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Fractional Quantum Hermite-Hadamard Type Inequalities

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Abstract

In this paper, Riemann-Liouville fractional quantum Hermite-Hadamard type inequalities are proved. Also, two identities for continuous functions in the form of Riemann-Liouville fractional quantum integral type are obtained. By using these identities, some Riemann-Liouville fractional quantum trapezoid and midpoint type inequalities for convex functions are given. The results of this paper generalize the results given in earlier works.

Keywords: *Convex functions,Fractional quantum integral, Hermite-Hadamard type inequalities, Quantum derivative, Quantum integral, Riemann-Liouville fractional integrals .*
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1. Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a,b \in I$ with $a < b$. Then the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hermite-Hadamard's inequality for convex functions. This double inequalities have been widely studied in recent years and so many generalizations have been given so far.
Dragomir and Agarwal gave the following identity to obtain trapezoid type inequalities for convex functions:

Lemma 1.1. *[3] Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^\circ$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{b-a}{2}\int_0^1 (1-2t)\, f'(ta+(1-t)b)\,dt. \tag{1.2}$$

Kırmacı gave the following identity to obtain midpoint type inequalities for convex functions.

Lemma 1.2. *[4] Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^\circ$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) = (b-a)\left[\int_0^{\frac{1}{2}} t f'(ta+(1-t)b)\,dt + \int_{\frac{1}{2}}^1 (t-1)\, f'(ta+(1-t)b)\,dt\right]. \tag{1.3}$$

The following properties of convex functions will be used in further proofs.

Theorem 1.3. *[12] If $f : I \to \mathbb{R}$ is convex, then f satisfies a Lipschitz condition on any closed interval $[a,b]$ contained in the interior I° of I. Consequently, f is absolutely continuous on $[a,b]$ and continuous on I°.*

Definition 1.4. *[12] A function f defined on I has a support at $x_0 \in I$ if there exists an affine function $A(x) = f(x_0) + m(x - x_0)$ such that $A(x) \leq f(x)$ for all $x \in I$. The graph of the support function A is called a line of support for f at x_0.*

Theorem 1.5. *[12] $f : (a,b) \to \mathbb{R}$ is a convex function if and only if there is at least one line of support for f at each $x_0 \in (a,b)$.*

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2. Riemann-Liouville Fractional Hermite-Hadamard Type Inequalities

Definition 2.1. *[9] Let $f \in L[a,b]$. Then the left and right Riemann-Liouville fractional integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \ x > a \quad \text{and} \quad J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$.

The following Riemann-Liouville fractional Hermite-Hadamard type inequalities given by Sarıkaya et al.

Theorem 2.2. *[13] Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{2.1}$$

with $\alpha > 0$.

Sarıkaya et al. gave the following identity to obtain Riemann-Liouville fractional trapezoid type inequalities for convex functions.

Lemma 2.3. *[13] Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right] = \frac{b-a}{2} \int_0^1 \left[(1-t)^{\alpha} - t^{\alpha}\right] f'(ta+(1-t)b)\,dt. \tag{2.2}$$

The following left Riemann-Liouville fractional Hermite-Hadamard type inequalities given by Kunt et al.

Theorem 2.4. *[8] Let $a,b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ be a convex function. If $f \in L[a,b]$, then the following inequalities for the left Riemann-Liouville fractional integrals holds:*

$$f\left(\frac{\alpha a+b}{\alpha+1}\right) \leq \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} J^{\alpha}_{a+}f(b) \leq \frac{\alpha f(a)+f(b)}{\alpha+1} \tag{2.3}$$

with $\alpha > 0$.

Kunt et al. gave the following identities to obtain the left Riemann-Liouville fractional trapezoid and midpoint type inequalities for convex functions.

Lemma 2.5. *[8] Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^{\circ}$ with $a < b$. If $f' \in L[a,b]$, then the following equality for the left Riemann-Liouville fractional integrals holds:*

$$\frac{\alpha f(a)+f(b)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} J^{\alpha}_{a+}f(b) = \frac{b-a}{\alpha+1} \int_0^1 \left[1-(\alpha+1)t^{\alpha}\right] f'(ta+(1-t)b)\,dt \tag{2.4}$$

with $\alpha > 0$.

Lemma 2.6. *[8] Let $f : I^{\circ} \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^{\circ}$ with $a < b$. If $f' \in L[a,b]$, then the following equality for the left Riemann-Liouville fractional integrals holds:*

$$\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} J^{\alpha}_{a+}f(b) - f\left(\frac{\alpha a+b}{\alpha+1}\right) = (b-a)\left[\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha} f'(ta+(1-t)b)\,dt + \int_{\frac{\alpha}{\alpha+1}}^1 (t^{\alpha}-1) f'(ta+(1-t)b)\,dt\right] \tag{2.5}$$

with $\alpha > 0$.

3. Quantum Hermite-Hadamard Type Inequalities

Throughout this paper, let $a,b \in \mathbb{R}$ with $a < b$ and $0 < q < 1$ be a constant. The following definitions and theorems for q- derivative (quantum derivative) and q- integral (quantum integral) of a function f on $[a,b]$ are given in [2, 16, 17].

Definition 3.1. *[16, 17] For a continuous function $f : [a,b] \to \mathbb{R}$ the q- derivative of f at $x \in [a,b]$ is characterized by the expression*

$$_aD_q f(x) = \frac{f(x) - f(qx+(1-q)a)}{(1-q)(x-a)}, \ x \neq a. \tag{3.1}$$

Since $f:[a,b]\to\mathbb{R}$ is a continuous function, thus we have ${}_aD_qf(a)=\lim_{x\to a}{}_aD_qf(x)$. The function f is said to be q- differentiable on $[a,b]$ if ${}_aD_qf(t)$ exists for all $x\in[a,b]$. If $a=0$ in (3.1), then ${}_0D_qf(x)=D_qf(x)$, where $D_qf(x)$ is familiar q- derivative of f at $x\in[a,b]$ defined by the expression (see [5])

$$D_qf(x)=\frac{f(x)-f(qx)}{(1-q)x},\ x\neq 0. \tag{3.2}$$

Definition 3.2. *[2, 16, 17] Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Then the q- definite integral on $[a,b]$ is delineated as*

$$\int_a^x f(t)\ {}_ad_qt=(1-q)(x-a)\sum_{i=0}^{\infty}q^if\left(q^ix+\left(1-q^i\right)a\right) \tag{3.3}$$

for $x\in[a,b]$.

If $a=0$ in (3.3), then $\int_0^x f(t)\ {}_0d_qt=\int_0^x f(t)\ d_qt$, where $\int_0^x f(t)\ d_qt$ is familiar q- definite integral on $[0,x]$ defined by the expression (see [5])

$$\int_0^x f(t)\ {}_0d_qt=\int_0^x f(t)\ d_qt=(1-q)x\sum_{i=0}^{\infty}q^if\left(q^ix\right). \tag{3.4}$$

If $c\in(a,x)$, then the q- definite integral on $[c,x]$ is expressed as

$$\int_c^x f(t)\ {}_ad_qt=\int_a^x f(t)\ {}_ad_qt-\int_a^c f(t)\ {}_ad_qt. \tag{3.5}$$

The following quantum Hermite-Hadamard type inequality was first seen in [2].In [19], Zhang et al. gave a shorter and more useful proof and alleviated the assumptions for f.

Theorem 3.3. *[2, 19] Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ and $0<q<1$. Then we have*

$$f\left(\frac{qa+b}{1+q}\right)\leq\frac{1}{b-a}\int_a^b f(t)\ {}_ad_qt\leq\frac{qf(a)+f(b)}{1+q}. \tag{3.6}$$

Alp et al., also, gave the following identity to obtain quantum midpoint type inequalities for convex functions.

Lemma 3.4. *[2] Let $f:[a,b]\to\mathbb{R}$ be a q- differentiable function on (a,b). If ${}_aD_qf$ is continuous and integrable on $[a,b]$, then the following identity holds:*

$$f\left(\frac{qa+b}{1+q}\right)-\frac{1}{(b-a)}\int_a^b f(x)\ {}_ad_qx=q(b-a)\left[\begin{array}{c}\int_0^{\frac{1}{1+q}} t\ {}_aD_qf(tb+(1-t)a)\ {}_0d_qt\\ +\int_{\frac{1}{1+q}}^1\left(t-\frac{1}{q}\right)\ {}_aD_qf(tb+(1-t)a)\ {}_0d_qt\end{array}\right]. \tag{3.7}$$

The authors gave the following identity to obtain quantum trapezoid type inequalities for convex functions.

Lemma 3.5. *[10, 14] Let $f:[a,b]\to\mathbb{R}$ be a continuous function. If ${}_aD_qf$ is an integrable function on (a,b), then the following equality holds:*

$$\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx-\frac{qf(a)+f(b)}{1+q}=\frac{q(b-a)}{1+q}\int_0^1(1-(1+q)t)\ {}_aD_qf(tb+(1-t)a)\ {}_0d_qt. \tag{3.8}$$

4. Riemann-Liouville Type Fractional Quantum Integrals

The concepts given here could be found in, for example [1, 5, 15, 18].

$$[m]_q=\frac{1-q^m}{1-q},\quad m\in\mathbb{R}. \tag{4.1}$$

The q- analog of the power function is defined by

$$(n-m)^{(0)}=1,(n-m)^{(k)}=\prod_{i=0}^{k-1}\left(n-q^im\right),\quad \text{here } k\in\mathbb{N},\ n,m\in\mathbb{R}. \tag{4.2}$$

More generally, if $\gamma\in\mathbb{R}$, then

$$(n-m)^{(\gamma)}=n^{\gamma}\prod_{i=0}^{\infty}\frac{n-q^im}{n-q^{\gamma+i}m},\quad n\neq 0. \tag{4.3}$$

If $m=0$, then $n^{(\gamma)}=n^{\gamma}$. Also $0^{(\gamma)}=0$ for $\gamma>0$. The q- gamma function is defined by

$$\Gamma_q(t)=\frac{(1-q)^{(t-1)}}{(1-q)^{t-1}},\quad t\in\mathbb{R}\backslash\{0,-1,-2,...\}. \tag{4.4}$$

It is clear that $\Gamma_q(t+1)=[t]_q\Gamma_q(t)$. Furthermore, $\lim_{q\to1^-}\Gamma_q(t)=\Gamma(t)$.
For any $s,t>0$, the q- beta function is defined by

$$B_q(s,t)=\int_0^1 u^{(s-1)}(1-qu)^{(t-1)}d_qu, \tag{4.5}$$

and

$$B_q(s,t)=\frac{\Gamma_q(s)\Gamma_q(t)}{\Gamma_q(s+t)}.$$

It is clear $\lim_{q\to1^-} B_q(s,t)=B(s,t)$ where $B(s,t)$ is classical beta function.
The q- analog of the Pochhammer symbol is defined by

$$(m;q)_0=1,\quad (m;q)_k=\prod_{i=0}^{k-1}\left(1-q^im\right),\quad \text{here } k\in\mathbb{N}\cup\{\infty\}. \tag{4.6}$$

Theorem 4.1. *[1, Theorem 1.8] Suppose $\lambda,\mu\in\mathbb{R}$, then*

$$\lim_{q\to1^-}\frac{(q^{\lambda}x;q)_\infty}{(q^{\mu}x;q)_\infty}=(1-x)^{\mu-\lambda}, \tag{4.7}$$

uniformly on $\{x\in\mathbb{C}: |x|\le1\}$, if $\mu\ge\lambda$, $\mu+\lambda\ge1$, and uniformly on compact subset of $\{x\in\mathbb{C}: |x|\le1,\ x\ne1\}$ for other choices of μ and λ.

The authors gave the following definitions and theorems for Riemann-Liouville type fractional quantum integral of a function f on $[a,b]$ (see [15, 18]).
In [15, 18], the authors defined q- shifting operator as

$${}_a\Phi_q(m)=qm+(1-q)a. \tag{4.8}$$

For any positive integer k, one has

$${}_a\Phi_q^k(m)={}_a\Phi_q^{k-1}\left({}_a\Phi_q(m)\right)\quad\text{and}\quad {}_a\Phi_q^0(m)=m. \tag{4.9}$$

The following properties for q- shifting operator could be hold by computing directly.

Properties 1. *[15, 18] For any $n,m\in\mathbb{R}$ and for all positive integer k,j, one has:*

(i) ${}_a\Phi_q^k(m)={}_a\Phi_{q^k}(m)$;
(ii) ${}_a\Phi_q^j\left({}_a\Phi_q^k(m)\right)={}_a\Phi_q^k\left({}_a\Phi_q^j(m)\right)={}_a\Phi_q^{j+k}(m)$;
(ii) ${}_a\Phi_q(a)=a$;
(iv) ${}_a\Phi_q^k(m)-a=q^k(m-a)$;
(v) $m-{}_a\Phi_q^k(m)=\left(1-q^k\right)(m-a)$;
(vi) ${}_a\Phi_q^k(m)=m\,{}_{\frac{a}{m}}\Phi_q^k(1)$ *for* $m\ne0$;
(vii) ${}_a\Phi_q(m)-{}_a\Phi_q^k(n)=q\left(m-{}_a\Phi_q^{k-1}(n)\right)$.

In [15, 18], the authors defined the power of q- shifting operator as

$${}_a(n-m)_q^{(0)}=1,\quad {}_a(n-m)_q^{(k)}=\prod_{i=0}^{k-1}\left(n-{}_a\Phi_q^i(m)\right),\quad\text{here } k\in\mathbb{N}. \tag{4.10}$$

More generally, if $\gamma\in\mathbb{R}$, then

$${}_a(n-m)_q^{(\gamma)}=(n-a)^{\gamma}\prod_{i=0}^{\infty}\frac{n-{}_a\Phi_q^i(m)}{n-{}_a\Phi_q^{\gamma+i}(m)}. \tag{4.11}$$

Properties 2. *[15, 18] For any $\gamma,n,m\in\mathbb{R}$ with $n\ne a$ and $k\in\mathbb{N}$, one has:*

(i) ${}_a(n-m)_q^{(k)}=(n-a)^k\left(\frac{m-a}{n-a};q\right)_k$;
(ii) ${}_a(n-m)_q^{(\gamma)}=(n-a)^{\gamma}\prod_{i=0}^{\infty}\frac{1-\frac{m-a}{n-a}q^i}{1-\frac{m-a}{n-a}q^{\gamma+i}}=(n-a)^{\gamma}\frac{\left(\frac{m-a}{n-a};q\right)_\infty}{\left(\frac{m-a}{n-a}q^{\gamma};q\right)_\infty}$;

(iii) $_{a}\left(n-\ {}_{a}\Phi_q^k(n)\ \right)_q^{(\gamma)} = (n-a)^\gamma \frac{(q^k;q)_\infty}{(q^{\gamma+k};q)_\infty}$.

Definition 4.2. *[15, 18] Let* $\alpha \geq 0$ *and* f *be a continuous function on* $[a,b]$. *Then the Riemann-Liouville type fractional quantum integral is given by* $\left({}_aI_q^0 f\right)(t) = f(t)$ *and*

$$\left({}_aI_q^\alpha f\right)(x) = \left({}_aI_q^\alpha f(t)\right)(x) = \frac{1}{\Gamma_q(\alpha)}\int_a^x {}_a\left(x-\ {}_a\Phi_q(t)\ \right)_q^{(\alpha-1)} f(t)\ {}_ad_qt \tag{4.12}$$

$$= \frac{(1-q)(x-a)}{\Gamma_q(\alpha)}\sum_{i=0}^{\infty} q^i\ {}_a\left(x-\ {}_a\Phi_q^{i+1}(x)\ \right)_q^{(\alpha-1)} f\left(\ {}_a\Phi_q^i(x)\ \right),$$

where $\alpha > 0$ *and* $x \in [a,b]$.

Sudsutad et al. gave the following Riemann-Liouville type fractional quantum Hermite-Hadamard inequalities.

Theorem 4.3. *[15] Let* $f : [a,b] \to \mathbb{R}$ *be a convex continuous function,* $0 < q < 1$ *and* $\alpha > 0$. *Then we have*

$$\frac{2}{\Gamma_q(\alpha+1)} f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)^\alpha}\left({}_aI_q^\alpha f(a+b-t)\right)(b) \leq \frac{1}{(b-a)^\alpha}\ \left({}_aI_q^\alpha f(t)\right)(b)\ \leq \frac{\left([\alpha+1]_q - 1\right) f(a) + f(b)}{\Gamma_q(\alpha+2)}. \tag{4.13}$$

If $f : [a,b] \to \mathbb{R}$ is a convex function, then it is already continuous on $[a,b]$, it means that in Theorem 4.3, it is not need to assume that f is continuous. In this paper, we will give more appropriate Riemann-Liouville fractional quantum Hermite-Hadamard type inequalities for convex functions than Theorem 4.3. Also, we will give the Riemann-Liouville fractional quantum trapezoid and midpoint type inequalities which generalize the results given in papers [2, 3, 4, 8, 10, 11, 14].

5. Main Results

The following Riemann-Liouville fractional quantum Hermite-Hadamard type inequalities for convex functions are more appropriate than Theorem 4.3.

Theorem 5.1. *Let* $f : [a,b] \to \mathbb{R}$ *be a convex function and* $\alpha > 0$. *Then we have*

$$f\left(\frac{\left([\alpha+1]_q - 1\right)a + b}{[\alpha+1]_q}\right) \leq \frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\ \left({}_aI_q^\alpha f\right)(b)\ \leq \frac{\left([\alpha+1]_q - 1\right) f(a) + f(b)}{[\alpha+1]_q}. \tag{5.1}$$

Proof. Since f is convex function on $[a,b]$, using Theorem 1.5, there is at least one line of support

$$A(x) = f\left(\frac{\left([\alpha+1]_q - 1\right)a + b}{[\alpha+1]_q}\right) + m\left(x - \frac{\left([\alpha+1]_q - 1\right)a + b}{[\alpha+1]_q}\right) \leq f(x) \tag{5.2}$$

for all $x \in [a,b]$ and $m \in \left[f'_-\left(\frac{([\alpha+1]_q-1)a+b}{[\alpha+1]_q}\right), f'_+\left(\frac{([\alpha+1]_q-1)a+b}{[\alpha+1]_q}\right)\right]$. From (5.2), we have

$$A((1-t)a+tb) = f\left(\frac{\left([\alpha+1]_q - 1\right)a + b}{[\alpha+1]_q}\right) + m\left((1-t)a+tb - \frac{\left([\alpha+1]_q - 1\right)a + b}{[\alpha+1]_q}\right) \tag{5.3}$$

$$\leq f((1-t)a+tb)$$

for all $t \in [0,1]$.
Multiplying both sides of (5.3) by $\frac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha)}\ {}_0\left(1-\ {}_0\Phi_q(t)\ \right)_q^{(\alpha-1)}$ and q- integration of order $\alpha > 0$ with respect to t over $[0,1]$, we have

$$\frac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha)}\int_0^1 A((1-t)a+tb)\ {}_0\left(1-\ {}_0\Phi_q(t)\ \right)_q^{(\alpha-1)}\ {}_0d_qt$$

$$= \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha)}\int_0^1 \left[f\left(\frac{\left([\alpha+1]_q - 1\right)a + b}{[\alpha+1]_q}\right) + m\left((1-t)a+tb - \frac{\left([\alpha+1]_q - 1\right)a + b}{[\alpha+1]_q}\right)\right]\ {}_0\left(1-\ {}_0\Phi_q(t)\ \right)_q^{(\alpha-1)}\ {}_0d_qt$$

$$= \left[\begin{array}{c} \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha)} f\left(\frac{([\alpha+1]_q-1)a+b}{[\alpha+1]_q}\right)\int_0^1 {}_0\left(1-\ {}_0\Phi_q(t)\ \right)_q^{(\alpha-1)}\ {}_0d_qt \\ +\Gamma_q(\alpha+1)m\left(\begin{array}{c} \frac{a}{\Gamma_q(\alpha)}\int_0^1 {}_0\left(1-\ {}_0\Phi_q(t)\ \right)_q^{(\alpha-1)}\ (1-t)\ {}_0d_qt + \frac{b}{\Gamma_q(\alpha)}\int_0^1 {}_0\left(1-\ {}_0\Phi_q(t)\ \right)_q^{(\alpha-1)}\ t\ {}_0d_qt \\ -\frac{1}{\Gamma_q(\alpha)}\frac{([\alpha+1]_q-1)a+b}{[\alpha+1]_q}\int_0^1 {}_0\left(1-\ {}_0\Phi_q(t)\ \right)_q^{(\alpha-1)}\ {}_0d_qt \end{array}\right) \end{array}\right]$$

$$= \left[\begin{array}{c} \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha)} f\left(\frac{\left([\alpha+1]_q-1\right)a+b}{[\alpha+1]_q}\right) B_q(1,\alpha) \\ +\Gamma_q(\alpha+1)\, m \left(a\left(\frac{1}{\Gamma_q(\alpha+1)} - \frac{1}{\Gamma_q(\alpha+2)}\right) + b\frac{1}{\Gamma_q(\alpha+2)} - \frac{1}{\Gamma_q(\alpha)} \frac{\left([\alpha+1]_q-1\right)a+b}{[\alpha+1]_q} B_q(1,\alpha)\right) \end{array} \right]$$

$$= f\left(\frac{\left([\alpha+1]_q-1\right)a+b}{[\alpha+1]_q}\right) + \Gamma_q(\alpha+1)\, m\left(a\left(\frac{1}{\Gamma_q(\alpha+1)} - \frac{1}{\Gamma_q(\alpha+2)}\right) + b\frac{1}{\Gamma_q(\alpha+2)} - \frac{\left([\alpha+1]_q-1\right)a+b}{[\alpha+1]_q}\frac{1}{\Gamma_q(\alpha+1)}\right)$$

$$= f\left(\frac{\left([\alpha+1]_q-1\right)a+b}{[\alpha+1]_q}\right) + \Gamma_q(\alpha+1)\, m\left(\frac{\left([\alpha+1]_q-1\right)a+b}{\Gamma_q(\alpha+2)} - \frac{\left([\alpha+1]_q-1\right)a+b}{\Gamma_q(\alpha+2)}\right)$$

$$= f\left(\frac{\left([\alpha+1]_q-1\right)a+b}{[\alpha+1]_q}\right)$$

$$\leq \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha)} \int_0^1 {}_0\left(1 - {}_0\Phi_q(t)\right)_q^{(\alpha-1)} f((1-t)a+tb)\ {}_0d_qt$$

$$= \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha)} (1-q) \sum_{i=0}^{\infty} q^i\ {}_0\left(1 - {}_0\Phi_q^{i+1}(1)\right)_q^{(\alpha-1)} f\left(\left(1 - {}_0\Phi_q^i(1)\right)a + {}_0\Phi_q^i(1)\ b\right)$$

(By using Properties 2-(iii), we have)

$$= \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha)} (1-q) \sum_{i=0}^{\infty} q^i \frac{\left(q^{i+1};q\right)_\infty}{\left(q^{(\alpha-1)+(i+1)};q\right)_\infty} f\left(\left(1 - {}_0\Phi_q^i(1)\right)a + {}_0\Phi_q^i(1)\ b\right)$$

$$= \frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha} \left(\frac{(1-q)(b-a)}{\Gamma_q(\alpha)} \sum_{i=0}^{\infty} q^i (b-a)^{\alpha-1} \frac{\left(q^{i+1};q\right)_\infty}{\left(q^{(\alpha-1)+(i+1)};q\right)_\infty} f\left({}_a\Phi_q^i(b)\right)\right)$$

$$= \frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha} \left(\frac{(1-q)(b-a)}{\Gamma_q(\alpha)} \sum_{i=0}^{\infty} q^i\ {}_a\left(b - {}_a\Phi_q^{i+1}(b)\right)_q^{(\alpha-1)} f\left({}_a\Phi_q^i(b)\right)\right)$$

$$= \frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha} \left(\frac{1}{\Gamma_q(\alpha)} \int_a^b {}_a\left(b - {}_a\Phi_q(t)\right)_q^{(\alpha-1)} f(t)\ {}_ad_qt\right)$$

$$= \frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha} \left({}_aI_q^\alpha f\right)(b)\ .$$

which gives the left hand side of (5.1).
On the other hand, using convexity of the function f, we have

$$f((1-t)a+tb) \leq (1-t)f(a) + tf(b) \tag{5.4}$$

for all $t \in [0,1]$. Similarly, multiplying both sides of (5.4) by $\frac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha)}\ {}_0\left(1 - {}_0\Phi_q(t)\right)_q^{(\alpha-1)}$ and q- integration of order $\alpha > 0$ with respect to t on $[0,1]$, we have

$$\frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha} \left({}_aI_q^\alpha f\right)(b) = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha)} \int_0^1 {}_0\left(1 - {}_0\Phi_q(t)\right)_q^{(\alpha-1)} f((1-t)a+tb)\ {}_0d_qt \leq \frac{\left([\alpha+1]_q-1\right)f(a)+f(b)}{[\alpha+1]_q}$$

which gives the right hand side of (5.1). Thus the proof is accomplished. □

Corollary 1. *In Theorem 5.1, one has the following:*

(i) If one takes $\alpha = 1$, then one has Theorem 3.3;
(ii) If one takes $q \to 1^-$, then one has Theorem 2.4;
(iii) If one takes $\alpha = 1$ and $q \to 1^-$, then one has (1.1) the Hermite-Hadamard's inequality.

Proof. (i) Let $\alpha = 1$ in (5.1), then $[\alpha+1]_q = q+1$, $\Gamma_q(\alpha+1) = 1$ and $\left({}_aI_q^1 f\right)(b) = \int_a^b f(t)\ {}_ad_qt$.
(ii) It is clear $[\alpha+1]_q \underset{q\to 1^-}{\longrightarrow} \alpha+1$ and $\Gamma_q(\alpha+1) \underset{q\to 1^-}{\longrightarrow} \Gamma(\alpha+1)$. Thus we have the following limit

$$\frac{\left([\alpha+1]_q-1\right)f(a)+f(b)}{[\alpha+1]_q} \underset{q\to 1^-}{\longrightarrow} \frac{\alpha f(a)+f(b)}{\alpha+1}. \tag{5.5}$$

Since f is convex on $[a,b]$, using Theorem 1.3, we have

$$f\left(\frac{\left([\alpha+1]_q-1\right)a+b}{[\alpha+1]_q}\right) \underset{q\to1^-}{\longrightarrow} f\left(\frac{\alpha a+b}{\alpha+1}\right). \tag{5.6}$$

On the other hand, from Properties 2-ii

$$_a\left(b-\ {}_a\Phi_q(t)\ \right)_q^{(\alpha-1)}=(b-a)^{\alpha-1}\frac{\left(\frac{{}_a\Phi_q(t)-a}{b-a};q\right)_\infty}{\left(\frac{{}_a\Phi_q(t)-a}{b-a}q^{\alpha-1};q\right)_\infty}=(b-a)^{\alpha-1}\frac{\left(\frac{t-a}{b-a}q;q\right)_\infty}{\left(\frac{t-a}{b-a}q^{\alpha};q\right)_\infty}.$$

Using the Theorem 4.1, if we take limit $q\longrightarrow1^-$, then we have

$$_a\left(b-\ {}_a\Phi_q(t)\ \right)_q^{(\alpha-1)}=(b-a)^{\alpha-1}\frac{\left(\frac{t-a}{b-a}q;q\right)_\infty}{\left(\frac{t-a}{b-a}q^{\alpha};q\right)_\infty}\underset{q\to1^-}{\longrightarrow}(b-a)^{\alpha-1}\left(1-\frac{t-a}{b-a}\right)^{\alpha-1}=(b-t)^{\alpha-1}.$$

Hence, we have the following limit

$$\left({}_aI_q^{\alpha}f\right)(b)=\frac{1}{\Gamma_q(\alpha)}\int_a^b {}_a\left(b-\ {}_a\Phi_q(t)\ \right)_q^{(\alpha-1)}\ f(t)\ \ {}_ad_qt \underset{q\to1^-}{\longrightarrow}\frac{1}{\Gamma(\alpha)}\int_a^b(b-t)^{\alpha-1}f(t)\,dt=J_{a+}^{\alpha}f(b). \tag{5.7}$$

If we take limit $q\to1^-$ in (5.1), using (5.5),(5.6) and (5.7), we have (2.3).

(iii) We clearly see that part (iii) can be derived from parts (i) and (ii) immediately
The proof is completed.

□

Now we will prove the following identity to obtain the Riemann-Liouville fractional quantum trapezoid type inequalities.

Lemma 5.2. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function and $\alpha>0$. If ${}_aD_qf$ is q- integrable on (a,b), then the following equality holds,*

$$\frac{\Gamma_q(\alpha+1)}{(b-a)^{\alpha}}\ \left({}_aI_q^{\alpha}f\right)(b)\ -\frac{\left([\alpha+1]_q-1\right)f(a)+f(b)}{[\alpha+1]_q} \tag{5.8}$$

$$=\frac{b-a}{[\alpha+1]_q}\int_0^1\left([\alpha+1]_q\ \ {}_0\left(1-\ {}_0\Phi_q(t)\ \right)_q^{(\alpha)}\ -1\right)\ {}_aD_qf((1-t)a+tb)\ \ {}_0d_qt\ .$$

Proof. Let we calculate the following integrals by using Definition 3.1, 3.2 and 4.2, we have

$$K_1=(b-a)\int_0^1 {}_0\left(1-\ {}_0\Phi_q(t)\ \right)_q^{(\alpha)}\ \ {}_aD_qf((1-t)a+tb)\ \ {}_0d_qt$$

$$=(b-a)\int_0^1 {}_0\left(1-\ {}_0\Phi_q(t)\ \right)_q^{(\alpha)}\ \frac{f((1-t)a+tb)-f((1-qt)a+qtb)}{(1-q)(b-a)t}\ \ {}_0d_qt$$

$$=\frac{1}{1-q}\int_0^1 {}_0\left(1-\ {}_0\Phi_q(t)\ \right)_q^{(\alpha)}\ \frac{f((1-t)a+tb)}{t}\ \ {}_0d_qt$$

$$-\frac{1}{1-q}\int_0^1 {}_0\left(1-\ {}_0\Phi_q(t)\ \right)_q^{(\alpha)}\ \frac{f((1-qt)a+qtb)}{t}\ \ {}_0d_qt$$

$$=\sum_{i=0}^{\infty}q^i\ {}_0\left(1-\ {}_0\Phi_q^{i+1}(1)\ \right)_q^{(\alpha)}\ \frac{f\left(\left(1-\ {}_0\Phi_q^{i}(1)\ \right)a+\ {}_0\Phi_q^{i}(1)\ b\right)}{{}_0\Phi_q^{i}(1)}$$

$$-\sum_{i=0}^{\infty}q^i\ {}_0\left(1-\ {}_0\Phi_q^{i+1}(1)\ \right)_q^{(\alpha)}\ \frac{f\left(\left(1-q\ {}_0\Phi_q^{i}(1)\ \right)a+q\ {}_0\Phi_q^{i}(1)\ b\right)}{{}_0\Phi_q^{i}(1)}$$

$$=\sum_{i=0}^{\infty}\frac{(q^{i+1};q)_\infty}{(q^{\alpha+i+1};q)_\infty}f\left(\left(1-q^i\right)a+q^ib\right)-\sum_{i=0}^{\infty}\frac{(q^{i+1};q)_\infty}{(q^{\alpha+i+1};q)_\infty}f\left(\left(1-q^{i+1}\right)a+q^{i+1}b\right)$$

$$=\sum_{i=0}^{\infty}\left(1-q^{\alpha+i}\right)\frac{(q^{i+1};q)_\infty}{(q^{\alpha+i};q)_\infty}f\left(\left(1-q^i\right)a+q^ib\right)-\sum_{i=0}^{\infty}\left(1-q^{i+1}\right)\frac{(q^{i+2};q)_\infty}{(q^{\alpha+i+1};q)_\infty}f\left(\left(1-q^{i+1}\right)a+q^{i+1}b\right)$$

$$=\sum_{i=0}^{\infty}\left(1-q^{\alpha+i}\right)\frac{(q^{i+1};q)_\infty}{(q^{\alpha+i};q)_\infty}f\left(\left(1-q^i\right)a+q^ib\right)-\sum_{i=0}^{\infty}\left(1-q^{i+1}\right)\frac{(q^{i+2};q)_\infty}{(q^{\alpha+i+1};q)_\infty}f\left(\left(1-q^{i+1}\right)a+q^{i+1}b\right)$$

$$= \sum_{i=0}^{\infty} \frac{(q^{i+1};q)_\infty}{(q^{\alpha+i};q)_\infty} f\left(\left(1-q^i\right)a+q^ib\right) - \sum_{i=0}^{\infty} \frac{(q^{i+2};q)_\infty}{(q^{\alpha+i+1};q)_\infty} f\left(\left(1-q^{i+1}\right)a+q^{i+1}b\right)$$

$$-\left[\sum_{i=0}^{\infty} q^{\alpha+i} \frac{(q^{i+1};q)_\infty}{(q^{\alpha+i};q)_\infty} f\left(\left(1-q^i\right)a+q^ib\right) - \sum_{i=0}^{\infty} q^{i+1} \frac{(q^{i+2};q)_\infty}{(q^{\alpha+i+1};q)_\infty} f\left(\left(1-q^{i+1}\right)a+q^{i+1}b\right)\right]$$

$$= \frac{(q^1;q)_\infty}{(q^\alpha;q)_\infty} f(b) - f(a) - \left[\sum_{i=0}^{\infty} q^{\alpha+i} \frac{(q^{i+1};q)_\infty}{(q^{\alpha+i};q)_\infty} f\left(\left(1-q^i\right)a+q^ib\right) - \sum_{i=1}^{\infty} q^i \frac{(q^{i+1};q)_\infty}{(q^{\alpha+i};q)_\infty} f\left(\left(1-q^i\right)a+q^ib\right)\right]$$

$$= \frac{(q^1;q)_\infty}{(q^\alpha;q)_\infty} f(b) - f(a) - \left[\sum_{i=0}^{\infty} q^{\alpha+i} \frac{(q^{i+1};q)_\infty}{(q^{\alpha+i};q)_\infty} f\left(\left(1-q^i\right)a+q^ib\right) - \sum_{i=0}^{\infty} q^i \frac{(q^{i+1};q)_\infty}{(q^{\alpha+i};q)_\infty} f\left(\left(1-q^i\right)a+q^ib\right) + \frac{(q^1;q)_\infty}{(q^\alpha;q)_\infty} f(b)\right]$$

$$= -f(a) + (1-q^\alpha) \sum_{i=0}^{\infty} q^i \frac{(q^{i+1};q)_\infty}{(q^{\alpha+i};q)_\infty} f\left(\left(1-q^i\right)a+q^ib\right)$$

$$= -f(a) + [\alpha]_q (1-q) \sum_{i=0}^{\infty} q^i \frac{(q^{i+1};q)_\infty}{(q^{\alpha+i};q)_\infty} f\left(\left(1-q^i\right)a+q^ib\right)$$

$$= -f(a) + \frac{[\alpha]_q \Gamma_q(\alpha)}{(b-a)^\alpha} \left(\frac{(1-q)(b-a)}{\Gamma_q(\alpha)} \sum_{i=0}^{\infty} q^i (b-a)^{\alpha-1} \frac{(q^{i+1};q)_\infty}{(q^{\alpha+i};q)_\infty} f\left(\left(1-q^i\right)a+q^ib\right)\right)$$

$$= -f(a) + \frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha} \left(\frac{(1-q)(b-a)}{\Gamma_q(\alpha)} \sum_{i=0}^{\infty} q^i (b-a)^{\alpha-1} \frac{(q^{i+1};q)_\infty}{(q^{(\alpha-1)+(i+1)};q)_\infty} f\left(\left(1-q^i\right)a+q^ib\right)\right)$$

$$= -f(a) + \frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha} \left(\frac{(1-q)(b-a)}{\Gamma_q(\alpha)} \sum_{i=0}^{\infty} q^i \; {}_a\left(b - {}_a\Phi_q^{i+1}(b)\right)_q^{(\alpha-1)} f\left({}_a\Phi_q^i(b)\right)\right)$$

$$= -f(a) + \frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha} \left(\frac{1}{\Gamma_q(\alpha)} \int_a^b {}_a\left(b - {}_a\Phi_q(t)\right)_q^{(\alpha-1)} f(t) \; {}_ad_qt\right)$$

$$= -f(a) + \frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha} \left({}_aI_q^\alpha f\right)(b), \tag{5.9}$$

and

$$K_2 = \frac{b-a}{[\alpha+1]_q} \int_0^1 {}_aD_q f((1-t)a+tb) \; {}_0d_qt$$

$$= \frac{b-a}{[\alpha+1]_q} \int_0^1 \frac{f((1-t)a+tb) - f((1-qt)a+qtb)}{(1-q)(b-a)t} \; {}_0d_qt$$

$$= \frac{1}{(1-q)[\alpha+1]_q} \int_0^1 \frac{f((1-t)a+tb)}{t} \; {}_0d_qt$$

$$- \frac{1}{(1-q)[\alpha+1]_q} \int_0^1 \frac{f((1-qt)a+qtb)}{t} \; {}_0d_qt$$

$$= \frac{1}{[\alpha+1]_q} \left[\sum_{i=0}^{\infty} f\left(\left(1-q^i\right)a+q^ib\right) - \sum_{i=0}^{\infty} f\left(\left(1-q^{i+1}\right)a+q^{i+1}b\right)\right]$$

$$= \frac{f(b)-f(a)}{[\alpha+1]_q}. \tag{5.10}$$

On the other hand, if we use (5.9) and (5.10) in the following integral, we have

$$\frac{b-a}{[\alpha+1]_q} \int_0^1 \left([\alpha+1]_q \; {}_0\left(1 - {}_0\Phi_q(t)\right)_q^{(\alpha)} - 1\right) {}_aD_q f((1-t)a+tb) \; {}_0d_qt$$

$$= (b-a) \int_0^1 {}_0\left(1 - {}_0\Phi_q(t)\right)_q^{(\alpha)} {}_aD_q f((1-t)a+tb) \; {}_0d_qt - \frac{b-a}{[\alpha+1]_q} \int_0^1 {}_aD_q f((1-t)a+tb) \; {}_0d_qt$$

$$= K_1 - K_2$$

$$= -f(a) + \frac{\Gamma_q(\alpha+1)}{(b-a)^{\alpha}} \left({}_aI_q^{\alpha} f\right)(b) - \frac{f(b)-f(a)}{[\alpha+1]_q}$$

$$= \frac{\Gamma_q(\alpha+1)}{(b-a)^{\alpha}} \left({}_aI_q^{\alpha} f\right)(b) - \frac{\left([\alpha+1]_q - 1\right) f(a) + f(b)}{[\alpha+1]_q}.$$

This completes the proof. □

Corollary 2. *In Lemma 5.2, one has the following:*

(i) If one takes $\alpha = 1$, then one has Lemma 3.5;
(ii) If one takes $q \to 1^-$, then one has Lemma 2.5;
(iii) If one takes $\alpha = 1$ and $q \to 1^-$, then one has Lemma 1.1.

Theorem 5.3. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function, $\alpha > 0$ and ${}_aD_q f$ be q- integrable on (a,b). If $\left|{}_aD_q f\right|$ is convex on $[a,b]$, then the following Riemann-Liouville fractional quantum trapezoid type inequality holds,*

$$\left| \frac{\Gamma_q(\alpha+1)}{(b-a)^{\alpha}} \left({}_aI_q^{\alpha} f\right)(b) - \frac{\left([\alpha+1]_q - 1\right) f(a) + f(b)}{[\alpha+1]_q} \right| \le \frac{b-a}{[\alpha+1]_q} \left(\left|{}_aD_q f(a)\right| M_1 + \left|{}_aD_q f(b)\right| M_2 \right) \tag{5.11}$$

where

$$M_1 = \int_0^1 \left| [\alpha+1]_q \; {}_0\left(1 - {}_0\Phi_q(t)\right)_q^{(\alpha)} - 1 \right| (1-t) \; {}_0d_qt \ ,$$

$$M_2 = \int_0^1 \left| [\alpha+1]_q \; {}_0\left(1 - {}_0\Phi_q(t)\right)_q^{(\alpha)} - 1 \right| t \; {}_0d_qt \ .$$

Proof. Using Lemma 5.2 and the convexity of $\left|{}_aD_q f\right|$, we have

$$\left| \frac{\Gamma_q(\alpha+1)}{(b-a)^{\alpha}} \left({}_aI_q^{\alpha} f\right)(b) - \frac{\left([\alpha+1]_q - 1\right) f(a) + f(b)}{[\alpha+1]_q} \right|$$

$$\le \frac{b-a}{[\alpha+1]_q} \int_0^1 \left| [\alpha+1]_q \; {}_0\left(1 - {}_0\Phi_q(t)\right)_q^{(\alpha)} - 1 \right| \left| {}_aD_q f((1-t)a + tb) \right| \; {}_0d_qt$$

$$\le \frac{b-a}{[\alpha+1]_q} \int_0^1 \left| [\alpha+1]_q \; {}_0\left(1 - {}_0\Phi_q(t)\right)_q^{(\alpha)} - 1 \right| \left[\left| {}_aD_q f(a) \right| (1-t) + \left| {}_aD_q f(b) \right| t \right] \; {}_0d_qt$$

$$\le \frac{b-a}{[\alpha+1]_q} \left[\begin{array}{c} \left| {}_aD_q f(a) \right| \int_0^1 \left| [\alpha+1]_q \; {}_0\left(1 - {}_0\Phi_q(t)\right)_q^{(\alpha)} - 1 \right| (1-t) \; {}_0d_qt \\ + \left| {}_aD_q f(b) \right| \int_0^1 \left| [\alpha+1]_q \; {}_0\left(1 - {}_0\Phi_q(t)\right)_q^{(\alpha)} - 1 \right| t \; {}_0d_qt \end{array} \right].$$

This completes the proof. □

Corollary 3. *In Theorem 5.3, one has the following:*

(i) If one takes $\alpha = 1$, then one has [14, Theorem 4.1];
(ii) If one takes $q \to 1^-$, then one has [8, Theorem 4];
(iii) If one takes $\alpha = 1$ and $q \to 1^-$, then one has [3, Theorem 2.2].

Theorem 5.4. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function, $\alpha > 0$ and ${}_aD_q f$ be q- integrable on (a,b). If $\left|{}_aD_q f\right|^r$ is convex on $[a,b]$ for $r \ge 1$, then the following Riemann-Liouville fractional quantum trapezoid type inequality holds,*

$$\left| \frac{\Gamma_q(\alpha+1)}{(b-a)^{\alpha}} \left({}_aI_q^{\alpha} f\right)(b) - \frac{\left([\alpha+1]_q - 1\right) f(a) + f(b)}{[\alpha+1]_q} \right| \le \frac{b-a}{[\alpha+1]_q} M_3^{1-\frac{1}{r}} \left(M_1 \left|{}_aD_q f(a)\right|^r + M_2 \left|{}_aD_q f(b)\right|^r \right)^{\frac{1}{r}} \tag{5.12}$$

where M_1,M_2 are the same as in the Theorem 5.3 and

$$M_3 = \int_0^1 \left| [\alpha+1]_q \; {}_0\left(1 - {}_0\Phi_q(t)\right)_q^{(\alpha)} - 1 \right| \; {}_0d_qt \ .$$

Proof. Using Lemma 5.2,the convexity of $\left| {}_aD_qf \right|^r$ and power mean inequality, we have

$$\left|\frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\left({}_aI_q^\alpha f\right)(b)-\frac{\left([\alpha+1]_q-1\right)f(a)+f(b)}{[\alpha+1]_q}\right|$$

$$\le \frac{b-a}{[\alpha+1]_q}\int_0^1\left|[\alpha+1]_q\ {}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}-1\right|\left|{}_aD_qf((1-t)a+tb)\right|\ {}_0d_qt$$

$$\le \frac{b-a}{[\alpha+1]_q}\left(\int_0^1\left|[\alpha+1]_q\ {}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}-1\right|\ {}_0d_qt\right)^{1-\frac{1}{r}}$$

$$\times\left(\int_0^1\left|[\alpha+1]_q\ {}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}-1\right|\left|{}_aD_qf((1-t)a+tb)\right|^r\ {}_0d_qt\right)^{\frac{1}{r}}$$

$$\le \frac{b-a}{[\alpha+1]_q}\left(\int_0^1\left|[\alpha+1]_q\ {}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}-1\right|\ {}_0d_qt\right)^{1-\frac{1}{r}}$$

$$\times\left(\int_0^1\left|[\alpha+1]_q\ {}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}-1\right|\left[\left|{}_aD_qf(a)\right|^r(1-t)+\left|{}_aD_qf(b)\right|^r t\right]\ {}_0d_qt\right)^{\frac{1}{r}}$$

$$\le \frac{b-a}{[\alpha+1]_q}\left(\int_0^1\left|[\alpha+1]_q\ {}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}-1\right|\ {}_0d_qt\right)^{1-\frac{1}{r}}$$

$$\left[\begin{array}{l}\left|{}_aD_qf(a)\right|^r\int_0^1\left|[\alpha+1]_q\ {}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}-1\right|(1-t)\ {}_0d_qt\\ +\left|{}_aD_qf(b)\right|^r\int_0^1\left|[\alpha+1]_q\ {}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}-1\right|t\ {}_0d_qt\end{array}\right]^{\frac{1}{r}}.$$

This completes the proof. □

Corollary 4. *In Theorem 5.4, one has the following:*

(i) If one takes $\alpha=1$, then one has [14, Theorem 4.2] and [10, Theorem 3.2];
(ii) If one takes $q\to1^-$,then one has [8, Theorem 5];
(iii) If one takes $\alpha=1$ and $q\to1^-$, then one has [11, Theorem 1].

Theorem 5.5. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function, $\alpha>0$ and ${}_aD_qf$ be q- integrable on (a,b). If $\left|{}_aD_qf\right|^r$ is convex on $[a,b]$ for $r>1$ and $\frac{1}{r}+\frac{1}{p}=1$, then the following Riemann-Liouville fractional quantum trapezoid type inequality holds,*

$$\left|\frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\left({}_aI_q^\alpha f\right)(b)-\frac{\left([\alpha+1]_q-1\right)f(a)+f(b)}{[\alpha+1]_q}\right|\le\frac{b-a}{[\alpha+1]_q}M_4^{\frac{1}{p}}\left(\frac{q\left|{}_aD_qf(a)\right|^r+\left|{}_aD_qf(b)\right|^r}{1+q}\right)^{\frac{1}{r}} \tag{5.13}$$

where

$$M_4=\int_0^1\left|[\alpha+1]_q\ {}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}-1\right|^p\ {}_0d_qt\ .$$

Proof. Using Lemma 5.2,the convexity of $\left|{}_aD_qf\right|^r$ and Hölder's inequality, we have

$$\left|\frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\left({}_aI_q^\alpha f\right)(b)-\frac{\left([\alpha+1]_q-1\right)f(a)+f(b)}{[\alpha+1]_q}\right|$$

$$\le \frac{b-a}{[\alpha+1]_q}\int_0^1\left|[\alpha+1]_q\ {}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}-1\right|\left|{}_aD_qf((1-t)a+tb)\right|\ {}_0d_qt$$

$$\le \frac{b-a}{[\alpha+1]_q}\left(\int_0^1\left|[\alpha+1]_q\ {}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}-1\right|^p\ {}_0d_qt\right)^{\frac{1}{p}}\left(\int_0^1\left|{}_aD_qf((1-t)a+tb)\right|^r\ {}_0d_qt\right)^{\frac{1}{r}}$$

$$\leq \frac{b-a}{[\alpha+1]_q}\left(\int_0^1\left|[\alpha+1]_q\ {}_0\left(1-\ {}_0\Phi_q(t)\ \right)_q^{(\alpha)}\ -1\right|^p\ {}_0d_qt\right)^{\frac{1}{p}}\left(\int_0^1\left[\left|\ {}_aD_qf(a)\ \right|^r(1-t)+\left|\ {}_aD_qf(b)\ \right|^r t\right]\ {}_0d_qt\right)^{\frac{1}{r}}$$

$$\leq \frac{b-a}{[\alpha+1]_q}\left(\int_0^1\left|[\alpha+1]_q\ {}_0\left(1-\ {}_0\Phi_q(t)\ \right)_q^{(\alpha)}\ -1\right|^p\ {}_0d_qt\right)^{\frac{1}{p}}\left(\frac{q\left|\ {}_aD_qf(a)\ \right|^r+\left|\ {}_aD_qf(b)\ \right|^r}{1+q}\right)^{\frac{1}{r}}.$$

This completes the proof. □

Corollary 5. *In Theorem 5.4, one has the following:*

(i) If one takes $\alpha=1$, then one has the following quantum trapezoid type inequality;

$$\left|\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx\ -\frac{qf(a)+f(b)}{1+q}\right|\leq\frac{q(b-a)}{1+q}\left(\int_0^1|1-(1+q)t|^p\ {}_0d_qt\right)^{\frac{1}{p}}\left(\frac{q\left|\ {}_aD_qf(a)\ \right|^r+\left|\ {}_aD_qf(b)\ \right|^r}{1+q}\right)^{\frac{1}{r}}$$

(ii) If one takes $q\to1^-$, then one has [8, Theorem 6];
(iii) If one takes $\alpha=1$ and $q\to1^-$, then one has [3, Theorem 2.3].

Now we will prove the following identity to obtain the Riemann-Liouville fractional quantum midpoint type inequalities.

Lemma 5.6. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function and $\alpha>0$. If ${}_aD_qf$ be q- integrable on (a,b), then the following equality holds,*

$$f\left(\frac{\left([\alpha+1]_q-1\right)a+b}{[\alpha+1]_q}\right)-\frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\ \left({}_aI_q^\alpha f\right)(b) \tag{5.14}$$

$$=(b-a)\left[\begin{array}{c}\int_0^{\frac{1}{[\alpha+1]_q}}\left(1-\ {}_0\left(1-\ {}_0\Phi_q(t)\ \right)_q^{(\alpha)}\ \right)\ {}_aD_qf((1-t)a+tb)\ \ {}_0d_qt\\ +\int_{\frac{1}{[\alpha+1]_q}}^1-\ {}_0\left(1-\ {}_0\Phi_q(t)\ \right)_q^{(\alpha)}\ \ {}_aD_qf((1-t)a+tb)\ \ {}_0d_qt\end{array}\right].$$

Proof. Let we calculate the following integrals by using Definition 3.1 and 3.2, we have

$$K_3=\int_0^{\frac{1}{[\alpha+1]_q}}\ {}_aD_qf((1-t)a+tb)\ \ {}_0d_qt \tag{5.15}$$

$$=\int_0^{\frac{1}{[\alpha+1]_q}}\frac{f((1-t)a+tb)-f((1-qt)a+qtb)}{(1-q)(b-a)t}\ {}_0d_qt$$

$$=\frac{1}{(1-q)(b-a)}\int_0^{\frac{1}{[\alpha+1]_q}}\frac{f((1-t)a+tb)}{t}\ {}_0d_qt\ -\frac{1}{(1-q)(b-a)}\int_0^{\frac{1}{[\alpha+1]_q}}\frac{f((1-qt)a+qtb)}{t}\ {}_0d_qt$$

$$=\frac{1}{(b-a)[\alpha+1]_q}\sum_{i=0}^\infty q^i\frac{f\left(\left(1-\frac{q^i}{[\alpha+1]_q}\right)a+\frac{q^i}{[\alpha+1]_q}b\right)}{\frac{q^i}{[\alpha+1]_q}}-\frac{1}{(b-a)[\alpha+1]_q}\sum_{i=0}^\infty q^i\frac{f\left(\left(1-\frac{q^{i+1}}{[\alpha+1]_q}\right)a+\frac{q^{i+1}}{[\alpha+1]_q}b\right)}{\frac{q^i}{[\alpha+1]_q}}$$

$$=\frac{1}{(b-a)}\left[\sum_{i=0}^\infty f\left(\left(1-\frac{q^i}{[\alpha+1]_q}\right)a+\frac{q^i}{[\alpha+1]_q}b\right)-\sum_{i=0}^\infty f\left(\left(1-\frac{q^{i+1}}{[\alpha+1]_q}\right)a+\frac{q^{i+1}}{[\alpha+1]_q}b\right)\right]$$

$$=\frac{1}{(b-a)}\left[f\left(\frac{\left([\alpha+1]_q-1\right)a+b}{[\alpha+1]_q}\right)-f(a)\right].$$

On the other hand, in (5.9)the following integral was calculated as

$$K_1=(b-a)\int_0^1\ {}_0\left(1-\ {}_0\Phi_q(t)\ \right)_q^{(\alpha)}\ \ {}_aD_qf((1-t)a+tb)\ \ {}_0d_qt \tag{5.16}$$

$$=-f(a)+\frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\ \left({}_aI_q^\alpha f\right)(b)\ .$$

At the end, if we use (5.15) and (5.16) in the following integral, we have

$$(b-a)\left[\begin{array}{c}\int_0^{\frac{1}{[\alpha+1]_q}}\left(1-{}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right){}_aD_qf((1-t)a+tb)\,{}_0d_qt\\ +\int_{\frac{1}{[\alpha+1]_q}}^{1}-{}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\,{}_aD_qf((1-t)a+tb)\,{}_0d_qt\end{array}\right]$$

$$=(b-a)\left[\int_0^{\frac{1}{[\alpha+1]_q}}{}_aD_qf((1-t)a+tb)\,{}_0d_qt-\int_0^1{}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\,{}_aD_qf((1-t)a+tb)\,{}_0d_qt\right]$$

$$=(b-a)\left[\frac{1}{(b-a)}\left[f\left(\frac{\left([\alpha+1]_q-1\right)a+b}{[\alpha+1]_q}\right)-f(a)\right]-\frac{1}{(b-a)}\left[-f(a)+\frac{\Gamma_q(\alpha+1)}{(b-a)^{\alpha}}\left({}_aI_q^{\alpha}f\right)(b)\right]\right]$$

$$=f\left(\frac{\left([\alpha+1]_q-1\right)a+b}{[\alpha+1]_q}\right)-\frac{\Gamma_q(\alpha+1)}{(b-a)^{\alpha}}\left({}_aI_q^{\alpha}f\right)(b)\,.$$

This completes the proof. □

Corollary 6. *In Lemma 5.6, one has the following:*

(i) If one takes $\alpha=1$, then one has Lemma 3.4;
(ii) If one takes $q\to 1^-$, then one has Lemma 2.6;
(iii) If one takes $\alpha=1$ and $q\to 1^-$, then one has Lemma 1.2.

Theorem 5.7. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function, $\alpha>0$ and ${}_aD_qf$ be q- integrable on (a,b). If $\left|{}_aD_qf\right|$ is convex on $[a,b]$, then the following Riemann-Liouville fractional quantum midpoint type inequality holds,*

$$\left|f\left(\frac{\left([\alpha+1]_q-1\right)a+b}{[\alpha+1]_q}\right)-\frac{\Gamma_q(\alpha+1)}{(b-a)^{\alpha}}\left({}_aI_q^{\alpha}f\right)(b)\right|\le(b-a)\left[\begin{array}{c}M_5\left|{}_aD_qf(a)\right|+M_6\left|{}_aD_qf(b)\right|\\+M_7\left|{}_aD_qf(a)\right|+M_8\left|{}_aD_qf(b)\right|\end{array}\right] \tag{5.17}$$

where

$$M_5=\left[\int_0^{\frac{1}{[\alpha+1]_q}}\left|1-{}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|(1-t)\,{}_0d_qt\right],$$

$$M_6=\left[\int_0^{\frac{1}{[\alpha+1]_q}}\left|1-{}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|t\,{}_0d_qt\right],$$

$$M_7=\int_{\frac{1}{[\alpha+1]_q}}^{1}\left|-{}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|(1-t)\,{}_0d_qt\,,$$

$$M_8=\int_{\frac{1}{[\alpha+1]_q}}^{1}\left|-{}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|t\,{}_0d_qt\,.$$

Proof. Using Lemma 5.6 and the convexity of $\left|{}_aD_qf\right|$, we have

$$\left|f\left(\frac{\left([\alpha+1]_q-1\right)a+b}{[\alpha+1]_q}\right)-\frac{\Gamma_q(\alpha+1)}{(b-a)^{\alpha}}\left({}_aI_q^{\alpha}f\right)(b)\right|$$

$$\le(b-a)\left[\begin{array}{c}\int_0^{\frac{1}{[\alpha+1]_q}}\left|1-{}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|\left|{}_aD_qf((1-t)a+tb)\right|{}_0d_qt\\+\int_{\frac{1}{[\alpha+1]_q}}^{1}\left|-{}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|\left|{}_aD_qf((1-t)a+tb)\right|{}_0d_qt\end{array}\right]$$

$$\leq (b-a)\left[\begin{array}{c}\int_0^{\frac{1}{[\alpha+1]_q}}\left|1-{}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|\left[\left|{}_aD_qf(a)\right|(1-t)+\left|{}_aD_qf(b)\right|t\right]{}_0d_qt\\ +\int_{\frac{1}{[\alpha+1]_q}}^1\left|-{}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|\left[\left|{}_aD_qf(a)\right|(1-t)+\left|{}_aD_qf(b)\right|t\right]{}_0d_qt\end{array}\right]$$

$$\leq (b-a)\left[\begin{array}{c}\left[\begin{array}{c}\int_0^{\frac{1}{[\alpha+1]_q}}\left|1-{}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|(1-t)\,{}_0d_qt\left|{}_aD_qf(a)\right|\\ +\int_0^{\frac{1}{[\alpha+1]_q}}\left|1-{}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|t\,{}_0d_qt\left|{}_aD_qf(b)\right|\end{array}\right]\\ +\left[\begin{array}{c}\int_{\frac{1}{[\alpha+1]_q}}^1\left|-{}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|(1-t)\,{}_0d_qt\left|{}_aD_qf(a)\right|\\ +\int_{\frac{1}{[\alpha+1]_q}}^1\left|-{}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|t\,{}_0d_qt\left|{}_aD_qf(b)\right|\end{array}\right]\end{array}\right]$$

This completes the proof. □

Corollary 7. *In Theorem 5.7, one has the following:*

(i) If one takes $\alpha=1$, then one has [2, Theorem 13];
(ii) If one takes $q\to1^-$, then one has [8, Theorem 7];
(iii) If one takes $\alpha=1$ and $q\to1^-$, then one has [4, Theorem 2.2].

Theorem 5.8. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function, $\alpha>0$ and ${}_aD_qf$ be q- integrable on (a,b). If $\left|{}_aD_qf\right|^r$ is convex on $[a,b]$ for $r\geq1$, then the following Riemann-Liouville fractional quantum midpoint type inequality holds,*

$$\left|f\left(\frac{\left([\alpha+1]_q-1\right)a+b}{[\alpha+1]_q}\right)-\frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\left({}_aI_q^\alpha f\right)(b)\right|\leq(b-a)\left[\begin{array}{c}M_9^{1-\frac{1}{r}}\left(M_5\left|{}_aD_qf(a)\right|^r+M_6\left|{}_aD_qf(b)\right|^r\right)^{\frac{1}{r}}\\ +M_{10}^{1-\frac{1}{r}}\left(M_7\left|{}_aD_qf(a)\right|^r+M_8\left|{}_aD_qf(b)\right|^r\right)^{\frac{1}{r}}\end{array}\right] \tag{5.18}$$

where M_5-M_8 are the same in the Theorem 5.7 and

$$M_9=\int_0^{\frac{1}{[\alpha+1]_q}}\left|1-{}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|{}_0d_qt\ ,$$

$$M_{10}=\int_{\frac{1}{[\alpha+1]_q}}^1\left|-{}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|{}_0d_qt\ .$$

Proof. Using Lemma 5.6, power mean inequality and the convexity of $\left|{}_aD_qf\right|^r$, we have

$$\left|f\left(\frac{\left([\alpha+1]_q-1\right)a+b}{[\alpha+1]_q}\right)-\frac{\Gamma_q(\alpha+1)}{(b-a)^\alpha}\left({}_aI_q^\alpha f\right)(b)\right|$$

$$\leq (b-a)\left[\begin{array}{c}\int_0^{\frac{1}{[\alpha+1]_q}}\left|1-{}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|\left|{}_aD_qf((1-t)a+tb)\right|{}_0d_qt\\ +\int_{\frac{1}{[\alpha+1]_q}}^1\left|-{}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|\left|{}_aD_qf((1-t)a+tb)\right|{}_0d_qt\end{array}\right]$$

$$\leq (b-a)\left[\begin{array}{c}\left(\int_0^{\frac{1}{[\alpha+1]_q}}\left|1-{}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|{}_0d_qt\right)^{1-\frac{1}{r}}\\ \times\left(\int_0^{\frac{1}{[\alpha+1]_q}}\left|1-{}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|\left|{}_aD_qf((1-t)a+tb)\right|^r{}_0d_qt\right)^{\frac{1}{r}}\\ +\left(\int_{\frac{1}{[\alpha+1]_q}}^1\left|-{}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|{}_0d_qt\right)^{1-\frac{1}{r}}\\ \times\left(\int_{\frac{1}{[\alpha+1]_q}}^1\left|-{}_0\left(1-{}_0\Phi_q(t)\right)_q^{(\alpha)}\right|\left|{}_aD_qf((1-t)a+tb)\right|^r{}_0d_qt\right)^{\frac{1}{r}}\end{array}\right]$$

$$\leq (b-a)\left[\begin{array}{c} \left(\int_0^{\frac{1}{[\alpha+1]_q}} \left|1-\ _0\left(1-\ _0\Phi_q(t)\ \right)_q^{(\alpha)}\ \right|\ _0d_qt\right)^{1-\frac{1}{r}} \\ \times\left(\int_0^{\frac{1}{[\alpha+1]_q}} \left|1-\ _0\left(1-\ _0\Phi_q(t)\ \right)_q^{(\alpha)}\ \right|\left[\left|\ _aD_qf(a)\ \right|^r(1-t)+\left|\ _aD_qf(b)\ \right|^r t\right]\ _0d_qt\right)^{\frac{1}{r}} \\ +\left(\int_{\frac{1}{[\alpha+1]_q}}^1 \left|-\ _0\left(1-\ _0\Phi_q(t)\ \right)_q^{(\alpha)}\ \right|\ _0d_qt\right)^{1-\frac{1}{r}} \\ \times\left(\int_{\frac{1}{[\alpha+1]_q}}^1 \left|-\ _0\left(1-\ _0\Phi_q(t)\ \right)_q^{(\alpha)}\ \right|\left[\left|\ _aD_qf(a)\ \right|^r(1-t)+\left|\ _aD_qf(b)\ \right|^r t\right]\ _0d_qt\right)^{\frac{1}{r}} \end{array}\right]$$

$$\leq (b-a)\left[\begin{array}{c} \left(\int_0^{\frac{1}{[\alpha+1]_q}} \left|1-\ _0\left(1-\ _0\Phi_q(t)\ \right)_q^{(\alpha)}\ \right|\ _0d_qt\right)^{1-\frac{1}{r}} \\ \times\left(\begin{array}{c} \left|\ _aD_qf(a)\ \right|^r \int_0^{\frac{1}{[\alpha+1]_q}} \left|1-\ _0\left(1-\ _0\Phi_q(t)\ \right)_q^{(\alpha)}\ \right|(1-t)\ _0d_qt \\ \left|\ _aD_qf(b)\ \right|^r \int_0^{\frac{1}{[\alpha+1]_q}} \left|1-\ _0\left(1-\ _0\Phi_q(t)\ \right)_q^{(\alpha)}\ \right|t\ _0d_qt \end{array}\right)^{\frac{1}{r}} \\ +\left(\int_{\frac{1}{[\alpha+1]_q}}^1 \left|-\ _0\left(1-\ _0\Phi_q(t)\ \right)_q^{(\alpha)}\ \right|\ _0d_qt\right)^{1-\frac{1}{r}} \\ \times\left(\begin{array}{c} \left|\ _aD_qf(a)\ \right|^r \int_{\frac{1}{[\alpha+1]_q}}^1 \left|-\ _0\left(1-\ _0\Phi_q(t)\ \right)_q^{(\alpha)}\ \right|(1-t)\ _0d_qt \\ \left|\ _aD_qf(b)\ \right|^r \int_{\frac{1}{[\alpha+1]_q}}^1 \left|-\ _0\left(1-\ _0\Phi_q(t)\ \right)_q^{(\alpha)}\ \right|t\ _0d_qt \end{array}\right)^{\frac{1}{r}} \end{array}\right]$$

This completes the proof. □

Corollary 8. *In Theorem 5.8, one has the following:*

(i) If one takes $\alpha = 1$, then one has [2, Theorem 16];
(ii) If one takes $q \to 1^-$, then one has [8, Theorem 8];
(iii) If one takes $\alpha = 1$ and $q \to 1^-$, then one has [8, Remark 9].

Theorem 5.9. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function, $\alpha > 0$ and $\ _aD_qf\ $ be q- integrable on (a,b). If $\left|\ _aD_qf\ \right|^r$ is convex on $[a,b]$ for $r > 1$ and $\frac{1}{r}+\frac{1}{p} = 1$, then the following Riemann-Liouville fractional quantum midpoint type inequality holds,*

$$\left|f\left(\frac{\left([\alpha+1]_q-1\right)a+b}{[\alpha+1]_q}\right)-\frac{\Gamma_q(\alpha+1)}{(b-a)^{\alpha}}\left(_aI_q^{\alpha}f\right)(b)\right| \tag{5.19}$$

$$\leq (b-a)\left[\begin{array}{c} M_{11}^{\frac{1}{p}}\left(\left|\ _aD_qf(a)\ \right|^r\left(\frac{(1+q)[\alpha+1]_q-1}{(1+q)\left([\alpha+1]_q\right)^2}\right)+\left|\ _aD_qf(b)\ \right|^r\left(\frac{1}{(1+q)\left([\alpha+1]_q\right)^2}\right)\right)^{\frac{1}{r}} \\ +M_{12}^{\frac{1}{p}}\left(\left|\ _aD_qf(a)\ \right|^r\left(\frac{q}{1+q}-\frac{(1+q)[\alpha+1]_q-1}{(1+q)\left([\alpha+1]_q\right)^2}\right)+\left|\ _aD_qf(b)\ \right|^r\left(\frac{1}{1+q}-\frac{1}{(1+q)\left([\alpha+1]_q\right)^2}\right)\right)^{\frac{1}{r}} \end{array}\right]$$

where

$$M_{11} = \int_0^{\frac{1}{[\alpha+1]_q}} \left|1-\ _0\left(1-\ _0\Phi_q(t)\ \right)_q^{(\alpha)}\ \right|^p\ _0d_qt\ ,$$

$$M_{12} = \int_{\frac{1}{[\alpha+1]_q}}^1 \left|-\ _0\left(1-\ _0\Phi_q(t)\ \right)_q^{(\alpha)}\ \right|^p\ _0d_qt\ .$$

Proof. Using Lemma 5.6, Hölder inequality and the convexity of $\left|\ _aD_qf\ \right|^r$, we have

$$\left|f\left(\frac{\left([\alpha+1]_q-1\right)a+b}{[\alpha+1]_q}\right)-\frac{\Gamma_q(\alpha+1)}{(b-a)^{\alpha}}\left(_aI_q^{\alpha}f\right)(b)\right|$$

$$\leq (b-a)\left[\begin{array}{l} \int_{0}^{\frac{1}{[\alpha+1]_q}} \left|1- {}_0\left(1- {}_0\Phi_q(t)\right)_q^{(\alpha)}\right| \left| {}_aD_q f((1-t)a+tb)\right| {}_0d_qt \\ + \int_{\frac{1}{[\alpha+1]_q}}^{1} \left|- {}_0\left(1- {}_0\Phi_q(t)\right)_q^{(\alpha)}\right| \left| {}_aD_q f((1-t)a+tb)\right| {}_0d_qt \end{array}\right]$$

$$\leq (b-a)\left[\begin{array}{l} \left(\int_{0}^{\frac{1}{[\alpha+1]_q}} \left|1- {}_0\left(1- {}_0\Phi_q(t)\right)_q^{(\alpha)}\right|^p {}_0d_qt\right)^{\frac{1}{p}} \left(\int_{0}^{\frac{1}{[\alpha+1]_q}} \left| {}_aD_q f((1-t)a+tb)\right|^r {}_0d_qt\right)^{\frac{1}{r}} \\ + \left(\int_{\frac{1}{[\alpha+1]_q}}^{1} \left|- {}_0\left(1- {}_0\Phi_q(t)\right)_q^{(\alpha)}\right|^p {}_0d_qt\right)^{\frac{1}{p}} \left(\int_{\frac{1}{[\alpha+1]_q}}^{1} \left| {}_aD_q f((1-t)a+tb)\right|^r {}_0d_qt\right)^{\frac{1}{r}} \end{array}\right]$$

$$\leq (b-a)\left[\begin{array}{l} \left(\int_{0}^{\frac{1}{[\alpha+1]_q}} \left|1- {}_0\left(1- {}_0\Phi_q(t)\right)_q^{(\alpha)}\right|^p {}_0d_qt\right)^{\frac{1}{p}} \left(\int_{0}^{\frac{1}{[\alpha+1]_q}} \left[\begin{array}{c} \left| {}_aD_q f(a)\right|^r (1-t) \\ +\left| {}_aD_q f(b)\right|^r t \end{array}\right] {}_0d_qt\right)^{\frac{1}{r}} \\ + \left(\int_{\frac{1}{[\alpha+1]_q}}^{1} \left|- {}_0\left(1- {}_0\Phi_q(t)\right)_q^{(\alpha)}\right|^p {}_0d_qt\right)^{\frac{1}{p}} \left(\int_{\frac{1}{[\alpha+1]_q}}^{1} \left[\begin{array}{c} \left| {}_aD_q f(a)\right|^r (1-t) \\ +\left| {}_aD_q f(b)\right|^r t \end{array}\right] {}_0d_qt\right)^{\frac{1}{r}} \end{array}\right]$$

$$\leq (b-a)\left[\begin{array}{c} \left(\int_{0}^{\frac{1}{[\alpha+1]_q}} \left|1- {}_0\left(1- {}_0\Phi_q(t)\right)_q^{(\alpha)}\right|^p {}_0d_qt\right)^{\frac{1}{p}} \\ \times\left(\left| {}_aD_q f(a)\right|^r \left(\frac{(1+q)[\alpha+1]_q-1}{(1+q)\left([\alpha+1]_q\right)^2}\right) + \left| {}_aD_q f(b)\right|^r \left(\frac{1}{(1+q)\left([\alpha+1]_q\right)^2}\right)\right)^{\frac{1}{r}} \\ + \left(\int_{\frac{1}{[\alpha+1]_q}}^{1} \left|- {}_0\left(1- {}_0\Phi_q(t)\right)_q^{(\alpha)}\right|^p {}_0d_qt\right)^{\frac{1}{p}} \\ \times\left(\left| {}_aD_q f(a)\right|^r \left(\frac{q}{1+q} - \frac{(1+q)[\alpha+1]_q-1}{(1+q)\left([\alpha+1]_q\right)^2}\right) + \left| {}_aD_q f(b)\right|^r \left(\frac{1}{1+q} - \frac{1}{(1+q)\left([\alpha+1]_q\right)^2}\right)\right)^{\frac{1}{r}} \end{array}\right]$$

This completes the proof. □

Corollary 9. *In Theorem 5.8, one has the following:*

(i) If one takes $\alpha = 1$, then one has [2, Theorem 25];
(ii) If one takes $q \to 1^-$, then one has [8, Theorem 9];
(iii) If one takes $\alpha = 1$ and $q \to 1^-$, then one has [4, Theorem 2.3].

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