}} dy.$$

Noticing that

$$\int_{\alpha x^\theta}^{2\alpha x^\theta} \frac{|f(x)|^p}{\alpha x^{1+\theta}} dy = \frac{|f(x)|^p}{\alpha x^{1+\theta}} \int_{\alpha x^\theta}^{2\alpha x^\theta} dy = \frac{|f(x)|^p}{x},$$

and integrating with respect to $x \in (0,a)$, we find that

$$\int_0^a \frac{|f(x)|^p}{x} dx \leq \max(2^{p-1},1)W + \max(2^{p-1},1)Z, \tag{4.3}$$

where

$$W := \int_0^a \int_{\alpha x^\theta}^{2\alpha x^\theta} \frac{|f(y)|^p}{\alpha x^{1+\theta}} dydx$$

and

$$Z := \int_0^a \int_{\alpha x^\theta}^{2\alpha x^\theta} \frac{|f(x)-f(y)|^p}{\alpha x^{1+\theta}} dydx.$$

Let us now bound W and Z successively.

For W, the Fubini-Tonelli integral theorem ensures the change in the order of integration, but we need to adjust the bounds of the integral by taking into account a. We find that

$$W = \int_0^{2\alpha a^\theta} \int_{[y/(2\alpha)]^{1/\theta}}^{\min\left[(y/\alpha)^{1/\theta},a\right]} \frac{|f(y)|^p}{\alpha x^{1+\theta}} dxdy.$$

Using $\min\left[(y/\alpha)^{1/\theta}, a\right] \leq (y/\alpha)^{1/\theta}$, Equation (4.2) which gives $2\alpha a^\theta \leq a$ and the fact that the integrated term is non-negative, we have

$$
\begin{aligned}
W &\leq \int_0^{2\alpha a^\theta} \int_{[y/(2\alpha)]^{1/\theta}}^{(y/\alpha)^{1/\theta}} \frac{|f(y)|^p}{\alpha x^{1+\theta}} dxdy \leq \int_0^a \int_{[y/(2\alpha)]^{1/\theta}}^{(y/\alpha)^{1/\theta}} \frac{|f(y)|^p}{\alpha x^{1+\theta}} dxdy \\
&= \int_0^a |f(y)|^p \left[\int_{[y/(2\alpha)]^{1/\theta}}^{(y/\alpha)^{1/\theta}} \frac{1}{\alpha x^{1+\theta}} dx\right] dy \\
&= \int_0^a |f(y)|^p \frac{1}{\alpha\theta} \left[\left(\frac{2\alpha}{y}\right) - \left(\frac{\alpha}{y}\right)\right] dy = \frac{1}{\theta} \int_0^a \frac{|f(y)|^p}{y} dy.
\end{aligned} \tag{4.4}
$$

On the other hand, for Z, using again Equation (4.2) which gives $y \in (\alpha x^\theta, 2\alpha x^\theta) \subseteq (0,a)$, we get

$$
\begin{aligned}
Z &= \frac{1}{\alpha} \int_0^a \int_{\alpha x^\theta}^{2\alpha x^\theta} \frac{|x^\theta - y|^{1+1/\theta}}{x^{1+\theta}} \times \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx \\
&\leq \frac{1}{\alpha} \left[\sup_{x\in(0,a)} \sup_{y\in(\alpha x^\theta, 2\alpha x^\theta)} \frac{|x^\theta - y|^{1+1/\theta}}{x^{1+\theta}}\right] \int_0^a \int_{\alpha x^\theta}^{2\alpha x^\theta} \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx \\
&\leq \frac{1}{\alpha} \left[\sup_{x\in(0,a)} \sup_{y\in(\alpha x^\theta, 2\alpha x^\theta)} \frac{|x^\theta - y|^{1+1/\theta}}{x^{1+\theta}}\right] \int_0^a \int_0^a \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx.
\end{aligned}
$$

We have

$$
\begin{aligned}
\sup_{x\in(0,a)} \sup_{y\in(\alpha x^\theta, 2\alpha x^\theta)} \frac{|x^\theta - y|^{1+1/\theta}}{x^{1+\theta}} &= \sup_{x\in(0,a)} \max\left[\frac{|x^\theta - \alpha x^\theta|^{1+1/\theta}}{x^{1+\theta}}, \frac{|x^\theta - 2\alpha x^\theta|^{1+1/\theta}}{x^{1+\theta}}\right] \\
&= \{\max[|1-\alpha|, |2\alpha - 1|]\}^{1+1/\theta}.
\end{aligned}
$$

We therefore obtain

$$
Z \leq \frac{1}{\alpha} \{\max[|1-\alpha|, |2\alpha-1|]\}^{1+1/\theta} \int_0^a \int_0^a \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx. \tag{4.5}
$$

It follows from Equations (4.3), (4.4) and (4.5) that

$$
\int_0^a \frac{|f(x)|^p}{x} dx \leq \frac{\max(2^{p-1},1)}{\theta} \int_0^a \frac{|f(x)|^p}{x} dx + \frac{\max(2^{p-1},1)}{\alpha} \{\max[|1-\alpha|, |2\alpha-1|]\}^{1+1/\theta} \int_0^a \int_0^a \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx,
$$

which is equivalent to

$$
\left[1 - \frac{\max(2^{p-1},1)}{\theta}\right] \int_0^a \frac{|f(x)|^p}{x} dx \leq \frac{\max(2^{p-1},1)}{\alpha} \{\max[|1-\alpha|, |2\alpha-1|]\}^{1+1/\theta} \int_0^a \int_0^a \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx.
$$

Since $\theta > \max(2^{p-1},1)$, we get

$$
\begin{aligned}
\int_0^a \frac{|f(x)|^p}{x} dx &\leq \frac{\max(2^{p-1},1)\theta}{\alpha\left[\theta - \max(2^{p-1},1)\right]} \{\max[|1-\alpha|, |2\alpha-1|]\}^{1+1/\theta} \int_0^a \int_0^a \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx \\
&= H_{\theta,p,\alpha} \int_0^a \int_0^a \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx,
\end{aligned}
$$

where $H_{\theta,p,\alpha}$ is given in Equation (4.1). The proof of Proposition 4.1 ends. □

Proposition 4.1 thus completes Proposition 3.1 by considering a finite integration interval. It shows that, contrary to Proposition 3.1, when considering the integration interval $(0,a)$, α cannot be chosen arbitrarily; the assumption in Equation (4.2) must be satisfied. In particular, if we take $a = 1$, we can choose $\alpha = 1/2$, so that the constant in Equation (4.1) becomes

$$
H_{\theta,p,\alpha} = \frac{\max(2^{p-1},1)\theta 2^{-1/\theta}}{\theta - \max(2^{p-1},1)}
$$

and the inequality in Proposition 4.1 gives

$$
\int_0^1 \frac{|f(x)|^p}{x} dx \leq \frac{\max(2^{p-1},1)\theta 2^{-1/\theta}}{\theta - \max(2^{p-1},1)} \int_0^1 \int_0^1 \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx.
$$

In the general case, note that the change of variables $z = x^\theta$ gives

$$
\int_0^a \int_0^a \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx = \frac{1}{\theta} \int_0^{a^\theta} \int_0^a \frac{|f(z^{1/\theta})-f(y)|^p}{|z-y|^{1+1/\theta}} z^{1/\theta - 1} dydz.
$$

Therefore, the inequality in Proposition 4.1 can be reformulated with the denominator term $|x-y|^{1+1/\theta}$ as follows:

$$
\int_0^a \frac{|f(x)|^p}{x} dx \leq \frac{1}{\theta} H_{\theta,p,\alpha} \int_0^{a^\theta} \int_0^a \frac{|f(x^{1/\theta})-f(y)|^p}{|x-y|^{1+1/\theta}} x^{1/\theta - 1} dydx.
$$

We can note that the integral of integration with respect to x is now $(0, a^\theta)$.

5. Conclusion

In this article, we have contributed to two key results in [23], which are [23, Lemma 2] and [23, Corollary 1]. In particular,

- we have refined the main constant in the main inequality in [23, Lemma 2], and with $p \in (0,+\infty)$ instead of $p \in (1,+\infty)$,
- we have provided solutions to open problems corresponding to the establishment of fractional-type inequalities with the following integrals as the left term:

$$\int_0^{+\infty} \frac{|f(x)|^p}{x} dx$$

or

$$\int_0^{a} \frac{|f(x)|^p}{x} dx,$$

where a is finite.

These inequalities have been proved with some significant modifications of the proof proposed in [23], which are more adaptable to the particular case under consideration. The article thus fills a theoretical gap and, in a sense, completes [23, Lemma 2] and [23, Corollary 1]. The techniques developed can certainly be reused to solve complex mathematical inequalities. This is the logical perspective of the article.

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References

[1] G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, Cambridge, 1934.
[2] E. F. Beckenbach, E. Bellman, *Inequalities*, Springer, Berlin, 1961.
[3] W. Walter, *Differential and Integral Inequalities*, Springer, Berlin, 1970.
[4] D. Bainov, P. Simeonov, *Integral Inequalities and Applications*, Mathematics and Its Applications, vol. 57. Kluwer Academic, Dordrecht, 1992.
[5] B.C. Yang, *Hilbert-Type Integral Inequalities*, Bentham Science Publishers, The United Arab Emirates, 2009.
[6] A. Kufner, L. E. Persson, N. Samko, *Weighted Inequalities of Hardy Type*, Second Edition, World Scientific, 2017.
[7] B. Opic, A. Kufner, *Hardy-Type Inequalities*, Pitman Research Notes in Mathematics Series, Longman Scientific & Technical, Harlow, Essex, 1990.
[8] G. H. Hardy, *Note on a Theorem of Hilbert*, Math. Z., **6**, (1920), 314-317.
[9] N. Levinson, *Generalizations of an inequality of Hardy*, Duke Math. J., **31** (1964), 389-394.
[10] H. Brezis, M. Marcus, *Hardy's inequalities revisited*, Ann. Sc. Norm. Super. Pisa, **25** (1997), 217-237.
[11] H. Triebel, *Sharp Sobolev embeddings and related Hardy inequalities: The sub-critical case*, Math. Nachr., **208** (1999), 167-178.
[12] S. Filippas, A. Tertikas, *Optimizing improved Hardy inequalities*, J. Funct. Anal., **192** (2002), 186-233.
[13] F. Gazzola, H. C. Grunau, E. Mitidieri, *Hardy inequalities with optimal constants and remainder terms*, Trans. Am. Math. Soc., **356** (2004), 2149-2168.
[14] H. Bahouri, J. Y. Chemin, I. Gallagher, *Refined Hardy inequalities*, Ann. Sc. Norm. Super. Pisa, **5** (2006), 375-391.
[15] R. L. Frank, R. Seiringer, *Non-linear ground state representations and sharp Hardy inequalities*, J. Funct. Anal., **255** (2008), 3407-3430.
[16] W. T. Sulaiman, *Some Hardy type integral inequalities*, Appl. Math. Lett., **25** (2012), 520-525.
[17] S. Machihara, T. Ozawa, H. Wadade, *Hardy type inequalities on balls*, Tohoku Math. J., **65** (2013), 321-330.
[18] B. Sroysang, *More on some Hardy type integral inequalities*, J. Math. Inequalities, **8** (2014), 497-501.
[19] B. Dyda, A. V. Vähäkangas, *Characterizations for fractional Hardy inequality*, Adv. Calc. Var., **8** (2015), 173-182.
[20] S. Wu, B. Sroysang, S. Li, *A further generalization of certain integral inequalities similar to Hardy's inequality*, J. Nonlinear Sci. Appl., **9** (2016), 1093-1102.
[21] K. Mehrez, *Some generalizations and refined Hardy type integral inequalities*, Afr. Mat., **28** (2016), 451-457.
[22] B. Devyver, Y. Pinchover, G. Psaradakis, *Optimal Hardy inequalities in cones*, Proc. R. Soc. Edinb. A, **147** (2017), 89-124.
[23] P. Mironescu, *The role of the Hardy type inequalities in the theory of function spaces*, Rev. Roum. Math. Pures Appl., **63** (2018), 447-525.
[24] B. Benaissa, M. Sarikaya, A. Senouci, *On some new Hardy-type inequalities*, Math. Methods Appl. Sci., **43** (2020), 8488-8495.
[25] S. Yin, Y. Ren, C. Liu, *A sharp L^p-Hardy type inequality on the n-sphere*, ScienceAsia, **46** (2020), 746-752.
[26] S. Thongjob, K. Nonlaopon, J. Tariboon, S. Ntouyas, *Generalizations of some integral inequalities related to Hardy type integral inequalities via (p,q)-calculus*, J. Inequalities Appl., **2021** (2021), 1-17.
[27] H. M. Rezk, M. E. Bakr, O. Balogun, A. Saied, *Exploring generalized Hardy-type inequalities via nabla calculus on time scales*, Symmetry, **15** (2023), 1-17.
[28] M. Aldovardi, J. Bellazzini, *A note on the fractional Hardy inequality*, Bolletino Unione Mat. Ital., **16** (2023), 667-676.
[29] H. M. Rezk, O. Balogun, M. Bakr, *Unified generalizations of Hardy-type inequalities through the nabla framework on time scales*, Axioms, **13** (2024), 1-16.
[30] E. M. Stein, R. Shakarchi, *Functional Analysis: Introduction to Further Topics in Analysis*, Princeton University Press, 2011.

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Research Article

Solvability of Infinite Systems of Third Order Differential Equations in a Sequence Space $n(\phi)$ via Measures of Non-Compactness

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Abstract

This paper establishes the necessary conditions for the existence of ω-periodic solutions in the sequence space $n(\phi)$ for an infinite system of third-order differential equations. The analysis utilizes the system's Green's function, the Meir-Keeler condensing operator, and measures of non-compactness. To illustrate our results, we provide relevant examples.

1. Introduction and Preliminaries

Measures of non-compactness refers to a function that determines to what extent a set is non-compact. This concept was pioneered by Kuratowski [1] in 1930, who introduced the function α. Other researchers used it as a base to come up with more measures of non compactness (see, [2–4]). Measures of non-compactness in combination with some fixed point theorem has been widely used to show the existence of solutions to various infinite system of equations. Banaś and Lecko [5] presented the existence theorems for infinite systems of first order differential equations by using the concept of measures of non compactness on Banach sequence spaces c_0, c and ℓ_1. On extension of this result, Mursaleen and Mohiuddine [6] gave conditions for existence of solutions to a similar system in a sequence space ℓ_p using techniques related with measures of non-compactness. Mursaleen [7] introduced the Hausdorff measure of non-compactness on the sequence space $n(\phi)$ and proved the theorem that validates there exist solutions to infinite systems of first order differential equations in this space. A theorem to support the existence of solutions to an infinite system of second order differential equations in the sequence spaces c_0 and ℓ_1 was developed by Mursaleen and Rizvi [8] . The authors established the existence theorems for an infinite system of second order differential equations [9] and [10] in $n(\phi)$. Green's function for third-order differential equations with constant coefficients was established by Chen *et al.* [11]. This result motivated Saadat *et al.* [12] to investigate wheather infinite system of third order differential equations are solvable. Using the obtained Green's function, measures of non-compactness, and Meir-Keeler condensing operators, they demonstrated that an infinite system of third-order differential equations can have ω-periodic solutions in the Banach sequence space c_0. This conclusion was expanded to another sequence space ℓ_p, by Pourhadi *et al.* [13]. Inspired by these results, the focal point of this study is to examine the necessary conditions for the ω-periodic solutions to exist in an infinite system of third order differential equations within a sequence space $n(\phi)$. Recently in [14–18], the solvability of infinite systems of fractional differential equations has been studied in tempered sequence spaces. One can see more results in [19–23] and the references therein.

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We consider a Banach space E with the norm $\|.\|$. We assume that $B(x,r)$ is the closed ball centred at x with a radius r and B_r represents the ball $B(\theta,r)$ where as θ is the zero element of the Banach space E. Let $\mathfrak{M}$ be a non-empty subset of a set E. In this context, the closure of $\mathfrak{M}$ is represented by $\bar{\mathfrak{M}}$, while the convex closure is denoted as Conv$\mathfrak{M}$. Further, we define M_E as the collection of all non-empty and bounded subsets of E, and N_E as its subset consisting of sets that are relatively compact. The set of real numbers is denoted by $\mathbb{R}$, the interval $[0,+\infty)$ is represented by $\mathbb{R}_+$ and $\mathbb{N}$ stands for the set of natural numbers. The axiomatic measures of non-compactness proposed by Banas and Goebel [24] is defined as follows:

Definition 1.1. *[24] If a mapping $\gamma : M_E \to \mathbb{R}_+$ satisfies the following conditions is reffered to be a measure of non-compactness in E*

i. *The family ker $\gamma = \{\mathfrak{M} \in M_E : \gamma(\mathfrak{M}) = 0\}$ is non-empty and ker $\gamma \subset N_E$.*
ii. $\mathfrak{M}_1 \subset \mathfrak{M}_2 \Rightarrow \gamma(\mathfrak{M}_1) \subset \gamma(\mathfrak{M}_2)$.
iii. $\gamma(\bar{\mathfrak{M}}) = \gamma(\mathfrak{M})$.
iv. γ *(Conv $\mathfrak{M}$)* $= \gamma(\mathfrak{M})$.
v. *For all $\lambda \in [0,1]$*
$\gamma(\lambda\mathfrak{M}_1 + (1-\lambda)\mathfrak{M}_2) \leq \lambda\gamma(\mathfrak{M}_1) + (1-\lambda)\gamma(\mathfrak{M}_2)$.
vi. *Suppose $\mathfrak{M}_n$ is a sequence of closed sets taken from M_E such that $\mathfrak{M}_{n+1} \subset \mathfrak{M}_n$ $\forall n \in \mathbb{N}$. If the limit as n approaches infinity of the measure of non-compactness, denoted by $\gamma(\mathfrak{M}_n)$, equals zero, then the intersection set $\mathfrak{M}_\infty = \bigcap_{n=1}^{\infty} \mathfrak{M}_n$ is guaranteed to be non-empty. A measure of non-compactness is classified as a regular measure if it satisfies the following additional conditions.*
vii. $\gamma(\mathfrak{M}_1 \cup \mathfrak{M}_2) = max\{\gamma(\mathfrak{M}_1), \gamma(\mathfrak{M}_2)\}$
viii. $\gamma(\mathfrak{M}_1 + \mathfrak{M}_2) \leq \gamma(\mathfrak{M}_1) + \gamma(\mathfrak{M}_2)$
ix. $\gamma(\lambda\mathfrak{M}) = |\lambda|\gamma(\mathfrak{M})$
x. *ker* $\gamma = N_E$.

The Haursdoff measure of non-compactness developed by Goldenstian *et al.* [2] and further researched by Goldenstian and Markus [3] is the most beneficial and convenient in terms of application among all measures of non-compactness.

Definition 1.2. *[25] Consider $(\mathscr{X},d)$ be a metric space and let $\mathfrak{M}$ be a bounded subset of $\mathscr{X}$. The Hausdorff measure of non-compactness of $\mathfrak{M}$, denoted as $(\chi(\mathfrak{M}))$, is the infimum of all real numbers $\varepsilon > 0$ such that $\mathfrak{M}$ can be covered by a finite number of balls with radii $< \varepsilon$. In other words,*
$\chi(\mathfrak{M}) = inf\,\{\varepsilon > 0 : \mathfrak{M}$ *has a finite* $\varepsilon-$*net in* $\mathscr{X}\}$.

Definition 1.3. *[25] Let F_1 and F_2 be Banach spaces and γ_1 and γ_2 be arbitrary measures of non-compactness on F_1 and F_2 respectively. An operator $\mathfrak{T}$ mapping from F_1 to F_2 is referred to as $(\gamma_1$-$\gamma_2)$ condensing operator if it satisfies two conditions*

i. *continuity and*
ii. *for every bounded non-compact set $\mathfrak{M}$ in F_1, the measure of non-compactness of the image set $\mathfrak{T}(\mathfrak{M})$ under $\mathfrak{T}$, denoted as $\gamma_2(\mathfrak{T}(\mathfrak{M}))$, is strictly smaller than the measure of non-compactness of $\mathfrak{M}$, denoted as $\gamma_1(\mathfrak{M})$.*

Remark: *If $F_1 = F_2$ and $\gamma_1 = \gamma_2 = \gamma$, then $\mathfrak{T}$ is known as γ-condensing operator.*

Darbo [26] developed fixed point theorem based on the idea of measures of non-compactness. The existence of solutions to numerous types of differential equations and integral equations has been proven using this theorem.

Theorem 1.4. *[26] Let H be a non-empty, closed, bounded, and convex subset of a Banach space F. Suppose $\mathfrak{T} : H \to H$ is a continuous mapping such that for any set $E \subset H$, $\gamma(\mathfrak{T}(H)) \leq k\gamma(H)$, where k is a constant in the range $[0,1)$. Then, the mapping $\mathfrak{T}$ has a fixed point in H.*

Meir and Keeler in 1969 [27], developed another contraction known as Meir-Keeler contraction with it's corresponding fixed point theorem .

Definition 1.5. *[27] Let $(\mathscr{X},d)$ be a complete metric space. A mapping $\mathfrak{T} : \mathscr{X} \to \mathscr{X}$ is said to be Meir-Keeler contraction if for any $\varepsilon > 0$ there exists $\delta > 0$ such that the following conditions holds,*
$\varepsilon \leq d(u,v) < \varepsilon + \delta \Rightarrow d(\mathfrak{T}u, \mathfrak{T}v) < \varepsilon$, $\forall u,v \in \mathscr{X}$.

Theorem 1.6. *[27] Let $(\mathscr{X},d)$ be a complete metric space. If $\mathfrak{T} : \mathscr{X} \to \mathscr{X}$ is a Meir-Keeler contraction, then $\mathfrak{T}$ has a unique fixed point.*

Aghajan *et al.* [28] generalized Darbo's fixed point theorems unto Meir-Keeler condensing operators fixed point theorem. This attracted numerous researchers as they turned their mathematical interest on this topic, due to the fact that the imposed conditions are significantly weakened. Aghajan *et al.* [28] extended Darbo's fixed point theorem to fixed point theorems for Meir-Keeler condensing operators.

Definition 1.7. *[28] Let H be a non empty subset of a Banach space F, and let γ be an arbitrary measure of non-compactness of F. An operator $\mathfrak{T} : H \to H$ is called a Meir Keeler condensing operator if for any $\varepsilon > 0$, there exists $\delta > 0$ such that the following condition is satisfied,*
$\varepsilon \leq \gamma(Y) < \varepsilon + \delta$ *implies* $\gamma(\mathfrak{T}(Y)) < \varepsilon$ *for any bounded subset Y of H.*

This theorem will be helpful in demonstrating our key finding.

Theorem 1.8. *[28] Let H be a non empty, bounded, closed and convex subset of a Banach space F and let γ be an arbitrary measure of non-compactness on F. If $\mathfrak{T} : H \to H$ is a continuous and Meir-Keeler condensing operator, then $\mathfrak{T}$ has at least one fixed point. Furthermore, the set of all fixed points of $\mathfrak{T}$ in H is compact.*

Definition 1.9. *[29] Let β stands for the space of finite sets of distinct positive integers. For any $\nu \in \beta$, we define the sequence $b(\nu)$ with*

$$b_n(\nu) = \begin{cases} 1 & if \quad n \in \nu, \\ 0 & if \quad n \notin \nu, \end{cases}$$

and

$$\beta_r = \left\{ \nu \in \beta : \sum_{n=1}^{\infty} b_n(\nu) \leq r \right\},$$

such that β_r is the set of ν whose support has cardinality at most r. The set Φ contains all sequences $(\phi_i)_{i=1}^{\infty}$ such that;
$\phi_1 > 0$, $\Delta\phi_i \geq 0$ and $\Delta(\frac{\phi_i}{i}) \leq 0$, for $i = (1,2,...)$.
For $\phi \in \Phi$, we have the following sequences

$$m(\phi) = \left\{ x = x_i : \|x\|_{m(\phi)} = \sup_{r\geq 1} \sup_{\nu \in \beta_r} \left(\frac{1}{\phi_r} \sum_{i\in\nu} |x_i| \right) < \infty \right\},$$

$$n(\phi) = \left\{ x = x_i : \|x\|_{n(\phi)} = \sup_{u\in S_x} \left(\sum_{i=1}^{\infty} |u_i| \Delta\phi_i \right) < \infty \right\},$$

where $\Delta\phi_i = \phi_i - \phi_{i-1}$, $\phi_0 = 0$ and $S(x)$ denotes the set of all sequences that are rearrangements of x.
***Remark**: For all $n \in \mathbb{N}$, if $\phi_n = 1$ then $m(\phi) = \ell_1$ and $n(\phi) = \ell_\infty$;*
and if $\phi_n = n$ then $m(\phi) = \ell_\infty$ and $n(\phi) = \ell_1$.

Mursaleen [7] introduced the Hausdorff measure of non-compactness on the sequence space $n(\phi)$. But this formula does not define a measure of non-compactness in $n(\phi)$ for the case $\phi_n = 1$. We redefine it as follows:

Theorem 1.10. *For any bounded subset M of $n(\phi)$, the Hausdorff measure of non-compactness of the set M is given by*

$$\chi(M) = \lim_{k\to\infty} \sup_{x\in M} \left(\sup_{u\in S(x)} \left(\sum_{n=k}^{\infty} |u_n| \Delta\phi_n \right) \right), \text{ for } \phi_n \neq 1;$$

$$= \lim_{k\to\infty} \left\{ \sup_{x\in M} \{ \sup[| x_n - x_m | : n,m \geq k] \} \right\}, \text{ for } \phi_n = 1.$$

Throughout this paper, we study the following infinite system of third order differential equations:

$$y_i''' + py_i'' + qy_i' + ry_i = h_i(\psi, y_1(\psi), y_2(\psi), ...) \tag{1.1}$$

where $h_i \in C(\mathbb{R} \times \mathbb{R}^\infty, \mathbb{R})$ with regard to the first coordinate ψ is ω- periodic and $p,q,r \in \mathbb{R}$ are constants.

Based on the theory of ordinary differential equations, the corresponding homogeneous equation of (1.1) is

$$y_i''' + py_i'' + qy_i' + ry_i = 0, i \in \mathbb{N}$$

as well as the corresponding characteristic equation is

$$\xi^3 + p\xi^2 + q\xi + r = 0. \tag{1.2}$$

The roots of the polynomial Equation of (1.2) take one of the following cases:

1. $\xi_1 \neq \xi_2 \neq \xi_3$
2. $\xi_1 = \xi_2 \neq \xi_3$
3. $\xi_1 = \xi_2 = \xi_3$
4. $\xi_1 = a + ib, \xi_2 = a - ib, \xi_3 = \xi$, where a, b, and ξ are real numbers.

The case $r = 0$ is not considered since the results can be easily extended to cover this special case. Therefore, the roots are assumed to be non-zero.
The main novelty of this work is to establish the necessary conditions for the existence of ω-periodic solutions in the sequence space $n(\phi)$ for an infinite system of third-order differential equations. The advantage of our results are that the space inhand $n(\phi)$ is more general than the classical seuence spaces c_0, c and ℓ_p.
This paper is organized into four sections. Section 1 provides an introduction and covers the necessary preliminaries and background for establishing the main results. Section 2 is divided into five subsections, presents four distinct cases for the theorem proved as the main result. Section 3 discusses two examples that validate the results of Section 2. Finally, Section 4 concludes the study with the suggestion for future study.

2. Main Results

2.1. Solvability in a Banach sequence space $n(\phi)$

In this section we provide the required conditions for existence of ω- periodic solution to the system (1.1).
Firstly, we recall the Fréchet space $\mathbb{R}^\infty$ which is the linear space of all real sequences equipped with the distance

$$d_{\mathbb{R}^\infty}(x,y) = \sup \left\{ \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|} : i \in \mathbb{N} \right\},$$

for $x = (x_i), y = (y_i) \in \mathbb{R}^\infty$.

The space of all continuous real functions on $\mathbb{R}$ is represented by $C(\mathbb{R},\mathbb{R}^\infty)$, and $C^3(\mathbb{R},\mathbb{R}^\infty)$ stands for the group of functions on $\mathbb{R}$ that have a third continuous derivative. A function $y\in C^3(\mathbb{R},\mathbb{R}^\infty)$ is known to be a solution of Equation (1.1) if and only if $y\in C(\mathbb{R},\mathbb{R}^\infty)$ is a solution of the following infinite system:

$$y_i(\psi)=\int_{\psi}^{\psi+\omega}G(\psi,\varsigma)h_i(\varsigma,y(\varsigma))d(\varsigma),(i\in\mathbb{N}),$$

where the Green's function will be specified in corresponding to different cases.

i. The functions $h_i:\mathbb{R}\times\mathbb{R}^\infty\to\mathbb{R}$ are supposed to be ω-periodic with regard to the first coordinate. The operator $h_i:\mathbb{R}\times n(\phi)\to n(\phi)$ is defined below

$$(\psi,y)\to(hy)(\psi)=(h_1(\psi,y),h_2(\psi,y),\ldots)$$

is such that the class of all functions $\{(hy)(\psi)\}_{\psi\in\mathbb{R}}$ is equicontinuous at each point of the space $n(\phi)$.

ii. The following inequality is true for any $i\in\mathbb{N}$

$$|h_n(\psi,y_1,y_2,\ldots)|\le u_n(\psi)+v_n(\psi)|y_i(\psi)|,$$

for $\psi\in\mathbb{R}$ and $y=y_i$ in $n(\phi)$. It is assumed that the functions $u_n(\psi)$ and $v_n(\psi)$ are continuous on $\mathbb{R}$, such that the mapping series

$$\sum_{k\ge1}|u_k|\Delta\phi_k$$

converges uniformly on $\mathbb{R}$, while the sequence $v_n(\psi)$ is equibounded on $\mathbb{R}$.

Suppose,

$$U=\sup_{\psi\in\mathbb{R}}\left\{\sum_{k\ge1}u_k\Delta\phi_k\right\},$$

$$V=\sup_{n\in\mathbb{N}}\{\,v_n(\psi)\},$$

and assume L is given as seen in [12] i.e

$$L=\frac{e^{(\omega|\xi_1|)}}{\left|(\xi_1-\xi_2)(\xi_1-\xi_3)(1-e^{(\xi_1\omega)})\right|}+\frac{e^{(\omega|\xi_2|)}}{\left|(\xi_2-\xi_1)(\xi_2-\xi_3)(1-e^{(\xi_2\omega)})\right|}+\frac{e^{(\omega|\xi_3|)}}{\left|(\xi_3-\xi_2)(\xi_3-\xi_1)(1-e^{(\xi_3\omega)})\right|}.$$

2.2. Solvability for case 1

In this part, we demonstrate the system's (1.1) solvability by assuming that the roots of Equation (1.2) are $\xi_1\neq\xi_2\neq\xi_3$. According to Chen *et al.* [11], the appropriate Green's function in this instance is as follows:

$$G_1(\psi,\varsigma)=\frac{e^{(\xi_1(\psi+\omega-\varsigma))}}{(\xi_1-\xi_2)(\xi_1-\xi_3)(1-e^{(\xi_1\omega)})}+\frac{e^{(\xi_2(\psi+\omega-\varsigma))}}{(\xi_2-\xi_1)(\xi_2-\xi_3)(1-e^{(\xi_2\omega)})}+\frac{e^{(\xi_3(\psi+\omega-\varsigma))}}{(\xi_3-\xi_2)(\xi_3-\xi_1)(1-e^{(\xi_3\omega)})}. \tag{2.1}$$

Theorem 2.1. *If the assumptions (i) and (ii) are true, the system (1.1) has at least one ω-periodic solution, $y(\psi)=y_i(\psi)$ whenever 0 ¡ ωLV ¡ 1, such that $y(\psi)\in n(\phi),\psi\in\mathbb{R}$. The set of all solutions is also compact.*

Proof. Assume that $S(y(\psi))$ is the collection of all sequences that are rearrangements of $y(\psi)$, and let assumption (ii) hold. Using relation (2.1) for any $\psi\in\mathbb{R}$,

$$\begin{aligned}
\|y(\psi)\|_{n(\phi)}&=\sup_{p\in S(y(\psi))}\left(\sum_{k=1}^{\infty}\left|\int_{\psi}^{\psi+\omega}G_1(\psi,\varsigma)h_k(\varsigma,p(\varsigma))d(\varsigma)\right|\Delta\phi_k\right)\\
&\le\sup_{p\in S(y(\psi))}\left(\sum_{k=1}^{\infty}\int_{\psi}^{\psi+\omega}|G_1(\psi,\varsigma)h_k(\varsigma,p(\varsigma))|d(\varsigma)\Delta\phi_k\right)\\
&\le\sup_{p\in S(y(\psi))}\left(\sum_{k=1}^{\infty}\int_{\psi}^{\psi+\omega}|G_1(\psi,\varsigma)|(u_k(\psi)+v_k(\psi)|p_k(\psi)|)d(\varsigma)\Delta\phi_k\right)\\
&=\sup_{p\in S(y(\psi))}\left(\sum_{k=1}^{\infty}\int_{\psi}^{\psi+\omega}G_1(\psi,\varsigma)u_k(\psi)\Delta\phi_k d(\varsigma)+\sum_{k=1}^{\infty}\int_{\psi}^{\psi+\omega}G_1(\psi,\varsigma)v_k(\psi)|p_k(\psi)|\Delta\phi_k d(\varsigma)\right)
\end{aligned}$$

$$\leq \sup_{p\in S(y(\psi))}\left(\int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)\sum_{k=1}^{\infty} u_k(\psi)\Delta\phi_k d(\varsigma) + V\int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)\sum_{k=1}^{\infty}|p_k(\psi)|\Delta\phi_k d(\varsigma)\right)$$

$$\leq \sup_{p\in S(y(\psi))} U\int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)d(\varsigma) + V\sup_{p\in S(y(\psi))}\int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)\sum_{k=1}^{\infty}|p_k(\psi)|\Delta\phi_k d(\varsigma)$$

$$\leq \omega LU + V\omega L\|y(\psi)\|_{n(\phi)}$$

$$\|y(\psi)\|_{n(\phi)} \leq \frac{\omega LU}{1-\omega LV}, = r.$$

This implies that y is a member of B_r where B_r denotes the closed ball with radius r centred at zero. So B_r is non empty, bounded, closed and convex subset of $n(\phi)$.
Here, we define the operator $\mathscr{J} = \mathscr{J}_i$ on $C(\mathbb{R}, B_r)$ as:

$$(\mathscr{J}y)(\psi) = (\mathscr{J}_i y)(\psi) = \left\{\int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)h_i(\varsigma, y(\varsigma))d(\varsigma)\right\}, \psi\in\mathbb{R}, \tag{2.2}$$

where $y(\psi) = y_i(\psi) \in B_r$ and $y_i(\psi) \in C(\mathbb{R},\mathbb{R})$, $\psi \in \mathbb{R}$. It is plainly known from the presumption (i) that $\mathscr{J}$ is continuous on $C((\mathbb{R}, n(\phi))$. Obviously since $y(\psi) = y_i(\psi) \in n(\phi)$, also $(\mathscr{J}y)(\psi) \in n(\phi)$ and $\mathscr{J}y$ is continuous function. Moreover, the function $(\mathscr{J}_i y)(\psi)$ is ω-periodic function whenever $y_i(\psi)$ is ω-periodic function.
Since $\|\mathscr{J}y(\psi)\|_{n(\phi)} \leq r$, thus $\mathscr{J}$ is a self mapping on B_r. We will now demonstrate that $\mathscr{J}$ is a Meir-Keeler condensing operator. Finding δ ¿ 0 such that for any given ε ¿ 0, $\varepsilon \leq \chi(B_r) < \varepsilon+\delta$ implies $\chi(\mathscr{J}(B_r)) < \varepsilon$. Assumption (ii) and Theorem 1.10 allow us to arrive at,

$$\chi(\mathscr{J}B_r) = \lim_{k\to\infty}\left\{\sup_{y(\psi)\in B}\left(\sup_{p\in S(y(\psi))}\left(\sum_{i=k}^{\infty}\left|\int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)h_i(\varsigma,p(\varsigma))d(\varsigma)\right|\Delta\phi_i\right)\right)\right\}$$

$$\leq \lim_{k\to\infty}\left\{\sup_{y(\psi)\in B}\left(\sup_{p\in S(y(\psi))}\left(\sum_{i=k}^{\infty}\int_{\psi}^{\psi+\omega}|G_1(\psi,\varsigma)h_i(\varsigma,p(\varsigma))|d(\varsigma)\Delta\phi_i\right)\right)\right\}$$

$$\leq \lim_{k\to\infty}\left\{\sup_{y(\psi)\in B}\left(\sup_{p\in S(y(\psi))}\left(\sum_{i=k}^{\infty}\int_{\psi}^{\psi+\omega}|G_1(\psi,\varsigma)|(u_i(\psi)+v_i(\psi)|p_i(\psi)|)d(\varsigma)\Delta\phi_i\right)\right)\right\}$$

$$= \lim_{k\to\infty}\left\{\sup_{y(\psi)\in B}\left(\sup_{p\in S(y(\psi))}\left(\sum_{k=1}^{\infty}\int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)u_i(\psi)\Delta\phi_i d(\varsigma) + \sum_{k=1}^{\infty}\int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)v_i(\psi)|p_i(\psi)|\Delta\phi_i d(\varsigma)\right)\right)\right\}$$

$$\leq \lim_{k\to\infty}\left\{\sup_{y(\psi)\in B}\left(\sup_{p\in S(y(\psi))}\left(\int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)\sum_{k=1}^{\infty}u_i(\psi)\Delta\phi_i d(\varsigma) + V\int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)\sum_{k=1}^{\infty}|p_i(\psi)|\Delta\phi_i d(\varsigma)\right)\right)\right\}$$

$$\leq V\lim_{k\to\infty}\left\{\sup_{y(\psi)\in B}\left(\sup_{p\in S(y(\psi))}\int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)\sum_{k=1}^{\infty}|p_i(\psi)|\Delta\phi_i d(\varsigma)\right)\right\}$$

$$\leq V\omega L\chi(B_r).$$

Therefore, for a given $\varepsilon > 0$, if we take $0 < \delta \leq \frac{(1-\omega LV)\varepsilon}{\omega LV}$ we get the following $\varepsilon \leq \chi(B_r) < \varepsilon+\delta \implies \chi(\mathscr{J}(B_r)) < \varepsilon$. Thus, $\mathscr{J}$ is a Meir-Keeler condensing operator on the set $B_r \subset n(\phi)$. As a result, according to Theorem 1.8, $\mathscr{J}$ has a fixed point in B_r that is a part ker χ. This is the needed solution for the system (1.1). □

Chen *et al.* ([11]) on their work introduced some bounds for Green's function $G_1(\psi,\varsigma)$ which may be used to restate the Theorem 2.1 by exchanging L by obtained upper bounds.
For more simplicity of notations let us consider,

$$f_1 := (\xi_2-\xi_3)e^{(\xi_3\omega)} + 2(\xi_1-\xi_3)e^{(\xi_2\omega)} + (\xi_1-\xi_2)e^{(\xi_1\omega)} + (\xi_1-\xi_3)e^{((\xi_1+\xi_2+\xi_3)\omega)},$$

$$g_1 := (\xi_1-\xi_3) + (\xi_1+\xi_2-2\xi_3)e^{((\xi_2+\xi_3)\omega)} + (2\xi_1-\xi_2-\xi_3)e^{((\xi_1+\xi_2)\omega)},$$

$$f_2 := (\xi_1+\xi_2-2\xi_3)e^{(\xi_1\omega)} + (2\xi_1-\xi_2-\xi_3)e^{(\xi_3\omega)} + (\xi_1-\xi_3)e^{((\xi_1+\xi_2+\xi_3)\omega)},$$

$$g_2 := (\xi_1-\xi_3) + (\xi_1-\xi_2)e^{((\xi_2+\xi_3)\omega)} + (\xi_2-\xi_3)e^{((\xi_1+\xi_2)\omega)} + 2(\xi_1-\xi_3)e^{((\xi_1+\xi_3)\omega)}$$

$$\mathscr{A}_3 = \frac{e^{(\omega\xi_1)}}{(\xi_1-\xi_2)(\xi_1-\xi_3)(1-e^{(\xi_1\omega)})} + \frac{1}{(\xi_2-\xi_1)(\xi_2-\xi_3)(1-e^{(\xi_2\omega)})} + \frac{e^{(\omega\xi_3)}}{(\xi_3-\xi_2)(\xi_3-\xi_1)(1-e^{(\xi_3\omega)})}$$

$$\mathscr{B}_3 = \frac{1}{(\xi 1-\xi_2)(\xi_1-\xi_3)(1-e^{(\xi_1\omega)})} + \frac{e^{(\xi_2\omega)}}{(\xi_2-\xi_1)(\xi_2-\xi_3)(1-e^{(\xi_2\omega)})} + \frac{1}{(\xi_3-\xi_2)(\xi_3-\xi_1)(1-e^{(\xi_3\omega)})}. \tag{2.3}$$

Then
($\mathscr{C}_1$) If $f_1 \leq g_1$, and one of the following conditions holds:

i. $0 < \xi_3 < \xi_2 < \xi_1$
ii. $\xi_1 > 0 > \xi_2 > \xi_3$

then $\mathscr{A}_3 \leq G_1(\psi,\varsigma) \leq \mathscr{B}_3 < 0$.
($\mathscr{C}_2$) If $f_2 > g_2$, and one of the following conditions holds:

i. $\xi_3 < \xi_2 < \xi_1 < 0$
ii. $\xi_3 < 0 < \xi_2 < \xi_1$

then $0 < \mathscr{A}_3 \leq G_1(\psi,\varsigma) \leq \mathscr{B}_3$.
Thus, one can quickly infer the following direct implication of Theorem 2.1 replacing L by the new boundaries by utilizing the recent findings made by Chen *et al.* [11] by using the above bounds, Theorem 2.1 may be changed to Theorem 2.1 and stated as follows:

Theorem 2.2. *Suppose that hypothesis $\mathscr{C}_1$ and the assumptions (i) - (ii) are true and $\omega V|\mathscr{A}_3| < 1$. Additionally, assume that $\mathscr{C}_2$ and the assumptions (i) - (ii) are true and $\omega V\mathscr{B}_3 < 1$. Then the infinite system (1.1) has at least one ω-periodic solution $y(\psi) = y_k(\psi)$ such that $y(\psi) \in n(\phi), \psi \in \mathbb{R}$. The set of all solutions is compact.*

Proof. On replacing L by $|\mathscr{A}_3|$ or $\mathscr{B}_3$ which are upper bounds of the Green's function $G_1(\psi,\varsigma)$ in the proof of Theorem 2.1, yields the intended outcome. □

2.3. Solvability for case 2

We shall present the solvability for the system (1.1), considering the roots corresponding to the homogenous part of the equation to be
$\xi_1 = \xi_2 \neq \xi_3$
in this section. The following is the Green's function for this instance:

$$G_2(\psi,\varsigma) = \frac{e^{(\xi_1(\psi+\omega-\varsigma))}\left[(1-e^{(\xi_1\omega)})((\psi-\varsigma)(\xi_3-\xi_1)-1)-(\xi_3-\xi_1)\omega\right]}{(\xi_1-\xi_3)^2(1-e^{(\xi_1\omega)})^2} + \frac{e^{(\xi_3(\psi+\omega-\varsigma))}}{(\xi_1-\xi_3)^2(1-e^{(\xi_3\omega)})}, \varsigma \in \{\psi, \psi+\omega\}.$$

We shall use the bounds concluded by Chen *et al.* [11] on proving the existence of solution for this case. Consider the following:

$$\mathscr{A}_4 := \frac{e^{(\xi_1\omega)}-1+(\xi_1-\xi_3)\omega}{(\xi_1-\xi_3)^2(1-e^{(\xi_1\omega)})^2} + \frac{e^{(\xi_3\omega)}}{(\xi_1-\xi_3)^2(1-e^{(\xi_3\omega)})},$$

$$\mathscr{A}_5 := \frac{e^{(2\xi_1\omega)}-e^{(\xi_1\omega)}+(\xi_1-\xi_3)\omega e^{(2\xi_1\omega)}}{(\xi_1-\xi_3)^2(1-e^{(\xi_1\omega)})^2} + \frac{e^{(\xi_3\omega)}}{(\xi_1-\xi_3)^2(1-e^{(\xi_3\omega)})},$$

$$\mathscr{A}_6 := \frac{e^{(\xi_1\omega)}-1+(\xi_1-\xi_3)\omega e^{(\xi_1\omega)}}{(\xi_1-\xi_3)^2(1-e^{(\xi_1\omega)})^2} + \frac{e^{(\xi_3\omega)}}{(\xi_1-\xi_3)^2(1-e^{(\xi_3\omega)})},$$

$$\mathscr{B}_4 := \frac{e^{(2\xi_1\omega)}-e^{(\xi_1\omega)}+(\xi_1-\xi_3)\omega e^{(2\xi_1\omega)}}{(\xi_1-\xi_3)^2(1-e^{(\xi_1\omega)})^2} + \frac{1}{(\xi_1-\xi_3)^2(1-e^{(\xi_3\omega)})},$$

$$\mathscr{B}_5 := \frac{e^{(\xi_1\omega)}-1+(\xi_1-\xi_3)\omega}{(\xi_1-\xi_3)^2(1-e^{(\xi_1\omega)})^2} + \frac{1}{(\xi_1-\xi_3)^2(1-e^{(\xi_3\omega)})},$$

$$\mathscr{B}_6 := \frac{e^{(2\xi_1\omega)}-e^{(\xi_1\omega)}+(\xi_1-\xi_3)\omega e^{(\xi_1\omega)}}{(\xi_1-\xi_3)^2(1-e^{(\xi_1\omega)})^2} + \frac{1}{(\xi_1-\xi_3)^2(1-\exp(\xi_3\omega))},$$

$$g_3 = e^{(\xi_1\omega)} + (\xi_1-\xi_3)\omega + (e^{(\xi_1\omega)}-3)e^{((\xi_1+\xi_3)\omega)} + (2+(\xi_3-\xi_1)\omega)e^{(\xi_3\omega)},$$

$$g_4 = (3-(\xi_1-\xi_3)\omega)e^{(\xi_1\omega)} + ((\xi_1-\xi_3)\omega-1)e^{((\xi_1+\xi_3)\omega)} + (e^{(\xi_3\omega)}-2)e^{(2\xi_1\omega)}.$$

Then from the results obtained by Chen *et al.* [11] we have that
($\mathscr{C}_3$) $0 < \mathscr{A}_4 \leq G_2(\psi,\varsigma) \leq \mathscr{B}_4$ whenever $\xi_3 < 0 < \xi_1 = \xi_2$,
($\mathscr{C}_4$) $\mathscr{A}_4 \leq G_2(\psi,\varsigma) \leq \mathscr{B}_4 < 0$ whenever $\xi_1 = \xi_2 < 0 < \xi_3$,
($\mathscr{C}_5$) $\mathscr{A}_5 \leq G_2(\psi,\varsigma) \leq \mathscr{B}_5 < 0$ whenever $0 < \xi_1 = \xi_2 < \xi_3$, and $e^{(\xi_1\omega)} < 1+(\xi_3-\xi_1)\omega$,
($\mathscr{C}_6$) $0 < \mathscr{A}_4 \leq G_2(\psi,\varsigma) \leq \mathscr{B}_4$ whenever $\xi_1 = \xi_2 < \xi_3 < 0$ and $g_3 > 1$,
($\mathscr{C}_7$) $\mathscr{A}_6 \leq G_2(\psi,\varsigma) \leq \mathscr{B}_6 < 0$ whenever $0 < \xi_3 < \xi_2 = \xi_1$, and $g_4 < 1$.

Theorem 2.3. *Assume that (i)-(ii) and hypothesis $\mathscr{C}_3$ (respcively $\mathscr{C}_4, \mathscr{C}_5, \mathscr{C}_6$, and $\mathscr{C}_7$) hold. Consider the case when $\omega V\mathscr{B}_4 < 1$ (respecively $\omega V|\mathscr{A}_4| < 1, \omega V|\mathscr{A}_5| < 1, \omega V\mathscr{B}_4 < 1, \omega V|\mathscr{A}_6| < 1$). The infinite system (1.1) then has at least one ω-periodic solution $y(\psi) = y_k(\psi)$ such that $y(\psi) \in n(\phi), \psi \in \mathbb{R}$. Also, the set of all solutions is compact.*

Proof. We get desired results by changing the Green's function from $G_1(\psi,\varsigma)$ to $G_2(\psi,\varsigma)$ from Theorem 2.1, since we are in case 2 where the associated Green's function is $G_2(\psi,\varsigma)$. And also replacing $L\omega V < 1$ from Theorem 2.1 by $\omega V\mathscr{B}_4 < 1, \omega V|\mathscr{A}_4| < 1, \omega V|\mathscr{A}_5| < 1, \omega V\mathscr{B}_4 < 1, \omega V|\mathscr{A}_6| < 1$ respectively, which are upper bounds of the Green's function $G_2(\psi,\varsigma)$. □

2.4. Solvability for case 3

In this section, we present the theorem for existence of ω-periodic solution to the system (1.1) considering the roots of (1.2) to be $\xi_1 = \xi_2 = \xi_3$. From [11] the associated Green's function for this case is shown to be:

$$G_3(\psi,\varsigma) = \frac{\left[(\varsigma-\psi)e^{(\xi\omega)}+\omega-\varsigma+\psi\right]^2+\omega^2 e^{(\xi\omega)}}{2(1-e^{(\xi\omega)})^3}e^{(\xi(\psi+\omega-\varsigma))}, \varsigma \in \{\psi, \psi+\omega\}.$$

In this case,we establish the existence theorem based on the upper bounds given by Chen *et al.* [11]
For more simplicity denote

$$\mathscr{A}_7 = \frac{\omega^2 e^{(2\xi\omega)}(1+e^{(\xi\omega)})}{2(1-e^{(\xi\omega)})^3}, \mathscr{B}_7 = \frac{\omega^2(1+e^{(\xi\omega)})}{2(1-e^{(\xi\omega)})^3}.$$

($\mathscr{C}_8$) $\mathscr{A}_7 \le G_3(\psi,\varsigma) \le \mathscr{B}_7 < 0$ whenever $\xi > 0$,
($\mathscr{C}_9$) $0 < \mathscr{A}_7 \le G_3(\psi,\varsigma) \le \mathscr{B}_7$ whenever $\xi < 0$.

Theorem 2.4. *Suppose that the presumptions (i)-(ii) and $\mathscr{C}_8$ ($\mathscr{C}_9$ respectively) are true. Consider $\omega V|\mathscr{A}_7| < 1$ ($\omega V\mathscr{B}_7 < 1$ respectively). Then the infinite system (1.1) has at least one ω-periodic solution $y(\psi) = y_k(\psi)$ such that $y(\psi) \in n(\phi), \psi \in \mathbb{R}$. Moreover, the set of all solutions is compact.*

Proof. Considering $G_3(\psi,\varsigma)$ as the Green's function and exchanging L from the proof of Theorem 2.1 by $|\mathscr{A}_7|$ and $\mathscr{B}_7$ which are the upper bounds of the Green's function $G_3(\psi,\varsigma)$, we are able to achieve the required result. □

2.5. Solvability for case 4

In this section, we present the solvability of system (1.1) by considering the roots of equation (1.2) as $\xi_1 = a+ib, \xi_2 = a-ib, \xi_3 = \xi$. From [11], for this case the Green's function is as follows:

$$G_4(\psi,\varsigma) = \frac{e^{(a(\psi+\omega-\varsigma))}\left[(a-\xi)\mathscr{B}_2(\psi)-b\mathscr{A}_2(\psi)\right]}{b\left[(a-\xi)^2+b^2\right](1+e^{(2a\omega)}-2\cos(b\omega)e^{(a\omega)})} + \frac{e^{(\xi(\psi+\omega-\varsigma))}}{(1-e^{(\xi\omega)})\left[(a-\xi)^2+b^2\right]}, \varsigma \in \{\psi, \psi+\omega\}.$$

where,

$$\mathscr{A}_2(\psi) := \cos b(\psi+\omega-\varsigma) - e^{(a\omega)}\cos b(\psi-\varsigma),$$

$$\mathscr{B}_2(\psi) := \sin b(\psi+\omega-\varsigma) - e^{(a\omega)}\sin b(\psi-\varsigma).$$

We simplify the notations to

$$\mathscr{A}_8 = \frac{-e^{(a\omega)}}{b\sqrt{\left[(a-\xi)^2+b^2\right](1+e^{(2a\omega)}-2\cos(b\omega)e^{(a\omega)})}} + \frac{e^{(\xi\omega)}}{(1-e^{(\xi\omega)})\left[(a-\xi)^2+b^2\right]},$$

$$\mathscr{B}_8 = \frac{e^{(a\omega)}}{b\sqrt{\left[(a-\xi)^2+b^2\right](1+e^{(2a\omega)}-2\cos(b\omega)e^{(a\omega)})}} + \frac{1}{(1-e^{(\xi\omega)})\left[(a-\xi)^2+b^2\right]},$$

$$\mathscr{A}_9 = \frac{-1}{b\sqrt{\left[(a-\xi)^2+b^2\right](1+e^{(2a\omega)}-2\cos(b\omega)e^{(a\omega)})}} + \frac{e^{(\xi\omega)}}{(1-e^{(\xi\omega)})\left[(a-\xi)^2+b^2\right]},$$

$$\mathscr{B}_9 = \frac{1}{b\sqrt{\left[(a-\xi)^2+b^2\right](1+e^{(2a\omega)}-2\cos(b\omega)e^{(a\omega)})}} + \frac{1}{(1-e^{(\xi\omega)})\left[(a-\xi)^2+b^2\right]}.$$

From [11] we have,

($\mathscr{C}_{10}$) $0 < \mathscr{A}_8 \le G_4(\psi,\varsigma) \le \mathscr{B}_8$ whenever $\xi < 0 < a, b$ and

$$\frac{\left[(a-\xi)^2+b^2\right](1-e^{(\xi\omega)})^2}{b^2 e^{(2a\omega)}} < \frac{1+e^{(2a\omega)}-2\cos(b\omega)e^{(a\omega)}}{e^{(2a\omega)}},$$

($\mathscr{C}_{11}$) $0 < \mathscr{A}_9 \le G_4(\psi,\varsigma) \le \mathscr{B}_9$ whenever $a, \xi < 0 < b$ and

$$\frac{\left[(a-\xi)^2+b^2\right](1-e^{(\xi\omega)})^2}{b^2 e^{(2a\omega)}} < 1+e^{(2a\omega)}-2\cos(b\omega)e^{(a\omega)},$$

$(\mathscr{C}_{12})$ $0 < \mathscr{A}_8 \leq G_4(\psi,\varsigma) \leq \mathscr{B}_8$ whenever $a,b,\xi > 0$ and

$$\frac{\left[(a-\xi)^2+b^2\right](1-e^{(\xi\omega)})^2}{b^2} < \frac{1+e^{(2a\omega)}-2\cos(b\omega)e^{(a\omega)}}{e^{(2a\omega)}},$$

$(\mathscr{C}_{13})$ $0 < \mathscr{A}_9 \leq G_4(\psi,\varsigma) \leq \mathscr{B}_9$ whenever $a < 0 < b,\xi$ and

$$\frac{\left[(a-\xi)^2+b^2\right](1-e^{(\xi\omega)})^2}{b^2} < 1+e^{(2a\omega)}-2\cos(b\omega)e^{(a\omega)}.$$

Theorem 2.5. *Suppose that the assumptions (i)-(ii) and hypothesis $\mathscr{C}_{10}$ ($\mathscr{C}_{11}$,$\mathscr{C}_{12}$ and $\mathscr{C}_{13}$ respectively) hold. Let $\omega V\mathscr{B}_8 < 1$ ($\omega V\mathscr{B}_9 < 1$, $\omega V|\mathscr{A}_8| < 1$ and $\omega V|\mathscr{A}_9| < 1$ respectively). Then the infinite system has at least one ω-periodic solution $y(\psi) = y_k(\psi)$ such that $y(\psi) \in n(\phi)$, $\psi \in \mathbb{R}$. Besides, the set of all solutions is compact.*

Proof. In order to achieve the desired conclusion, we replace the Green's function from $G_1(\psi,\varsigma)$ to $G_4(\psi,\varsigma)$ from Theorem 2.1 and $L\omega V < 1$ by $\omega V\mathscr{B}_8 < 1$ ($\omega V\mathscr{B}_9 < 1$, $\omega V|\mathscr{A}_8| < 1$ and $\omega V|\mathscr{A}_9| < 1$ respectively). □

3. Examples

We present two examples in this section for cases 1 and 3 to validate the aforementioned theorems.

3.1. Example 1

Take into account of the following infinite system of differential equation of third order:

$$y_n'''(\psi)+2.9y_n''(\psi)+1.7y_n'(\psi)+0.2y_n(\psi) = \frac{1}{n^4}+\sum_{k=n}^{\infty}\frac{y_k(\psi)\cos\psi}{100n^2(k+1)^6}. \tag{3.1}$$

Consider:

$$h_n(\psi,y_k(\psi)) = \frac{1}{n^4}+\sum_{k=n}^{\infty}\frac{y_k(\psi)\cos\psi}{100n^2(k+1)^6}.$$

For $n \in \mathbb{N}$, the function $h_n(\psi,y_k(\psi))$ is seen to be continuous at every point on $\mathbb{R}$ and is 2π - periodic. Additionally, whenever $y(\psi) = y_n(\psi) \in n(\phi)$, $h_n(\psi,y_k(\psi)) \in n(\phi)$.

$$\begin{aligned}
\sum_{n=1}^{\infty}|h_n(\psi,y_k(\psi))| &= \sum_{n=1}^{\infty}\left|\frac{1}{n^4}+\sum_{k=n}^{\infty}\frac{\%y_k(\psi)\cos\psi}{100n^2(k+1)^6}\right| \\
&\leq \sum_{n=1}^{\infty}\frac{1}{n^4}+\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}\left|\frac{y_k(\psi)\cos\psi}{\%100n^2(k+1)^6}\right| \\
&\leq \frac{\pi^4}{90}+\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}\frac{1}{\%100n^2(k+1)^6}|y_k(\psi)| \\
&\leq \frac{\pi^4}{90}+\frac{1}{100}\sum_{n=1}^{\infty}\sum_{k=n}^{\%\infty}\frac{1}{n^2(1+k)^6}|y_k(\psi)| \\
&\leq \frac{\pi^4}{90}+\frac{\pi^6}{95400}\|y_k(\psi)\|_{n(\phi)} < \infty.
\end{aligned}$$

Now let us prove that the assumption (i) holds. Choose an arbitrary ε ¿ 0 and $y(\psi) = y_n(\psi), z(\psi) = z_n(\psi) \in n(\phi)$ such that,

$$\|y(\psi)-z(\psi)\|_{n(\phi)} < \delta(\varepsilon) := \frac{95400\varepsilon}{\pi^6}.$$

Then,

$$|h_n(\psi,y(\psi))-h_n(\psi,z(\psi))| = \sum_{k=n}^{\infty}\left|\frac{(y_k(\psi)-z_k(\psi))\cos\psi}{100n^2(k+1)^6}\right|$$

$$\leq \sum_{k=n}^{\infty} \frac{|y_k(\psi)-z_k(\psi)|}{100n^2(k+1)^6}$$

$$\leq \frac{\pi^6}{95400} \|y(\psi)-z(\psi)\|_{n(\phi)}$$

$$\leq \frac{\pi^6}{95400}\delta < \varepsilon.$$

This guarantee that, the function is continuous as assumed in (i). We now show the assumption (ii) hold

$$|h_n(\psi,y_k(\psi))| = \left|\frac{1}{n^4} + \sum_{k=n}^{\infty} \frac{y_k(\psi)\cos\psi}{100n^2(k+1)^6}\right|$$

$$\leq \frac{1}{n^4} + \sum_{k=n}^{\infty} \left|\frac{y_k(\psi)\cos\psi}{100n^2(k+1)^6}\right|$$

$$\leq \frac{1}{n^4} + \sum_{k=n}^{\infty} \frac{1}{100n^2(k+1)^6} |y_k(\psi)|$$

$$:= u_n(\psi) + v_n(\psi)|y_k(\psi)|.$$

The function $u_n(\psi)$ is continuous on $\mathbb{R}$ with $n \in \mathbb{N}$ and $\sum_{n\geq 1} u_n(\psi)$ converges uniformly to $\frac{\pi^4}{90}$. More also, the sequence $v_n(\psi)$ is equibonded on $\mathbb{R}$. Thus the assumption (ii) is fulfilled.
The roots of homogeneous equations which correspond to (3.1) are $\xi_1 = 2, \xi_2 = 1, \xi_3 = -0.1$. This demonstrates that the Green's function associated with (3.1) is a form of $G_1(\psi,\varsigma)$ and $f_2 = 1.7295 \times 1^8 > g_2 = 1.6955 \times 10^8$. Applying the formula (2.3) ,we have $0 < \mathscr{A}_3 = 0.0206 \leq G_1(\psi,\varsigma) \leq \mathscr{B}_3 = 1.8387$. Thus, the condition in $\mathscr{C}_2$ is satisfied. The value $\omega V \mathscr{B}_3 \approx 0.1164 < 1$, for $\omega = 2\pi$. This indicates that the infinite system (3.1) has atleast one 2π-periodic solution $y(\psi) = (y_n(\psi)) \in n(\phi)$ as all criteria of Theorem 2.1 are met.

3.2. Example 2

We now provide a further illustrative example to further elucidate our conclusion for the case 3.
Consider the infinite system of differential equation of third order below:

$$y_n'''(\psi) - 3y_n''(\psi) - y_n'(\psi) - 1 = \frac{\cos(\psi)}{n^2} + \sum_{k=n}^{\infty} \frac{y_k(\psi)\sin\psi}{512(1+n^2)(k+1)^2}. \tag{3.2}$$

Consider:

$$h_n(\psi,y_k(\psi)) = \frac{\cos(\psi)}{n^2} + \sum_{k=n}^{\infty} \frac{y_k(\psi)\sin\psi}{512(1+n^2)(k+1)^2}.$$

We observe that, the function $h_n(\psi,y_k(\psi))$ is continuous at every points on $\mathbb{R}$ and is 2π - periodic for $n \in \mathbb{N}$.
The system (3.2) is a particular case of the considered system (1.1). Moreover, $h_n(\psi,y_k(\psi)) \in n(\phi)$ whenever $y(\psi) = y_n(\psi) \in n(\phi)$ as we have

$$\sum_{n=1}^{\infty} |h_n(\psi,y_k(\psi))| = \sum_{n=1}^{\infty} \left|\frac{cos(\psi)}{n^2} + \sum_{k=n}^{\infty} \frac{y_k(\psi)\sin\psi}{512(1+n^2)(k+1)^2}\right|$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} \left|\frac{y_k(\psi)\sin\psi}{512(1+n^2)(k+1)^2}\right|$$

$$\leq \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} \frac{1}{512(1+n^2)(k+1)^2} |y_k(\psi)|$$

$$\leq \frac{\pi^2}{6} + \frac{1}{512}\sum_{n=1}^{\infty}\sum_{k=n}^{\infty} \frac{1}{(1+n^2)(k+1)^2} |y_k(\psi)|$$

$$\leq \frac{\pi^2}{6} + \frac{1}{512} \times \frac{\pi^2}{6} \|y_k(\psi)\|_{n(\phi)} < \infty.$$

Now let us prove that the assumption (i) is satisfied. Consider any ε ¿ 0 and $y(\psi) = y_n(\psi), z(\psi) = z_n(\psi) \in n(\phi)$ such that,

$$\|y(\psi)-z(\psi)\|_{n(\phi)} < \delta(\varepsilon) := \frac{3072\varepsilon}{\pi^2}.$$

We have that

$$
\begin{aligned}
|h(\psi,y(\psi))-h(\psi,z(\psi))| &= \sum_{k=n}^{\infty}\left|\frac{(y_k(\psi)-z_k(\psi))\sin\psi}{512(1+n^2)(k+1)^2}\right| \\
&\leq \sum_{k=n}^{\infty}\frac{|y_k(\psi)-z_k(\psi)|}{512(1+n^2)(k+1)^2} \\
&\leq \frac{\pi^2}{6}\frac{1}{512}\|y(\psi)-z(\psi)\|_{n(\phi)} \\
&\leq \frac{\pi^2}{3072}\delta<\varepsilon,
\end{aligned}
$$

which ensures the desired continuity as assumed in (i). We now show the assumption (ii) hold

$$
\begin{aligned}
|h_n(\psi,y_k(\psi))| &= \left|\frac{\cos(\psi)}{n^2}+\sum_{k=n}^{\infty}\frac{y_k(\psi)\sin\psi}{512(1+n^2)(k+1)^2}\right| \\
&\leq \frac{1}{n^2}+\sum_{k=n}^{\infty}\left|\frac{y_k(\psi)\sin\psi}{512(1+n^2)(k+1)^2}\right| \\
&\leq \frac{1}{n^2}+\frac{1}{512}\sum_{k=n}^{\infty}\frac{1}{(1+n^2)(k+1)^2}|y_k(\psi)| \\
&:= u_n(\psi)+v_n(\psi)|y_k(\psi)|.
\end{aligned}
$$

The function $u_n(\psi)$ is continuous on $\mathbb{R}$ with $n\in\mathbb{N}$ and $\sum_{n\geq 1}u_n(\psi)$ converges uniformly to $\frac{\pi^2}{6}$. Furthermore, the sequence $v_n(\psi)$ is equibonded on $\mathbb{R}$. Thus the assumption (ii) is satisfied.
Using the notations from the preceding section, we can observe that the roots of the related homogeneous equation of (3.2) are $\xi_1=\xi_2=\xi_3=1$. Using the concept of $\mathscr{C}_8$ and the aforementioned roots, we find, $\mathscr{A}_7=-19.8873\leq G_3(\psi,\varsigma)\leq\mathscr{B}_7=-6.9354\times10^{-5}<0$, for $\omega=2\pi$ and $\omega V|\mathscr{A}_7|\approx 0.40145<1$.
All the hypothesis of Theorem 2.3 are satisfied, because for $n\in\mathbb{N}$, the function $h_n(\psi)$ is 2π- periodic with regard to first coordinate. The infinite system (3.2) therefore has a 2π-periodic, $y(\psi)=(y_n(\psi))\in n(\phi)$.

4. Conclusion

In our work, we have presented the conditions for existence of ω-periodic solution to an infinite system of third order differential equations in a sequence space $n(\phi)$ are given. Our conclusion was supported by the Meir-Keeler condensing operator and the notion of measures of non-compactness. To help illustrate the outcome, we have also included examples. More investigations is still needed to determine the required conditions for the existence of solutions to an infinite system of similar type in different Banach spaces.
For some related future work, we suggest that such type of differential equations of order higher than three can be studied in different sequence spaces, like c_0, c, ℓ_p, $m^\beta(\phi)$, $m^\beta(\phi,p)$, etc..

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References

[1] K. Kuratowski, *Sur les espaces complets,* Fundamenta Mathematicae, **15(1)**(1930), 301-309.
[2] L. S. Goldenstein, I. C. Gohberg, A. S. Markus, *Investigation of some properties of bounded sets and linear operators in connection with their q-norms,* Uch. Zap. Kishiner. Gos. Univ., **29** (1957), 29-36.
[3] L. S. Goldenstein, A. S. Markus, *On a measure of non-compactness of bounded sets and linear operators,* Studies in Algebra and Mathematical Analysis, Kishinev, (1965) 45-54.
[4] V. Istrătescu, *On a measure of noncompactness,* Bulletin Mathématique de la Société des Sciences Mathematiques de la République Socialiste de Roumanie, **16**(1972), 195-197.
[5] J. Banaś, M. Lecko, *Solvability of infinite systems of differential equations in Banach sequence spaces,* J. Comput. Appl. Math., **137**(2) (2001), 363-375.
[6] M. Mursaleen, S. Mohiuddine, *Applications of measures of noncompactness to the infinite system of differential equations in ℓ_p spaces,* Nonlinear Anal., **75**(4) (2012), 2111-2115.
[7] M. Mursaleen, *Application of measure of noncompactness to infinite systems of differential equations*, Canad. Math. Bull., **56**(2) (2013), 388-394.

[8] M. Mursaleen, S. Rizvi, *Solvability of infinite systems of second order differential equations in c_0 and ℓ_1 by Meir-Keeler condensing operators,* Proc. Amer. Math. Soc., **144**(10)(2016), 4279-4289.
[9] A. Alotaibi, M. Mursaleen, B.A. Alamri, *Solvability of second order linear differential equations in the sequence space $n(\phi)$*, Adv. Difference Equ., **377** (2018), 1-8.
[10] B. Hazarika, A. Das, R. Arab, M. Rabban, *Applications of fixed point theorem to solve the infinite system of second order differential equations in the Banach space $n(\phi), c$ and $\ell_p(1 \leq p \leq \infty)$,* Afrika Mat., (2017).
[11] Y. Chen, R. Jingli, S. Stefan, *Green's function for third-order differential equations,* Rocky Mountain J. Math., **41**(5) (2011), 1417-1448. https://doi.org/10.1216/RMJ-2011-41-5-1417
[12] R. Saadati, E. Pourhadi, M. Mursaleen, *Solvability of infinite systems of third-order differential equations in c_0 by Meir-Keeler condensing operators,* J. Fixed Point Theory Appl., **21**(2) (2019), 1-16.
[13] E. Pourhadi, M. Mursaleen, R. Saadati, *On a class of infinite system of third-order differential equations in l_p via measure of noncompactness,* Filomat, **34**(11) (2020), 3861-3870.
[14] M. Mursaleen, B. Bilalov, S.M.H. Rizvi, *Applications of measures of noncompactness to infinite system of fractional differential equations,* Filomat, **31**(11) (2017), 3421-3432.
[15] M. Mursaleen, V. Rakočević, *A survey on measures of noncompactness with some applications in infinite systems of differential equations,* Aequationes Mathematicae, **96** (2022), 489–514. https://doi.org/10.1007/s00010-021-00848-0
[16] I. Haque, J. Ali, M. Mursaleen, *Existence of solutions for an infinite system of Hilfer fractional boundary value problems in tempered sequence spaces,* Alexandria Eng. J., **65** (2023), 575-583.
[17] I. Haque, J. Ali, M. Mursaleen, *Solvability of infinite system of Langevin fractional differential equation in a new tempered sequence space,* Fract. Calc. Appl. Anal., **26** (2023), 1894–1915. https://doi.org/10.1007/s13540-023-00175-y
[18] M. Mursaleen, E. Savaş, *Solvability of an infinite system of fractional differential equations with p-Laplacian operator in a new tempered sequence space,* J. Pseudo-Differ. Oper. Appl., **14** (2023), 57.
[19] H. A. Kayvanloo, M. Khanehgir, R. Allahyari, *A family of measures of noncompactness in the Hölder space $C^{n,\gamma}(R_+)$ and its application to some fractional differential equations and numerical methods*, J. Comput. Appl. Math., **363** (2020) 256-272.
[20] F. P. Najafabadi, J.J. Nieto, H. A. Kayvanloo, *Measure of noncompactness on weighted Sobolev space with an application to some nonlinear convolution type integral equations*, J. Fixed Point Theory Appl., **22**(3)(2020), 1-15.
[21] H. Mehravaran, H. A. Kayvanloo, M. Mursaleen, *Solvability of infinite systems of fractional differential equations in the double sequence space* $2^c(\triangle)$, Fract. Calc. Appl. Anal., **25**(6) (2022), 2298-2312.
[22] H. Mehravaran, H. A. Kayvanloo, *Solvability of infinite system of nonlinear convolution type integral equations in the tempered sequence space* $m^\beta(\phi, p)$, Asian-European J. Math., **16**(1) (2022). https://doi.org/10.1142/S1793557123500043
[23] H. Mehravaran, H. A. Kayvanloo, R. Allahyari, *Solvability of infinite systems of fractional differential equations in the space of tempered sequence space $m^\beta(\phi)$*, Int. J. Nonlinear Anal. Appl., **13**(1) (2022), 1023-1034.
[24] J. Banaś, *On measures of noncompactness in Banach spaces,* Comment. Math. Univ. Carolin., **21**(1) (1980), 131-143.
[25] J. Banaś, M. Mursaleen, *Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations,* Springer, New Delhi, (2014).
[26] G. Darbo, *Punti uniti in trasformazioni a codominio non compatto,* Rendiconti del Seminario Matematico della Universitá di Padova, **24**(1955), 284-292.
[27] E. M. Keeler, A. Meir, *A theorem on contraction mappings.* J. Math. Anal. Appl., **28(1)** (1969), 326-329.
[28] A. Aghajani, M. Mursaleen, H. A. Shole, *Fixed point theorems for Meir-Keeler condensing operators via measure of noncompactness,* Acta Math. Sci., **35(3)** (2015), 552-566.
[29] W. L. C. Sargent, *Some sequence spaces related to the lp spaces.* J. London Math. Soc., **35** (1960) 161-171.

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Research Article

New Analytical Wave Structures for the (2+1)-Dimensional Chaffee-Infante Equation

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Abstract

The focus of this paper is the (2+1)-dimensional Chaffee-Infante equation (CIE). The model describes the diffusion of a gas in a homogeneous medium, which makes it an important tool in the research of mathematics and physics. The modified extended Tanh expansion method is employed. Many soliton solutions have been obtained by rigorous analysis and calculation. This method can generate various types of solutions including trigonometric, trigonometric-hyperbolic, rational, kink, singular, and periodic singular solitons. We also present some of the obtained solutions' 3D, contour, and 2D plots. In order to tackle complex nonlinear issues, the solutions are dependable, efficient, and manageable, and the generated results provide a basis for further research. The study's method used in this paper is characterised by its ability to generate simple, reliable and original solutions to nonlinear partial differential equations (NLPDEs) in mathematical physics. To the best of our knowledge, no such work has been done before for this problem. The Maple software has been used to check the correctness of each solution found.

1. Introduction

NLPDEs are frequently used to describe complex physical events in the disciplines of chemistry, biology, mechanics, and physics [1–5] . It's an exciting attempt to find accurate solutions to NLPDEs, and academics have made great strides in this direction [6] . Many techniques have been developed over time to obtain analytical solutions for these kinds of issues. Because of their innately unpredictable behaviors, NLPDEs continue to provide substantial management and control issues despite these developments [7, 8] . A system can change quickly even with tiny modifications to some of the influencing variables. Therefore, scientists from different disciplines are investigating analytical form solutions of nonlinear equations to understand and investigate complex processes [9–16] . These solutions aid in our understanding of the behaviors of many nonlinear occurrences by illuminating their conceptual and visual connections. Therefore, in order to obtain deeper understanding, scholars who are interested in nonlinear phenomena, whether in engineering and other scientific fields, have been examining these analytic-form solutions [17, 18] .

The (2+1)-dimensional CIE is a NLEE that was first developed for use in combustion chemistry and combustion physics research. Characterizing the kinetics of chemical interactions during combustion processes is essential, especially when premixed flames are involved. Since its development, the CIE has been extensively studied and applied in the field of science. In many other disciplines, including electrical science, nuclear physics, ecology, fluid dynamics, and others, it has been widely used to explain the physical processes of mass movement and particle dispersion. In mathematics and physics, the (2+1)-dimensional CIE is primarily studied because it offers a valuable model for examining the diffusion phenomenon that occurs a gas in a homogeneous medium.

In this paper, motivated by other studies, we used the modified extended tanh expansion method (METEM) approach to study the (2+1)-dimensional CIE and get soliton solutions. Our ability to represent different wave patterns of complex physical events in scientific disciplines is made possible by these innovative discoveries. The physical processes of mass transport and particle diffusion can be described using the well-known reaction diffusion equation known as the CI equation. Numerous scientific and technological domains, including as fluid dynamics, plasma physics, ion-acoustic waves in the plasma, sound waves, electromagnetic waves, and signal processing through optical cables, use this equation. It is now known as the Chafee–Infante equation, and it was initially proposed by Nathaniel Chafee and Ettore Infante.

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The study and identification of different types of optical solitons is crucial to the investigation and application of this equation. In this context, researchers have carried out several studies. Sulaiman et al. obtained new lump solutions for the (2+1)-dimensional CIE using the Hirota bilinear form (HBF) [19] . Zhu et al. obtained analytical solutions to the main equation by the application of two methods involving the solutions of the trigonometric and hyperbolic functions [20] . To find closed-form solitary wave solutions for (2+1)-dimensional CIE, Akbar et al. used the first integral method [21] . Ay and Yaşar Painlevé constructed Bäcklund transformations and the other symmetries in non-local structures by using the shortened expansion approach. They put these symmetries in place and constructed the corresponding single variable Lie conversion group using an extended system. Also they proposed novel correct solution profiles in this transformation group by combining other simple accurate solution structures [22] .
In this paper, we consider the (2+1)-dimensional CIE in the form [19–22] :

$$V_{xt}+\left(-V_{xx}-\sigma V^{3}-\sigma V\right)_{x}+\eta V_{yy}=0. \tag{1.1}$$

Here $V(x,y,t)$ is the function which describes the intricate wave profile σ, η are the coefficient for diffusion and degradation, respectively. The primary motivation for studying the (2+1)-dimensional CIE in mathematics and physics is that it offers a practical model for examining the diffusion happening of a gas in a homogeneous medium. Obtaining several families of analytical soliton solutions and illustrating the dynamics of solitonic structures of the (2+1)-dimensional CIE will be the main goals of this study. These new discoveries enable us to depict different wave patterns of complex physical occurrences in a variety of scientific fields.
The rest of the paper is structured as follows: Description of the METEM is given in §2. The implementation of the proposed method are provided in §3. The dynamic behaviors for the various solutions is shown in 3D, contour, and 2D graphs in §4, and the findings are explained. Lastly, §5 some conclusions are presented.

2. Description of Applied Method

Assume that the presence of a NLPDE in the following form [23] :

$$N(V,V_x,V_y,V_t,V_{xx},V_{yy},V_{xy},V_{xt},\dots)=0, \tag{2.1}$$

where N stands for both the polynomial in space and time and its partial derivatives of $V(x,y,t)$. In order to solve Eq. (2.1), the subsequent wave transforms are used:

$$V(x,y,t)=U(\xi), \quad \xi=ax+by-ct, \tag{2.2}$$

where $U(\xi)$ represents the pulse's form and a, b, c are the non-zero real constants.
The following ordinary differential equation (ODE) is derived from Eq. (2.1):

$$O(U,U',U'',U''',\dots)=0. \tag{2.3}$$

The main components of METEM are summarized below.
Step 1. To solve the ODE, take Eq. (2.3) in the series form as:

$$U(\xi)=A_0+\sum_{i=1}^{n}\left(A_i\Phi^{i}(\xi)+B_i\Phi^{-i}(\xi)\right), \tag{2.4}$$

where A_i, B_i are the ordinary constant parameters to be determined later. The function $\Phi(\xi)$, which will be determined later, satisfies the auxiliary equation as follows:

$$\frac{d\Phi(\xi)}{d\xi}=\varphi+(\Phi(\xi))^{2}. \tag{2.5}$$

In Eq. (2.5), are constants discovered be later. In subsequent, the use of Eq. (2.5) is shown.
Family 1. For $\varphi<0$, the general solutions for the ansatz Eq. (2.5), given the following hyperbolic solutions:

$$\Phi_1(\xi)=-\sqrt{-\varphi}\tanh\left(\sqrt{-\varphi}(\xi+\xi_0)\right), \tag{2.6}$$

$$\Phi_2(\xi)=-\sqrt{-\varphi}\coth\left(\sqrt{-\varphi}(\xi+\xi_0)\right), \tag{2.7}$$

$$\Phi_3(\xi)=-\sqrt{-\varphi}\left(\tanh\left(2\sqrt{-\varphi}(\xi+\xi_0)\right)+i\varepsilon\,\mathrm{sech}\left(2\sqrt{-\varphi}(\xi+\xi_0)\right)\right), \tag{2.8}$$

$$\Phi_4(\xi)=\frac{-\sqrt{-\varphi}\tanh(\sqrt{-\varphi}(\xi+\xi_0))+\varphi}{\sqrt{-\varphi}\tanh(\sqrt{-\varphi}(\xi+\xi_0))+1}, \tag{2.9}$$

$$\Phi_5(\xi)=\frac{\sqrt{-\varphi}(-4\cosh(2\sqrt{-\varphi}(\xi+\xi_0))+5)}{(4\sinh(2\sqrt{-\varphi}(\xi+\xi_0))+3)}, \tag{2.10}$$

$$\Phi_6(\xi) = \frac{\varepsilon\sqrt{-w\left(a^2+b^2\right)} - a\sqrt{-\varphi}\cosh\left(2\sqrt{-\varphi}\left(\xi+\xi_0\right)\right)}{a\sinh\left(2\sqrt{-\varphi}\left(\xi+\xi_0\right)\right)+b}, \tag{2.11}$$

$$\Phi_7(\xi) = \varepsilon\sqrt{-\varphi}\left[1-\frac{2a}{a+\cosh\left(2\sqrt{-\varphi}\left(\xi+\xi_0\right)\right)-\varepsilon\sinh\left(2\sqrt{-\varphi}\left(\xi+\xi_0\right)\right)}\right]. \tag{2.12}$$

Family 2. For $\varphi > 0$, the general solutions for the ansatz Eq. (2.5), given the following trigonometric solutions:

$$\Phi_8(\xi) = \sqrt{\varphi}\tan\left(\sqrt{\varphi}\left(\xi+\xi_0\right)\right), \tag{2.13}$$

$$\Phi_9(\xi) = -\sqrt{\varphi}\cot\left(\sqrt{\varphi}\left(\xi+\xi_0\right)\right), \tag{2.14}$$

$$\Phi_{10}(\xi) = \sqrt{\varphi}\left(\tan\left(2\sqrt{\varphi}\left(\xi+\xi_0\right)\right)+\varepsilon\sec\left(2\sqrt{\varphi}\left(\xi+\xi_0\right)\right)\right), \tag{2.15}$$

$$\Phi_{11}(\xi) = -\frac{\sqrt{\varphi}\left(1-\tan\left(\sqrt{\varphi}\left(\xi+\xi_0\right)\right)\right)}{\left(1+\tan\left(\sqrt{\varphi}\left(\xi+\xi_0\right)\right)\right)}, \tag{2.16}$$

$$\Phi_{12}(\xi) = \frac{\sqrt{\varphi}\left(-5\cos\left(2\sqrt{\varphi}\left(\xi+\xi_0\right)\right)+4\right)}{\left(5\sin\left(2\sqrt{\varpi}\left(\xi+\xi_0\right)\right)+3\right)}, \tag{2.17}$$

$$\Phi_{13}(\xi) = \frac{\varepsilon\sqrt{\varphi\left(a^2+b^2\right)} - a\sqrt{\varphi}\cos\left(2\sqrt{\varphi}\left(\xi+\xi_0\right)\right)}{a\sin\left(2\sqrt{\varphi}\left(\xi+\xi_0\right)\right)+b}, \tag{2.18}$$

$$\Phi_{14}(\xi) = i\varepsilon\sqrt{\varphi}\left[1-\frac{2a}{a+\cos\left(2\sqrt{\varphi}\left(\xi+\xi_0\right)\right)-i\varepsilon\sin\left(2\sqrt{\varphi}\left(\xi+\xi_0\right)\right)}\right]. \tag{2.19}$$

Family 3. For $\varphi = 0$, the general solutions for the Eq. (2.5), the following rational solution is given:

$$\Phi_{15}(\xi) = -\frac{1}{\xi+\xi_0}. \tag{2.20}$$

Here, the real arbitrary parameters are $\varepsilon = \pm 1,\ a,\ b,\ \varphi,\ \xi_0$.
Step 2. In order to balance the nonlinear terms in Eq. (2.3) with the highest order derivative, we determine n for Eq. (2.4).
Step 3. Inserting Eq. (2.4) and its derivatives in Eq. (2.3) with regard to Eq. (2.5), we get a polynomial for $U(\xi)$. By taking the coefficients of each power to zero, we yield a system of equations with unknown parameters $\varphi,\ A_i,\ B_i\ (i = 1,2,3,\ldots,n)$ and solving this system we obtain the analytic solutions of Eq. (2.3).
Step 4. Finally, we get several analytical solutions to Eq. (2.1) by applying the transformation to Eq. (2.2) and using the solutions to Eq. (2.3). Through the consideration of the above three families, we have acquired the analytical solutions for Eq. (1.1).

3. Analysis of Solitons for the (2+1)-Dimensional CIE

In this section of the study, we build different soliton solutions for Eq. (1.1) while taking the METEM into consideration.
The (2+1)-dimensional CIE is consider as:

$$V(x,y,t) = U(\xi), \quad \xi = (ax+by-ct). \tag{3.1}$$

When we apply in Eq. (3.1) to Eq. (1.1), we get

$$-a^3U'''(\xi)+b^2\eta U''(\xi)-caU''(\xi)-a\sigma U'(\xi)+3a\sigma U'(\xi)U^2(\xi)=0. \tag{3.2}$$

After we integrate once with respect to ξ, Eq. (3.2) changes to as follows:

$$-a^3U''(\xi)+\left(b^2\eta-ca\right)U'(\xi)-a\sigma U(\xi)+a\sigma U^3(\xi)=0, \tag{3.3}$$

in which we obtain $n = 1$ by balancing $U^3(\xi)$ with $U''(\xi)$. The general solution to Eq. (3.3) is as follows:

$$U(\xi) = A_0 + A_1\Phi(\xi) + B_1\frac{1}{\Phi(\xi)}. \tag{3.4}$$

By inserting Eq. (3.4) together with Eq. (3.3) into Eq. (2.5) and setting the coefficients to zero for various powers of $\Phi(\xi)$, we have a system of equations. And we solve using the Maple software program to get the subsequent equation system:

$$\begin{cases}\Phi(\xi)^3: a\sigma A_1^3-2a^3A_1=0,\\ \Phi(\xi)^2: -\left((-3\sigma A_0A_1+c)a-b^2\eta\right)A_1=0,\\ \Phi(\xi)^1: 3A_1a\sigma\left(B_1A_1+A_0^2-\frac{1}{3}\right)-2a^3A_1\varphi=0,\\ \Phi(\xi)^0: \left((6B_1\sigma A_0-c\varphi)A_1+\sigma A_0^3+cB_1-\sigma A_0\right)a-b^2\eta\left(-A_1\varphi+B_1\right)=0,\\ \Phi(\xi)^{-1}: 3B_1a\left(B_1A_1+A_0^2-\frac{1}{3}\right)\sigma-2a^3B_1\varphi=0,\\ \Phi(\xi)^{-2}: \left((3B_1\sigma A_0+c\varphi)a-b^2\eta\varphi\right)B_1=0,\\ \Phi(\xi)^{-3}: -2a^3B_1\varphi^2+a\sigma B_1^3=0.\end{cases}$$

By solving above system for B_1, A_1, A_0, c, and φ, we get these set solutions:
Set 1:

$$B_1=0,\quad A_1=\sqrt{\frac{2}{\sigma}}a,\quad A_0=0,\quad c=\frac{b^2\eta}{a},\quad \varphi=-\frac{\sigma}{2a^2}.$$

Set 2:

$$B_1=0,\quad A_1=\sqrt{\frac{2}{\sigma}}a,\quad A_0=\frac{1}{2},\quad c=\frac{3a^2\sigma\sqrt{\frac{2}{\sigma}}+2b^2\eta}{2a},\quad \varphi=-\frac{\sigma}{8a^2}.$$

Set 3:

$$B_1=\frac{\sqrt{2\sigma}}{2a},\quad A_1=0,\quad A_0=0,\quad c=\frac{b^2\eta}{a},\quad \varphi=-\frac{\sigma}{2a^2}.$$

Set 4:

$$B_1=\frac{3\sqrt{2\sigma}+\sqrt{\sigma}}{16a},\quad A_1=\sqrt{\frac{2}{\sigma}}a,\quad A_0=0,\quad c=\frac{b^2\eta}{a},\quad \varphi=\frac{4\sigma}{16a^2}.$$

Set 5:

$$B_1=\frac{\sqrt{2\sigma}}{32a},\quad A_1=\sqrt{\frac{2}{\sigma}}a,\quad A_0=\frac{1}{2},\quad c=\frac{3a^2\sqrt{2\sigma}+2b^2\eta}{2a},\quad \varphi=-\frac{\sigma}{32a^2}.$$

Set 6:

$$B_1=\frac{\sqrt{2\sigma}}{8a},\quad A_1=0,\quad A_0=-\frac{1}{2},\quad c=\frac{-3a^2\sigma\sqrt{2}+2\sqrt{\sigma}b^2\eta}{2a\sqrt{\sigma}},\quad \varphi=-\frac{\sigma}{8a^2}.$$

For **Set 1**, substituting the values of constants into Eq. (3.4) and Eq. (3.1) along with Eq. (2.6)-Eq. (2.20) and simplifying, yields next solutions:
Family 1: When $\varphi<0$ and the hyperbolic solutions of Eq. (1.1) are given as follows:

$$V_{1,1}(x,y,t)=-\tanh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right), \tag{3.5}$$

$$V_{1,2}(x,y,t)=-\coth\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right),$$

$$V_{1,3}(x,y,t)=-\left(\tanh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+i\,\mathrm{sech}\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right),$$

$$V_{1,4}(x,y,t)=\frac{\sqrt{\frac{2}{\sigma}}a\left(-\frac{\sigma}{2a^2}-\sqrt{\frac{\sigma}{2a^2}}\tanh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{\left(1+\sqrt{\frac{\sigma}{2a^2}}\tanh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}, \tag{3.6}$$

$$V_{1,5}(x,y,t)=\frac{5-4\cosh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)}{\left(3+4\sinh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},$$

$$V_{1,6}(x,y,t)=\frac{\sqrt{\frac{2a^2}{\sigma}}\left(\sqrt{\frac{2\sigma(a^2+b^2)}{a^2}}-\sqrt{2\sigma}\cosh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{\left(2a\sinh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+b\right)},$$

$$V_{1,7}(x,y,t)=1-\frac{2a}{\left(a+\cosh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)-\sinh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}.$$

Family 2: Given $\varphi>0$, using the offered method obtained the following trigonometric solutions for Eq. (1.1):

$$V_{1,8}(x,y,t)=i\tan\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right),$$

$$V_{1,9}(x,y,t)=-i\cot\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right),$$

$$V_{1,10}(x,y,t)=i\left(\tan\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+\sec\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right),$$

$$V_{1,11}(x,y,t)=-\frac{i\left(1-\tan\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{\left(1+\tan\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},$$

$$V_{1,12}(x,y,t)=\frac{i\left(4-5\cos\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{\left(3+5\sin\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},$$

$$V_{1,13}(x,y,t)=\frac{\sqrt{\frac{2a^2}{\sigma}}\left(\sqrt{-\frac{\sigma(a^2-b^2)}{2a^2}}-\sqrt{-\frac{\sigma}{2}}\cos\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{\left(a\sin\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+b\right)},$$

$$V_{1,14}(x,y,t)=1-\frac{2a}{\left(a+\cos\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)-i\sin\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}.$$

Family 3: When $\varphi=0$, then Eq. (1.1) has rational solution:

$$V_{1,15}(x,y,t)=-\frac{\sqrt{\frac{2a^2}{\sigma}}}{ax+by-\frac{b^2\eta t}{a}}.$$

For **Set 2**, substituting the values of constants into Eq. (3.4) and Eq. (3.1) along with Eq. (2.6)-Eq. (2.20) and simplifying, yields following solutions, respectively:

Family 1: When $\varphi<0$ and the hyperbolic solutions of Eq. (1.1) are given as follows:

$$V_{2,1}(x,y,t)=\frac{1}{2}\left(1-\tanh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right),$$

$$V_{2,2}(x,y,t)=\frac{1}{2}\left(1-\coth\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right),$$

$$V_{2,3}(x,y,t)=\frac{1}{2}\left(1-\left(\tanh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)+i\,\mathrm{sech}\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)\right),$$

$$V_{2,4}(x,y,t)=\frac{1}{2}+\frac{\sqrt{\frac{2}{\sigma}}a\left(-\frac{\sigma}{8a^2}-\sqrt{\frac{\sigma}{8a^2}}\tanh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{\left(1+\sqrt{\frac{\sigma}{8a^2}}\tanh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)},$$

$$V_{2,5}(x,y,t)=\frac{1}{2}\left(1+\frac{5-4\cosh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)}{3+4\sinh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)}\right),$$

$$V_{2,6}(x,y,t)=\frac{1}{2}+\frac{\sqrt{\frac{2a^2}{\sigma}}\left(\sqrt{\frac{\sigma(a^2+b^2)}{8a^2}}-\sqrt{\frac{\sigma}{8}}\cosh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{\left(a\sinh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)+b\right)},$$

$$V_{2,7}(x,y,t)=1-\frac{a}{\left(\begin{array}{l}a+\cosh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\ -\sinh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\end{array}\right)}.$$

Family 2: Given $\varphi>0$, using the method obtained the following trigonometric solutions for Eq. (1.1):

$$V_{2,8}(x,y,t)=\frac{1}{2}\left(1+i\tan\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right),$$

$$V_{2,9}(x,y,t)=\frac{1}{2}\left(1-i\cot\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right),$$

$$V_{2,10}(x,y,t)=\frac{1}{2}\left(1+i\left(\begin{array}{l}\tan\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\ +\sec\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\end{array}\right)\right),$$

$$V_{2,11}(x,y,t)=\frac{1}{2}\left(1-\frac{i\left(1-\tan\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{1+\tan\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)}\right),$$

$$V_{2,12}(x,y,t)=\frac{1}{2}\left(1+\frac{i\left(4-5\cos\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{3+5\sin\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)}\right),$$

$$V_{2,13}(x,y,t)=\frac{1}{2}+\frac{\sqrt{\frac{2a^2}{\sigma}}\left(\sqrt{-\frac{\sigma(a^2-b^2)}{8a^2}}-\sqrt{-\frac{\sigma}{8}}\cos\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{\left(a\sin\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)+b\right)},$$

$$V_{2,14}(x,y,t)=1-\frac{a}{\left(\begin{array}{l}a+\cos\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\ -i\sin\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\end{array}\right)}.$$

Family 3: When $\varphi=0$, then Eq. (1.1) has rational solution:

$$V_{2,15}(x,y,t)=\frac{1}{2}-\frac{\sqrt{\frac{2a^2}{\sigma}}}{ax+by-\frac{3a^2\sigma\sqrt{\frac{2}{\sigma}}+2b^2\eta t}{2a}}.$$

For **Set 3**, substituting the values of constants into Eq. (3.4) and Eq. (3.1) along with Eq. (2.6)-Eq. (2.20) and simplifying, yields following solutions, respectively:

Family 1: When $\varphi<0$ and the hyperbolic solutions of Eq. (1.1) are given as follows:

$$V_{3,1}(x,y,t)=-\frac{1}{\tanh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)},$$

$$V_{3,2}(x,y,t)=-\frac{1}{\coth\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)},$$

$$
\begin{aligned}
V_{3,3}(x,y,t) &= -\frac{1}{\tanh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+i\,\mathrm{sech}\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)},\\
V_{3,4}(x,y,t) &= \frac{\sqrt{2\sigma}\left(1+\sqrt{\frac{\sigma}{2a^2}}\tanh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{2a\left(-\frac{\sigma}{2a^2}-\sqrt{\frac{\sigma}{2a^2}}\tanh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},\\
V_{3,5}(x,y,t) &= \frac{3+4\sinh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)}{5-4\cosh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)},\\
V_{3,6}(x,y,t) &= \frac{\sqrt{2\sigma}\left(a\sinh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+b\right)}{a\left(\sqrt{\frac{2\sigma(a^2+b^2)}{a^2}}-\sqrt{2\sigma}\cosh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}, \qquad (3.7)\\
V_{3,7}(x,y,t) &= 1-\frac{2a}{\left(a+\cosh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)-\sinh\left(\sqrt{\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}.
\end{aligned}
$$

Family 2: Given $\varphi > 0$, using the method obtained the following trigonometric solutions for Eq. (1.1):

$$
\begin{aligned}
V_{3,8}(x,y,t) &= \frac{1}{i\tan\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)},\\
V_{3,9}(x,y,t) &= -\frac{1}{i\cot\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)},\\
V_{3,10}(x,y,t) &= \frac{1}{i\left(\tan\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+\sec\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},\\
V_{3,11}(x,y,t) &= -\frac{\left(1+\tan\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{i\left(1-\tan\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},\\
V_{3,12}(x,y,t) &= \frac{3+5\sin\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)}{i\left(4-5\cos\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},\\
V_{3,13}(x,y,) &= \frac{\sqrt{2\sigma}\left(a\sin\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+b\right)}{a\left(\sqrt{-\frac{2\sigma(a^2-b^2)}{a^2}}-\sqrt{-2\sigma}\cos\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},\\
V_{3,14}(x,y,t) &= -1+\frac{2a}{\left(a+\cos\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)-i\sin\left(\sqrt{-\frac{2\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}.
\end{aligned}
$$

Family 3: When $\varphi = 0$, then Eq. (1.1) has rational solution:

$$
V_{3,15}(x,y,t) = -\frac{\sqrt{2\sigma}}{2a}\left(ax+by-\frac{b^2\eta t}{a}\right).
$$

For **Set 4**, substituting the values of constants into Eq. (3.4) and Eq. (3.1) along with Eq. (2.6)-Eq. (2.20) and simplifying, yields following solutions, respectively:

Family 1: When $\varphi < 0$ and the hyperbolic solutions of Eq. (1.1) are given as follows:

$$
\begin{aligned}
V_{4,1}(x,y,t) = &-\frac{\sqrt{-2}}{2}\tanh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\\
&-\frac{1}{\sqrt{-2}\tanh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)},
\end{aligned}
$$

$$V_{4,2}(x,y,t)=-\frac{\sqrt{-2}}{2}\coth\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)-\frac{1}{\sqrt{-2}\coth\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)},$$

$$V_{4,3}(x,y,t)=-\frac{\sqrt{-2}}{2}\left(\begin{array}{c}\tanh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\\+i\,\mathrm{sech}\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\end{array}\right)-\frac{1}{\sqrt{-2}\left(\begin{array}{c}\tanh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\\+i\,\mathrm{sech}\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\end{array}\right)},$$

$$V_{4,4}(x,y,t)=-\frac{\sqrt{\frac{2a^2}{\sigma}}\left(\frac{\sigma}{4a^2}-\sqrt{-\frac{\sigma}{4a^2}}\tanh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{1+\sqrt{-\frac{\sigma}{4a^2}}\tanh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)}+\frac{\sqrt{2\sigma}\left(1+\sqrt{-\frac{\sigma}{4a^2}}\tanh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{4a\left(\frac{\sigma}{4a^2}-\sqrt{-\frac{\sigma}{4a^2}}\tanh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},$$

$$V_{4,5}(x,y,t)=\frac{\sqrt{-2}\left(5-4\cosh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{2\left(3+4\sinh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}+\frac{\left(3+4\sinh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{\sqrt{-2}\left(5-4\cosh\left(\sqrt{-\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},$$

$$V_{4,6}(x,y,t)=\frac{\sqrt{\frac{2a^2}{\sigma}}\left(\sqrt{-\frac{\sigma(a^2+b^2)}{4a^2}}-\sqrt{-\frac{\sigma}{4}}\cosh\left(\sqrt{-\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{a\sinh\left(\sqrt{-\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+b}+\frac{\sqrt{2\sigma}\left(a\sinh\left(\sqrt{-\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+b\right)}{4a\left(\sqrt{-\frac{\sigma(a^2+b^2)}{4a^2}}-\sqrt{-\frac{\sigma}{4}}\cosh\left(\sqrt{-\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},$$

$$V_{4,7}(x,y,t)=\sqrt{-\frac{1}{2}}-\frac{\sqrt{-\frac{1}{2}}2a}{\left(\begin{array}{c}a+\cosh\left(\sqrt{-\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\\-\sinh\left(\sqrt{-\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\end{array}\right)}+\frac{1}{\sqrt{-2}}-\frac{\frac{1}{\sqrt{-2}}2a}{\left(\begin{array}{c}a+\cosh\left(\sqrt{-\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\\-\sinh\left(\sqrt{-\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\end{array}\right)}.$$

Family 2: Given $\varphi>0$, using the method obtained the following trigonometric solutions for Eq. (1.1):

$$V_{4,8}(x,y,t)=\frac{1}{\sqrt{2}}\tan\left(\sqrt{\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+\frac{1}{\sqrt{2}\tan\left(\sqrt{\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)},$$

$$V_{4,9}(x,y,t)=-\frac{1}{\sqrt{2}}\cot\left(\sqrt{\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)-\frac{1}{\sqrt{2}\cot\left(\sqrt{\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)},$$

$$V_{4,10}(x,y,t)=\frac{1}{\sqrt{2}}\begin{pmatrix}\tan\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\\+\sec\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\end{pmatrix}+\frac{1}{\sqrt{2}\begin{pmatrix}\tan\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\\+\sec\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\end{pmatrix}}, \tag{3.8}$$

$$V_{4,11}(x,y,t)=-\frac{\sqrt{2}\left(1-\tan\left(\sqrt{\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{2\left(1+\tan\left(\sqrt{\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}-\frac{\left(1+\tan\left(\sqrt{\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{\sqrt{2}\left(1-\tan\left(\sqrt{\frac{\sigma}{4a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},$$

$$V_{4,12}(x,y,t)=\frac{\sqrt{2}\left(4-5\cos\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{2\left(3+5\sin\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}+\frac{\left(3+5\sin\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{\sqrt{2}\left(4-5\cos\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},$$

$$V_{4,13}(x,y,t)=\frac{\sqrt{\frac{2a^2}{\sigma}}\left(\sqrt{\frac{\sigma(a^2-b^2)}{4a^2}}-\sqrt{\frac{\sigma}{4a^2}}\cos\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)}{\sqrt{2\sigma}\left(a\sin\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+b\right)}$$
$$+\frac{\sqrt{2\sigma}\left(a\sin\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)+b\right)}{4a\left(\sqrt{\frac{\sigma(a^2-b^2)}{4a^2}}-\sqrt{\frac{\sigma}{4a^2}}\cos\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)\right)},$$

$$V_{4,14}(x,y,t)=\sqrt{-\frac{1}{2}}\left(1-\frac{2a}{a+\cos\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)-i\sin\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)}\right)$$
$$-\sqrt{\frac{1}{2}}\left(1-\frac{2a}{a+\cos\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)-i\sin\left(\sqrt{\frac{\sigma}{a^2}}\left(ax+by-\frac{b^2\eta t}{a}\right)\right)}\right).$$

Family 3: When $\varphi=0$, then Eq. (1.1) has rational solution:

$$V_{4,15}(x,y,t)=-\frac{\sqrt{\frac{2a^2}{\sigma}}}{\left(ax+by-\frac{b^2\eta t}{a}\right)}-\frac{\sqrt{2\sigma}}{4a}\left(ax+by-\frac{b^2\eta t}{a}\right).$$

For **Set 5**, substituting the values of constants into Eq. (3.4) and Eq. (3.1) along with Eq. (2.6)-Eq. (2.20) and simplifying, yields following solutions, respectively:

Family 1: When $\varphi<0$ and the hyperbolic solutions of Eq. (1.1) are given as follows:

$$V_{5,1}(x,y,t)=\frac{1}{2}-\frac{1}{4}\tanh\left(\sqrt{\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)$$
$$-\frac{1}{4\tanh\left(\sqrt{\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)},$$

$$V_{5,2}(x,y,t)=\frac{1}{2}-\frac{1}{4}\coth\left(\sqrt{\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)$$
$$-\frac{1}{4\coth\left(\sqrt{\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)},$$

$$V_{5,3}(x,y,t)=\frac{1}{2}-\frac{1}{4}\begin{pmatrix}\tanh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\+i\,\mathrm{sech}\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\end{pmatrix}$$
$$-\frac{1}{4\begin{pmatrix}\tanh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\+i\,\mathrm{sech}\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\end{pmatrix}}, \tag{3.9}$$

$$V_{5,4}(x,y,t)=\frac{1}{2}-\frac{\sqrt{\frac{2a^2}{\sigma}}\left(-\frac{\sigma}{32a^2}-\sqrt{\frac{\sigma}{32a^2}}\tanh\left(\sqrt{\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{\left(1+\sqrt{\frac{\sigma}{32a^2}}\tanh\left(\sqrt{\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}$$
$$-\frac{\sqrt{2\sigma}\left(1+\sqrt{\frac{\sigma}{32a^2}}\tanh\left(\sqrt{\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{32a\left(-\frac{\sigma}{32a^2}-\sqrt{\frac{\sigma}{32a^2}}\tanh\left(\sqrt{\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)},$$

$$V_{5,5}(x,y,t)=\frac{1}{2}+\frac{\left(5-4\cosh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{4\left(3+4\sinh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}$$
$$+\frac{\left(3+4\sinh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{4\left(5-4\cosh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)},$$

$$V_{5,6}(x,y,t)=\frac{1}{2}+\frac{\sqrt{\frac{2a^2}{\sigma}}\left(\sqrt{\frac{\sigma(a^2+b^2)}{32a^2}}-\sqrt{\frac{\sigma}{32}}\cosh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{\left(a\sinh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)+b\right)}$$
$$+\frac{\sqrt{2\sigma}\left(a\sinh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)+b\right)}{32a\left(\sqrt{\frac{\sigma(a^2+b^2)}{32a^2}}-\sqrt{\frac{\sigma}{32}}\cosh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)},$$

$$V_{5,7}(x,y,t)=\frac{1}{2}+\frac{1}{4}-\frac{\frac{1}{2}a}{\left(\begin{array}{l}a+\cosh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\-\sinh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\end{array}\right)}$$
$$-\frac{1}{4-\frac{8a}{\left(\begin{array}{l}a+\cosh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\-\sinh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\end{array}\right)}}.$$

Family 2: Given $\varphi>0$, using the method obtained the following trigonometric solutions for Eq. (1.1):

$$V_{5,8}(x,y,t)=\frac{1}{2}+\sqrt{-\frac{1}{16}}\tan\left(\sqrt{-\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)$$
$$-\frac{1}{4i\tan\left(\sqrt{-\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)},$$

$$V_{5,9}(x,y,t)=\frac{1}{2}+\sqrt{-\frac{1}{16}}\cot\left(\sqrt{-\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)$$
$$-\frac{1}{4i\cot\left(\sqrt{-\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)},$$

$$V_{5,10}(x,y,t)=\frac{1}{2}+\sqrt{-\frac{1}{16}}\left(\begin{array}{l}\tan\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\+\sec\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\end{array}\right)$$
$$+\frac{1}{4i\left(\begin{array}{l}\tan\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\+\sec\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\end{array}\right)},$$

$$V_{5,11}(x,y,t)=\frac{1}{2}-\frac{\sqrt{-\frac{1}{16}}\left(1+\tan\left(\sqrt{-\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{\left(1-\tan\left(\sqrt{-\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}-\frac{\left(1-\tan\left(\sqrt{-\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{4i\left(1-\tan\left(\sqrt{-\frac{\sigma}{32a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)},$$

$$V_{5,12}(x,y,t)=\frac{1}{2}+\frac{\sqrt{-\frac{1}{16}}\left(4-5\cos\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{\left(3+5\sin\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}+\frac{\left(3+5\sin\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{4i\left(4-5\cos\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)},$$

$$V_{5,13}(x,y,t)=\frac{1}{2}+\frac{\sqrt{\frac{2a^2}{\sigma}}\left(\sqrt{-\frac{\sigma(a^2-b^2)}{32a^2}}-\sqrt{-\frac{\sigma}{32}}\cos\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)}{\left(a\sin\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)+b\right)}+\frac{\sqrt{2\sigma}\left(a\sin\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)+b\right)}{32a\left(\sqrt{-\frac{\sigma(a^2-b^2)}{32a^2}}-\sqrt{-\frac{\sigma}{32}}\cos\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\right)},$$

$$V_{5,14}(x,y,t)=\frac{1}{2}-\sqrt{\frac{1}{16}}+\frac{\frac{1}{2}a}{\left(\begin{array}{l}a+\cos\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\-i\sin\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\end{array}\right)}-\frac{1}{4-\frac{8a}{\left(\begin{array}{l}a+\cos\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\\-i\sin\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)\right)\end{array}\right)}}.$$

Family 3: When $\varphi=0$, then Eq. (1.1) has rational solution:

$$V_{5,15}(x,y,t)=\frac{1}{2}-\frac{\sqrt{\frac{2a^2}{\sigma}}}{\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right)}-\frac{\sqrt{2\sigma}}{32a}\left(ax+by-\frac{3a^2\sqrt{2\sigma}+2b^2\eta t}{2a}\right).$$

For **Set 6**, substituting the values of constants into Eq. (3.4) and Eq. (3.1) along with Eq. (2.6)-Eq. (2.20) and simplifying, yields following solutions, respectively:
Family 1: When $\varphi<0$ and the hyperbolic solutions of Eq. (1.1) are given as follows:

$$V_{6,1}(x,y,t)=-\frac{1}{2}-\frac{1}{2\tanh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta)t}{2\sqrt{\sigma}a}\right)\right)},$$

$$V_{6,2}(x,y,t)=-\frac{1}{2}-\frac{1}{2\coth\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta)t}{2\sqrt{\sigma}a}\right)\right)},$$

$$V_{6,3}(x,y,t)=-\frac{1}{2}-\frac{1}{2\left(\begin{array}{l}\tanh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\\+i\,\mathrm{sech}\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\end{array}\right)},$$

$$V_{6,4}(x,y,t)=-\frac{1}{2}+\frac{\sqrt{2\sigma}\left(1+\sqrt{\frac{\sigma}{8a^2}}\tanh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\right)}{8a\left(-\frac{\sigma}{8a^2}-\sqrt{\frac{\sigma}{8a^2}}\tanh\left(\sqrt{\frac{\sigma}{8a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\right)},$$

$$V_{6,5}(x,y,t)=-\frac{1}{2}+\frac{\left(3+4\sinh\left(\sqrt{\frac{\sigma}{4a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\right)}{2\left(5-4\cosh\left(\sqrt{\frac{\sigma}{4a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\right)},$$

$$V_{6,6}(x,y,t)=-\frac{1}{2}+\frac{\sqrt{2\sigma}\left(a\sinh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)+b\right)}{8a\left(\sqrt{\frac{\sigma(a^2+b^2)}{8a^2}}-\sqrt{\frac{\sigma}{8}}\cosh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\right)},$$

$$V_{6,7}(x,y,t)=-\frac{1}{2}+\frac{1}{2-\frac{4a}{\left(\begin{array}{l}a+\cosh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\\ \quad-\sinh\left(\sqrt{\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\end{array}\right)}}.$$

Family 2: Given $\varphi>0$, using the method obtained the following trigonometric solutions for Eq. (1.1):

$$V_{6,8}(x,y,t)=-\frac{1}{2}+\frac{1}{2i\tan\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)},$$

$$V_{6,9}(x,y,t)=-\frac{1}{2}-\frac{1}{2i\cot\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)},$$

$$V_{6,10}(x,y,t)=-\frac{1}{2}+\frac{1}{2i\left(\begin{array}{l}\tan\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\\ +\sec\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\end{array}\right)},$$

$$V_{6,11}(x,y,t)=-\frac{1}{2}-\frac{\left(1+\tan\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\right)}{2i\left(1-\tan\left(\sqrt{-\frac{\sigma}{8a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\right)},$$

$$V_{6,12}(x,y,t)=-\frac{1}{2}+\frac{\left(3+5\sin\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\right)}{2i\left(4-5\cos\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\right)},$$

$$V_{6,13}(x,y,t)=-\frac{1}{2}+\frac{\sqrt{2\sigma}\left(a\sin\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)+b\right)}{2a\left(\sqrt{-\frac{\sigma(a^2-b^2)}{8a^2}}-\sqrt{-\frac{\sigma}{8}}\cos\left(\sqrt{-\frac{\sigma}{2a^2}}\left(ax+by-\frac{\left(-3\sqrt{2}a^2\sigma+2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a}\right)\right)\right)},$$

$$V_{6,14}(x,y,t) = -\frac{1}{2} - \frac{1}{2 - \frac{4a}{\left(a + \cos\left(\sqrt{-\frac{\sigma}{2a^2}} \left(ax + by - \frac{\left(-3\sqrt{2}a^2\sigma + 2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a} \right) \right) - i\sin\left(\sqrt{-\frac{\sigma}{2a^2}} \left(ax + by - \frac{\left(-3\sqrt{2}a^2\sigma + 2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a} \right) \right) \right)}}.$$

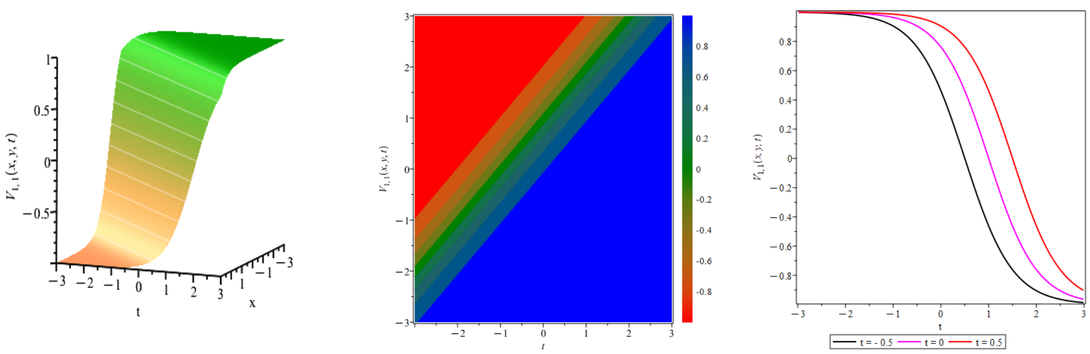
Family 3: When $\varphi = 0$, then Eq. (1.1) has rational solution:

$$V_{6,15}(x,y,t) = -\frac{1}{2} - \frac{\sqrt{2\sigma}}{8a\left(ax + by - \frac{\left(-3\sqrt{2}a^2\sigma + 2\sqrt{\sigma}b^2\eta\right)t}{2\sqrt{\sigma}a} \right)}.$$

4. Discussion of the results achieved and graphical representation

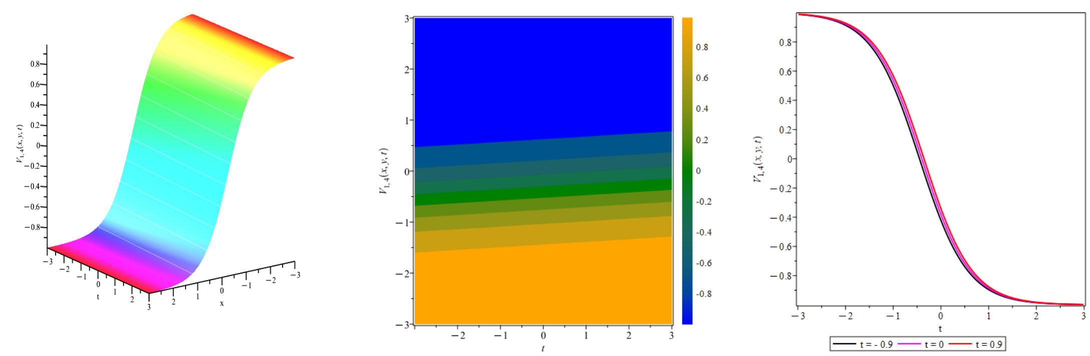
The (2+1)-dimensional CIE is solved using the canonical-like transformation approach and the trial equation method [24]. Sulaiman et al.'s is focused on finding new lump solutions to significant equations with variable coefficients, specifically the (2+1)-dimensional CIE [19]. In this study, METEM is utilized for obtaining several soliton solutions for the (2+1)-dimensional CIE, like trigonometric, hyperbolic, and rational solutions. METEM offers deeper analysis and gives an easier and powerful framework for researching complex occurrences like impact interactions. 3D, contour, and 2D graphs are also provided to help comprehend the patterns of these solutions.

A kink soliton is usually a type of soliton that moves at a certain speed and conserves its energy. Its graph shows a sharp transition from an initial low value to a high value (or vice versa). This can show the movement of a particle in a potential trough or a phase change in a field. Bright solitons are solutions in which the intensity is sharply higher, with a pronounced peak in the centre, while dark solitons are solutions in which the intensity decreases in the centre. The difference between the two types of soliton is the behaviour of their density in the centre. Singular solitons are typically more complex and sometimes theoretically significant solutions. Such solutions usually have infinite values in a given region. Solutions with a singularity are usually cases where a wave or field grows very rapidly, indicating a breakdown point at the centre.



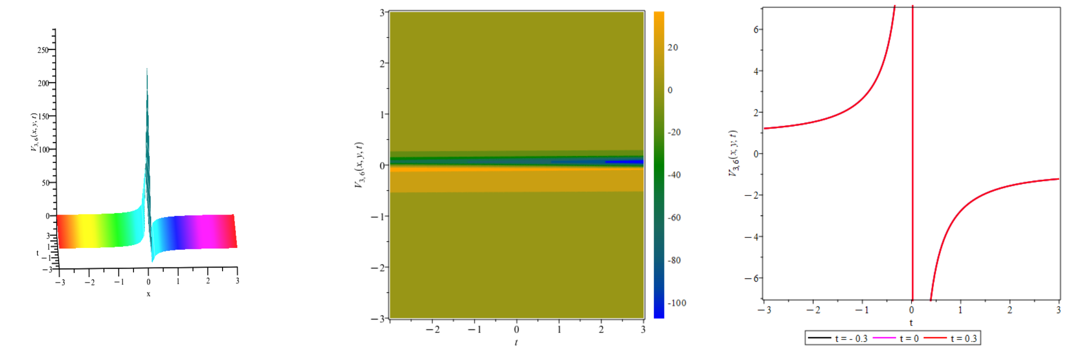
(a) 3D (b) Contour (c) 2D

Figure 4.1: The graphical explanation for $V_{1,1}(x,y,t)$ to Eq. (3.5) when $\sigma = 2$, $a = 1$, $b = -1$, $\eta = 1$, $y = 1$.



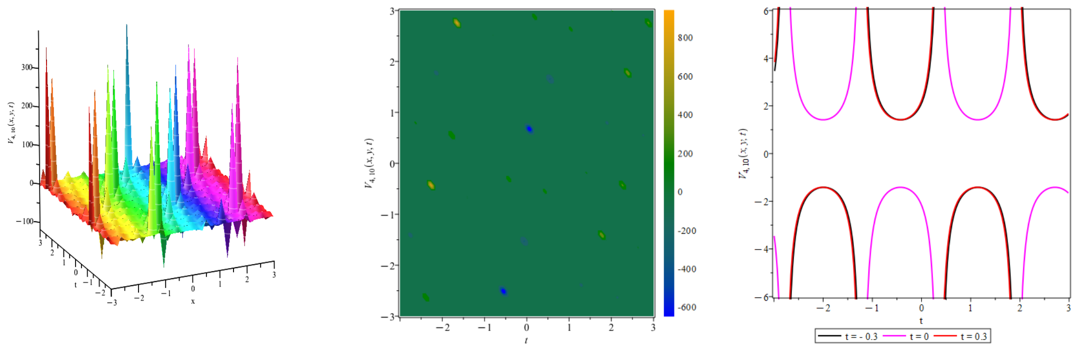
(a) 3D (b) Contour (c) 2D

Figure 4.2: The physical explanation for $V_{1,4}(x,y,t)$ to Eq. (3.6) when $\sigma = 2$, $a = 5$, $b = 1$, $\eta = 1.3$, $y = 1$.



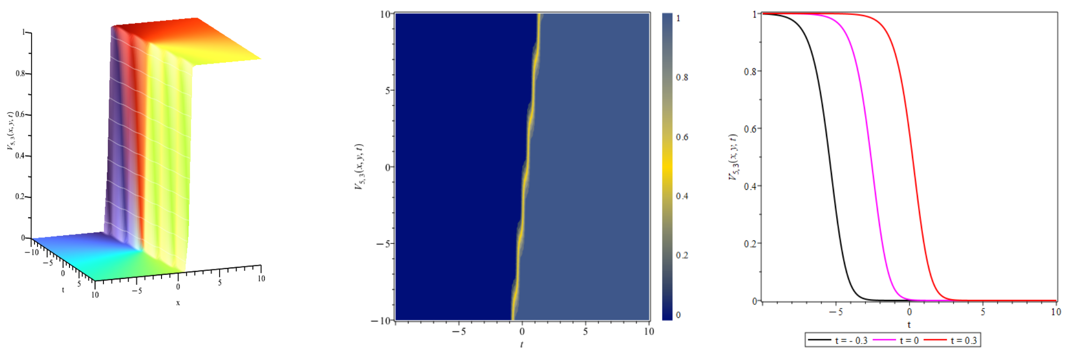
(a) 3D (b) Contour (c) 2D

Figure 4.3: The physical explanation for $V_{3,6}(x,y,t)$ to Eq. (3.7) when $\sigma = 0.3$, $a = 0.5$, $b = 0.04$, $\eta = 0.73$, $y = 1$.



(a) 3D (b) Contour (c) 2D

Figure 4.4: The graphical representation for $V_{4,10}(x,y,t)$ to Eq. (3.8) when $\sigma = 4$, $a = 1$, $b = 2$, $\eta = 1.3$, $y = 1$.



(a) 3D (b) Contour (c) 2D

Figure 4.5: The physical representation for $V_{5,3}(x,y,t)$ to Eq. (3.9) when $\sigma = 4$, $a = 1$, $b = 2$, $\eta = 1.3$, $y = 1$.

5. Conclusion

In this paper, various analytical solutions for (2+1)-dimensional CIE were found using an efficient methodology. In contrast to conventional methods used up to this point, METEM exhibits its capacity to generate novel and broadly applicable precise solutions. This manifestation demonstrates the great promise and effectiveness of the technique in solving difficult single-wave issues that are frequently encountered in mathematical physics. The approach employed here yields analytical solutions, including trigonometric, rational, and hyperbolic function solutions, to the (2+1)-dimensional CIE. Numerous phenomena, including periodic waves, kink-wave patterns, and bright and dark solitons, have been reported in relation to the (2+1)-dimensional CIE. To further explain the dynamic behavior of resource solutions, graphical representations have been created. The unique dynamic structures and features of these solutions can be fully understood through the use of 3D, contour, and 2D graphs. In the field of mathematical physics, these functions are useful to solve PDEs and offer a handy way to illustrate periodic solutions [25–28] . This specific approach has the ability to address a multitude of higher-dimensional nonlinear issues that arise in the fields of mathematics and the applied sciences [23] . As a result, it is expected to contribute to the comprehensive study and investigation of future research. The new results obtained from a wide range of dynamical structures and arbitrary parameters are expected to provide important new insights into the behavior of the gas diffusion equations in a homogeneous medium. The accuracy of these results has been ensured and extensively verified using Maple symbolic computing software.

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References

[1] H. Wang, X. Wu, X. Zheng, X. Yuan, *Model predictive current control of nine-phase open-end winding PMSMs with an online virtual vector synthesis strategy*, IEEE Trans. on Indust. Electronics, **70**(3) (2022), 2199-2208.
[2] W. B. Rabie, H. M. Ahmed, A. R. Seadawy, A. Althobaiti, *The higher-order nonlinear Schrödinger's dynamical equation with fourth-order dispersion and cubic-quintic nonlinearity via dispersive analytical soliton wave solutions*, Opt. Quantum Electron., **53** (2021), 1-25.
[3] M. A. Ravindran, K. Nallathambi, P. Vishnuram, R. S. Rathore, M. Bajaj, I. Rida, A. Alkhayyat, *A novel technological review on fast charging infrastructure for electrical vehicles: Challenges, solutions, and future research directions*, Alex. Eng. J., **82** (2023), 260-290.
[4] N. Farah, A. R. Seadawy, S. Ahmad, S. T. Rizvi, *Butterfly, S and W-shaped, parabolic, and other soliton solutions to the improved perturbed nonlinear Schrödinger equation*, Opt. Quantum Electron., **55**(1) (2023), 99.
[5] Y. Li, H. Wang, M. Trik, *Design and simulation of a new current mirror circuit with low power consumption and high performance and output impedance*, Analog Integr. Circuits Signal Process., **119**(1) (2024), 29-41.
[6] J. Y. Gao, J. Liu, H. M. Yang, H. S. Liu, G. Zeng, B. Huang, *Anisotropic medium sensing controlled by bound states in the continuum in polarization-independent metasurfaces*, Opt. Express, **31**(26) (2023), 44703-44719.
[7] D. Baleanu, S. M. Aydogn, H. Mohammadi, S. Rezapour, *On modelling of epidemic childhood diseases with the Caputo-Fabrizio derivative by using the Laplace Adomian decomposition method*, Alex. Eng. J., **59**(5) (2020), 3029-3039.
[8] F. Jarad, A. Jhangeer, J. Awrejcewicz, M. B. Riaz, M. Junaid-U-Rehman, *Investigation of wave solutions and conservation laws of generalized Calogero–Bogoyavlenskii–Schiff equation by group theoretic method*, Results Phys., **37** (2022), 105479.
[9] Y. Pandir, A. Ekin, *New solitary wave solutions of the Korteweg-de Vries (KdV) equation by new version of the trial equation method*, Electron. J. Appl. Math., **1**(1) (2023), 101-113.
[10] Y. Yıldırım, A. Biswas, A. Dakova, P. Guggilla, S. Khan, H. M. Alshehri, M. R. Belic, *Cubic-quartic optical solitons having quadratic-cubic nonlinearity by sine-Gordon equation approach*, Ukr. J. Phys. Opt., **22**(4) (2021), 255-269.
[11] B. Kopçasız, *Unveiling New Exact Solutions of the Complex-Coupled Kuralay System Using the Generalized Riccati Equation Mapping Method*, J. Math. Sci. Model., **7**(3) (2024), 146-156.
[12] F. N. K. Sağlam, S. Malik, *Various traveling wave solutions for (2+ 1)-dimensional extended Kadomtsev–Petviashvili equation using a newly created methodology*, Chaos Solitons Fractals, **186** (2024), 115318.
[13] B. Kopçasız, E. Yaşar, *Solitonic structures and chaotic behavior in the geophysical Korteweg–de Vries equation: A μ-symmetry and g'-expansion approach*, Mod. Phys. Lett. B., (2024), 2450419.
[14] B. Kopçasız, E. Yaşar, *Adaptation of Caputo residual power series scheme in solving nonlinear time fractional Schrödinger equations*, Optik, **289** (2023), 171254.
[15] H. Durur, A. Kurt, O. Tasbozan, *New travelling wave solutions for KdV6 equation using sub equation method*, Appl. Math. Nonlinear Sci., **5**(1) (2020), 455-460.
[16] V. Ala, G. Shaikhova, *Analytical solutions of nonlinear Beta fractional Schrödinger equation via Sine-Cosine method*, Lobachevskii J. Math., **43**(11) (2022), 3033-3038.
[17] B. Inan, M. S. Osman, T. Ak, D. Baleanu, *Analytical and numerical solutions of mathematical biology models: The Newell-Whitehead-Segel and Allen-Cahn equations*, Math. Methods Appl. Sci., **43**(5) (2020), 2588-2600.
[18] J. Lenells, *Traveling wave solutions of the Camassa–Holm equation*, J. Differential Equations, **217**(2) (2005), 393-430.
[19] T. A. Sulaiman, A. Yusuf, M. Alquran, *Dynamics of lump solutions to the variable coefficients (2+ 1)-dimensional Burger's and Chaffee-infante equations*, J. Geom. Phys., **168** (2021), 104315.
[20] C. Zhu, M. Al-Dossari, S. Rezapour, B. Gunay, *On the exact soliton solutions and different wave structures to the (2+ 1) dimensional Chaffee–Infante equation*, Results Phys., **57** (2024), 107431.
[21] M. A. Akbar, N. H. M. Ali, J. Hussain, *Optical soliton solutions to the (2+ 1)-dimensional Chaffee–Infante equation and the dimensionless form of the Zakharov equation*, Adv. Difference Equ., **2019**(1) (2019), 446.
[22] N. G. Ay, E. Yaşar, *The residual symmetry, Bäcklund transformations, CRE integrability and interaction solutions:(2+ 1)-dimensional Chaffee–Infante equation*, Commun. Theor. Phys., **75**(11) (2023), 115004.
[23] H. Esen, M. Ozisik, A. Secer, M. Bayram, *Optical soliton perturbation with Fokas–Lenells equation via enhanced modified extended tanh-expansion approach*, Optik, **267** (2022), 169615.
[24] Y. Mao, *Exact solutions to (2+ 1)-dimensional Chaffee–Infante equation*, Pramana, **91**(1) (2018), 9.
[25] D. Chen, Q. Wang, Y. Li, Y. Li, H. Zhou, Y. Fan, *A general linear free energy relationship for predicting partition coefficients of neutral organic compounds*, Chemosphere, **247** (2020), 125869.
[26] C. Zhu, M. Al-Dossari, N. S. A. El-Gawaad, S. A. M. Alsallami, S. Shateyi, *Uncovering diverse soliton solutions in the modified Schrödinger's equation via innovative approaches*, Results Phys., **54** (2023), 107100.
[27] D. Chen, Y. Li, X. Li, X. Hong, X. Fan, T. Savidge, *Key difference between transition state stabilization and ground state destabilization: increasing atomic charge densities before or during enzyme–substrate binding,* Chem. Sci., **13**(27) (2022), 8193-8202.
[28] M. Bilal, J. Iqbal, R. Ali, F. A. Awwad, E. A. Ismail, *Exploring families of solitary wave solutions for the fractional coupled Higgs system using modified extended direct algebraic method*, Fractal Fract., **7**(9) (2023), 653.