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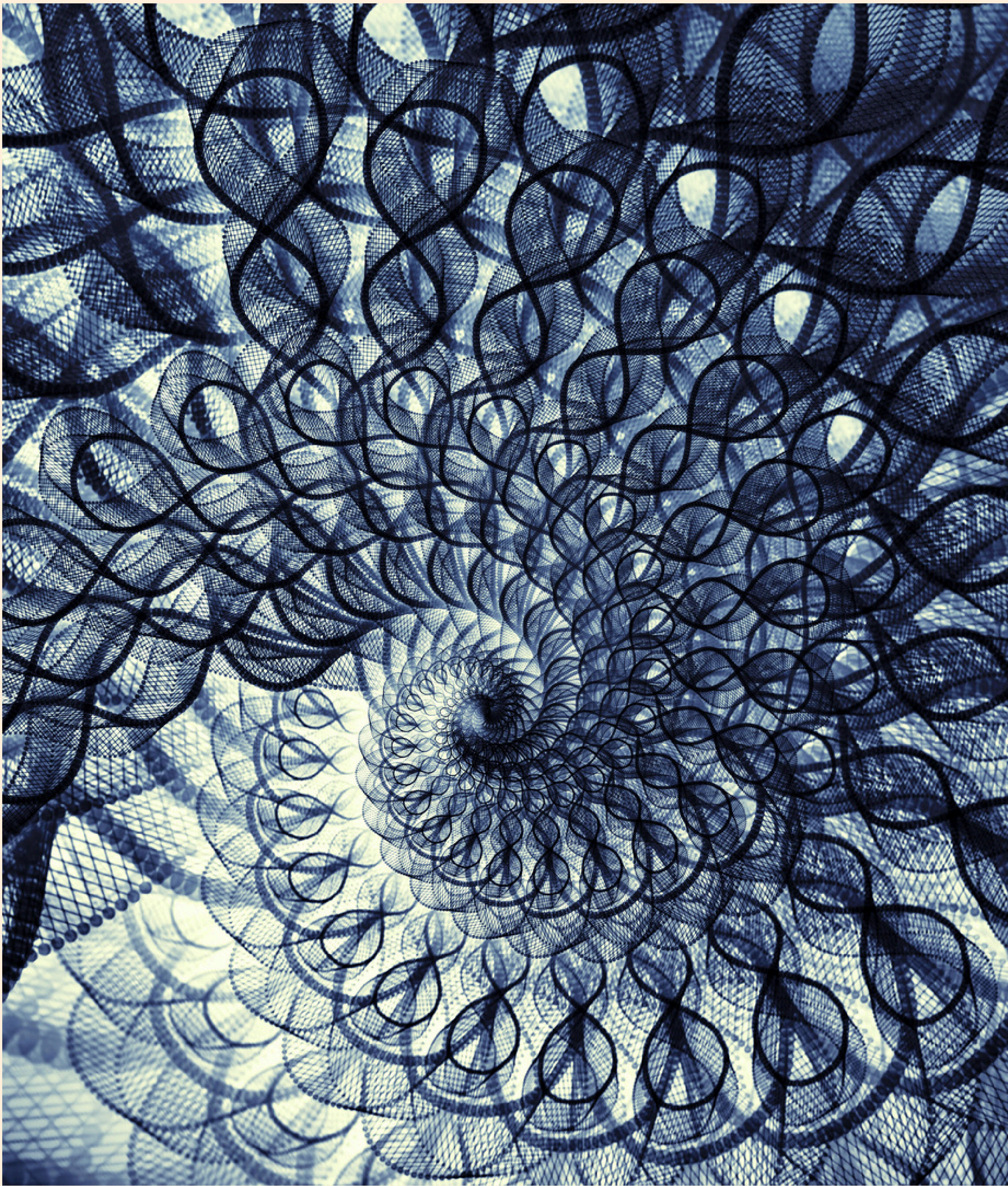

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The Generalized Binomial Transform of the Bivariate Fibonacci and Lucas p-Polynomials

Yasemin Alp

Abstract
The generalized binomial transforms of the bivariate Fibonacci $p-$polynomials and Lucas $p-$polynomials are introduced in this study. Furthermore, the generating functions of these polynomials are provided. Moreover, some relations are found for them. All results obtained are reduced to the $k-$binomial, falling binomial, rising binomial, and binomial transforms of the Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas, Fibonacci, and Lucas numbers.

Keywords: *Binomial transform, Bivariate Fibonacci $p-$polynomials, Bivariate Lucas $p-$polynomials, Generating function*

AMS Subject Classification (2020): *11B37; 11B39; 11B83; 05A15*

1. Introduction

Integer sequences play a crucial role in many fields, including mathematics. Furthermore, several special integer sequences have been studied in recent years. Especially, Fibonacci and Lucas sequences are among the number sequences widely studied. Their properties are provided in many papers [1, 2].

The generalization of these number sequences is another one of the most studied topics by researchers. Catalini introduced the generalization of bivariate Fibonacci and Lucas polynomials in [3]. Moreover, the bivariate Fibonacci and Lucas $p-$ polynomials are considered, and some properties of them are investigated in [4].

The bivariate Fibonacci $p-$polynomials are defined by the following recurrence relation:

$$F_{p,n}(x,y) = xF_{p,n-1}(x,y) + yF_{p,n-p-1}(x,y) \tag{1.1}$$

where $n > p$. In addition, the initial values are

$$F_{p,0}(x,y) = 0, F_{p,1}(x,y) = 1, F_{p,2}(x,y) = x, F_{p,3}(x,y) = x^2, \ldots, F_{p,p}(x,y) = x^{p-1}.$$

Similarly, the bivariate Lucas $p-$polynomials are presented by the following recurrence relation:

$$L_{p,n}(x,y) = xL_{p,n-1}(x,y) + yL_{p,n-p-1}(x,y) \tag{1.2}$$

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where $n > p$. Furthermore, the initial values are

$$L_{p,0}(x,y) = p+1, L_{p,1}(x,y) = x, L_{p,2}(x,y) = x^2, L_{p,3}(x,y) = x^3, \ldots, L_{p,p}(x,y) = x^p.$$

The generating functions of these polynomials are as follows:

$$g_F(t) = \sum_{i=0}^{\infty} F_{p,i}(x,y)t^i = \frac{t}{1-xt-yt^{p+1}} \tag{1.3}$$

and

$$g_L(t) = \sum_{i=0}^{\infty} L_{p,i}(x,y)t^i = \frac{1+p(1-xt)}{1-xt-yt^{p+1}} \tag{1.4}$$

respectively. The relation between them is

$$L_{p,n}(x,y) = F_{p,n+1}(x,y) + pyF_{p,n-p}(x,y)$$

in [4]. Detailed information on Fibonacci numbers, Lucas numbers, and their generalizations can be found in [1–5].

Another topic of study in mathematics is the binomial transformation; using the binomial coefficient, a new sequence can be obtained from a sequence. An integer sequence $\{a_n\}$ has the following binomial transformation.

$$b_n = \sum_{i=0}^{n} \binom{n}{i} a_i$$

in [6]. In addition, a generalization of the binomial transform is

$$b_n = \sum_{i=0}^{n} \binom{n}{i} u^{n-i} s^i a_i \tag{1.5}$$

where u and s are complex numbers. Let $f(t)$ and $h(t)$ be the generating functions of the sequences $\{a_n\}$ and $\{b_n\}$. Then, the generating function of the sequence $\{b_n\}$ is

$$h(t) = \left(\frac{1}{1-ut}\right) f\left(\frac{st}{1-ut}\right) \tag{1.6}$$

in [7]. The authors have considered special cases of the equation (1.5), which are the $k-$binomial transform, the rising $k-$binomial transform, and the falling $k-$binomial transform in [8]. The binomial transform and its properties are provided in [7, 9, 10].

Many authors have applied binomial transforms to special integer sequences. In addition, they have given numerous properties of them. In [11, 12], the authors defined the binomial transforms of the generalized $k-$Fibonacci and $k-$Fibonacci numbers and provided some identities. The falling binomial, rising binomial, $k-$binomial and binomial transforms of the modified $k-$Fibonacci-like sequence are considered in [13]. In addition, the binomial transform of the balancing polynomials is investigated, and the generating functions and summation formulas are given in [14]. Detailed information on the binomial transforms of special number sequences can be found in [15–17].

Motivated by the above papers, we apply the generalized binomial transform to the bivariate Fibonacci $p-$polynomials and Lucas $p-$polynomials. We obtain a new polynomial sequence from bivariate Fibonacci $p-$polynomials using the generalized binomial transform in Section 2. It is reduced to the falling, rising, $k-$binomial, and binomial transforms of the Jacobsthal, Pell, and Fibonacci numbers. In addition, we provide the generating function and some identities for this new polynomial sequence. In Section 3, we introduce the generalized binomial transform of the bivariate Lucas $p-$polynomials, which are reduced to the falling, rising, $k-$binomial and binomial transforms of Jacobsthal-Lucas, Pell-Lucas, and Lucas numbers. Furthermore, we find the generating function and some identities of them.

2. Main results

We utilize the generalized binomial transform on bivariate Fibonacci $p-$polynomials in this part of the study. Moreover, we present some results for this transform.

Definition 2.1. Let $w_{p,n}(x,y)$ be the generalized binomial transform of the bivariate Fibonacci $p-$polynomial. The definition is as follows:

$$w_{p,n}(x,y)=\sum_{i=0}^{n}\binom{n}{i}u^{n-i}s^{i}F_{p,i}(x,y) \tag{2.1}$$

for $u\neq 0$, $s\neq 0$ or $n\neq 0$ and $w_{p,0}(x,y)=0$.

In the following table, we can see some special cases of the generalized binomial transform of the bivariate Fibonacci $p-$polynomials.

Table 1. Special cases of $w_{p,n}(x,y)$

u	s	p	x	y	Special cases
1	1	1	x	y	The binomial transform of bivariate Fibonacci polynomials
1	1	p	x	1	The binomial transform of Fibonacci $p-$polynomials
1	1	1	x	1	The binomial transform of Fibonacci polynomials
1	1	p	1	1	The binomial transform of Fibonacci $p-$numbers
k	1	1	1	1	The falling binomial transform of Fibonacci numbers
1	k	1	1	1	The rising binomial transform of Fibonacci numbers
1	1	1	1	1	The binomial transform of Fibonacci numbers
1	1	1	$2x$	y	The binomial transform of bivariate Pell polynomials
1	1	p	$2x$	1	The binomial transform of Pell $p-$polynomials
1	1	1	$2x$	1	The binomial transform of Pell polynomials
1	1	p	2	1	The binomial transform of Pell $p-$numbers
k	1	1	2	1	The falling binomial transform of Pell numbers
1	k	1	2	1	The rising binomial transform of Pell numbers
1	1	1	2	1	The binomial transform of Pell numbers
1	1	1	x	$2y$	The binomial transform of bivariate Jacobsthal polynomials
1	1	p	1	$2y$	The binomial transform of Jacobsthal $p-$polynomials
1	1	1	1	$2y$	The binomial transform of Jacobsthal polynomials
1	1	p	1	2	The binomial transform of Jacobsthal $p-$numbers
k	1	1	1	2	The falling binomial transform of Jacobsthal numbers
1	k	1	1	2	The rising binomial transform of Jacobsthal numbers
1	1	1	1	2	The binomial transform of Jacobsthal numbers

Proposition 2.1. *The generating function for the generalized binomial transform of the bivariate Fibonacci $p-$polynomials is derived as follows:*

$$\sum_{n=0}^{\infty}w_{p,n}(x,y)t^{n}=\frac{st}{(ut-1)\left(ut+sxt+syt\left(\frac{st}{1-ut}\right)^{p}-1\right)}.$$

Proof. By using (1.6), we find

$$\sum_{n=0}^{\infty}w_{p,n}(x,y)t^{n}=\left(\frac{1}{1-ut}\right)g_{F}\left(\frac{st}{1-ut}\right).$$

The result is obtained by the equation (1.3). □

Proposition 2.2. *For $n\geq 0$, the generalized binomial transform of the bivariate Fibonacci $p-$polynomials verifies the subsequent relationship:*

$$w_{p,n+1}(x,y)=\sum_{i=0}^{n}\binom{n}{i}u^{n-i}s^{i}\left(uF_{p,i}(x,y)+sF_{p,i+1}(x,y)\right).$$

Proof. Using (2.1), we have

$$w_{p,n+1}(x,y)=\sum_{i=0}^{n+1}\binom{n+1}{i}u^{n+1-i}s^{i}F_{p,i}(x,y).$$

From the following binomial equality

$$\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1} \tag{2.2}$$

we find

$$w_{p,n+1}(x,y) = \sum_{i=1}^{n+1} \left(\binom{n}{i} + \binom{n}{i-1} \right) u^{n+1-i} s^i F_{p,i}(x,y).$$

Then

$$w_{p,n+1}(x,y) = u \sum_{i=0}^{n} \binom{n}{i} u^{n-i} s^i F_{p,i}(x,y) + s \sum_{i=0}^{n} \binom{n}{i} u^{n-i} s^i F_{p,i+1}(x,y).$$

The result can be obtained from here. □

Corollary 2.1. *n is a nonnegative integer. The generalized binomial transform of the bivariate Fibonacci $p-$polynomials satisfies the equation below:*

$$w_{p,n+1}(x,y) - u w_{p,n}(x,y) = s \sum_{i=0}^{n} \binom{n}{i} u^{n-i} s^i F_{p,i+1}(x,y). \tag{2.3}$$

Proposition 2.3. *The given equality is satisfied for $n \geq 0$.*

$$w_{p,n+1}(x,y) - (u+sx) w_{p,n}(x,y) - su^n = sy \sum_{i=1}^{n} \binom{n}{i} u^{n-i} s^i F_{p,i-p}(x,y) \tag{2.4}$$

where $w_{p,n}(x,y)$ denotes the generalized binomial transform of the bivariate Fibonacci $p-$polynomials.

Proof. Using the equality (2.2), we obtain

$$w_{p,n+1}(x,y) = \sum_{i=1}^{n} \binom{n}{i} u^{n-i} s^i \left(u F_{p,i}(x,y) + s F_{p,i+1}(x,y) \right) + su^n.$$

From (1.1), we determine

$$w_{p,n+1}(x,y) = \sum_{i=1}^{n} \binom{n}{i} u^{n-i} s^i \left((u+sx) F_{p,i}(x,y) + sy F_{p,i-p}(x,y) \right) + su^n.$$

When the last equation is adjusted, the result is found. □

This part of the study focuses on the generalized binomial transform of the bivariate Lucas $p-$polynomials. We also present some results for this new polynomial sequence.

Definition 2.2. Let $W_{p,n}(x,y)$ represent the generalized binomial transform of the bivariate Lucas $p-$polynomial. The definition is as follows:

$$W_{p,n}(x,y) = \sum_{i=0}^{n} \binom{n}{i} u^{n-i} s^i L_{p,i}(x,y) \tag{2.5}$$

for $u \neq 0$, $s \neq 0$ or $n \neq 0$ and $W_{p,0}(x,y) = p+1$.

In the table below, we provide some particular cases of the generalized binomial transform for the bivariate Lucas $p-$polynomials.

Table 2. Special cases of $W_{p,n}(x,y)$

u	s	p	x	y	Special cases
1	1	1	x	y	The binomial transform of bivariate Lucas polynomials
1	1	p	x	1	The binomial transform of Lucas $p-$polynomials
1	1	1	x	1	The binomial transform of Lucas polynomials
1	1	p	1	1	The binomial transform of Lucas $p-$numbers
k	1	1	1	1	The falling binomial transform of Lucas numbers
1	k	1	1	1	The rising binomial transform of Lucas numbers
1	1	1	1	1	The binomial transform of Lucas numbers
1	1	1	$2x$	y	The binomial transform of bivariate Pell-Lucas polynomials
1	1	p	$2x$	1	The binomial transform of Pell-Lucas $p-$polynomials
1	1	1	$2x$	1	The binomial transform of Pell-Lucas polynomials
1	1	p	2	1	The binomial transform of Pell-Lucas $p-$numbers
k	1	1	2	1	The falling binomial transform of Pell-Lucas numbers
1	k	1	2	1	The rising binomial transform of Pell-Lucas numbers
1	1	1	2	1	The binomial transform of Pell-Lucas numbers
1	1	1	x	$2y$	The binomial transform of bivariate Jacobsthal-Lucas polynomials
1	1	p	1	$2y$	The binomial transform of Jacobsthal-Lucas $p-$polynomials
1	1	1	1	$2y$	The binomial transform of Jacobsthal-Lucas polynomials
1	1	p	1	2	The binomial transform of Jacobsthal-Lucas $p-$numbers
k	1	1	1	2	The falling binomial transform of Jacobsthal-Lucas numbers
1	k	1	1	2	The rising binomial transform of Jacobsthal-Lucas numbers
1	1	1	1	2	The binomial transform of Jacobsthal-Lucas numbers

Proposition 2.4. *The generating function for the generalized binomial transform of $L_{p,n}(x,y)$ is*

$$\sum_{n=0}^{\infty} W_{p,n}(x,y)t^n = \frac{(u+pu+psx)t-p-1}{(1-ut)\left(ut+sxt+syt\left(\frac{st}{1-ut}\right)^p-1\right)}.$$

Proof. Applying (1.6), it follows that

$$\sum_{n=0}^{\infty} W_{p,n}(x,y)t^n = \left(\frac{1}{1-ut}\right) g_L\left(\frac{st}{1-ut}\right).$$

The result is determined using the equation (1.4). □

Proposition 2.5. *The subsequent relation holds for $n \geq 0$.*

$$W_{p,n+1}(x,y) = \sum_{i=0}^{n} \binom{n}{i} u^{n-i} s^i \left(uL_{p,i}(x,y) + sL_{p,i+1}(x,y)\right).$$

Proof. Based on (2.5), we have

$$W_{p,n+1}(x,y) = \sum_{i=0}^{n+1} \binom{n+1}{i} u^{n+1-i} s^i L_{p,i}(x,y).$$

By using (2.2), we find

$$W_{p,n+1}(x,y) = u^{n+1}(p+1) + \sum_{i=1}^{n+1} \left(\binom{n}{i} + \binom{n}{i-1}\right) u^{n+1-i} s^i L_{p,i}(x,y).$$

Thus

$$W_{p,n+1}(x,y) = u\sum_{i=0}^{n} \binom{n}{i} u^{n-i} s^i L_{p,i}(x,y) + s\sum_{i=0}^{n} \binom{n}{i} u^{n-i} s^i L_{p,i+1}(x,y).$$

The result can be derived from the last step. □

Corollary 2.2. *For $n \geq 0$, the following relationship of the generalized binomial transform of the bivariate Lucas $p-$polynomials holds.*

$$W_{p,n+1}(x,y) - uW_{p,n}(x,y) = s\sum_{i=0}^{n}\binom{n}{i}u^{n-i}s^{i}L_{p,i+1}(x,y).$$

Proposition 2.6. *n is a nonnegative integer and $W_{p,n}(x,y)$ denotes the generalized binomial transform of the bivariate Lucas $p-$polynomials. In that case, we obtain*

$$W_{p,n+1}(x,y) - (u+sx)W_{p,n}(x,y) + xpsu^{n} = sy\sum_{i=1}^{n}\binom{n}{i}u^{n-i}s^{i}L_{p,i-p}(x,y).$$

Proof. Considering (2.5), we get

$$W_{p,n+1}(x,y) = \sum_{i=1}^{n}\binom{n}{i}u^{n-i}s^{i}\left(uL_{p,i}(x,y) + sL_{p,i+1}(x,y)\right) + u^{n}(u(p+1)+sx).$$

From (1.2), we have

$$\begin{aligned} W_{p,n+1}(x,y) = &\sum_{i=1}^{n}\binom{n}{i}u^{n-i}s^{i}\left((u+sx)L_{p,i}(x,y) + syL_{p,i-p}(x,y)\right) \\ &+ u^{n}(u(p+1)+sx). \end{aligned}$$

When the last equation is adjusted, the result is found. □

The relation between $w_{p,n}(x,y)$ and $W_{p,n}(x,y)$ is given in the following proposition.

Proposition 2.7. *Let $w_{p,n}(x,y)$ and $W_{p,n}(x,y)$ denote the generalized binomial transforms of the bivariate Fibonacci and Lucas $p-$polynomials. Then, we find*

$$sW_{n}(x,y) = (p+1)w_{n+1}(x,y) - (u+pu+psx)w_{n}(x,y).$$

Proof. If both sides of the equality (2.4) are multiplied by p, we get

$$pw_{p,n+1}(x,y) - p\left((u+sx)w_{p,n}(x,y) + su^{n}\right) = psy\sum_{i=1}^{n}\binom{n}{i}u^{n-i}s^{i}F_{p,i-p}(x,y).$$

From (1), we obtain

$$pw_{p,n+1}(x,y) - p\left((u+sx)w_{p,n}(x,y) + su^{n}\right) = s\sum_{i=1}^{n}\binom{n}{i}u^{n-i}s^{i}(L_{p,i}(x,y) - F_{p,i+1}(x,y)).$$

Hence

$$\begin{aligned} pw_{p,n+1}(x,y) - p(u+sx)w_{p,n}(x,y) = &s\sum_{i=0}^{n}\binom{n}{i}u^{n-i}s^{i}L_{p,i}(x,y) \\ &-s\sum_{i=0}^{n}\binom{n}{i}u^{n-i}s^{i}F_{p,i+1}(x,y). \end{aligned}$$

The result can be derived using the equality (2.3).

□

3. Conclusion

In the paper presented, we focus on applying the generalized binomial transforms to the bivariate Fibonacci and Lucas $p-$polynomials. Moreover, the generating functions and some identities are obtained for these transforms. Note that all results obtained are reduced to the falling, rising, $k-$binomial, and binomial transforms of the Jacobsthal, Jacobsthal-Lucas, Pell, Pell-Lucas, Fibonacci, and Lucas numbers. These particular cases are shown in Table 1 and Table 2. It would be an intriguing study to investigate the Hankel and Catalan transformations of the bivariate Fibonacci and Lucas $p-$polynomials. Additionally, relations among them can be found.

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Hankel Determinants of Logarithmic Coefficients for the Class of Bounded Turning Functions Associated with Lune Domain

Bilal Şeker*, Bilal Çekiç, Sevtap Sümer and Onur Akçiçek

Abstract
In this paper, we first obtained some initial logarithmic coefficient bounds on a subclass of bounded turning functions $\mathcal{R}_{\mathcal{L}}$ related to a lune domain. For functions belonging to this class, we determined the sharp bounds for the second Hankel determinant of logarithmic coefficients $H_{2,1}(F_f/2)$ of bounded turning functions related to a lune domain. Finally, we calculated the bounds of third Hankel determinant of logarithmic coefficients $H_{3,1}(F_f/2)$ of bounded turning functions associated with a lune domain.

Keywords: *Bounded turning functions, Hankel determinant, Logarithmic coefficients*

AMS Subject Classification (2020): *30C45; 30C50*

1. Introduction

The study of analytic and univalent functions in the unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ remains a foundational and dynamic area within geometric function theory. Denoted typically by $\mathcal{A}$, the class of analytic functions in $\mathbb{U}$ can be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

where the normalization conditions $f(0) = f'(0) - 1 = 0$ are satisfied. A function $f \in \mathcal{A}$ that is also injective in $\mathbb{U}$ belongs to the distinguished class $\mathcal{S}$ of univalent functions. The exploration of univalent functions plays a central role in complex analysis due to their rich structural properties and applications in diverse mathematical and physical theories.

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A fundamental concept in this domain is the notion of subordination: a function $f(z)$ is subordinate to $g(z)$, written $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ on $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$. Subordination has been instrumental in defining various significant subclasses of $\mathcal{S}$, such as starlike, convex, and bounded turning functions.

Of particular interest is the Carathéodory class $\mathcal{P}$, consisting of functions of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k, \qquad (z \in \mathbb{U}) \tag{1.2}$$

which satisfy $Rep(z) > 0$ in $\mathbb{U}$ and $p(0) = 1$. Functions in $\mathcal{P}$ often serve as comparison functions in subordination relations.

In their seminal work, Ma and Minda [1] introduced generalized subclasses of $\mathcal{S}$ through the use of subordination with a function φ mapping $\mathbb{U}$ onto domains symmetric about the real axis and starlike with respect to $\varphi(0) = 1$. These generalized classes include:

- The Ma-Minda starlike functions $\mathcal{S}^*(\varphi)$,
- The Ma-Minda convex functions $\mathcal{C}(\varphi)$, and
- The Ma-Minda bounded turning functions $\mathcal{R}(\varphi)$,

with respective subordination conditions involving $\frac{zf'(z)}{f(z)}$, $\frac{(zf'(z))'}{f'(z)}$, and $f'(z)$.

Recently, many authors [2, 3] have described the Ma-Minda type functions or subordinations with different regions such as cardioid, lune and nephroid [4–6].

The Logarithmic coefficients, introduced through the expansion

$$F_f(z) := \log \frac{f(z)}{z} = 2\sum_{k=1}^{\infty} \gamma_k z^k, \qquad (z \in \mathbb{U}) \tag{1.3}$$

have attracted considerable attention for their role in characterizing the geometric properties of univalent functions. γ_k are called logarithmic coefficients [7]. Although for the Koebe function $k(z) = z(1-z)^{-2}$ the coefficients satisfy $\gamma_k = \frac{1}{k}$, it is known that for general $f \in \mathcal{S}$, sharp bounds for γ_k are not completely determined beyond γ_1, γ_2 and γ_3. Over the past few years, numerous researchers [8–10] have sought to establish upper bounds for the logarithmic coefficients associated with select subclasses within the class of univalent functions.

The Hankel determinant of $f \in \mathcal{A}$ the function for $q, n \in \mathbb{N}$, denoted by $H_{q,n}(f)$, is defined as follows:

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}$$

have emerged as a significant object of study due to their connections to the coefficient problem and geometric function properties. In particular, the second and third-order Hankel determinants, $H_{2,1}(f)$ and $H_{3,1}(f)$, are closely related to coefficient functionals that measure the deviation from extremal functions.

The Hankel determinant

$$H_{2,1}(f) = a_3 - a_2^2$$

is the recognized as the Fekete-Szegö functional [11]. The second Hankel determinant $H_{2,2}(f)$ is represented by

$$H_{2,2}(f) = a_2 a_4 - a_3^2.$$

Determining the upper bound of $|H_{q,n}(f)|$ for various subclasses of $\mathcal{A}$ is a fascinating and well-studied problem in the field of Geometric Function Theory. Several authors have successfully derived sharp upper bounds for $|H_{2,2}(f)|$ and $|H_{3,1}(f)|$ within specific subclasses of analytic functions, as the referenced in [12–23].

The logarithmic coefficients of analytic univalent functions are crucial for understanding the behaviour of a function in the boundary region where it is defined. The logarithmic coefficients of analytic univalent functions are employed in the investigation of a range of properties, including the sharp bounds of the determinants of Hankel matrices, growth estimates for the moduli of these functions and their derivatives.

Several authors have contributed to this growing field. Duren and Leung[7] initially investigated logarithmic coefficients and their impact on univalent function theory. Subsequent studies, such as those by Girela[24], Obradović [25] , and Ponnusamy [26], refined the understanding of logarithmic coefficients and established bounds in various contexts. More recent work by Kowalczyk and Lecko [27, 28] has expanded the theory to include subclasses associated with special geometric domains, such as the cardioid and nephroid, thereby providing broader applicability and sharper estimates for Hankel determinants. The question of computing sharp bounds for strongly starlike and strongly convex functions was addressed by Eker et al.[29]. Additionally, upper bounds for the second Hankel determinant of logarithmic coefficients for various subclasses of $\mathcal{S}$ were obtained by et al. [30], Eker et al. [31], Shi et al.[32] and Mandal et al. [6].

For a function $f \in \mathcal{S}$, as defined in equation (1.1), differentiating equation (1.3) allows the logarithmic coefficients to be determined.

$$\gamma_1 = \frac{1}{2}a_2, \tag{1.4}$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \tag{1.5}$$

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right), \tag{1.6}$$

$$\gamma_4 = \frac{1}{2}\left(a_5 - a_2a_4 + a_2^2a_3 - \frac{1}{2}a_3^2 - \frac{1}{4}a_2^4\right), \tag{1.7}$$

$$\gamma_5 = \frac{1}{2}\left(a_6 - a_2a_5 - a_3a_4 + a_2a_3^2 + a_2^2a_4 - a_2^3a_3 + \frac{1}{5}a_2^5\right). \tag{1.8}$$

In light of the ideas presented above, we propose the study of the Hankel determinant, whose entries are logarithmic coefficients of f, namely

$$H_{q,n}(f) := \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2(q-1)} \end{vmatrix}.$$

From qth-order Hankel determinant $H_{q,n}(F_f/2)$ whose entries are the logarithmic coefficients of f, one can easily deduce that

$$H_{2,1}(F_f/2) = \gamma_1\gamma_3 - \gamma_2^2, \tag{1.9}$$
$$H_{2,2}(F_f/2) = \gamma_2\gamma_4 - \gamma_3^2,$$

and

$$H_{3,1}(F_f/2) = \gamma_3\left(\gamma_2\gamma_4 - \gamma_3^2\right) - \gamma_4\left(\gamma_1\gamma_4 - \gamma_2\gamma_3\right) + \gamma_5\left(\gamma_1\gamma_3 - \gamma_2^2\right). \tag{1.10}$$

Morever, if $f \in \mathcal{S}$, then for $f_\theta \in \mathcal{S}$, $\theta \in \mathbb{R}$, defined as

$$f_\theta(z) := e^{-i\theta}f(e^{i\theta}z) \qquad (z \in \mathbb{U}),$$

we find that

$$H_{2,1}\left(F_{f_\theta}/2\right) = e^{4i\theta}H_{2,1}\left(F_f/2\right)$$

and

$$H_{2,2}\left(F_{f_\theta}/2\right) = e^{6i\theta}H_{2,2}\left(F_f/2\right).$$

Given the importance of bounded turning functions, a particular subclass $\mathcal{R}_{\mathcal{L}}$ has been defined, wherein functions satisfy

$$\mathcal{R}_{\mathcal{L}} := \left\{f \in \mathcal{A} : f'(z) \prec z + \sqrt{1+z^2}, \quad z \in \mathbb{U}\right\}, \tag{1.11}$$

where branch of the square root is selected such that $\wp(0) = 1$. Geometrically, $\mathcal{R}_{\mathcal{L}}$ functions are associated with a domain bounded by a lune shape, and they naturally generalize the notion of bounded turning to a setting influenced by complex geometric regions.

The motivation of this paper is twofold. First, we aim to establish new sharp bounds for the second Hankel determinant $|H_{2,1}(F_f/2)|$ of logarithmic coefficients for functions belonging to $\mathcal{R}_{\mathcal{L}}$. Second, we endeavor to determine precise estimates for the third Hankel determinant $|H_{3,1}(F_f/2)|$ within the same class. Such results contribute both to the ongoing exploration of Hankel determinants and to the broader understanding of logarithmic coefficients in function theory.

Moreover, we build upon existing lemmas related to the structure of $\mathcal{P}$, utilize inequalities concerning the coefficients c_k , and employ techniques from complex analysis to develop our main theorems. By enriching the analytical framework and refining the known bounds, this study adds a novel perspective to the intricate relationship between geometric function theory and coefficient-based functionals. For our consideration we need the next lemmas.

For our consideration we need the next lemmas.

Lemma 1.1. *[33] If $p \in \mathcal{P}$ is of the form (1.2) with $c_1 \geq 0$, then*

$$\begin{aligned} c_1 &= 2d_1, \\ c_2 &= 2d_1^2 + 2\left(1-d_1^2\right)d_2, \\ c_3 &= 2d_1^3 + 4\left(1-d_1^2\right)d_1d_2 - 2\left(1-d_1^2\right)d_1d_2^2 + 2\left(1-d_1^2\right)\left(1-|d_2|^2\right)d_3 \end{aligned} \tag{1.12}$$

for some $d_1 \in [0,1]$ and $d_2, d_3 \in \overline{\mathbb{U}} := \{z \in \mathbb{C} : |z| \leq 1\}$.

For $d_1 \in \mathbb{U}$ and $d_2 \in \partial\mathbb{U} := \{z \in \mathbb{C} : |z| = 1\}$ there is a unique function $p \in \mathcal{P}$ with c_1 and c_2 as in (1.12), namely

$$p(z) = \frac{1 + \left(\overline{d_1}d_2 + d_1\right)z + d_2z^2}{1 + \left(\overline{d_1}d_2 - d_1\right)z + d_2z^2}, \qquad (z \in \mathbb{U}).$$

Lemma 1.2. *If $p \in \mathcal{P}$ is of the form (1.2) then the following inequalities hold*

$$|c_n| \leq 2 \quad \text{for} \quad n \geq 1, \tag{1.13}$$

$$|c_{n+k} - \mu c_n c_k| < 2 \quad \text{for} \quad 0 \leq \mu \leq 1, \tag{1.14}$$

$$|c_m c_n - c_k c_l| \leq 4 \quad \text{for} \quad m+n = k+l, \tag{1.15}$$

$$|c_{n+2k} - \mu c_n c_k^2| \leq 2(1+2\mu) \quad \text{for} \quad \mu \in \mathbb{R}, \tag{1.16}$$

and for complex number λ, we have

$$|c_2 - \lambda c_1^2| \leq 2\max(1, |\lambda - 1|). \tag{1.17}$$

For the inequalities in (1.13), (1.14), (1.15) and (1.16), we refer to [34]. Also, see [35] for the inequality (1.17).

Lemma 1.3. *[36] Let $p \in \mathcal{P}$ and has the form (1.2), then*

$$|Kc_1^3 - Lc_1c_2 + Mc_3| \leq 2|K| + 2|L - 2K| + 2|K - L + M|.$$

Lemma 1.4. *[37] Given real numbers A, B, C, let*

$$Y(A,B,C) := \max\left\{\left|A + Bz + Cz^2\right| + 1 - |z|^2 : z \in \overline{\mathbb{U}}\right\}.$$

I. *If $AC \geq 0$, then*

$$Y(A,B,C) = \begin{cases} |A| + |B| + |C|, & |B| \geq 2(1-|C|), \\ 1 + |A| + \dfrac{B^2}{4(1-|C|)}, & |B| < 2(1-|C|). \end{cases}$$

II. *If $AC < 0$, then*

$$Y(A,B,C) = \begin{cases} 1 - |A| + \dfrac{B^2}{4(1-|C|)}, & -4AC(\frac{1}{C^2} - 1) \leq B^2 \wedge |B| < 2(1-|C|), \\ 1 + |A| + \dfrac{B^2}{4(1+|C|)}, & B^2 < \min\{4(1+|C|)^2, -4AC(\frac{1}{C^2} - 1)\}, \\ R(A,B,C), & \text{otherwise}, \end{cases}$$

where

$$R(A,B,C) := \begin{cases} |A|+|B|-|C|, & |C|(|B|+4|A|) \le |AB|, \\ -|A|+|B|+|C|, & |AB| \le |C|(|B|-4|A|), \\ (|A|+|C|)\sqrt{1-\dfrac{B^2}{4AC}}, & \text{otherwise.} \end{cases}$$

The objective of this paper to provide the sharp bounds for $|H_{2,1}(F_f/2)|$ for bounded turning functions in the open unit disk. In addition, we intended to find the sharp bound of $|H_{3,1}(F_f/2)|$ for the class bounded turning functions.

The present work seeks to address a gap in the literature concerning bounded turning functions associated with non-circular symmetric domains, particularly the lune domain. The lune, characterized by its asymmetry and curvature variation, offers a richer geometric environment than simple circular or cardioidal regions, necessitating a more nuanced analysis of subordination and coefficient behavior. The determination of sharp bounds for second and third-order Hankel determinants within this framework not only enhances theoretical understanding but also supports practical computations in applied sciences. In engineering fields such as signal processing, the stability and distortion behavior of systems modeled by analytic functions can benefit directly from tight coefficient bounds. Similarly, in quantum mechanics and potential theory, conformal mappings associated with bounded domains are critical, and accurate coefficient bounds ensure better physical modeling. Future directions include the study of higher-order determinants, variations under perturbations of the domain, and extension to more complex multi-connected regions, paving the way for deeper theoretical insights and cross-disciplinary applications.

2. Logarithmic coefficients for bounded turning function associated with a lune domain

Theorem 2.1. *If $f \in \mathcal{R}_{\mathcal{L}}$ and it has the form given in (1.1), then*

$$|\gamma_1| \le \frac{1}{4},$$

$$|\gamma_2| \le \frac{1}{6},$$

$$|\gamma_3| \le \frac{1}{8}, \tag{2.1}$$

$$|\gamma_4| \le \frac{607}{2304}, \tag{2.2}$$

$$|\gamma_5| \le \frac{1973}{5760}. \tag{2.3}$$

The functions listed below illustrate the sharpness of the aforementioned first three inequalities:

$$f_1(z) = \int_0^z \left(t + \sqrt{1+t^2}\right)dt$$
$$f_2(z) = \int_0^z \left(t^2 + \sqrt{1+t^4}\right)dt$$
$$f_3(z) = \int_0^z \left(t^3 + \sqrt{1+t^6}\right)dt.$$

Proof. Let $f \in \mathcal{R}_{\mathcal{L}}$ and then, by the definitions of subordination, there exists a Schwarz function $w(z)$ with the properties that

$$w(0) = 0 \quad and \quad |w(z)| \le 1$$

such that

$$f'(z) = w(z) + \sqrt{1+w^2(z)}. \tag{2.4}$$

Define the function

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots .$$

It is clear that $p(z) \in \mathcal{P}$. This implies that

$$\begin{aligned} w(z) =& \frac{p(z)-1}{p(z)+1} = \frac{1+c_1z+c_2z^2+c_3z^3+\cdots}{2+c_1z+c_2z^2+c_3z^3+\cdots} \\ =& \frac{1}{2}c_1z + \Big(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\Big)z^2 + \Big(\frac{1}{8}c_1^3 - \frac{1}{2}c_1c_2 + \frac{1}{2}c_3\Big)z^3 + \cdots . \end{aligned}$$

Now, from (2.4), we have

$$f'(z) = 1 + 2a_2z + 3a_3z^2 + 4a_4z^3 + \cdots \tag{2.5}$$

and

$$w(z)+\sqrt{1+w(z)^2} = 1+\frac{1}{2}c_1z+\frac{1}{2}\Big(c_2-\frac{1}{4}c_1^2\Big)z^2+\frac{1}{2}\Big(c_3-\frac{1}{2}c_1c_2\Big)z^3+\frac{1}{128}\Big(64c_4-16c_2^2+3c_1^4-32c_1c_3\Big)z^4+\cdots . \tag{2.6}$$

Comparing (2.5) and (2.6), we achieve

$$a_2 = \frac{1}{4}c_1, \tag{2.7}$$

$$a_3 = \frac{1}{6}\Big(c_2 - \frac{1}{4}c_1^2\Big),$$

$$a_4 = \frac{1}{8}\Big(c_3 - \frac{1}{2}c_1c_2\Big),$$

$$a_5 = \frac{1}{640}\Big(64c_4 - 16c_2^2 + 3c_1^4 - 32c_1c_3\Big),$$

$$a_6 = \frac{1}{384}\left(32c_5 + 6c_1^3c_2 - c_1^5 - 16c_1c_4 - 16c_2c_3\right). \tag{2.8}$$

Now, from (1.4) to (1.8) and (2.7) to (2.8), we obtain

$$\gamma_1 = \frac{1}{8}c_1, \tag{2.9}$$

$$\gamma_2 = \frac{1}{192}\left(16c_2 - 7c_1^2\right), \tag{2.10}$$

$$\gamma_3 = \frac{1}{384}\left(24c_3 + 3c_1^3 - 20c_1c_2\right), \tag{2.11}$$

$$\gamma_4 = \frac{1}{92160}\left(4608c_4 + 1520c_1^2c_2 - 1792c_2^2 + 11c_1^4 - 3744c_1c_3\right), \tag{2.12}$$

$$\gamma_5 = \frac{1}{92160}\Big(1176c_1^2c_3 - 115c_1^5 - 3072c_1c_4 + 3840c_5 + 140c_1^3c_2 + 1088c_1c_2^2 - 2880c_2c_3\Big). \tag{2.13}$$

Applying (1.13) to (2.9), we get

$$|\gamma_1| \leq \frac{1}{4}.$$

From (2.10) and using (1.17), we have

$$|\gamma_2| = \frac{1}{12}\left|c_2 - \frac{7}{16}c_1^2\right| \leq \frac{1}{6}\max\left\{1, \left|\frac{7}{16}-1\right|\right\} = \frac{1}{6}.$$

Applying Lemma 1.3 to the equation (2.11), we get

$$|\gamma_3| \leq \frac{1}{8}.$$

From (2.12), it follows that

$$\gamma_4 = \frac{1}{20}\Big(c_4 - \frac{1792}{4608}c_2^2\Big) + \frac{c_1}{92160}\Big(11c_1^3 + 1520c_1c_2 - 3744c_3\Big)$$

By making use of (1.14) and Lemma (1.3), along with the triangle inequality, we get

$$|\gamma_4| \leq \frac{607}{2304}.$$

If we revise the equation in (2.13), we obtain

$$\gamma_5 = \frac{1}{92160}\Big(1176c_1^2\big(c_3 - \frac{115}{1176}c_1c_1^2\big) + 3840\big(c_5 - \frac{3072}{3840}c_1c_4\big) + c_2\big(140c_1^3 + 1088c_1c_2 - 2880c_3\big)\Big).$$

Using triangle inequality along with (1.13), (1.14), (1.16) and Lemma (1.3), we get

$$|\gamma_5| \leq \frac{1973}{5760}.$$

Since

$$f_1(z) = \int_0^z \big(t + \sqrt{1+t^2}\big)dt = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots,$$

$$f_2(z) = \int_0^z \big(t^2 + \sqrt{1+t^4}\big)dt = z + \frac{1}{3}z^3 + \cdots$$

and

$$f_3(z) = \int_0^z \big(t^3 + \sqrt{1+t^6}\big)dt = z + \frac{1}{4}z^4 + \cdots,$$

from the equations (1.4) and (1.5) and (1.6) , it is easily obtained that the first three results given in the theorem are sharp. □

3. Second Hankel determinant of logarithmic coefficients for bounded turning with a lune domain

Theorem 3.1. *If $f \in \mathcal{R}_{\mathcal{L}}$, then*

$$|H_{2,1}(F_f/2)| \leq \frac{1}{36}. \tag{3.1}$$

The inequality in (3.1) is sharp.

Proof. Let $f \in \mathcal{R}_{\mathcal{L}}$ be of the form (1.1). Then by (1.11) we have

$$f'(z) = w(z) + \sqrt{1 + w^2(z)}, \qquad (z \in \mathbb{U}). \tag{3.2}$$

for some function $p \in \mathcal{P}$ of the form (1.2). So equating coefficients we obtain

$$\begin{aligned} a_2 &= \frac{1}{4}c_1, \\ a_3 &= \frac{1}{6}\Big(c_2 - \frac{1}{4}c_1^2\Big), \\ a_4 &= \frac{1}{8}\Big(c_3 - \frac{1}{2}c_1c_2\Big). \end{aligned} \tag{3.3}$$

Since the class $\mathcal{R}_{\mathcal{L}}$ and $|H_{2,1}(F_f/2)|$ are rotationally invariant, without loss of generality we may assume that $a_2 \geq 0$, so $c = c_1 \in [0, 2]$ (i.e., in view of (1.12) that $d_1 \in [0, 1]$). By using (1.5)-(1.7) and (1.9) we obtain

$$\begin{aligned} \gamma_1\gamma_3 - \gamma_2^2 &= \frac{1}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right) \\ &= -\frac{1}{2304}\Big[13d_1^4 - 8\left(1-d_1^2\right)d_1^2d_2 + 8\left(8+d_1^2\right)\left(1-d_1^2\right)d_2^2 - 72d_1\left(1-d_1^2\right)\left(1-|d_2|^2\right)d_3\Big]. \end{aligned} \tag{3.4}$$

Now, we may have the following cases on d_1.

Case 1. Suppose that $d_1 = 1$. Then by (3.4) we obtain

$$|\gamma_1\gamma_3 - \gamma_2^2| = \frac{13}{2304}.$$

Case 2. Suppose that $d_1 = 0$. Then by (3.4) we obtain

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| = \frac{1}{36}\left|d_2\right|^2 \leq \frac{1}{36}.$$

Case 3. Suppose that $d_1 \in (0, 1)$. By the fact that $|d_3| \leq 1$, applying the triangle inequality to (3.4) we can write

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \leq \frac{1}{32}\left(1 - d_1^2\right) d_1 \left(\left|A + Bd_2 + Cd_2^2\right| + 1 - \left|d_2\right|^2\right)$$

where

$$A := \frac{13d_1^3}{72\left(1 - d_1^2\right)} > 0, \quad B := -\frac{d_1}{9} < 0 \quad \text{and} \quad C := \frac{8 + d_1^2}{9d_1} > 0.$$

Since $AC > 0$, we apply the part I of Lemma 1.3.

We consider the following sub-case. Note that

$$\begin{aligned}
|B| - 2\left(1 - |C|\right) &= \frac{d_1}{9} - 2\left(1 - \frac{8 + d_1^2}{9d_1}\right) \\
&= \frac{3d_1^2 - 18d_1 + 16}{9d_1} \\
&\geq \frac{2\left(d_1 - 8\right)\left(d_1 - 1\right)}{9d_1} > 0.
\end{aligned}$$

Applying Lemma 1.3, we obtain

$$\begin{aligned}
\left|\gamma_1\gamma_3 - \gamma_2^2\right| &\leq \frac{1}{32}(1 - d_1^2)d_1\left(|A| + |B| + |C|\right) \\
&= \frac{1}{32}(1 - d_1^2)d_1\left(\frac{13d_1^3}{72(1 - d_1^2)} + \frac{2d_1}{9} + \frac{d_1^2 + 8}{9d_1}\right) \\
&= \frac{1}{2304}\left(64 - 48d_1^2 - 3d_1^4\right) \\
&\leq \frac{1}{2304}64 = \frac{1}{36}.
\end{aligned}$$

Summarizing parts from Case 1-3, it follows that the inequality (3.1) is true.

To show the sharpness, consider the function as follows

$$p\left(z\right) := \frac{1 + z^2}{1 - z^2}.$$

It is obvious that the function p is in $\mathcal{R}_{\mathcal{L}}$ with $c_1 = c_3 = 0$ and $c_2 = 2$. The corresponding function $f \in \mathcal{R}_{\mathcal{L}}$ is described by (3.2). Hence by (3.3) it follows that $a_2 = a_4 = 0$ and $a_3 = \frac{1}{3}$. From (3.4) we obtain

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| = \frac{1}{36}.$$

This completes the proof. □

4. Third Hankel determinant of logarithmic coefficients for bounded turning with a lune domain

Theorem 4.1. *If $f \in \mathcal{R}_{\mathcal{L}}$, then*

$$\left|\gamma_2\gamma_4 - \gamma_3^2\right| \leq \frac{4247}{69120}. \tag{4.1}$$

Proof. From (2.10), (2.11) and (2.12)

$$\gamma_2\gamma_4-\gamma_3^2=\frac{1}{17694720}\Big(-3936c_1^4c_2+1157c_1^6+28672c_2^3+11136c_1^2c_2^2 73728c_2c_4+32256c_1^2c_4-8928c_1^3c_3-55296c_1c_2c_3+69120c_3^2\Big)$$

Rearranging the above and applying triangle inequality, we get

$$\begin{aligned}\left|\gamma_2\gamma_4-\gamma_3^2\right| \leq \frac{1}{17694720}\Big(&3936|c_1|^4\Big|c_2-\frac{1157}{3936}c_1^2\Big|+28672|c_2|^2\Big|c_2+\frac{11136}{28672}c_1^2\Big|\\ &+73728|c_4|\Big|c_2-\frac{32256}{73728}c_1^2\Big|+|c_3|\Big|-8928c_1^3-55296c_1c_2+69120c_3\Big|\Big)\end{aligned}$$

Using (1.13), (1.17) and Lemma 1.3, we get the required result. □

Theorem 4.2. *If $f\in\mathcal{R}_{\mathcal{L}}$, then*

$$\left|\gamma_1\gamma_4-\gamma_2\gamma_3\right|<\frac{103}{1920}. \tag{4.2}$$

Proof. From (2.10), (2.11) and (2.12)

$$\gamma_1\gamma_4-\gamma_2\gamma_3=\frac{1}{737280}\Big(-360c_1^3c_2+221c_1^5+4608c_1c_4-2064c_1^2c_3-3840c_2c_3+1408c_1c_2^2\Big)$$

Rearranging the above and applying triangle inequality, we get

$$\left|\gamma_1\gamma_4-\gamma_2\gamma_3\right|\leq\frac{1}{737280}\Big(360|c_1|^3\Big|c_2-\frac{221}{360}c_1^2\Big|+4608|c_1|\Big|c_4-\frac{2064}{4608}c_1c_3\Big|+3840|c_2|\Big|c_3-\frac{1408}{3840}c_1c_2\Big|\Big)$$

Using (1.13), (1.14) (1.17) and Lemma 1.3, we get the required result. □

Theorem 4.3. *If $f\in\mathcal{R}_{\mathcal{L}}$, then*

$$\left|H_{3,1}(f)\right|\leq\frac{415763}{13271040}.$$

Proof. Since (1.10), it follows that

$$\left|H_{3,1}(f)\right|\leq|\gamma_3|\left|\gamma_2\gamma_4-\gamma_3^2\right|+|\gamma_4|\left|\gamma_1\gamma_4-\gamma_2\gamma_3\right|+|\gamma_5|\left|\gamma_1\gamma_3-\gamma_2^2\right|.$$

From the values of (2.1-2.3), (3.1), (4.1) and (4.2), we achieve the required result. □

5. Conclusion

In this study, we have examined the Hankel determinants of logarithmic coefficients for a specific subclass of bounded turning functions associated with the lune domain. By deriving sharp upper bounds for $|H_{2,1}(F_f/2)|$ and $|H_{3,1}(F_f/2)|$, we contribute meaningful advancements to the field of geometric function theory. Our analysis extends the methodologies established by previous scholars and offers novel insights into the behavior of logarithmic coefficients within this structured context.

The results presented reinforce the intricate relationship between geometric properties of univalent functions and their associated coefficient functionals. Furthermore, the techniques and lemmas utilized here pave the way for broader applications, suggesting that similar methods could be adapted for more complex domains or higher-order Hankel determinants.

Future research directions may include extending these results to subclasses defined by different geometric constraints, such as nephroid or symmetric cardioid domains, and exploring applications of these bounds in related fields such as signal processing or complex dynamical systems. The interplay between geometry, coefficient behavior, and function theory remains a fertile ground for ongoing and impactful mathematical discovery.

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Korovkin Type Theorem for Modified Bernstein Operators via A-Statistical Convergence and Power Summability Method

Tuğba Değirmenci and Merve İlkhan Kara*

Abstract
In this study, we investigate the approximation properties of modified Bernstein operators through the lens of A-statistical convergence and power summability methods. Our main objective is to establish a Korovkin type approximation theorem in this generalized setting. By incorporating statistical convergence, we aim to provide broader and more powerful approximation results that can be applied in various contexts where classical convergence criteria may fail or be insufficient.

Keywords: *A-statistical convergence, Bernstein operators, Korovkin type theorem, Power summability method*

AMS Subject Classification (2020): *40G10; 40C15*

1. Introduction and preliminaries

One of the most significant milestones in approximation theory is undoubtedly the renowned Weierstrass approximation theorem, which asserts that any continuous function on a closed interval can be uniformly approximated by polynomials. In the years that followed, numerous proofs of this theorem were developed. Among them, Bernstein's proof stands out due to its elegance and its introduction of the Bernstein operator, which laid the foundation for the theory of linear positive operators. The significance of linear positive operators soon became evident, as they provide a straightforward and constructive means to approximate functions.

In this context, Bohman established that if a sequence of linear positive operators L_n satisfies $L_n(1;x) \rightrightarrows 1$, $L_n(t;x) \rightrightarrows x$ and $L_n(t^2;x) \rightrightarrows x^2$, then for any continuous function f, it follows that $L_n(f(t);x) \rightrightarrows f(x)$. Subsequently, Korovkin extended this result to integral-type operators, leading to what is now known as Korovkin's theorem. The contributions of Bohman and Korovkin have significantly advanced the theory of linear positive operators.

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As a result, numerous researchers have since introduced and investigated various types of operators that fulfill the conditions of this theorem. Consequently, the theory of function approximation via linear positive operators remains a vibrant and continually evolving area of mathematical analysis.

The Bernstein polynomials are defined as

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(\frac{k}{n})$$

for every bounded function on $[0,1]$.

Let $C[0,1]$ be the Banach space of all real-valued continuous functions on $[0,1]$ endowed with the sup norm

$$\|f\| = \sup_{x\in[0,1]} |f(x)|.$$

We shall denote by $\mathbb{N}$ the set of all natural numbers. The sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$, the set $K_\varepsilon = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has natural density zero [1], i.e. for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case, we write $L = st - \lim x$. Note that every convergent sequence is statistically convergent, but not conversely. In what follows we will use the definition of the A-statistical convergence. Let $A = (a_{nj})$ be a summability matrix and $x = (x_j)$ be a sequence. If the series

$$(Ax)_n = \sum_j a_{nj} x_j$$

converges for every $n \in \mathbb{N}$, then we say that $((Ax)_n)$ is the A-transform of the sequence $x = (x_n)$. And if the $((Ax)_n)$ converges to a number L, we say that x is A-summable to L. The summability matrix A is regular whenever $\lim_j x_j = L$, then $\lim_n (Ax)_n = L$. A be a non-negative regular summability matrix. The sequence $x = (x_j)$ is said to be A-statistically convergent (see [2]) to real number a if for any $\varepsilon > 0$

$$\lim_n \sum_{j:|x_j - a| \geq \varepsilon} a_{nj} = 0.$$

In this case we write $st_A - limx = a$.

The A-statistical convergence is a generalization of the statistical convergence and it is proven in the Example given in paper [3]. The second summability method used in this paper is power summability method.

Let (p_j) be a real sequence with $p_0 > 0$ and $p_1, p_2, p_3, ... \geq 0$ such that the corresponding power series $p(t) = \sum_{j=0}^{\infty} p_j t^j$ has radius of convergence R with $0 < R \leq \infty$. If, for all $t \in (0,R)$,

$$\lim_{t\to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} x_j p_j t^j = L$$

exists, then we say that $x = (x_j)$ is convergent in the sense of power summability method (see [4, 5]). Power summability method includes many known summability methods such as Abel and Borel. Both methods have in common that their definitions are based on power series and they are not matrix methods (see [6–8]). That power summability method is more effective than ordinary convergence, as shown in the example given in [9].

Note that the power summability method is regular if and only if

$$\lim_{t\to R^-} \frac{p_j t^j}{p(t)} = 0$$

holds for each $j \in \{0,1,2,3,...\}$ ([10]). Throughout the paper we assume that power series method is regular.

The theory of Korovkin type theorems was studied in several function spaces, and further details the reader can find in those papers (see [11–15]).

In this paper, we will prove the Korovkin-type theorem for the modified Bernstein operators via A-statistical convergence and the power summability method. To this end, we define a new sequence of modified Bernstein operators tailored to preserve certain structural properties of the approximated functions. We then analyze their behavior under the A-statistical convergence scheme, which is governed by a regular summability matrix A, and combine this approach with power summability techniques to obtain stronger results concerning the convergence of the operators to the target function. The theoretical results presented in this paper extend and generalize classical Korovkin-type theorems, offering new insights into approximation theory in the context of summability methods.

2. Main results

In a recent paper, Usta [16] has defined the new modification of Bernstein type operators which fix a constant and preserves Korovkin's other test functions. The modification of the operator is defined for function $f \in C(0,1)$ as follows:

$$B_n^*(f,x) = \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}(k-nx)^2 x^{k-1}(1-x)^{n-k-1} f(\frac{k}{n}). \tag{2.1}$$

It is observed that this new operator is positive and linear. The following lemma is required to prove our main results.

Lemma 2.1. *[16] Let $e_i = t^i$ for $i = 0,1,2$. Then the following equations hold:*

$$B_n^*(e_0,x) = 1$$

$$B_n^*(e_1,x) = \frac{n-2}{n}x + \frac{1}{n}$$

$$B_n^*(e_2,x) = \frac{n^2-7n+6}{n^2}x^2 + \frac{5n-6}{n^2}x + \frac{1}{n^2}.$$

Here and what follows

$$C_b[0,\infty) = \{f | f : [0,\infty) \to \mathbb{R} \text{ continuous and bounded}\}.$$

Let $A = (a_{jn})$ be a non negative regular summability matrix and the condition

$$\lim_{j\to\infty}\sum_{n=1}^{\infty} a_{jn}\frac{1}{n} = 0 \tag{2.2}$$

hold. We now define a new operator as a modification of the operator introduced in (2.1), originally proposed in [17] as follows:

$$B_j^{**}(f,x) = \sum_{n=1}^{\infty} a_{jn} B_n^*(f,x).$$

We first get the next approximation result.

Theorem 2.1. *Given any $f \in C_b[0,\infty)$*

$$\lim_{j\to\infty} B_j^{**}(f,x) = f(x)$$

holds uniformly on compact subsets of $[0,\infty)$.

Proof. Since the matrix $A = (a_{jn})$ is regular we have

$$\left|B_j^{**}(e_0,x) - e_0(x)\right| = \left|\sum_{n=1}^{\infty} a_{jn} - 1\right| \longrightarrow 0$$

for $j \to \infty$.

By using the inequality

$$|B_n^*(e_1,x) - x| = \left|-\frac{2}{n}x + \frac{1}{n}\right| \le \frac{2}{n}x + \frac{1}{n},$$

the regularity of the matrix $A=(a_{jn})$ with the condition (2.2), we obtain

$$\begin{aligned}\left|B_j^{**}(e_1,x)-e_1(x)\right| &= \left|\sum_{n=1}^{\infty} a_{jn}B_n^*(e_1,x)-e_1(x)\right| \\ &= \left|\sum_{n=1}^{\infty} a_{jn}\left[B_n^*(e_1,x)-e_1(x)\right]+\sum_{n=1}^{\infty} a_{jn}e_1(x)-e_1(x)\right| \\ &= \left|\sum_{n=1}^{\infty} a_{jn}\left[B_n^*(e_1,x)-e_1(x)\right]+e_1(x)\left[\sum_{n=1}^{\infty} a_{jn}-1\right]\right| \\ &\le \sum_{n=1}^{\infty} a_{jn}\left|B_n^*(e_1,x)-e_1(x)\right|+x\left|\sum_{n=1}^{\infty} a_{jn}-1\right| \\ &\le \sum_{n=1}^{\infty} a_{jn}\left(\frac{2}{n}x+\frac{1}{n}\right)+x\left|\sum_{n=1}^{\infty} a_{jn}-1\right| \\ &= 2x\sum_{n=1}^{\infty}\frac{a_{jn}}{n}+\sum_{n=1}^{\infty}\frac{a_{jn}}{n}+x\left|\sum_{n=1}^{\infty} a_{jn}-1\right| \longrightarrow 0\end{aligned}$$

for $j\to\infty$.

Similarly, using the inequality

$$\begin{aligned}\left|B_n^*(e_2,x)-x^2\right| &= \left|-\frac{7}{n}x^2+\frac{6}{n^2}x^2+\frac{5}{n}x-\frac{6}{n^2}x+\frac{1}{n^2}\right| \\ &\le \frac{7}{n}x^2+\frac{6}{n^2}x^2+\frac{5}{n}x+\frac{6}{n^2}x+\frac{1}{n^2},\end{aligned}$$

the regularity of the matrix $A=(a_{jn})$ with the condition (2.2), we obtain

$$\begin{aligned}\left|B_j^{**}(e_2,x)-e_2(x)\right| &= \left|\sum_{n=1}^{\infty} a_{jn}B_n^*(e_2,x)-e_2(x)\right| \\ &= \left|\sum_{n=1}^{\infty} a_{jn}\left[B_n^*(e_2,x)-e_2(x)\right]+e_2(x)\left[\sum_{n=1}^{\infty} a_{jn}-1\right]\right| \\ &\le \sum_{n=1}^{\infty} a_{jn}\left|B_n^*(e_2,x)-e_2(x)\right|+x^2\left|\sum_{n=1}^{\infty} a_{jn}-1\right| \\ &\le \sum_{n=1}^{\infty} a_{jn}\left(\frac{7}{n}x^2+\frac{6}{n^2}x^2+\frac{5}{n}x+\frac{6}{n^2}x+\frac{1}{n^2}\right)+x^2\left|\sum_{n=1}^{\infty} a_{jn}-1\right| \\ &= (7x^2+5x)\sum_{n=1}^{\infty}\frac{a_{jn}}{n}+(6x^2+6x+1)\sum_{n=1}^{\infty}\frac{a_{jn}}{n^2}+x^2\left|\sum_{n=1}^{\infty} a_{jn}-1\right| \longrightarrow 0\end{aligned}$$

for $j\to\infty$. □

Now we give the Korovkin type theorem for A-statistical convergence.

Theorem 2.2. *Let $A=(a_{jn})$ be a non negative regular summability matrix and*

$$st_A-\lim_n \|B_n^* e_i-e_i\|=0$$

hold for $i=0,1,2$. Then given any $f\in C[0,1]$

$$st_A-\lim_n \|B_n^* f-f\|=0$$

holds, where $\|f\|=\sup_{t\in[0,1]}|f(t)|$.

Proof. The inequality

$$\|B_n^* e_2 - e_2\| \le \sup_{x\in[0,1]} \left\{ \left| \frac{n^2-7n+6}{n^2}x^2 + \frac{5n-6}{n^2}x + \frac{1}{n^2} - x^2 \right| \right\}$$
$$\le \frac{7n+6}{n^2} + \frac{5n+6}{n^2} + \frac{1}{n^2}$$

holds. If

$$M = \{n : \|B_n^* e_2 - e_2\| \ge \varepsilon\},$$
$$M_1 = \left\{ \frac{7n+6}{n^2} \ge \frac{\varepsilon}{3} \right\},$$
$$M_2 = \left\{ \frac{5n+6}{n^2} \ge \frac{\varepsilon}{3} \right\},$$
$$M_3 = \left\{ \frac{1}{n^2} \ge \frac{\varepsilon}{3} \right\}$$

then we have $M \subset M_1 \cup M_2 \cup M_3$. Hence we conclude that

$$st_A - \lim_n \|B_n^* e_2 - e_2\| = 0$$

holds. □

Now we will prove the Korovkin-type theorem for the modified Bernstein operators by the power summability method.

By the aid of sequence of operators B_n^* let define the power series $\sum_{n=0}^{\infty} B_n^*(f,x)p_n t^n$. For every $t \in (0,R)$, if the limit

$$\lim_{t\to R^-} \frac{1}{p(t)} \sum_{n=0}^{\infty} B_n^*(f,x)p_n t^n$$

exists then we say that sequence of operators B_n^* converges in the sense of power series.

Theorem 2.3. *Let B_n^* be a sequence of positive linear operators from $C[0,1]$ into $B[0,1]$. For every $f \in C[0,R]$*

$$\lim_{t\to R^-} \frac{1}{p(t)} \left\| \sum_{n=0}^{\infty} (B_n^* f - f)p_n t^n \right\| = 0 \tag{2.3}$$

if and only if

$$\lim_{t\to R^-} \frac{1}{p(t)} \left\| \sum_{n=0}^{\infty} (B_n^* e_i - e_i)p_n t^n \right\| = 0 \tag{2.4}$$

for $i = 0,1,2$.

Proof. From the equality (2.3) we have the equality (2.4).

For the converse assume that the equality (2.4) holds. Let $f \in C[0,1]$. For all $t \in [0,1]$, there exists a real number $M > 0$ such that $|f(t)| \le M$ holds. Hence, for $t,x \in [0,1]$ we have

$$|f(t) - f(x)| \le 2M. \tag{2.5}$$

Since f is continuous, given any $\varepsilon > 0$ there exists $\delta > 0$ such that $|t-x| < \delta$ implies

$$|f(t) - f(x)| \le \varepsilon. \tag{2.6}$$

Let $\Psi(t,x) = (t-x)^2$. $|t-x| \ge \delta$ implies that

$$|f(t) - f(x)| \le \frac{2M}{\delta^2}\Psi(t,x). \tag{2.7}$$

By using the inequalities (2.5)-(2.7), we have

$$|f(t)-f(x)|<\varepsilon+\frac{2M}{\delta^2}\Psi(t,x).$$

Hence we write

$$-\varepsilon-\frac{2M}{\delta^2}\Psi(t,x)<f(t)-f(x)<\varepsilon+\frac{2M}{\delta^2}\Psi(t,x).$$

Since $B_n^*(1,x)$ is monoton and linear

$$B_n^*(1,x)\left(-\varepsilon-\frac{2M}{\delta^2}\Psi(t,x)\right)<B_n^*(1,x)\left(f(t)-f(x)\right)<B_n^*(1,x)\left(\varepsilon+\frac{2M}{\delta^2}\Psi(t,x)\right)$$

and so we have

$$-\varepsilon B_n^*(1,x)-\frac{2M}{\delta^2}B_n^*(\Psi(t),x)<B_n^*(f,x)-f(x)B_n^*(1,x)<\varepsilon B_n^*(1,x)+\frac{2M}{\delta^2}B_n^*(\Psi(t),x). \tag{2.8}$$

By using the following equality with the inequality (2.8)

$$B_n^*(f,x)-f(x)=B_n^*(f,x)-f(x)B_n^*(1,x)+f(x)\left(B_n^*(1,x)-1\right)$$

we obtain

$$B_n^*(f,x)-f(x)<\varepsilon B_n^*(1,x)+\frac{2M}{\delta^2}B_n^*(\Psi(t),x)+f(x)\left(B_n^*(1,x)-1\right). \tag{2.9}$$

From the equality

$$B_n^*(\Psi(t),x)=B_n^*((t-x)^2,x)=B_n^*((x^2-2xt+t^2),x)=x^2B_n^*(1,x)-2xB_n^*(t,x)+B_n^*(t^2,x)$$

and the inequality (2.9) we have

$$\begin{aligned}
B_n^*(f,x)-f(x)&<\frac{2M}{\delta^2}\left\{x^2\left(B_n^*(1,x)-1\right)-2x\left(B_n^*(t,x)-x\right)+\left(B_n^*(t^2,x)-x^2\right)\right\}\\
&\quad+\varepsilon B_n^*(1,x)+f(x)\left(B_n^*(1,x)-1\right)\\
&=\varepsilon+\varepsilon\left(B_n^*(1,x)-1\right)+f(x)\left(B_n^*(1,x)-1\right)\\
&\quad+\frac{2M}{\delta^2}\left\{x^2\left(B_n^*(1,x)-1\right)-2x\left(B_n^*(t,x)-x\right)+\left(B_n^*(t^2,x)-x^2\right)\right\}.
\end{aligned}$$

It follows that

$$|B_n^*(f,x)-f(x)|\le\varepsilon+\left(\varepsilon+M+\frac{2M}{\delta^2}\right)|B_n^*(1,x)-1|+\frac{4M}{\delta^2}|B_n^*(t,x)-x|+\frac{2M}{\delta^2}\left|B_n^*(t^2,x)-x^2\right|$$

holds. From the last inequality and the linearity of the operator $B_n^*(f,x)$ we conclude that

$$\begin{aligned}
\frac{1}{p(t)}\left\|\sum_{n=0}^{\infty}(B_n^*(f,x)-f(x))p_nt^n\right\|&\le\varepsilon+\left(\varepsilon+M+\frac{2M}{\delta^2}\right)\frac{1}{p(t)}\left\|\sum_{n=0}^{\infty}(B_n^*(1,x)-1)p_nt^n\right\|\\
&\quad+\frac{4M}{\delta^2}\frac{1}{p(t)}\left\|\sum_{n=0}^{\infty}(B_n^*(t,x)-x)p_nt^n\right\|\\
&\quad+\frac{2M}{\delta^2}\frac{1}{p(t)}\left\|\sum_{n=0}^{\infty}(B_n^*(t^2,x)-x^2)p_nt^n\right\|.
\end{aligned}$$

This completes the proof. □

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Serret-Frenet Formula of Quaternionic Framed Curves

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Abstract

In this study, we present an approach by introducing the quaternionic structure of framed curves. Furthermore, we derive Serret-Frenet formulas and give specific results for quaternionic framed curves. Initially, we focus on the moving frame and its curvatures corresponding to the frame T, N, B along the quaternionic framed base curve in three-dimensional Euclidean space $\mathbb{R}^3$. Then, we establish the Serret-Frenet type formulas of quaternionic framed curves. We then generalize these formulas and the definition of quaternionic framed curves to four-dimensional Euclidean space, highlighting the relationship between the curvatures in both 3-dimensional and four-dimensional Euclidean spaces. In addition, the theorems are supported by examples, demonstrating the applicability of the proposed results.

Keywords: *Framed curves, Serret-Frenet formulas, Singular point, Quaternion*

AMS Subject Classification (2020): *53A04; 58K05; 53A40; 11R52*

1. Introduction

The theory of curves is one of the most fundamental subjects of differential geometry and has been studied for many years. Different frame constructions for curves have also been investigated, as first introduced in [1]. Furthermore, associated curves and their geometric properties have been examined in the Frenet frame context, offering valuable insights into curve behavior [2]. In the theory of curves, the Serret-Frenet formulas of a curve are given by defining the constant quantities of curvatures and torsions of the curve. The harmonic curvatures of the Frenet curve have been researched by K. Arslan and H.H. Hacısalihoğlu [3]. The generalized helix concept, which is an important curve making a constant angle with a constant vector with a fixed direction, has been studied in [4]. Various types of curves, such as rectifying, osculating, and normal curves, have been studied with the help of the frameworks of the curves, known as Serret-Frenet elements, where their changes can be observed instantly. A study on these curves can be found in [5]. The important point here is that by defining Frenet elements with the help of regular curves, many properties of the curve, such as the geometric structure of the relevant space, are investigated. However, if space curves contain singular points, constructing the Frenet frame becomes impossible to examine the structure of these curves. As it is known, the fact that the tangent vectors are zero at singular points creates some difficulties. Because the principal normal and binormal vectors cannot be normalized by known

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methods. However, thanks to the recently defined generalized tangent, principal normal vector, special curves with singular points can also be studied at non-singular points [6]. Framed curves, defined as space curves equipped with a moving frame that may include singular points, have been described to be both a generalization of Legendre curves and a generalization of regular curves with linearly independent conditions, and also Serret-Frenet type formulas of framed curves have been given by Honda and Takahashi [7]. In addition, spinors in terms of framed Manheim curves and framed Bertrand curves are discussed in [8, 9]. Framed curves have been studied in recent years, both as special curve couples in Lie groups [10] and as surfaces formed in Euclidean space [11], making important contributions to this field. Apart from these framed curves have been studied in the Euclidean space [12, 13]. Some geometric properties of framed curves in all four dimensions have been investigated [14, 15].

During the study of curves, quaternionic curves have been studied. The properties of smooth quaternionic curves in spatial and non-spatial quaternion spaces were studied by Baharatti and Nagaraj [16]. In addition, Serret-Frenet formulas are also given in this study. The Serret-Frenet formulas are given by Tuna for quaternionic curves in semi-Euclidean space [17]. In recent years, studies on quaternionic curves in differential geometry [18, 19] have come to the fore. For basic notions of differential geometry, we refer to [20]. In order to strengthen the theoretical background and provide a broader context, we have incorporated relevant studies on quaternionic frames and curves [21–25].

In this study, we derive the well-known Serret-Frenet formulas of differential geometry in terms of framed curves using quaternion algebra. We begin by presenting the fundamental definitions and algebraic properties of quaternions, along with the concept of framed curves in three-dimensional Euclidean space. Next, we introduce a notion of quaternionic framed curves in three-dimensional Euclidean space. We then rigorously prove the Serret-Frenet-type formulas for quaternionic framed curves and provide their matrix representations. Following this, we extend these formulas and the definition of quaternionic framed curves to four-dimensional Euclidean space, establishing a relationship between the curvatures in both 3-dimensional and 4-dimensional Euclidean spaces. This innovative framework, encompassing the definitions of quaternionic framed curves and their associated theorems offers a fresh perspective on the geometry of curves. Classical methods in elementary differential geometry do not provide a direct approach for relating a curve $\boldsymbol{\gamma}$ in $\mathbb{R}^3$ to its corresponding a curve $\tilde{\boldsymbol{\gamma}}$ in $\mathbb{R}^4$. However, through the use of quaternions, we are able to achieve this connection. The study is further substantiated by examples in both three-dimensional and four-dimensional spaces, demonstrating the practical application of the developed theory.

2. Preliminary

In this section, some definitions and theorems that are useful in our study are given. The set of real quaternions is defined as:

$$H = \left\{ q : q = a_0\mathbf{e}_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3,\ a_0, a_1, a_2, a_3 \in \mathbb{R},\ \mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbb{R}^4 \right\}.$$

Here, $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are quaternion base elements and satisfy the following multiplication rules:

$$\begin{aligned}
&{\mathbf{e}_0}^2 = 1,\ {\mathbf{e}_1}^2 = {\mathbf{e}_2}^2 = {\mathbf{e}_3}^2 = \mathbf{e}_1 \times \mathbf{e}_2 \times \mathbf{e}_3 = -1,\\
&\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3,\ \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1,\ \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2,\\
&\mathbf{e}_2 \times \mathbf{e}_1 = -\mathbf{e}_3,\ \mathbf{e}_3 \times \mathbf{e}_2 = -\mathbf{e}_1,\ \mathbf{e}_1 \times \mathbf{e}_3 = -\mathbf{e}_2.
\end{aligned}$$

Here, the symbol $\times$ denotes the quaternionic product. A quaternion q with the scalar part is denoted by S_q and the vector part is denoted by $\boldsymbol{V}_q$. Here $S_q = a_0$ and $\boldsymbol{V}_q = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ are written as $q = S_q + \boldsymbol{V}_q$. Then the quaternionic products of two quaternions q and p are given by:

$$q \times p = S_q S_p - \langle \boldsymbol{V}_q, \boldsymbol{V}_p \rangle + S_q \boldsymbol{V}_p + S_p \boldsymbol{V}_q + \boldsymbol{V}_q \wedge \boldsymbol{V}_p\,.$$

Here the symbols $\langle , \rangle$ and $\wedge$ denote the Euclidean dot product and the vector product in $\mathbb{R}^3$, respectively. The conjugate of a quaternion $q \in H$ is defined as $\alpha q = S_q - \boldsymbol{V}_q$. Accordingly, the product of the quaternion and its conjugate is written as follows:

$$q \times \alpha q = (a_0\mathbf{e}_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) \times (a_0\mathbf{e}_0 - a_1\mathbf{e}_1 - a_2\mathbf{e}_2 - a_3\mathbf{e}_3) = {a_0}^2 + {a_1}^2 + {a_2}^2 + {a_3}^2.$$

Also, the properties related the conjugate of the quaternion q and p for $a, b \in \mathbb{R}$ are given by:

$$\begin{aligned}
\alpha\left(aq + bp\right) &= a\alpha\left(q\right) + b\alpha\left(p\right),\\
\alpha\left(q \times p\right) &= \alpha\left(p\right) \times \alpha\left(q\right),\\
\alpha\left(\alpha q\right) &= q.
\end{aligned}$$

The inner product for real quaternions $p, q \in H$ is defined as:

$$h : H \times H \to \mathbb{R},$$
$$(p,q) \to h(p,q) = \frac{1}{2}(p \times \alpha q + q \times \alpha p)$$

where h is real-valued, symmetric, and bilinear. The norm of the quaternion $q \in H$ defined by:

$$\|\| : H \to \mathbb{R},$$
$$q \to \|q\|,$$
$$\|q\|^2 = h(q,q) = {a_0}^2 + {a_1}^2 + {a_2}^2 + {a_3}^2.$$

If the quaternion $q \in H$ satisfies the equality

$$q + \alpha q = 0, \tag{2.1}$$

then it is called a spatial quaternion. If for the quaternion $q \in H$

$$q - \alpha q = 0$$

then it is called a temporal quaternion. In general, the quaternion $q \in H$ can be written as

$$q = \frac{1}{2}(q + \alpha q + q - \alpha q) = \frac{1}{2}(q + \alpha q) + \frac{1}{2}(q - \alpha q).$$

On the first hand, Honda and Takahashi stated that a framed curve is a smooth space curve with a moving frame with singular points [7]. In order to construct a framed curve, a structure known as a "Stiefel Manifold" in the literature is defined as follows:

$$\begin{aligned}\Delta_{n-1} &= \{\boldsymbol{v} = (v_1, v_2, ..., v_{n-1}) \in \mathbb{R}^n \times ... \times \mathbb{R}^n : \langle v_i, v_j \rangle = \delta_{ij},\ i,j = 1, ..., n-1\} \\ &= \left\{\boldsymbol{v} = (v_1, v_2, ..., v_{n-1}) \in S^{n-1} \times ... \times S^{n-1} : \langle v_i, v_j \rangle = 0,\ i \neq j,\ i,j = 1, ..., n-1\right\}.\end{aligned}$$

This manifold is $\frac{n(n-1)}{2}$-dimensional and smooth. If taken as $\boldsymbol{v} = (v_1, v_2, ..., v_{n-1}) \in \Delta_{n-1}$, the unit vector $\boldsymbol{\mu} \in \mathbb{R}^n$ can be defined as $\boldsymbol{\mu} = v_1 \wedge v_2 ... \wedge v_{n-1}$. It can be seen from here that $(\boldsymbol{v}, \boldsymbol{\mu}) \in \Delta_{n-1}$ and $\det(\boldsymbol{v}, \boldsymbol{\mu}) = 1$.

Definition 2.1. $(\boldsymbol{\gamma}, \boldsymbol{\nu}) : I \to \mathbb{R}^n \times \Delta_{n-1}$ is called a framed curve if $h(\dot{\boldsymbol{\gamma}}(t), \nu_i(t)) = 0$ for all $t \in I$ and $i = 1, 2, ...n-1$.$\boldsymbol{\gamma} : I \to \mathbb{R}^n$ is said that a framed base curve if there exists $\boldsymbol{\nu} : I \to \Delta_{n-1}$ such that $(\boldsymbol{\gamma}, v_1, v_2, ..., v_{n-1})$ is a framed curve [7].

Along the framed base curve $\gamma(t)$, the moving frame $\{\boldsymbol{\mu}(t), \boldsymbol{v}(t)\}$ can be defined and the Serret-Frenet type formulas are given as

$$\begin{bmatrix} \dot{\boldsymbol{v}}(t) \\ \dot{\boldsymbol{\mu}}(t) \end{bmatrix} = A(t) \begin{bmatrix} \boldsymbol{v}(t) \\ \boldsymbol{\mu}(t) \end{bmatrix}.$$

Here $i, j = 1, 2, ..., n$ is for each $A(t) = (\sigma_{ij}(t)) \in o(n)$ and $o(n)$ is the set of anti-symmetric matrices. Also the smooth function $\sigma : I \to \mathbb{R}$ given as

$$\dot{\boldsymbol{\gamma}}(t) = \sigma(t)\, \boldsymbol{\mu}(t).$$

In addition, the necessary and sufficient condition for the point t to be a singular point of the curve γ is that it is $\sigma(t) = 0$. Here the functions $(\sigma_{ij}(t), \sigma(t))$ are called the curvatures of the framed curve. Now we will give the definition of framed curves, especially in the case of $n = 3$. The definition of framed curves in $\mathbb{R}^3 \times \Delta_2$ is given as follows:

Let be $(\gamma, v_1, v_2) : I \to \mathbb{R}^3 \times \Delta_2$ a framed curve and $\boldsymbol{\mu} = v_1 \wedge v_2$. In this case, the following conditions are satisfied

$$\langle \dot{\boldsymbol{\gamma}}(t), v_i(t) \rangle = 0,\ \forall t \in I,\ i = 1, 2.$$

Here it is expressed as

$$\Delta_2 = \left\{\boldsymbol{v} = (v_1, v_2) \in \mathbb{R}^3 \times \mathbb{R}^3 : \langle v_1, v_2 \rangle = 0, \langle v_1, v_1 \rangle = 1, \langle v_2, v_2 \rangle = 1\right\}.$$

The Serret-Frenet type formulas are given by

$$\{\boldsymbol{\mu}(t), \boldsymbol{v}(t)\} \in \Delta_2\,,\ \begin{bmatrix} \dot{v}_1(t) \\ \dot{v}_2(t) \\ \dot{\boldsymbol{\mu}}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & l(t) & m(t) \\ -l(t) & 0 & n(t) \\ -m(t) & -n(t) & 0 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ \boldsymbol{\mu}(t) \end{bmatrix}$$

where the curvatures are $l(t) = \langle \dot{v}_1(t), v_2(t) \rangle$, $m(t) = \langle \dot{v}_1(t), \boldsymbol{\mu}(t) \rangle$,
$n(t) = \langle \dot{v}_2(t), \boldsymbol{\mu}(t) \rangle$ and $\sigma(t) = \langle \dot{\gamma}(t), \boldsymbol{\mu}(t) \rangle$ [7].

3. Serret-Frenet formulas for quaternionic framed curves

In this section, the definition of a spatial quaternionic framed curve in 3-dimensional Euclidean space $\mathbb{R}^3$ is given. Then, the Serret-Frenet type formula is obtained for these quaternionic framed curves. Also, the relationship between their curvatures is presented. Serret-Frenet type formula for spatial quaternionic framed curve and framed curvatures is supported with an example and illustrated.

Definition 3.1. We define $(\boldsymbol{\gamma}, \boldsymbol{\nu}) : I \to H \times \Delta_2$ is a quaternionic framed curve in three dimensional Euclidean space $\mathbb{R}^3$ if $h(\dot{\boldsymbol{\gamma}}(t), \nu_i(t)) = 0$ for all $t \in I$ and $i = 1, 2$. In other words, we say that $\boldsymbol{\gamma} : I \to H$ is a quaternionic framed base curve in $\mathbb{R}^3$ if there exists $\boldsymbol{\nu} : I \to \Delta_2$ such that $(\boldsymbol{\gamma}, v_1, v_2)$ is a quaternionic framed curve.

Theorem 3.1. *Let $(\boldsymbol{\gamma}, v_1, v_2) : I \to H \times \Delta_2$ be a quaternionic framed curve in three dimensional Euclidean space $\mathbb{R}^3$ and $\boldsymbol{\mu} = v_1 \wedge v_2$ with the space of spatial quaternions $\{\gamma \in H : \gamma + \alpha(\gamma) = 0\}$ which the quaternionic framed base curve $\boldsymbol{\gamma} : I \subset \mathbb{R} \to H$ defining by $t \to \boldsymbol{\gamma}(t) = \sum_{i=1}^{3} \boldsymbol{\gamma}_i(t) e_i$ for all $t \in I = [0, 1] \subset \mathbb{R}$. Then, Serret-Frenet type formula of the quaternionic framed curve $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ at point $\boldsymbol{\gamma}(t)$ is given by*

$$\begin{bmatrix} \dot{v}_1(t) \\ \dot{v}_2(t) \\ \dot{\boldsymbol{\mu}}(t) \end{bmatrix} = \begin{bmatrix} 0 & l(t) & m(t) \\ -l(t) & 0 & n(t) \\ -m(t) & -n(t) & 0 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ \boldsymbol{\mu}(t) \end{bmatrix}$$

where the curvatures of the quaternionic framed curve are $l(t) = h(\dot{v}_1(t), v_2(t))$, $m(t) = h(\dot{v}_1(t), \boldsymbol{\mu}(t))$, $n(t) = h(\dot{v}_2(t), \boldsymbol{\mu}(t))$ and $\sigma(t) = h(\dot{\boldsymbol{\gamma}}(t), \boldsymbol{\mu}(t))$.

Proof. Let $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ be the quaternionic framed curve and $\boldsymbol{\mu}(t) = v_1(t) \wedge v_2(t)$ for all $t \in I = [0, 1] \subset \mathbb{R}$. Since

$$\|\boldsymbol{\mu}(t)\|^2 = h(\boldsymbol{\mu}(t), \boldsymbol{\mu}(t)) = \frac{1}{2}(\boldsymbol{\mu}(t) \times \alpha(\boldsymbol{\mu}(t)) + \boldsymbol{\mu}(t) \times \alpha(\boldsymbol{\mu}(t))) = \boldsymbol{\mu}(t) \times \alpha(\boldsymbol{\mu}(t)) = 1.$$

$\boldsymbol{\mu}(t)$ has unit length. Taking the derivative of both sides of the above equation is obtained

$$\dot{\boldsymbol{\mu}}(t) \times \alpha\boldsymbol{\mu}(t) + \boldsymbol{\mu}(t) \times (\alpha\dot{\boldsymbol{\mu}})(t) = 0. \tag{3.1}$$

We get from the equation (3.1) as follows:

$$h(\dot{\boldsymbol{\mu}}(t), \boldsymbol{\mu}(t)) = \frac{1}{2}(\dot{\boldsymbol{\mu}}(t) \times \alpha\boldsymbol{\mu}(t) + \boldsymbol{\mu}(t) \otimes \alpha\dot{\boldsymbol{\mu}}(t)) = 0. \tag{3.2}$$

In this case it is $\dot{\boldsymbol{\mu}} \perp \boldsymbol{\mu}$. This shows the meaning of quaternionic orthogonality. If the equation (3.1) is arranged according to the conjugate rules, then we have $(\dot{\boldsymbol{\mu}} \times \alpha\boldsymbol{\mu}) + \alpha(\boldsymbol{\mu} \times \alpha\dot{\boldsymbol{\mu}}) = 0$. Hence, $\dot{\boldsymbol{\mu}} \times \alpha\boldsymbol{\mu}$ is a spatial quaternion from equation (2.1). Since $\boldsymbol{\gamma}$ is a framed curve, there exists a smooth function $\sigma : I \to \mathbb{R}$ such that $\dot{\boldsymbol{\gamma}}(t) = \sigma(t)\boldsymbol{\mu}(t)$ and unit vector $\boldsymbol{\mu}$. Accordingly $\dot{\boldsymbol{\mu}}(t) \in Sp\{\boldsymbol{\mu}(t), v_1(t), v_2(t)\}$ and since $\dot{\boldsymbol{\mu}}$ is a spatial quaternion it can be written as follows:

$$\dot{\boldsymbol{\mu}}(t) = a_{31}(t)v_1(t) + a_{32}(t)v_2(t) + a_{33}(t)\boldsymbol{\mu}(t), \tag{3.3}$$

where $a_{31}, a_{32}, a_{33} : I \to \mathbb{R}$ are smooth functions. Right now, we compute quaternionic inner product that is $h-$ inner product of the vectors $v_1, v_2, \boldsymbol{\mu}$ respectively, as follows:

$$h(v_1(t), \boldsymbol{\mu}(t)) = \frac{1}{2}(v_1(t) \times \alpha\boldsymbol{\mu}(t) + \boldsymbol{\mu}(t) \times \alpha v_1(t)) = \frac{1}{2}(v_2(t) + \alpha v_2(t)) = 0,$$

$$h(v_2(t), \boldsymbol{\mu}(t)) = \frac{1}{2}(v_2(t) \times \alpha\boldsymbol{\mu}(t) + \boldsymbol{\mu}(t) \times \alpha v_2(t)) = \frac{1}{2}(\alpha v_1(t) + v_1(t)) = 0.$$

It is seen that $\boldsymbol{\mu}$,v_1 and v_2 are orthogonal quaternions. Also v_1 and v_2 are the unit spatial quaternions. It is $a_{33}(t) = 0$ from the equation (3.2) and the last two equations. Thus the equation (3.3) becomes as follows:

$$\dot{\boldsymbol{\mu}}(t) = a_{31}(t)v_1(t) + a_{32}(t)v_2(t). \tag{3.4}$$

On the other hand, since it is $\dot{v}_1(t) \in Sp\{v_1(t), v_2(t), \boldsymbol{\mu}(t)\}$ a combination of the vectors, v_1, v_2 and $\boldsymbol{\mu}$ can be written as

$$\dot{v}_1(t) = a_{11}(t)v_1(t) + a_{12}(t)v_2(t) + a_{13}(t)\mu(t). \tag{3.5}$$

From here, by $h-$ inner producting with the vectors v_1, v_2 and $\boldsymbol{\mu}$ respectively, for the vector $\dot{v}_1$ we have

$$h\left(\dot{v}_1\left(t\right),v_1\left(t\right)\right)=a_{11}(t)h\left(v_1\left(t\right),v_1\left(t\right)\right)+a_{12}(t)h\left(v_2\left(t\right),v_1\left(t\right)\right)+a_{13}(t)h\left(\boldsymbol{\mu}\left(t\right),v_1\left(t\right)\right), \tag{3.6}$$

$$h\left(\dot{v}_1\left(t\right),v_2\left(t\right)\right)=a_{11}(t)h\left(v_1\left(t\right),v_2\left(t\right)\right)+a_{12}(t)h\left(v_2\left(t\right),v_2\left(t\right)\right)+a_{13}(t)h\left(\boldsymbol{\mu}\left(t\right),v_2\left(t\right)\right), \tag{3.7}$$

$$h\left(\dot{v}_1\left(t\right),\boldsymbol{\mu}\left(t\right)\right)=a_{11}(t)h\left(v_1\left(t\right),\boldsymbol{\mu}\left(t\right)\right)+a_{12}(t)h\left(v_2\left(t\right),\boldsymbol{\mu}\left(t\right)\right)+a_{13}(t)h\left(\boldsymbol{\mu}\left(t\right),\boldsymbol{\mu}\left(t\right)\right).$$

In addition, we obtain by the simple computing with inner products in the equation (3.6) as follows:

$$h\left(v_1\left(t\right),v_1\left(t\right)\right)=\frac{1}{2}\left(v_1\left(t\right)\times\alpha v_1\left(t\right)+v_1\left(t\right)\times\alpha v_1\left(t\right)\right)=\frac{1}{2}2\left(v_1\left(t\right)\times -v_1\left(t\right)\right)=1$$

and similarly, we get the following equations as

$$\begin{aligned}h\left(v_2\left(t\right),v_1\left(t\right)\right)&=\tfrac{1}{2}\left(v_2\left(t\right)\times\alpha v_1\left(t\right)+v_1\left(t\right)\times\alpha v_2\left(t\right)\right)=\left(v_2\left(t\right)\times -v_1\left(t\right)\right)+\left(v_1\left(t\right)\times -v_2\left(t\right)\right)\\&=\tfrac{1}{2}\left(-\left\langle v_2\left(t\right),-v_1\left(t\right)\right\rangle+v_2\left(t\right)\wedge v_1\left(t\right)+\left(-\left\langle v_1\left(t\right),-v_2\left(t\right)\right\rangle\right)+v_1\left(t\right)\wedge v_2\left(t\right)\right)\\&=\tfrac{1}{2}\left(\alpha\boldsymbol{\mu}+\boldsymbol{\mu}\right)=0,\end{aligned}$$

$$\begin{aligned}h\left(\boldsymbol{\mu}\left(t\right),v_1\left(t\right)\right)&=\tfrac{1}{2}\left(\boldsymbol{\mu}\left(t\right)\times\alpha v_1\left(t\right)+v_1\left(t\right)\times\alpha\boldsymbol{\mu}\left(t\right)\right)=\tfrac{1}{2}\left(\left(\boldsymbol{\mu}\left(t\right)\times -v_1\left(t\right)\right)+\left(v_1\left(t\right)\times -\boldsymbol{\mu}\left(t\right)\right)\right)\\&=\tfrac{1}{2}\left(-\left\langle\boldsymbol{\mu},-v_1\left(t\right)\right\rangle+\boldsymbol{\mu}\left(t\right)\wedge -v_1\left(t\right)+\left(-\left\langle v_1\left(t\right),-\boldsymbol{\mu}\left(t\right)\right\rangle\right)+v_1\left(t\right)\wedge -\boldsymbol{\mu}\left(t\right)\right)\\&=\tfrac{1}{2}\left(\alpha v_2+v_2\right)=0.\end{aligned}$$

It is seen that $\boldsymbol{\mu}\perp v_1$ and $v_2\perp v_1$ are orthogonal unit quaternions. Here is

$$\begin{aligned}\boldsymbol{\mu}\times v_1&=-v_1\times\boldsymbol{\mu}=v_2,\\\boldsymbol{\mu}\times v_2&=-v_2\times\boldsymbol{\mu}=-v_1\end{aligned}$$

and by using quaternionic inner product we get

$$\begin{aligned}h\left(\dot{v}_1\left(t\right),v_1\left(t\right)\right)&=a_{11}(t),\\h\left(\dot{v}_1\left(t\right),v_2\left(t\right)\right)&=a_{12}(t),\\h\left(\dot{v}_1\left(t\right),\boldsymbol{\mu}\left(t\right)\right)&=a_{13}(t).\end{aligned}$$

Also, by taking the derivative of both sides of the expression $h\left(v_1\left(t\right),v_1\left(t\right)\right)=1$ for each $t\in I$ we get

$$h\left(\dot{v}_1\left(t\right),v_1\left(t\right)\right)+h\left(v_1\left(t\right),\dot{v}_1\left(t\right)\right)=0.$$

The last equation gives $h\left(v_1\left(t\right),\dot{v}_1\left(t\right)\right)=0$. In this case, it is $\dot{v}_1\perp v_1$. In this case, it is obtained as $a_{11}(t)=0$. Therefore the equation (3.5) can be written as

$$\dot{v}_1\left(t\right)=a_{12}(t)v_2\left(t\right)+a_{13}(t)\boldsymbol{\mu}\left(t\right). \tag{3.8}$$

The $h-$ inner products in the equation (3.7) are obtained as follows:

$$h\left(v_2\left(t\right),v_2\left(t\right)\right)=\frac{1}{2}\left(v_2\left(t\right)\times\alpha v_2\left(t\right)+v_2\left(t\right)\times\alpha v_2\left(t\right)\right)=1,$$

and

$$h\left(\boldsymbol{\mu}\left(t\right),v_2\left(t\right)\right)=\frac{1}{2}\left(\left(\boldsymbol{\mu}\left(t\right)\times\alpha v_2\left(t\right)\right)+\left(v_2\left(t\right)\times\alpha\boldsymbol{\mu}\left(t\right)\right)\right)=0.$$

Hence, the vector v_2 is unit and $\boldsymbol{\mu}\perp v_1$. Now we take the derivative of both sides of the equation $h\left(v_2\left(t\right),v_1\left(t\right)\right)=0$ and by writing $h\left(\dot{v}_1\left(t\right),v_2\left(t\right)\right)=l\left(t\right)$ we obtain the below equations: $a_{12}=l\left(t\right)$ and $h\left(v_1\left(t\right),\dot{v}_2\left(t\right)\right)=-l\left(t\right)$. On the other hand, by derivativing of both sides of the equation $h\left(v_1\left(t\right),\boldsymbol{\mu}\left(t\right)\right)=0$ for each $t\in I$ we obtain

$$h\left(\dot{v}_1\left(t\right),\boldsymbol{\mu}\left(t\right)\right)+h\left(v_1\left(t\right),\dot{\boldsymbol{\mu}}\left(t\right)\right)=0$$

and this results as $h\left(\dot{v}_1\left(t\right),\boldsymbol{\mu}\left(t\right)\right)+h\left(v_1\left(t\right),\dot{\boldsymbol{\mu}}\left(t\right)\right)=0$ with by writing $m\left(t\right):=h\left(\dot{v}_1\left(t\right),\boldsymbol{\mu}\left(t\right)\right)$. Hence also $a_{13}(t)=m\left(t\right)$. Then, using the given values, the equation (3.5) can be re-written as

$$\dot{v}_1\left(t\right)=l\left(t\right)v_2\left(t\right)+m\left(t\right)\boldsymbol{\mu}\left(t\right). \tag{3.9}$$

Now, considering that is $\dot{v}_2\left(t\right)=Sp\left\{v_1\left(t\right),v_2\left(t\right),\boldsymbol{\mu}\left(t\right)\right\}$ the following equation can be written as

$$\dot{v}_2\left(t\right)=a_{21}(t)v_1\left(t\right)+a_{22}(t)v_2\left(t\right)+a_{23}(t)\boldsymbol{\mu}\left(t\right).$$

From here, by $h-$ inner product with v_1, v_2 and $\boldsymbol{\mu}$ respectively, we have for the quaternion v_1

$$h\left(\dot{v}_2\left(t\right), v_1\left(t\right)\right) = a_{21}(t) h\left(v_1\left(t\right), v_1\left(t\right)\right) + a_{22}(t) h\left(v_2\left(t\right), v_1\left(t\right)\right) + a_{23}(t) h\left(\boldsymbol{\mu}\left(t\right), v_1\left(t\right)\right)$$

where $v_1\left(t\right) \times v_1\left(t\right) = -1$, $v_2\left(t\right) \times v_1\left(t\right) = 0$ and $\mu\left(t\right) \times v_1\left(t\right) = 0$ from the above equation can be written as $-l\left(t\right) = a_{21}(t)$.
Similarly, $h-$ inner product for the quaternion v_2 it can be obtained as

$$h\left(\dot{v}_2\left(t\right), v_2\left(t\right)\right) = a_{21}(t) h\left(v_1\left(t\right), v_2\left(t\right)\right) + a_{22}(t) h\left(v_2\left(t\right), v_2\left(t\right)\right) + a_{23}(t) h\left(\mu\left(t\right), v_2\left(t\right)\right).$$

Hence, it can be seen that $h\left(\dot{v}_2\left(t\right), \boldsymbol{\mu}\left(t\right)\right) = a_{23}(t)$. Also, by taking the derivative of both sides in the equation $h\left(v_2\left(t\right), \boldsymbol{\mu}\left(t\right)\right) = a_{23}(t)$ for each $t \in I$ we can write the following equation:

$$h\left(\dot{v}_2\left(t\right), \boldsymbol{\mu}\left(t\right)\right) + h\left(v_2\left(t\right), \dot{\boldsymbol{\mu}}\left(t\right)\right) = 0.$$

By writing $n\left(t\right) := a_{23}(t) = h\left(\dot{v}_2\left(t\right), \boldsymbol{\mu}\left(t\right)\right)$ in the last equation we have

$$h\left(v_2\left(t\right), \dot{\boldsymbol{\mu}}\left(t\right)\right) = -n\left(t\right)$$

and hence the equation (3.8) yields

$$\dot{v}_2\left(t\right) = -l\left(t\right) v_1\left(t\right) + n\left(t\right) \boldsymbol{\mu}\left(t\right). \tag{3.10}$$

To summarize, by considering these equations (3.4), (3.9) and (3.10) we get

$$\begin{aligned} \dot{\boldsymbol{\mu}}(t) &= -m(t) v_1(t) - n(t) v_2(t), \\ \dot{v}_1(t) &= l(t) v_2(t) + m(t) \boldsymbol{\mu}(t), \\ \dot{v}_2(t) &= -l(t) v_1\left(t\right) + n(t) \boldsymbol{\mu}(t) \ . \end{aligned}$$

Hence,

$$\begin{bmatrix} \dot{v}_1\left(t\right) \\ \dot{v}_2\left(t\right) \\ \dot{\boldsymbol{\mu}}\left(t\right) \end{bmatrix} = \begin{bmatrix} 0 & l\left(t\right) & m\left(t\right) \\ -l\left(t\right) & 0 & n\left(t\right) \\ -m\left(t\right) & -n\left(t\right) & 0 \end{bmatrix} \begin{bmatrix} v_1\left(t\right) \\ v_2\left(t\right) \\ \boldsymbol{\mu}\left(t\right) \end{bmatrix}. \tag{3.11}$$

□

Now the formulas given in equation (3.11) can be better understood with the following example.

Example 3.1.

$$\boldsymbol{\gamma}\left(t\right) = \left(3t^2, t^3, -t^3\right),$$

$$v_1\left(t\right) = \frac{1}{\sqrt{t^2+4}}\left(t, 0, 2\right),$$

$$v_2\left(t\right) = \frac{1}{\sqrt{t^2+4}\sqrt{2t^2\left(t^2+2\right)}}\left(2t^2, -4t-t^3, -t^3\right)$$

since $h\left(\dot{\boldsymbol{\gamma}}\left(t\right), v_1\left(t\right)\right) = 0$, $h\left(\dot{\boldsymbol{\gamma}}\left(t\right), v_2\left(t\right)\right) = 0$ and $h\left(v_1\left(t\right), v_2\left(t\right)\right) = 0$ for all $t \in I$ then $\left(\boldsymbol{\gamma}, v_1, v_2\right) : I \to H \times \Delta_2$ is quaternionic framed curve and $\boldsymbol{\mu} = v_1 \wedge v_2$ therefore we can calculate the vector $\boldsymbol{\mu}$ as perpendicular both the vector v_1 and the vector v_2 which

$$\boldsymbol{\mu}\left(t\right) = \frac{1}{\sqrt{t^2+2}}\left(2, t, -t\right).$$

It is clear from here that the point $t = 0$ is a singular point of the curve. In other words, since there exists $\boldsymbol{\mu}$ for the spatial quaternionic framed fundamental curve, Serret-Frenet type formulas and framed curvatures can be given in a quaternionic sense. Here, the curvatures are obtained as follows:

$$\begin{aligned} l\left(t\right) &= h\left(\dot{v}_1\left(t\right), v_2\left(t\right)\right) = \tfrac{\sqrt{2}t^2}{\left(t^2+4\right)\sqrt{t^2\left(t^2+2\right)}}, \\ m\left(t\right) &= h\left(\dot{v}_1\left(t\right), \boldsymbol{\mu}\left(t\right)\right) = \tfrac{2}{\sqrt{t^2+2}\sqrt{t^2+4}}, \\ n\left(t\right) &= h\left(\dot{v}_2\left(t\right), \boldsymbol{\mu}\left(t\right)\right) = \tfrac{2\sqrt{2}t}{\sqrt{t^2+2}\sqrt{\left(t^2+4\right)t^2\left(t^2+2\right)}}, \\ \sigma(t) &= 6t\sqrt{t^2+2}. \end{aligned}$$

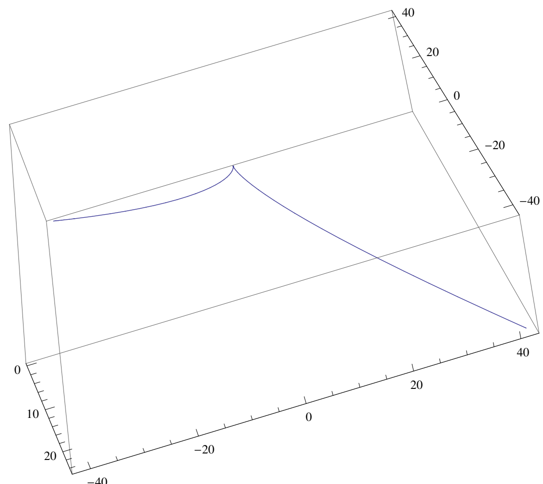

Figure 1. $\boldsymbol{\gamma}(t) = (3t^2, t^3, -t^3)$ quaternionic framed base curve.

Example 3.2. The curve $\boldsymbol{\gamma}(t) = \left(\frac{t^5}{5}, \sin t^4, \cos t^4\right)$ is a quaternionic Frenet-type framed curve. Indeed, the first derivative of the curve is taken, and the part that provides the singularity is separate,d and the other vector unit is united to obtain as

$$\sigma(t) = t^3\sqrt{16+t^2},$$

$$\boldsymbol{\mu}(t) = \frac{1}{\sqrt{16t^6+t^8}}\left(t^4, 4t^3\cos t^4, -4t^3\sin t^4\right)$$

framed elements for each $t \in I$. Therefore, the frame can be established over this vector.

$$v_1(t) = \frac{1}{\sqrt{\sin t^4 + \cos t^4}}\left(0, \sin t^4, \cos t^4\right),$$

$$v_2(t) = \frac{1}{\sqrt{\sin t^4 + \cos t^4}}\left(4t^3, -t^4\cos t^4, t^4\sin t^4\right)$$

since $h(\dot{\boldsymbol{\gamma}}(t), v_1(t)) = 0$, $h(\dot{\boldsymbol{\gamma}}(t), v_2(t)) = 0$ and $h(v_1(t), v_2(t)) = 0$ for all $t \in I$ then $(\boldsymbol{\gamma}, v_1, v_2) : I \to H \times \Delta_2$ is quaternionic framed curve. Also the vector $\boldsymbol{\mu}$, since $\boldsymbol{\mu} = v_1 \wedge v_2$ as perpendicular both the vector v_1 and the vector v_2. It is clear from here that the point $t = 0$ is a singular point of the curve. In other words, since there exists $\boldsymbol{\mu}$ for the spatial quaternionic framed fundamental curve, Serret-Frenet type formulas and framed curvatures can be given in a quaternionic sense. Here, the curvatures are obtained as follows:

$$l(t) = h(\dot{v}_1(t), v_2(t)) = -\frac{4t^7}{\sqrt{t^8+16t^6}},$$

$$m(t) = h(\dot{v}_1(t), \boldsymbol{\mu}(t)) = \frac{16t^6}{\sqrt{t^8+16t^6}},$$

$$n(t) = h(\dot{v}_2(t), \boldsymbol{\mu}(t)) = -\frac{4}{t^2+16},$$

$$\sigma(t) = h(\dot{\boldsymbol{\gamma}}(t), \boldsymbol{\mu}(t)) = \frac{t^8+16t^6}{t^3\sqrt{16+t^2}} = t^3\sqrt{16+t^2}.$$

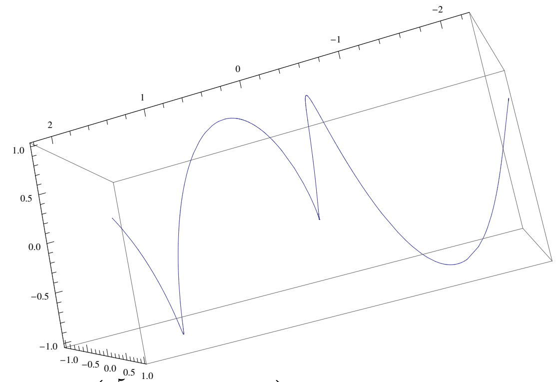

Figure 2. $\gamma\left(t\right)=\left(\dfrac{t^{5}}{5},\sin t^{4},\cos t^{4}\right)$ quaternionic framed base curve

4. Quaternionic framed curves for $\mathbb{R}^4$

In this section, the relationship between a four-dimensional matrix and the curvatures of Serret-Frenet type formulas for quaternionic framed curves in four-dimensional Euclidean space is given. Then, in the light of the information obtained here, Serret-Frenet type formulas and framed curvatures in the quaternionic sense are reinforced with an example.

Theorem 4.1. *Let $\mathbb{R}^4$ denote the four dimensional Euclidean space and let $(\tilde{\boldsymbol{\gamma}},\tilde{v}_1,\tilde{v}_2,\tilde{v}_3):I=[0,1]\subset\mathbb{R}\rightarrow H\times\Delta_3$ be a quaternionic framed curve. Then the quaternionic framed basis curve is defined by $t\rightarrow\tilde{\boldsymbol{\gamma}}\left(t\right)=\sum_{i=0}^{3}\tilde{\boldsymbol{\gamma}}_i\left(t\right)\boldsymbol{e}_i,\ \boldsymbol{e}_0=1$ where $t\in I$ is any parameter. In this case there exist curvature functions $L,M,(-l+N),N,(-n-M),(m+L),\tilde{\sigma}(t):I\rightarrow\mathbb{R}$ such that the Serret-Frenet type formulas at point $\tilde{\boldsymbol{\gamma}}\left(t\right)$ of the quaternionic frame curve $(\tilde{\boldsymbol{\gamma}},\tilde{\boldsymbol{\nu}})$ are given as follows:*

$$\begin{bmatrix}\dot{\tilde{v}}_1\\ \dot{\tilde{v}}_2\\ \dot{\tilde{v}}_3\\ \dot{\tilde{\boldsymbol{\mu}}}\end{bmatrix}=\begin{bmatrix}0 & L & M & -l+N\\ -L & 0 & N & -n-M\\ -M & -N & 0 & m+L\\ l-N & n+M & -m-L & 0\end{bmatrix}\begin{bmatrix}\tilde{v}_1\\ \tilde{v}_2\\ \tilde{v}_3\\ \tilde{\boldsymbol{\mu}}\end{bmatrix}$$

where curvatures are
$L\left(t\right)=h\left(\dot{\tilde{v}}_1\left(t\right),\tilde{v}_2\left(t\right)\right)$, $M\left(t\right)=h\left(\dot{\tilde{v}}_1\left(t\right),\tilde{v}_3\left(t\right)\right)$, $\left(-l+N\right)\left(t\right)=h\left(\dot{\tilde{v}}_1\left(t\right),\boldsymbol{\mu}\left(t\right)\right)$, $N\left(t\right)=h\left(\dot{\tilde{v}}_2\left(t\right),\tilde{v}_3\left(t\right)\right)$, $\left(-n-M\right)\left(t\right)=h\left(\dot{\tilde{v}}_2\left(t\right),\tilde{\boldsymbol{\mu}}\left(t\right)\right)$, $\left(m+L\right)\left(t\right)=h\left(\dot{\tilde{v}}_3\left(t\right),\tilde{\boldsymbol{\mu}}\left(t\right)\right)$, $\tilde{\sigma}\left(t\right)=h\left(\dot{\tilde{\boldsymbol{\gamma}}}\left(t\right),\tilde{\boldsymbol{\mu}}\left(t\right)\right)$.

Proof. Let $(\tilde{\boldsymbol{\gamma}},\tilde{\boldsymbol{\nu}})$ be a quaternionic framed curve and $\tilde{\boldsymbol{\mu}}\left(t\right)=\tilde{v}_1\left(t\right)\wedge\tilde{v}_2\left(t\right)\wedge\tilde{v}_3\left(t\right)$ unit vector for each $t\in I$ in $\mathbb{R}^4$. On the other hand, there is the smooth function $\tilde{\sigma}:I\rightarrow\mathbb{R}$ that satisfies the condition $\dot{\tilde{\boldsymbol{\gamma}}}\left(t\right)=\tilde{\sigma}\left(t\right)\tilde{\boldsymbol{\mu}}\left(t\right)$. Also, since $\tilde{\boldsymbol{\mu}}$ is the unit vector,

$$\left\|\tilde{\boldsymbol{\mu}}\left(t\right)\right\|^2=h\left(\tilde{\boldsymbol{\mu}}\left(t\right),\tilde{\boldsymbol{\mu}}\left(t\right)\right)=\frac{1}{2}\left(\tilde{\boldsymbol{\mu}}\left(t\right)\times\alpha\tilde{\boldsymbol{\mu}}\left(t\right)+\tilde{\boldsymbol{\mu}}\left(t\right)\times\alpha\tilde{\boldsymbol{\mu}}\left(t\right)\right)=\tilde{\boldsymbol{\mu}}\left(t\right)\times\alpha\tilde{\boldsymbol{\mu}}\left(t\right)=1$$

can be written. From here, by taking the derivative of both sides in the last expression, it becomes

$$\dot{\tilde{\boldsymbol{\mu}}}\left(t\right)\times\alpha\tilde{\boldsymbol{\mu}}\left(t\right)+\tilde{\boldsymbol{\mu}}\left(t\right)\times\alpha\dot{\tilde{\boldsymbol{\mu}}}\left(t\right)=0.$$

In this case $h\left(\dot{\tilde{\boldsymbol{\mu}}}\left(t\right),\tilde{\boldsymbol{\mu}}\left(t\right)\right)=\frac{1}{2}\left(\dot{\tilde{\boldsymbol{\mu}}}\left(t\right)\times\alpha\tilde{\boldsymbol{\mu}}\left(t\right)+\tilde{\boldsymbol{\mu}}\left(t\right)\times\alpha\dot{\tilde{\boldsymbol{\mu}}}\left(t\right)\right)=0$. Then $\dot{\tilde{\boldsymbol{\mu}}}\left(t\right)$ and $\tilde{\boldsymbol{\mu}}\left(t\right)$ are orthogonal. Since

$$\dot{\tilde{\boldsymbol{\mu}}}\left(t\right)\times\alpha\tilde{\boldsymbol{\mu}}\left(t\right)+\tilde{\boldsymbol{\mu}}\left(t\right)\times\alpha\dot{\tilde{\boldsymbol{\mu}}}\left(t\right)=0$$

then, according to conjugate properties we can write

$$\dot{\tilde{\boldsymbol{\mu}}}\left(t\right)\times\alpha\tilde{\boldsymbol{\mu}}\left(t\right)+\alpha\left(\tilde{\boldsymbol{\mu}}\left(t\right)\times\alpha\dot{\tilde{\boldsymbol{\mu}}}\left(t\right)\right)=0.$$

Therefore $\dot{\tilde{\boldsymbol{\mu}}}\left(t\right)\times\alpha\tilde{\boldsymbol{\mu}}\left(t\right)$ is spatial quaternion. Also since $\boldsymbol{\mu}$ are spatial and unit then

$$\left\|\boldsymbol{\mu}\left(t\right)\right\|=\boldsymbol{\mu}\left(t\right)\times\alpha\boldsymbol{\mu}\left(t\right)=\dot{\tilde{\boldsymbol{\mu}}}\left(t\right)\times\alpha\tilde{\boldsymbol{\mu}}\left(t\right)\times\alpha\left(\dot{\tilde{\boldsymbol{\mu}}}\left(t\right)\times\alpha\tilde{\boldsymbol{\mu}}\left(t\right)\right)=\dot{\tilde{\boldsymbol{\mu}}}\left(t\right)\times\alpha\tilde{\boldsymbol{\mu}}\left(t\right)\times\tilde{\boldsymbol{\mu}}\left(t\right)\times\alpha\dot{\tilde{\boldsymbol{\mu}}}\left(t\right)=\dot{\tilde{\boldsymbol{\mu}}}\left(t\right)\times\alpha\dot{\tilde{\boldsymbol{\mu}}}\left(t\right)=1.$$

On the other hand since $\dot{\tilde{\boldsymbol{\mu}}}(t) \in Sp\left\{\tilde{v}_1(t), \tilde{v}_2(t), \tilde{v}_3(t)\right\}$ we get

$$\dot{\tilde{\boldsymbol{\mu}}}(t) = a_{41}(t)\tilde{v}_1(t) + a_{42}(t)\tilde{v}_2(t) + a_{43}(t)\tilde{v}_3(t) + a_{44}(t)\tilde{\boldsymbol{\mu}}(t). \tag{4.1}$$

Here, by using quaternionic inner product with $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3$ and $\tilde{\boldsymbol{\mu}}$ respectively,

$$h\left(\dot{\tilde{\boldsymbol{\mu}}}, \tilde{v}_1\right) = a_{41}, \quad h\left(\dot{\tilde{\boldsymbol{\mu}}}, \tilde{v}_2\right) = a_{42}, \quad h\left(\dot{\tilde{\boldsymbol{\mu}}}, \tilde{v}_3\right) = a_{43}, \quad h\left(\dot{\tilde{\boldsymbol{\mu}}}, \tilde{\mu}\right) = a_{44}.$$

Also, since $h(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\mu}}) = 1$ then $h\left(\dot{\tilde{\boldsymbol{\mu}}}, \tilde{\boldsymbol{\mu}}\right) + h\left(\tilde{\boldsymbol{\mu}}, \dot{\tilde{\boldsymbol{\mu}}}\right) = 0$. Thus, considering equation (4.1), it is seen that it is $a_{44}(t) = 0$. On the other hand, considering equation (4.1), we can write following equations

$$\begin{aligned}
\dot{\tilde{v}}_1(t) &= a_{11}(t)\tilde{v}_1(t) + a_{12}(t)\tilde{v}_2(t) + a_{13}(t)\tilde{v}_3(t) + a_{14}(t)\tilde{\mu}(t), \\
\dot{\tilde{v}}_2(t) &= a_{21}(t)\tilde{v}_1(t) + a_{22}(t)\tilde{v}_2(t) + a_{23}(t)\tilde{v}_3(t) + a_{24}(t)\tilde{\mu}(t), \\
\dot{\tilde{v}}_3(t) &= a_{31}(t)\tilde{v}_1(t) + a_{32}(t)\tilde{v}_2(t) + a_{33}(t)\tilde{v}_3(t) + a_{34}(t)\tilde{\mu}(t).
\end{aligned}$$

By using quaternionic inner product with $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3$ and $\tilde{\boldsymbol{\mu}}$ respectively,

$$\begin{aligned}
h\left(\dot{\tilde{v}}_1, \tilde{v}_1\right) &= a_{11}\ h\left(\dot{\tilde{v}}_2, \tilde{v}_1\right) = a_{21}\ h\left(\dot{\tilde{v}}_3, \tilde{v}_1\right) = a_{31}, \\
h\left(\dot{\tilde{v}}_1, \tilde{v}_2\right) &= a_{12}\ h\left(\dot{\tilde{v}}_2, \tilde{v}_2\right) = a_{22}\ h\left(\dot{\tilde{v}}_3, \tilde{v}_2\right) = a_{32}, \\
h\left(\dot{\tilde{v}}_1, \tilde{v}_3\right) &= a_{13}\ h\left(\dot{\tilde{v}}_2, \tilde{v}_3\right) = a_{23}\ h\left(\dot{\tilde{v}}_3, \tilde{v}_3\right) = a_{33}, \\
h\left(\dot{\tilde{v}}_1, \tilde{\boldsymbol{\mu}}\right) &= a_{14}\ h\left(\dot{\tilde{v}}_2, \tilde{\boldsymbol{\mu}}\right) = a_{24}\ h\left(\dot{\tilde{v}}_3, \tilde{\boldsymbol{\mu}}\right) = a_{34}.
\end{aligned}$$

Considering that there are $h(\tilde{v}_1, \tilde{v}_1) = 1$, $h(\tilde{v}_2, \tilde{v}_2) = 1$ and $h(\tilde{v}_1, \tilde{v}_2) = 0$ then we get $a_{11} = a_{22} = 0$. It is also seen that since $h(\tilde{v}_1, \tilde{v}_2) = 0$ then $a_{12} + a_{21} = 0$. Similarly, it can also be shown that $a_{13} + a_{31} = 0$. Suppose now that $a_{12} = L$ and $a_{13} = M$ then we have $a_{21} = -L$ and $a_{31} = -M$. Similarly, in the case that $a_{23} = N$, $a_{14} = F$, $a_{24} = G$ and $a_{34} = H$ we have $a_{32} = -N$, $a_{41} = -F$, $a_{42} = -G$ and $a_{43} = -H$. Using the given values, we get

$$\begin{aligned}
\dot{\tilde{\boldsymbol{\mu}}}(t) &= -F(t)\tilde{v}_1(t) - G(t)\tilde{v}_2(t) - H(t)\tilde{v}_3(t), \\
\dot{\tilde{v}}_1(t) &= L(t)\tilde{v}_2(t) + M(t)\tilde{v}_3(t) + F(t)\tilde{\boldsymbol{\mu}}(t) \\
\dot{\tilde{v}}_2(t) &= -L(t)\tilde{v}_1(t) + N(t)\tilde{v}_3(t) + G(t)\tilde{\boldsymbol{\mu}}(t), \\
\dot{\tilde{v}}_3(t) &= -M(t)\tilde{v}_1(t) - N(t)\tilde{v}_2(t) + H(t)\tilde{\boldsymbol{\mu}}(t).
\end{aligned} \tag{4.2}$$

On the other hand since $h(\tilde{v}_1, \tilde{\boldsymbol{\mu}}) = \frac{1}{2}(\tilde{v}_1 \times \alpha\tilde{\boldsymbol{\mu}} + \tilde{\boldsymbol{\mu}} \times \alpha\tilde{v}_1) = 0$ and

$$\dot{\tilde{\boldsymbol{\mu}}} \times \alpha\tilde{\boldsymbol{\mu}} + \alpha\left(\dot{\tilde{\boldsymbol{\mu}}} \times \alpha\tilde{\boldsymbol{\mu}}\right) = 0$$

then

$$-F(t)\tilde{v}_1(t) - G(t)\tilde{v}_2(t) - H(t)\tilde{v}_3(t) \times \alpha\tilde{\boldsymbol{\mu}}(t) + \tilde{\boldsymbol{\mu}}(t) \times \alpha\left(-F(t)\tilde{v}_1(t) - G(t)\tilde{v}_2(t) - H(t)\tilde{v}_3(t)\right) = 0.$$

If the above equation is arranged for each $t \in I$, then we get

$$-F(\tilde{v}_1 \times \alpha\tilde{\boldsymbol{\mu}} + \tilde{\boldsymbol{\mu}} \times \alpha\tilde{v}_1) - G(\tilde{v}_2 \times \alpha\tilde{\boldsymbol{\mu}} + \tilde{\boldsymbol{\mu}} \times \alpha\tilde{v}_2) - H(\tilde{v}_3 \times \alpha\tilde{\boldsymbol{\mu}} + \tilde{\boldsymbol{\mu}} \times \alpha\tilde{v}_3) = 0.$$

If F, G, and H are different from zero, then the following qualities are written

$$\begin{aligned}
\tilde{v}_1 \times \alpha\tilde{\boldsymbol{\mu}} + \alpha(\tilde{v}_1 \times \alpha\tilde{\boldsymbol{\mu}}) &= 0, \\
\tilde{v}_2 \times \alpha\tilde{\boldsymbol{\mu}} + \alpha(\tilde{v}_2 \times \alpha\tilde{\boldsymbol{\mu}}) &= 0, \\
\tilde{v}_3 \times \alpha\tilde{\boldsymbol{\mu}} + \alpha(\tilde{v}_3 \times \alpha\tilde{\boldsymbol{\mu}}) &= 0.
\end{aligned} \tag{4.3}$$

Thus $\tilde{v}_1 \times \alpha\tilde{\boldsymbol{\mu}}, \tilde{v}_2 \times \alpha\tilde{\boldsymbol{\mu}}, \tilde{v}_3 \times \alpha\tilde{\boldsymbol{\mu}}$ become spatial quaternions. Also, since $\boldsymbol{\mu}$ is spatial and unit,

$$\|\boldsymbol{\mu}\|^2 = \boldsymbol{\mu} \times \alpha\boldsymbol{\mu} = \tilde{v}_1 \times \alpha\tilde{\boldsymbol{\mu}} \times \alpha(\tilde{v}_1 \times \alpha\tilde{\boldsymbol{\mu}}) = \tilde{v}_1 \times \alpha\tilde{\boldsymbol{\mu}} \times \tilde{\boldsymbol{\mu}} \times \alpha\tilde{v}_1 = \tilde{v}_1 \times \alpha\tilde{v}_1 = 1.$$

Since $\boldsymbol{\mu} = \tilde{v}_1 \times \alpha\tilde{\boldsymbol{\mu}}$ along the curve $\tilde{\boldsymbol{\gamma}}$, the vector $\tilde{v}_1$ can be written as

$$\boldsymbol{\mu} \times \tilde{\boldsymbol{\mu}} = \tilde{v}_1. \tag{4.4}$$

On the other hand, from equation (4.3) we get

$$\|v_1\|^2 = v_1 \times \alpha v_1 = \tilde{v}_2 \times \alpha\tilde{\boldsymbol{\mu}} \times \alpha\left(\tilde{v}_2 \times \alpha\tilde{\boldsymbol{\mu}}\right) = \tilde{v}_2 \times \alpha\tilde{\boldsymbol{\mu}} \times \tilde{\boldsymbol{\mu}} \times \alpha\tilde{v}_2 = \tilde{v}_2 \times \alpha\tilde{v}_2 = 1.$$

From the last equation, since $v_1 = \tilde{v}_2 \times \alpha\tilde{\boldsymbol{\mu}}$ along the curve and $\tilde{\boldsymbol{\gamma}}$ the vector $\tilde{v}_3$ can be written as

$$\|v_2\|^2 = v_2 \times \alpha v_2 = \tilde{v}_3 \times \alpha\tilde{\mu} \times \alpha\left(\tilde{v}_3 \times \alpha\tilde{\mu}\right) = \tilde{v}_3 \times \alpha\tilde{\mu} \times \tilde{\mu} \times \alpha\tilde{v}_3 = 1.$$

Hence,

$$v_2 \times \tilde{\boldsymbol{\mu}} = \tilde{v}_3$$

can be written. Taking the derivative of both sides of equation (4.4), we get

$$(-mv_1 - nv_2) \times \tilde{\boldsymbol{\mu}} + \boldsymbol{\mu} \times (-F\tilde{v}_1 - G\tilde{v}_2 - H\tilde{v}_3) = \dot{\tilde{v}}_1.$$

By making the necessary arrangements to the above equation we obtain

$$\begin{aligned} -m + H &= L \Rightarrow H = L + m, \\ -n - G &= M \Rightarrow G = -n - M. \end{aligned}$$

On the other hand, by using the equation $v_1 \times \tilde{\boldsymbol{\mu}} = \tilde{v}_2$ and by differentiating we get

$$(\dot{v}_1 \times \tilde{\boldsymbol{\mu}} + v_1) \times \dot{\tilde{\boldsymbol{\mu}}} = \dot{\tilde{v}}_2.$$

This leads to the following equation:

$$(lv_2 + m\boldsymbol{\mu}) \times \tilde{\boldsymbol{\mu}} + v_1 \times (-F\tilde{v}_1 - G\tilde{v}_2 - H\tilde{v}_3) = -L\tilde{v}_1 + N\tilde{v}_3 + G\tilde{\boldsymbol{\mu}}.$$

By using the equations, $v_2 \times \tilde{\boldsymbol{\mu}} = \tilde{v}_3$, $\boldsymbol{\mu} \times \tilde{\boldsymbol{\mu}} = \tilde{v}_1$, $v_1 \times \boldsymbol{\mu} = -v_2$ and $v_1 \times v_2 = \boldsymbol{\mu}$ in the above equation

$$\begin{aligned} l + F &= N \Rightarrow F = -l + N, \\ m - H &= -L \Rightarrow H = m + L \end{aligned}$$

are obtained. By differentiating $v_2 \times \tilde{\boldsymbol{\mu}} = \tilde{v}_3$ we get

$$\dot{v}_2 \times \tilde{\boldsymbol{\mu}} + v_2 \times \dot{\tilde{\boldsymbol{\mu}}} = \dot{\tilde{v}}_3.$$

Considering the equations (4.2), (3.10), the above equation can be written as

$$(-lv_1 + n\boldsymbol{\mu}) \times \tilde{\boldsymbol{\mu}} + v_2 \times (-F\tilde{v}_1 - G\tilde{v}_2 - H\tilde{v}_3) = -M\tilde{v}_1 - N\tilde{v}_3 + H\tilde{\mu}.$$

If we use the $\tilde{v}_2$ following

$$\begin{aligned} v_1 \times \tilde{\boldsymbol{\mu}} &= \tilde{v}_2, \\ \boldsymbol{\mu} \times \tilde{\boldsymbol{\mu}} &= \tilde{v}_1, \\ \tilde{v}_3 &= v_2 \times \tilde{\boldsymbol{\mu}}. \end{aligned}$$

Then we get the following relations

$$\begin{aligned} -l - F &= -n \Rightarrow F = -l + N, \\ n + G &= -M \Rightarrow G = -M - n. \end{aligned}$$

Thus, the matrix representation of the Serret-Frenet formula is given by

$$\begin{bmatrix} \dot{\tilde{v}}_1 \\ \dot{\tilde{v}}_2 \\ \dot{\tilde{v}}_3 \\ \dot{\tilde{\boldsymbol{\mu}}} \end{bmatrix} = \begin{bmatrix} 0 & L & M & -l+N \\ -L & 0 & N & -n-M \\ -M & -N & 0 & m+L \\ l-N & n+M & -m-L & 0 \end{bmatrix} \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{v}_3 \\ \tilde{\boldsymbol{\mu}} \end{bmatrix}. \tag{4.5}$$

Therefore, the proof is completed. □

Here, a geometric interpretation can be provided. The quaternionic framed basis curve $\tilde{\boldsymbol{\gamma}}$ is selected such that its unit vector $\boldsymbol{\mu}$ is defined by the relation $\boldsymbol{\mu} = \tilde{v}_1 \times \alpha\tilde{\boldsymbol{\mu}}$. In this context, the curvatures of a quaternionic framed curve are associated with those in three-dimensional and four-dimensional Euclidean spaces. Consequently, the third curvature of $\tilde{\boldsymbol{\gamma}}$ is determined as the sum of the negative of the first curvature of $\boldsymbol{\gamma}$ and the positive of the fourth curvature of $\tilde{\boldsymbol{\gamma}}$. Similarly, the fifth curvature of $\tilde{\boldsymbol{\gamma}}$ is the sum of the negative sign of the second curvature of $\tilde{\boldsymbol{\gamma}}$ and the third curvature of $\boldsymbol{\gamma}$. Furthermore, the sixth curvature of $\tilde{\boldsymbol{\gamma}}$ is the sum of the first curvature of $\tilde{\boldsymbol{\gamma}}$ and the second curvature of $\boldsymbol{\gamma}$ (See, 4.5).

Example 4.1. $\tilde{\boldsymbol{\gamma}}(t) = \left(\frac{1}{5}\sin(5t) - t\cos(5t), \frac{1}{5}\cos(5t) + t\sin(5t), 2t\cos(2t) - \sin(2t), 2t\sin(2t) + \cos(2t)\right)$

$$\tilde{v}_1(t) = (-\cos(5t), \sin(5t), 0, 0),$$

$$\tilde{v}_2(t) = (0, 0, \cos(2t), \sin(2t)),$$
$$\tilde{v}_3(t) = \frac{5}{\sqrt{41}}\left(\frac{4}{5}\sin(5t), \frac{4}{5}, \cos(5t), \sin(2t), -\cos(2t)\right)$$

and for all $t \in I$ since $h\left(\dot{\tilde{\boldsymbol{\gamma}}}(t), \tilde{v}_1(t)\right) = 0$, $h\left(\dot{\tilde{\boldsymbol{\gamma}}}(t), \tilde{v}_2(t)\right) = 0$, $h(\tilde{v}_1(t), \tilde{v}_2(t)) = 0$, $h(\tilde{v}_1(t), \tilde{v}_3(t)) = 0$, $h(\tilde{v}_2(t), \tilde{v}_3(t)) = 0$, then $(\tilde{\boldsymbol{\gamma}}, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3) : I \to H \times \Delta_3$ is a quaternionic framed curve and there exists $\tilde{\boldsymbol{\mu}}(t) = \tilde{v}_1(t) \wedge \tilde{v}_2(t) \wedge \tilde{v}_3(t)$ such that

$$\begin{aligned}
\tilde{\boldsymbol{\mu}}(t) &= \frac{1}{\sqrt{41}}(5\sin(5t), 5\cos(5t), -4\sin(2t), 4\cos(2t)),\\
\tilde{\sigma}(t) &= \sqrt{(41)}t.
\end{aligned}$$

Here, the quaternions are expressed in vector form. The unit quaternion $\tilde{\boldsymbol{\mu}}$ is computed as orthogonal to the quaternion $\tilde{v}_1$, the quaternion $\tilde{v}_2$ and the quaternion $\tilde{v}_3$. It is clear from here that point $t = 0$ is a singular point of the curve $\tilde{\gamma}$. In other words, since there exists $\tilde{\boldsymbol{\mu}}$ for the quaternionic framed fundamental curve $\tilde{\gamma}$, Serret-Frenet type formulas and framed curvatures can be given in a quaternionic sense. Hence, the curvatures of the curve $\tilde{\gamma}$ are

$$\begin{aligned}
L(t) &= h\left(\dot{\tilde{v}}_1(t), \tilde{v}_2(t)\right) = 0,\\
M(t) &= h\left(\dot{\tilde{v}}_1(t), \tilde{v}_3(t)\right) = \frac{20\sqrt{41}}{41},\\
(N - l)(t) &= h\left(\dot{\tilde{v}}_1(t), \tilde{\boldsymbol{\mu}}(t)\right) = \frac{25}{\sqrt{41}},\\
N(t) &= h\left(\dot{\tilde{v}}_2(t), \tilde{v}_3(t)\right) = -\frac{10\sqrt{41}}{41},\\
(-n - M)(t) &= h\left(\dot{\tilde{v}}_2(t), \tilde{\boldsymbol{\mu}}(t)\right) = \frac{8\sqrt{41}}{41},\\
(m + L)(t) &= h\left(\dot{\tilde{v}}_3(t), \tilde{\boldsymbol{\mu}}(t)\right) = 0,\\
\tilde{\sigma}(t) &= h\left(\dot{\tilde{\gamma}}(t), \tilde{\boldsymbol{\mu}}(t)\right) = \sqrt{41}t.
\end{aligned}$$

Also, in this example we can find the spatial quaternionic framed curve γ related to curve $\tilde{\gamma}$ and its Frenet elements and curvatures. Using the proof of Theorem 4.1, we get

$$\begin{aligned}
\boldsymbol{\gamma}(t) &= \frac{\sqrt{41}}{82}(10\cos(t), -\sin(4t) + 2\sin(2t), -\cos(4t) + 2\cos(2t))\\
v_1 &= -\frac{\sqrt{41}}{41}(-4, 5\sin(3t), 5\cos(3t)),\\
v_2 &= (0, \cos(3t), -\sin(3t)),\\
\boldsymbol{\mu}(t) &= (5, 4\sin(3t), 4\cos(3t)),\\
\alpha(t) &= \sin(t).
\end{aligned}$$

Thus, the curvatures of the spatial quaternionic framed curve γ are

$$l(t) = h(\dot{v}_1(t), v_2(t)) = -\frac{15\sqrt{41}}{41},$$

$$m(t) = h(\dot{v}_1(t), \boldsymbol{\mu}(t)) = 0,$$

$$n(t) = h(\dot{v}_2(t), \boldsymbol{\mu}(t)) = -\frac{12\sqrt{41}}{41}$$

$$\sigma(t) = h(\dot{\boldsymbol{\gamma}}(t), \boldsymbol{\mu}(t)) = \sin(\mathrm{t}).$$

Based on this example, the theorem has been verified, and corresponding relations have been successfully obtained.

5. Results

As is known, a framed curve is a smooth space curve with a moving frame with singular points. In this study, the quaternionic structure of framed curves is discussed. Firstly, quaternionic framed curves are defined in three-dimensional Euclidean space with the help of framed curve definitions, which are defined in [7]. Then, Serret-Frenet formulas and curvatures for quaternionic framed curves in three-dimensional Euclidean space $\mathbb{R}^3$ are given in Theorem 3.1. In addition, these formulas is supported by examples. It is seen in the matrix expression given in (3.11) that the first curvature of $\boldsymbol{\gamma}$ is equal to the principal curvature given for regular curves and the third curvature of $\boldsymbol{\gamma}$ is equal to the torsion given for regular curves. Then, for non-spatial quaternionic framed curves in 4-dimensional Euclidean space, Theorem 4.1 and Serret-Frenet type new formulas and curvatures are obtained as in equation (4.5). Matrix representations of these new formulas are given. If we make some geometric deductions from equation (4.5), Frenet elements and Serret-Frenet formulas for quaternionic framed base curve $\tilde{\gamma}$ are obtained by using Serret-Frenet formulas for quaternionic framed base curve $\boldsymbol{\gamma}$ in $\mathbb{R}^3$ (See 4.5). Here, the quaternionic framed basis curve $\tilde{\gamma}$ is chosen such that the unit vector $\boldsymbol{\mu}$ of the curve $\tilde{\gamma}$ is given by the relation $\boldsymbol{\mu} = \tilde{v}_1 \times \alpha\tilde{\boldsymbol{\mu}}$. Thus, the 3th curvature of $\tilde{\gamma}$ is the sum of the negative sign of the 1st curvature of $\boldsymbol{\gamma}$ and the positive sign of 4th curvature of $\tilde{\gamma}$, and the 5th curvature of $\tilde{\gamma}$ is the sum of the negative sign of 2th curvature of $\tilde{\gamma}$ and 3nd curvature of $\boldsymbol{\gamma}$, and the 6th curvature of $\tilde{\gamma}$ is the sum of the 1th curvature of $\tilde{\gamma}$ and 2nd curvature of $\boldsymbol{\gamma}$(See, 4.5). Finally, the theoretical framework is illustrated through examples, offering a quaternion-based perspective on framed curves.

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Statistical Convergence of Matrix Sequences and Eigenvalue Influences

Prasenjit Bal*, Bikram Sutradhar and Gourab Pal

Abstract
This paper delves into the statistical convergence of sequences of square matrices with entries in the real or complex domain. Matrix sequence convergence is traditionally examined through two distinct lenses: element-wise convergence and norm convergence. We explore both paradigms, unraveling their interconnections through illustrative examples. Furthermore, we shed light on the intrinsic nature of matrix sequence convergence, emphasizing its intricate dependence on the eigenvalues of the matrices involved.

Keywords: *Eigenvalue, Matrix sequence, Statistical convergence*

AMS Subject Classification (2020): *15A60; 40C05; 40C15*

1. Introduction

The concept of natural density (also called asymptotic density) originates from number theory and has been used for studying the distribution of subsets of natural numbers [1]. It was first introduced in the 19th and early 20th centuries as mathematicians explored the frequency of prime numbers and other arithmetic sequences. The idea of natural density became an essential tool in probabilistic number theory, where mathematicians sought to understand how frequently certain types of numbers appear within the natural numbers.

For a subset A of natural number $\mathbb{N}$, the natural density of A, if it exist, is given by

$$\delta(A) = \lim_{n\to\infty} \frac{|A \cap \{1, 2, 3, ...n\}|}{n}.$$

This measures the proportion of numbers in A relative to all natural numbers up to n.

The concept of statistical convergence was introduced by Fast [2] in the year 1951 and later extended by Schoenberg [3] in the year 1959 and others in the field of summability theory. The idea emerged from the need to study convergence beyond classical pointwise limits, especially in probability theory and real analysis [4–11]. Some

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deeper discussions of recent related works in statistical convergence and matrix analysis can be found in [12, 13]. A matrix norm is a function that assigns a non-negative real number to a matrix, representing its size or magnitude. Just like vector norms measure the length of a vector, matrix norms provide a measure of how a matrix transforms vectors. A well-defined matrix norm satisfies properties such as non-negativity, homogeneity, subadditivity (triangle inequality), and the property that the norm is zero if and only if the matrix itself is the zero matrix [1]. On the other side, eigenvalues play a fundamental role in understanding linear transformations and systems. They help in determining the stability of a system, as seen in differential equations and control theory, where the nature of eigenvalues indicates whether a system will remain stable or diverge over time. The importance of eigenvalues extends across multiple disciplines, making them indispensable in various applications. Moreover, the nature of a matrix determines the nature of its eigenvalues, and the nature of an eigenvalue can also indicate the nature of the corresponding matrix. This motivation drives us to explore diverse forms of statistical convergence in matrix sequences and to examine the pivotal role of eigenvalues in shaping their statistical convergence behavior.

2. Preliminaries

In this section, we discuss key definitions and foundational concepts pertaining to statistical convergence, statistical convergence of order α, norms, and eigenvalues, which serve as the cornerstone for our study. In this paper we adopt the following notions:

$s-\lim$	Statistical limit.
δ	Natural density or asymptotic density.
$M^{(k)}$	k-th element in a matrix sequence, k is an index.
$m_{ij}^{(k)}$	The element of the matrix $M^{(k)}$ at the i-th row and j-th column.
$\varepsilon, \varepsilon'$	Very small positive numbers.

For other usual notions and symbols, we follow [14].

Statistical convergence is a generalization of the usual notion of convergence in mathematical analysis.

Definition 2.1. A real or complex sequence $\{x_n\}$ is said to be statistically convergent to L if for every $\varepsilon > 0$, the set $S(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has natural density zero, i.e.,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

This means that the proportion of terms in the sequence that deviate from the limit L by at least ε, becomes arbitrarily small as n increases [15]. It is known that every convergent sequence statistically converges to the same limit, statistical limits are unique, subsequences of a statistically convergent sequence may not be statistically convergent to the same limit, if $\{x_n\}$ and $\{y_n\}$ are statistically convergent to L and M respectively, then for scalars a, b the sequence $\{ax_n + by_n\}$ is statistically convergent to $aL + bM$, a statistically convergent sequence may not be bounded.

Statistical convergence of order α $(0 < \alpha \leq 1)$ is an extension of statistical convergence.

Definition 2.2. A sequence $\{x_n\}$ is said to be statistically convergent of order α to L if

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

This definition refines the concept of statistical convergence by introducing a parameter α, which controls the rate at which non-convergent terms diminish [16, 17].

Definition 2.3. A sequence $\{M^{(k)} : k \in \mathbb{N}\}$ (where $M^{(k)} \in M_n(\mathbb{C})$, the set of all $n \times n$ complex matrices) is said to element wise convergent to $M \in M_n(\mathbb{C})$ if $m_{ij}^{(k)} \to m_{ij}$ as $k \to \infty$ for each pair of sub-scripts i, j $(1 \leq i, j \leq n)$.

Example 2.1. Let $M^{(k)} = \begin{bmatrix} 0 & \frac{2k+1}{k} \\ \frac{1+6k}{2k} & 0 \end{bmatrix}$, then the sequence $\{M^{(k)} : k \in \mathbb{N}\}$ converges to $\begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$.

Definition 2.4. A norm is a function $\|\cdot\|$ that assigns a non-negative length or size to a vector in a vector space. A norm satisfies the following properties for all vectors u, v and scalar λ :

1. $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$.
2. $\|\lambda v\| = |\lambda|\|v\|$.
3. $\|u+v\| \leq \|u\| + \|v\|$.

For a matrix $A = [a_{i,j}]$ with dimensions $m \times n$, the following norms can be found in the literature:

- 1-Norm (Column Sum Norm):$\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}|$.
- Infinity-Norm (Row Sum Norm):$\|A\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|$.
- Euclidean Norm:$\|A\|_E = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij})^2}$.
- Frobenius Norm: $\|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2}$.
- Spectral Norm: $\|A\|_2 = \max_{1\leq i\leq n} |\lambda_i|$($\lambda_i$'s are the eigenvalues of A).

In this paper, we intend to discover some properties of the statistical convergence of matrix sequences.

3. Statistical convergence of matrix sequences

Statistical convergence of matrix sequences using the norm was first discussed by Yildirim. Motivated by his work, we introduce the following concept.

Definition 3.1. A sequence $\{M^{(k)} : k \in \mathbb{N}\}$ (where $M^{(k)} \in M_n(\mathbb{C})$, the set of all $n \times n$ complex matrices) is said to element wise statistical convergent to $M \in M_n(\mathbb{C})$ if for every $0 \leq i, j \leq 1$, $m_{ij}^{(k)}$ statistically converges to m_{ij}, i.e., $m_{ij}^{(k)} \overset{s-\lim}{\rightarrow} m_{ij}$ as $k \to \infty$.

Example 3.1. The sequence $\{M^{(k)} : k \in \mathbb{N}\}\}$, where

$$M^{(k)} = \begin{cases} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, & \text{if } k = a^3 \text{ for some } a \in \mathbb{N}, \\ \\ \begin{bmatrix} 0 & \frac{2k+1}{k} \\ \frac{1+6k}{2k} & 0 \end{bmatrix}, & \text{otherwise.} \end{cases}$$

statistically convergent to $\begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$ but not convergent at all.

Theorem 3.1. *For the sequences $\{A^{(k)} : k \in \mathbb{N}\}$ and $\{B^{(k)} : k \in \mathbb{N}\}$ of $n \times n$ complex matrices, if $A^{(k)} \overset{s-\lim}{\rightarrow} A$ and $B^{(k)} \overset{s-\lim}{\rightarrow} B$ then $A^{(k)} + B^{(k)} \overset{s-\lim}{\rightarrow} A + B$.*

Proof. Let $A^{(k)} \overset{s-\lim}{\rightarrow} A$ and $B^{(k)} \overset{s-\lim}{\rightarrow} B$. So, $a_{ij}^{(k)} \overset{s-\lim}{\rightarrow} a_{ij}$ and $b_{ij}^{(k)} \overset{s-\lim}{\rightarrow} b_{ij}$, for all $1 \leq i, j \leq n$. So, for every $\varepsilon > 0$,

$$\delta(C_{ij} = \{K \in \mathbf{N} : |a_{ij}^{(k)} - a_{ij}| \geq \frac{\varepsilon}{2}\}) = 0, \quad \text{for all } 1 \leq i, j \leq n.$$

and

$$\delta(D_{ij} = \{K \in \mathbf{N} : |b_{ij}^{(k)} - b_{ij}| \geq \frac{\varepsilon}{2}\}) = 0, \quad \text{for all } 1 \leq i, j \leq n.$$

Therefore, $\delta(C_{ij} \cup D_{ij}) = 0$ for all $1 \leq i,j \leq n$. Now, let $p \notin (C_{ij} \cup D_{ij})$ for specific $1 \leq i,j \leq n$. Therefore, $|a_{ij}^{(p)} - a_{ij}| < \frac{\varepsilon}{2}$ and $|b_{ij}^{(p)} - b_{ij}| < \frac{\varepsilon}{2}$ for that $1 \leq i,j \leq n$. So, $|(a_{ij}^{(p)} + b_{ij}^{(p)}) - (a_{ij} + b_{ij})| < \varepsilon$ for the same $1 \leq i,j \leq n$.

$$\text{Therefore, } \{k \in \mathbb{N} : |(a_{ij}^{(k)} + b_{ij}^{(k)}) - (a_{ij} + b_{ij})| \geq \varepsilon\} \subseteq (C_{ij} \cup D_{ij}) \text{ for that } 1 \leq i,j \leq n.$$

$$\text{Thus, } \delta(\{k \in \mathbb{N} : |(a_{ij}^{(k)} + b_{ij}^{(k)}) - (a_{ij} + b_{ij})| \geq \varepsilon\}) = 0 \text{ for all } 1 \leq i,j \leq n.$$

So, $A^{(k)} + B^{(k)} \overset{s-\lim}{\to} A + B$.

□

Theorem 3.2. *For the sequences $\{A^{(k)} : k \in \mathbb{N}\}$ and $\{B^{(k)} : k \in \mathbb{N}\}$ of $n \times n$ complex matrices, if $A^{(k)} \overset{s-\lim}{\to} A$ and $B^{(k)} \overset{s-\lim}{\to} B$, then $A^{(k)}B^{(k)} \overset{s-\lim}{\to} AB$.*

Proof. Let $A^{(k)} \overset{s-\lim}{\to} A$ and $B^{(k)} \overset{s-\lim}{\to} B$. So, $a_{ij}^{(k)} \overset{s-\lim}{\to} a_{ij}$ and $b_{ij}^{(k)} \overset{s-\lim}{\to} b_{ij}$, for all $1 \leq i,p,j \leq n$. So, for very small $\varepsilon' > 0$,

$$\delta(C_{ij} = \{k \in \mathbb{N} : |a_{ij}^{(k)} - a_{ij}| \geq \frac{\varepsilon'}{2n}\}) = 0, \quad \text{for all } 1 \leq i,j \leq n.$$

and

$$\delta(D_{jp} = \{k \in \mathbb{N} : |a_{ij}^{(k)} - a_{ij}| \geq \frac{\varepsilon'}{2n}\}) = 0, \quad \text{for all } 1 \leq i,j \leq n.$$

Therefore, $\delta(C_{ij} \cup D_{jp}) = 0$ for all $1 \leq i,j,p \leq n$. We choose $\varepsilon > 0$ so small such that

$$\varepsilon'. \sup_{(k \in C_{ij}^c)} |a_{ij}^{(k)}| < \frac{\varepsilon}{2n} \text{ and } \varepsilon'.(|b_{jp}|) < \frac{\varepsilon}{2n} \text{ for all } 1 \leq i,j,p \leq n.$$

Let $q \notin C_{ij} \cup D_{jp}$ for all $1 \leq i,j,p \leq n$.

$$\text{Therefore, } |a_{ij}^{(q)} \cdot b_{jp}^{(q)} - a_{ij} \cdot b_{jp}| = |a_{ij}^{(q)} \cdot b_{jp}^{(q)} - a_{ij}^{(q)} \cdot b_{jp} + a_{ij}^{(q)} \cdot b_{jp} - a_{ij} \cdot b_{jp}|$$

$$\leq |a_{ij}^{(q)}| \cdot |b_{jp}^{(q)} - b_{jp}| + |b_{jp}| \cdot |a_{ij}^{(q)} - a_{ij}| < \frac{\varepsilon}{2n\varepsilon'} \cdot \varepsilon' + \frac{\varepsilon}{2n\varepsilon'} \cdot \varepsilon' = \frac{\varepsilon}{n}.$$

Thus for all $q \notin C_{ij} \cup D_{jp}$ for all $1 \leq i,p \leq n$.

$$\text{So, } |\sum_{j=1}^{n} a_{ij}^{(q)} \cdot b_{jp}^{(q)} - \sum_{j=1}^{n} a_{ij} \cdot b_{jp}| \leq \sum_{j=1}^{n} |a_{ij}^{(q)} \cdot b_{jp}^{(q)} - a_{ij} \cdot b_{jp}| < n \cdot \frac{\varepsilon}{n} = \varepsilon.$$

$$\text{Therefore, } \{k \in \mathbb{N} : |\sum_{j=1}^{n} a_{ij}^{(k)} \cdot b_{jp}^{(k)} - \sum_{j=1}^{n} a_{ij} \cdot b_{jp}|\} \subseteq C_{ij} \cup D_{jp}, \text{ for all } 1 \leq i,p \leq n.$$

$$\text{So, } \delta(\{k \in \mathbb{N} : |\sum_{j=1}^{n} a_{ij}^{(k)} \cdot b_{jp}^{(k)} - \sum_{j=1}^{n} a_{ij} \cdot b_{jp}| \geq \varepsilon\}) = 0, \text{ for all } 1 \leq i,p \leq n.$$

So, $A^{(k)}.B^{(k)} \overset{s-\lim}{\to} A \cdot B$. □

Theorem 3.3. *Let $\{A^{(k)} : k \in \mathbb{N}\}$ be a sequence of $n \times n$ complex matrices. If $A^{(k)} \xrightarrow{s-\lim} A$, then $(A^{(k)})^T \xrightarrow{s-\lim} A^T$.*

Proof. The proof is a direct consequence of the Definition 3.1. □

Theorem 3.4. *Let $\{A^{(k)} : k \in \mathbb{N}\}$ be a sequence of $n \times n$ complex orthogonal matrices. If $A^{(k)} \xrightarrow{s-\lim} A$, then A is an orthogonal matrix.*

Proof. Let $\{A^{(k)} : k \in \mathbb{N}\}$ be a sequence of $n \times n$ complex orthogonal matrices and $A^{(k)} \xrightarrow{s-\lim} A$.

$$\text{So, for every } \varepsilon > 0, \quad \delta(\Big\{k \in \mathbb{N} : |a_{ij}^{(k)} - a_{ij}| \geq \varepsilon\Big\}) = 0, \quad \text{for all } 1 \leq i, j \leq n.$$

Therefore, $a_{ij}^{(k)} \xrightarrow{s-\lim} a_{ij}$. Since each $A^{(k)}$ is an orthogonal matrix, $(A^{(k)})^T A^{(k)} = I_n$.

$$\text{Thus,} \sum_{r=1}^{n} a_{ri}^{(k)} \cdot a_{rj}^{(k)} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \quad \text{for all } 1 \leq i, j \leq n.$$

$$\text{Since } A^{(k)} \xrightarrow{s-\lim} A, \quad a_{rj}^{(k)} \xrightarrow{s-\lim} a_{rj} \quad \text{and} \quad a_{ri}^{(k)} \xrightarrow{s-\lim} a_{ri}.$$

$$\text{So, } a_{ri}^{(k)} a_{rj}^{(k)} \xrightarrow{s-\lim} a_{ri} a_{rj} \ \text{ i.e., } \sum_{r=0}^{n} a_{ri}^{(k)} a_{rj}^{(k)} \xrightarrow{s-\lim} \sum_{r=0}^{n} a_{ri} a_{rj}.$$

$$\text{So, } \sum_{r=0}^{n} (a_{ri} a_{rj}) = s\text{-}\lim_{k \to \infty} \sum_{r=0}^{n} a_{ri}^{(k)} a_{rj}^{(k)} = s\text{-}\lim_{k \to \infty} \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

$$\text{Therefore, } \sum_{r=0}^{n} a_{ri} a_{rj} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \text{for all } 1 \leq i, j \leq n.$$

So, $A^T A = I_n$. Thus, A is orthogonal. □

Example 3.2. The statistical limit of a sequence of invertible matrices may not be invertible.
Consider the sequence $\{A^{(k)} : k \in \mathbb{N}\}$ of $n \times n$ complex matrices, where

$$A^{(k)} = \begin{cases} \begin{bmatrix} \frac{3k+2}{k+1} & \frac{2k+1}{k} \\ \frac{6k+1}{k} & \frac{4k-3}{k+1} \end{bmatrix}, & \text{if } k = a^2 \text{ for some } a \in \mathbb{N}, \\ \\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, & \text{otherwise.} \end{cases}.$$

$$\text{For each } k \in \mathbb{N}, \ det(A^{(k)}) = \frac{-33k^3 - 35k^2 - 10k - 1}{k^2.(k+1)^2} \neq 0.$$

So, for each $k \in \mathbb{N}$, $A^{(k)}$ is invertible. But the sequence $\{A^{(k)} : k \in \mathbb{N}\}$ is element-wise statistical convergent to $A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$ and $det(A) = 0$, i.e., A is not invertible.

Theorem 3.5. *For a sequence $\{A^{(k)} : k \in \mathbb{N}\}$ of $n \times n$ complex matrices, if $A^{(k)} \xrightarrow{s-\lim} A$, then for any scalar $c \in \mathbb{C}$, $cA^{(k)} \xrightarrow{s-\lim} cA$.*

Proof. The proof follows directly. □

Theorem 3.6. *If $\{A^{(k)} : k \in \mathbb{N}\}$ be a sequence of $n \times n$ stochastic matrices such that $A^{(k)} \xrightarrow{s-\lim} A$, then A is a stochastic matrix.*

Proof. Let $\{A^{(k)} : k \in \mathbb{N}\}$ be a sequence of $n \times n$ stochastic matrices and $A^{(k)} \xrightarrow{s-\lim} A$.

$$\text{So, } a_{ij}^{(k)} \geq 0, \quad \sum_{j=1}^{n} a_{ij}^{(k)} = 1 \quad \text{and} \quad a_{ij}^{(k)} \xrightarrow{s-\lim} a_{ij} \text{ for all } 1 \leq i, j \leq n.$$

Therefore, for every $\varepsilon > 0$,

$$\delta\Big(\Big\{k \in \mathbb{N} : |a_{ij}^{(k)} - a_{ij}| \geq \varepsilon\Big\}\Big) = 0 \quad \text{for all } 1 \leq i, j \leq n.$$

Since, $a_{ij}^{(k)} \geq 0$ for all $1 \leq i, j \leq n, k \in \mathbb{N}$, must be $a_{ij} \geq 0$.

Now, $|a_{ij}^{(k)} - a_{ij}| \xrightarrow{s-\lim} 0$ for all $1 \leq i, j \leq n$. Thus, $\sum_{j=1}^{n} |a_{ij}^{(k)} - a_{ij}| \to 0$ for all $1 \leq i, j \leq n$.

$$\text{So, } \lim_{n\to\infty} \sum_{j=1}^{n} a_{ij}^{(k)} = \sum_{j=1}^{n} a_{ij} \quad \text{for all } 1 \leq i \leq n.$$

Since, $\sum_{j=1}^{n} a_{ij}^{(k)} = 1$ for all $1 \leq i \leq n$, we get, $\sum_{j=1}^{n} a_{ij} = 1$ for all $1 \leq i \leq n$.

$$\text{So, } a_{ij} \geq 0 \text{ and } \sum_{j} a_{ij} = 1 \quad \text{for all } 1 \leq i \leq n.$$

Thus, A itself is a stochastic matrix. □

4. Equivalence of norm s-convergence and element wise s-convergence of matrix sequences

Definition 4.1. A sequence $\{M^{(k)} : k \in \mathbb{N}\}$ $(where M^{(k)} \in M_n(\mathbb{C})$, the set of all $n \times n$ complex matrices) is said to be statistically norm convergent to $M \in M_n(\mathbb{C})$ if for every $\varepsilon > 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} \left| \{k \leq n : \|M^{(k)} - M\| \geq \varepsilon\} \right| = 0.$$

It is known that for finite-dimensional matrices, all norms are equivalent. Now, in this section, we want to show that element-wise statistical convergence of matrix sequences is equivalent to the statistical convergence defined by norms.

Theorem 4.1. *The element wise statistically convergence of a sequence of square matrices of finite order is equivalent to statistical norm convergence.*

Proof. Let $\{M_n^{(k)} : n \in \mathbb{N}\}$ is element-wise statistical convergent to a matrix M_n. Therefore, $m_{ij}^{(k)} \overset{s-\lim}{\to} m_{ij}$ for all $1 \leq i, j \leq n$. So, $\delta(\{k \in \mathbb{N} : |m_{ij}^{(k)} - m_{ij}| \geq \frac{\varepsilon}{n}\}) = 0$ for all $1 \leq i, j \leq n$. Let $p \in \mathbb{N}$ be such that $|m_{ij}^{(p)} - m_{ij}| < \frac{\varepsilon}{n}$ for all $1 \leq i, j \leq n$.

$$\text{Therefore, } |\sum_{j=1}^{n} (m_{ij}^{(p)} - m_{ij})| \leq \sum_{j=1}^{n} |m_{ij}^{(p)} - m_{ij}| < \sum_{j=1}^{n} \frac{\varepsilon}{n} = \varepsilon \text{ for all } 1 \leq i \leq n.$$

$$\text{So, } |\sum_{j=1}^{n} (m_{ij}^{(p)} - m_{ij})| < \varepsilon \text{ for all } 1 \leq i \leq n.$$

$$\text{Therefore, } \max_{1\leq i\leq n} |\sum_{j=1}^{n} (m_{ij}^{(p)} - m_{ij})| < \varepsilon \quad \text{or} \quad ||M^{(p)} - M||_\infty < \varepsilon.$$

$$\text{So, } \{p : |m_{ij}^{(p)} - m_{ij}| < \frac{\varepsilon}{n}\} \subseteq \{p : ||M^{(p)} - M||_\infty < \varepsilon\} \text{ for all } 1 \leq i, j \leq n.$$

$$\text{Therefore, } 1 = \delta(\{p : |m_{ij}^{(p)} - m_{ij}| < \frac{\varepsilon}{n}\}) \leq \delta(\{p : ||M^{(p)} - M||_\infty < \varepsilon\}).$$

$$\text{So, } \delta(\{p : ||M^{(p)} - M||_\infty < \varepsilon\}) = 1 \text{ i.e., } \delta(\{p : ||M^{(p)} - M||_\infty \geq \varepsilon\}) = 0.$$

Therefore, $M^{(k)}$ is statistically norm convergent to M.

Conversely, let $\{M_n^{(k)} : k \in \mathbb{N}\}$ is statistically norm convergent to M. So, for every $\varepsilon > 0$, $\delta(\{k \in \mathbb{N} : ||M^{(k)} - M||_\infty \geq \varepsilon\}) = 0$.

$$\text{Therefore, } \delta(\{k \in \mathbb{N} : \max_{1\leq i\leq n} \sum_{j=1}^{n} |m_{ij}^{(k)} - m_{ij}| \geq \varepsilon\}) = 0.$$

$$\text{For each } k \in \mathbb{N}, \quad |m_{ij}^{(k)} - m_{ij}| \leq \sum_{j=1}^{n} |m_{ij}^{(k)} - m_{ij}|, \text{ for all } 1 \leq i \leq n.$$

$$\text{So, } \{k \in \mathbb{N} : \sum_{j=1}^{n} |m_{ij}^{(k)} - m_{ij}| \geq \frac{\varepsilon}{n}\} \subseteq \{k \in \mathbb{N} : \max_{1 \leq i \leq n} \sum_{j=1}^{n} |m_{ij}^{(k)} - m_{ij}| \geq n \cdot \frac{\varepsilon}{n}\}, \quad \text{for all } 1 \leq i \leq n.$$

$$\text{So, } \delta(\{k \in \mathbb{N} : \sum_{j=1}^{n} |m_{ij}^{(k)} - m_{ij}| \geq \frac{\varepsilon}{n}\}) \leq \delta(\{k \in \mathbb{N} : \sum_{j=1}^{n} |m_{ij}^{(k)} - m_{ij}| \geq \varepsilon\}), \quad \text{for all } 1 \leq i \leq n.$$

$$\text{Therefore, } \delta(\{k \in \mathbb{N} : \sum_{j=1}^{n} |m_{ij}^{(k)} - m_{ij}| \geq \frac{\varepsilon}{n}\}) \leq \delta(\{k \in \mathbb{N} : ||m_{ij}^{(k)} - m_{ij}|| \geq \varepsilon\}), \quad \text{for all } 1 \leq i \leq n.$$

$$\text{Therefore, } \delta(\{k \in \mathbb{N} : |m_{ij}^{(k)} - m_{ij}| \geq \frac{\varepsilon}{n}\}) = 0, \quad \text{for all } 1 \leq i, j \leq n.$$

$$\text{Therefore, } m_{ij}^{(k)} \overset{s-\lim}{\rightarrow} m_{ij}, \quad \text{for all } 1 \leq i, j \leq n.$$

So, $\{M_n^{(k)} : k \in \mathbb{N}\}$ is element-wise statistically convergent to M. □

Theorem 4.2. *Let $\{A^{(k)} : k \in \mathbb{N}\}$ be a sequence of $n \times n$ complex Hermitian matrices. If $A^{(k)} \xrightarrow{s-\lim} A$, then A is also a Hermitian matrix.*

Proof. Let $\{A^{(k)} : k \in \mathbb{N}\}$ be a sequence of $n \times n$ complex Hermitian matrices and $A^{(k)} \xrightarrow{s-\lim} A$. So, $a_{ij}^{(k)} \xrightarrow{s-\lim} a_{ij}$, for all $1 \leq i, j \leq n$. Since $A^{(k)}$ is Hermitian, $a_{ji}^{(k)} = \overline{(a_{ij}^{(k)})}$ for all $1 \leq i, j \leq n$. Therefore, $a_{ji} = s - \lim_{k\to\infty} a_{ji}^{(k)} = s - \lim_{k\to\infty} \overline{(a_{ij}^{(k)})} = \overline{a_{ij}}$, for all $1 \leq i, j \leq n$. Therefore, A is Hermitian. □

Theorem 4.3. *Let $\{A^{(k)} : k \in \mathbb{N}\}$ be a sequence of $n \times n$ complex Skew-Hermitian matrices. If $A^{(k)} \xrightarrow{s-\lim} A$, then A is also a Skew-Hermitian matrix.*

Proof. The proof is similar to the proof of Theorem 4.2. □

Theorem 4.4. *Let $\{A^{(k)} : k \in \mathbb{N}\}$ be a sequence of $n \times n$ nilpotent matrices. If $A^{(k)}$ is statistically convergent, then its statistical limit is the zero matrix.*

Proof. Let $\{A^{(k)} : k \in \mathbb{N}\}$ be a sequence of $n \times n$ nilpotent matrices and $A^{(k)} \xrightarrow{s-\lim} A$. Since $A^{(k)}$ is nilpotent, there exists an integer $m \leq n$ such that ${A^{(k)}}^m = \mathbf{0}$ for each $k \in \mathbb{N}$. So, ${A^{(k)}}^n = \mathbf{0}$ for each $k \in \mathbb{N}$.

$$\text{Therefore, } \{s - \lim_{k\to\infty} A^{(k)}\}^n = s - \lim_{k\to\infty} {A^{(k)}}^n = \mathbf{0}.$$

$$\text{So, } s - \lim_{k\to\infty} A^{(k)} = \mathbf{0}.$$

□

5. Influence of eigenvalues in s-convergence of matrix sequences

Theorem 5.1. *Let $\{A^{(k)} : k \in \mathbb{N}\}$ be a sequence of Hermitian positive definite $n \times n$ complex matrices such that $\{A^{(k)} : k \in \mathbb{N}\}$ statistically converges to the zero matrix then the product of eigenvalues of $A^{(k)}$ statistically converges to zero.*

Proof. Let $\{A^{(k)} : k \in \mathbb{N}\}$ be a sequence of Hermitian positive definite $n \times n$ complex matrices, and $A^{(k)} \xrightarrow{s-\lim} 0$. So, for every $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : ||A^{(k)}||_2 \geq \varepsilon^{\frac{1}{n}}\}) = 0.$$

Since $A^{(k)}$ is Hermitian positive definite, eigenvalues of $A^{(k)}$ are $\lambda_1^{(k)}, \lambda_2^{(k)}, \ldots \lambda_n^{(k)}$ (say), which are positive for each $k \in \mathbb{N}$.

For each $k \in \mathbb{N}$, let $M^{(k)} = \max_{1 \leq i \leq n} \lambda_i^{(k)}$ and the geometric mean of eigenvalues of $A^{(k)}$ be

$$G^{(k)} = (\prod_{i=1}^{n} \lambda_i^{(k)})^{\frac{1}{n}}.$$

We know that for positive real numbers, the geometric mean is less than or equal to the arithmetic mean. So,

$$G^{(k)} \leq \frac{1}{n}\sum_{i=1}^{n}\lambda_i^{(k)} \leq \frac{1}{n}.nM^{(k)} = M^{(k)} = ||A^{(k)}||_2 \text{ for each } k \in \mathbb{N}.$$

$$\text{Now, for every } \varepsilon > 0, \{k \in \mathbb{N} : ||A^{(k)}|| < \varepsilon^{\frac{1}{n}}\} \subseteq \{k \in \mathbb{N} : G^{(k)} < \varepsilon^{\frac{1}{n}}\}.$$

$$\text{i.e., } \{k \in \mathbb{N} : ||A^{(k)}|| \geq \varepsilon^{\frac{1}{n}}\} \supseteq \{k \in \mathbb{N} : G^{(k)} \geq \varepsilon^{\frac{1}{n}}\}$$

$$= \{k \in \mathbb{N} : \prod_{i=1}^{n}\lambda_i^{(k)} \geq \varepsilon\}.$$

$$\text{Thus, } \delta(\{k \in \mathbb{N} : \prod_{i=1}^{n}\lambda_i^{(k)} \geq \varepsilon\}) \leq \delta(\{k \in \mathbb{N} : ||A^{(k)}|| \geq \varepsilon^{\frac{1}{n}}\}) = 0.$$

$$\text{Hence, } \prod_{i=1}^{n}\lambda_i^{(k)} \xrightarrow{s-\lim} 0.$$

□

Example 5.1. The converse of Theorem 5.1 does not necessarily hold. Consider a sequence $\{A^{(k)} : k \in \mathbb{N}\}$ of $n \times n$ ($n \geq 3$) diagonal matrices

$$A^{(k)} = \text{diag}(\lambda_1^{(k)}, \lambda_2^{(k)}, \ldots, \lambda_n^{(k)}),$$

where the eigenvalues are given by:

$$\lambda_i^{(k)} = \begin{cases} k, & i = 1, \\ \frac{1}{k}, & 2 \leq i \leq n \end{cases} \text{ for each } k \in \mathbb{N}.$$

$$\text{So, } ||A||_2 = \|A^{(k)}\|_\infty = \max\left\{\frac{1}{k}, k\right\} = k, \text{ for each } k \in \mathbb{N}.$$

Hence, the sequence $\{A^{(k)} : k \in \mathbb{N}\}$ does not statistically converge to zero. However,

$$\prod_{i=1}^{n}\lambda_i^{(k)} = \frac{1}{k^{n-2}}, \text{ for each } k \in \mathbb{N}.$$

So, the sequence $\{\prod_{i=1}^{n}\lambda_i^{(k)} : k \in \mathbb{N}\}$ statistically converges to zero.

Theorem 5.2. *Let $\{A^{(k)} : k \in \mathbb{N}\}$ be a sequence of $n \times n$ complex matrices. If the sequence $\{A^{(k)} : k \in \mathbb{N}\}$ statistically converges to zero matrix, then the sum of eigenvalues of $A^{(k)}$ statistically converges to the zero.*

Proof. Let $\{A^{(k)} : k \in \mathbb{N}\}$ be a sequence of $n \times n$ complex matrices, and $A^{(k)} \xrightarrow{s-\lim} 0$. So, for every $k \in \mathbb{N}$ and $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : ||A^{(k)}||_2 \geq \varepsilon\}) = 0.$$

$$\text{i.e., } \delta(\{k \in \mathbb{N} : \max_{1 \leq i \leq n} |\lambda_i^{(k)}| \geq \varepsilon\}) = 0.$$

$$\text{So, } \delta(\{k \in \mathbb{N} : |\lambda_i^{(k)}| \geq \varepsilon\}) = 0, \text{ for } 1 \leq i \leq n.$$

$$\text{Thus, } \lambda_i^{(k)} \xrightarrow{s-\lim} 0 \text{ for } 1 \leq i \leq n.$$

$$\text{Hence, } \sum_{i=1}^{n}\lambda_i^{(k)} \xrightarrow{s-\lim} 0.$$

□

Example 5.2. The converse of Theorem 5.2 does not necessarily hold. Consider a sequence $\{A^{(k)} : k \in \mathbb{N}\}$ of 3×3 diagonal matrices, where,

$$A^{(k)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{k} \end{bmatrix}.$$

$$\text{So, } \sum_{i=1}^{n} \lambda_i^{(k)} = 1 + (-1) + \frac{1}{k} = \frac{1}{k} \text{ for each } k \in \mathbb{N}.$$

Hence, $\{\sum_{i=1}^{n} \lambda_i^{(k)} : k \in \mathbb{N}\}$ statistically converges to zero. But, $\|A^{(k)}\|_\infty = \max\{|1|, |-1|, |\frac{1}{k}|\} = 1$. So, $\{A^{(k)} : k \in \mathbb{N}\}$ does not statistically converge to zero.

6. Conclusion

In this paper, we explored the statistical convergence of matrix sequences, focusing on element-wise s-convergence and the concept of norm s-convergence in matrix sequences. It has been established that element-wise convergence of matrix entries is equivalent to convergence of the entire matrix sequence in the sense of norm, and statistical convergence of matrix sequences is preserved under matrix addition and matrix multiplication. The statistical limit of s-convergent sequences of orthogonal matrices is an orthogonal matrix. But this does not hold for invertible matrices. We proved that the nature of s-convergence of the sum and product of eigenvalues for a sequence of matrices depends on the nature of the matrices, as well as on the s-convergence of the matrix sequences. These results provided a deeper understanding of the statistical convergence of matrix sequences and their applications in various fields.

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