Konuralp Journal of Mathematics, 8 (1) (2020) 165-174

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

On Hermite-Hadamard Type Inequalities with Respect to the Generalization of Some Types of s-Convexity

Sercan Turhan[1*], Mehmet Kunt[2] and İmdat İşcan[1]

[1] *Department of Mathematics, Faculty of Arts and Sciences, Giresun University, 28200, Giresun, Turkey.*
[2] *Department of Mathematics, Faculty of Arts and Sciences, Karadeniz Technical University, 61080, Trabzon, Turkey.*
[*] *Corresponding Author*

Abstract

In this paper, the authors give a new concept which is a generalization of the concepts s-convexity, $GA-s$-convexity, harmonically s-convexity and (p,s)-convexity establish some new Hermite-Hadamard type inequalities for this class of functions. Some natural applications to special means of real numbers are also given.

Keywords: *$M_{\varphi}A-s$-convex function, Hermite-Hadamard type inequality.*
2010 Mathematics Subject Classification: *Primary 26D15; Secondary 26A51.*

1. Introduction

Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a convex function defined on the interval I of real numbers and $a,b\in I$ with $a<b$. The following inequality

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2} \tag{1.1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave. For some results which generalize, improve and extend the inequalities (1.1) we refer the reader to the recent papers (see [4, 6, 7, 10, 11, 14, 16, 20, 23, 24, 25]).
For $r\in\mathbb{R}$ the power mean $M_r(a,b)$ of order r of two positive numbers a and b is defined by

$$M_r=M_r(a,b)=\begin{cases}\left(\frac{a^r+b^r}{2}\right)^{1/r}, & r\neq 0\\ \sqrt{ab}, & r=0\end{cases}.$$

It is well-known that $M_r(a,b)$ is continuous and strictly increasing with respect to $r\in\mathbb{R}$ for fixed $a,b>0$ with $a\neq b$.
Let $L=L(a,b)=(b-a)/(\ln b-\ln a)$, $I=I(a,b)=\frac{1}{e}\left(a^a/b^b\right)^{1/a-b}$, $A=A(a,b)=(a+b)/2$, $G=G(a,b)=\sqrt{ab}$ and $H=H(a,b)=2ab/(a+b)$ be the logarithmic, identric, arithmetic, geometric, and harmonic means of two positive real numbers a and b with $a\neq b$, respectively. Then

$$\begin{aligned}\min\{a,b\} &< H(a,b)=M_{-1}(a,b)<G(a,b)=M_0(a,b)<L(a,b)\\ &< I(a,b)<A(a,b)=M_1(a,b)<\max\{a,b\}.\end{aligned}$$

Let $\mathfrak{M}$ be the family of all mean values of two numbers in $\mathbb{R}_+=(0,\infty)$. Given $M,N\in\mathfrak{M}$, we say that a function $f:\mathbb{R}_+\to\mathbb{R}_+$ is (M,N)-convex if $f(M(x,y))\leq N(f(x),f(y))$ for all $x,y\in\mathbb{R}_+$. The concept of (M,N)-convexity has been studied extensively in the literature from various points of view (see e.g. [2, 3, 5, 26]),
Let $A(a,b;t)=ta+(1-t)b$, $G(a,b;t)=a^tb^{1-t}$, $H(a,b;t)=ab/(ta+(1-t)b)$ and $M_p(a,b;t)=(ta^p+(1-t)b^p)^{1/p}$ be the weighted arithmetic, geometric, harmonic , power of order p means of two positive real numbers a and b with $a\neq b$ for $t\in[0,1]$, respectively.

Email addresses: **sercan.turhan@giresun.edu.tr (Sercan Turhan), mkunt@ktu.edu.tr (Mehmet Kunt), imdat.iscan@giresun.edu.tr (İmdat İşcan)**

The most used class of means is quasi-arithmetic mean, which are associated to a continuous and strictly monotonic function $\varphi : I \to \mathbb{R}$ by the formula

$$M_\varphi(x,y) = \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right), \text{ for } x,y \in I.$$

Weighted quasi-arithmetic mean is given by the formula

$$M_\varphi(x,y;t) = \varphi^{-1}\left(t\varphi(x)+(1-t)\varphi(y)\right), \text{ for } x,y \in I,\, t \in [0,1].$$

Here $t \in (0,1)$ and $x < y$ always implies $x < M_\varphi(x,y;t) < y$. The function φ is called *Kolmogoroff-Naguma function of* M. Of special interest are the power means M_p on $\mathbb{R}_+$, defined by

$$\varphi_p(x) := \begin{cases} x^p, & p \neq 0 \\ \ln x, & p = 0 \end{cases}.$$

For $p = 1$, we get the arithmetic mean $A = M_1$, for $p = 0$, we get the geometric mean $G = M_0$ and for $p = -1$, we get the harmonic mean $H = M_{-1}$.
For any two quasi-arithmetic means M,N (with *Kolmogoroff-Naguma function* φ,ψ defined on intervals I,J, respectively), a function $f : I \to J$ can be called $\left(M_\varphi, M_\psi\right)$-convex if it satisfies

$$f(M_\varphi(x,y;t)) \leq M_\psi(f(x),f(y);t) \tag{1.2}$$

for all $x,y \in I$ and $t \in [0,1]$. Unless (1.2) is inequality, then f is said to be $\left(M_\varphi, M_\psi\right)$-concave. If $\psi : \mathbb{R} \to \mathbb{R}$, $\psi(x) = x$, (i.e., $M_\psi(f(x),f(y);t) = A\left(f(x),f(y);t\right)$), then we just say that f is $M_\varphi A$-convex.
Let f be a $M_\varphi A$-convex.
i) If we take $\varphi : I \subset \mathbb{R} \to \mathbb{R}$, $\varphi(x) = x$, then $M_\varphi A$-convexity deduce usual convexity.
ii) If we take $\varphi : I \subset (0,\infty) \to \mathbb{R}$, $\varphi(x) = \ln x$, then $M_\varphi A$-convexity deduce GA-convexity. (see [27, 28])
iii) If we take $\varphi : I \subset (0,\infty) \to \mathbb{R}$, $\varphi(x) = x^{-1}$, then $M_\varphi A$-convexity deduce Harmonically convexity. (see [13])
iv) If we take $\varphi : I \subset (0,\infty) \to \mathbb{R}$, $\varphi(x) = x^p$, $p \in \mathbb{R}\backslash\{0\}$, then $M_\varphi A$-convexity deduce p-convexity. (see [18]).
The theory of $\left(M_\varphi, M_\psi\right)$-convex functions can be deduced from the theory of usual convex functions.

Lemma 1.1 (Aczél [1])**.** *If φ and ψ are two continuous and strictly monotonic functions (on intervals I and J respectively) and ψ is increasing then a function $f : I \to J$ is $\left(M_\varphi, M_\psi\right)$-convex if and only if $\psi \circ f \circ \varphi^{-1}$ is convex on $\varphi(I)$ in the usual sense.*

The following concept was introduced by Orlicz in [29]:

Definition 1.2. *Let $0 < s \leq 1$. A function $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ where $\mathbb{R}_0 = [0,\infty)$, is said to be s-convex in the first sense if*

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x,y \in I$ and $\alpha,\beta \geq 0$ with $\alpha^s + \beta^s = 1$. We denote this class of real functions by K_s^1.

In [9], Hudzik and Maligranda considered the following class of functions:

Definition 1.3. *A function $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ where $\mathbb{R}_0 = [0,\infty)$, is said to be s-convex in the second sense if*

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x,y \in I$ and $\alpha,\beta \geq 0$ with $\alpha + \beta = 1$ and s fixed in $(0,1]$. They denoted this by K_s^2.

It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0,\infty)$.
In [8], Dragomir and Fitzpatrick proved a variant of Hermite-Hadamard inequality which holds for the s-convex functions.

Theorem 1.4. *Suppose that $f : \mathbb{R}_0 \to \mathbb{R}_0$ is an s-convex function in the second sense, where $s \in (0,1]$ and let $a,b \in [0,\infty)$, $a < b$. If $f \in L[a,b]$, then the following inequalities hold*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1.3}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.3).

The main purpose of this paper is to introduce the concepts $M_\varphi A$-s-convex function in the first sense and the second sense and give the Hermite-Hadamard's inequality for these classes of functions. Morever, in this paper we establish a new identity and a consequence of the identity is that we obtain some new general integral inequalities.

2. Definitions of $M_\varphi A$-s-convex functions in the first and second sense

Definition 2.1. *Let I be a real interval, $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotonic function and $s \in (0,1]$.*
i) A function $f : I \to \mathbb{R}$ is said to be $M_\varphi A$-s-convex in the first sense, if

$$f\left(\varphi^{-1}\left(t\varphi(x)+(1-t)\varphi(y)\right)\right) \le t^s f(x) + (1-t^s) f(y) \tag{2.1}$$

for all $x,y \in I$ and $t \in [0,1]$. If the inequality in (2.1) is reversed, then f is said to be $M_\varphi A$-s-concave in the first sense.
ii) A function $f : I \to \mathbb{R}$ is said to be $M_\varphi A$-s-convex in the second sense, if

$$f\left(\varphi^{-1}\left(t\varphi(x)+(1-t)\varphi(y)\right)\right) \le t^s f(x) + (1-t)^s f(y) \tag{2.2}$$

for all $x,y \in I$ and $t \in [0,1]$. If the inequality in (2.2) is reversed, then f is said to be $M_\varphi A$-s-concave in the second sense.

It can be easily seen that:
i) For $\varphi : I \to \mathbb{R}$, $\varphi(x) = mx+n$, $m \in \mathbb{R}\backslash\{0\}, n \in \mathbb{R}$, $M_\varphi A$-s-convexity (in the first sense or second sense) reduces to ordinary s convexity on I.
ii) For $\varphi : I \subset (0,\infty) \to \mathbb{R}$, $\varphi(x) = \ln x$, then $M_\varphi A$-s-convexity deduce GA-s-convexity.
iii) For $\varphi : I \subset (0,\infty) \to \mathbb{R}$, $\varphi(x) = x^{-1}$, then $M_\varphi A$-s-convexity deduce Harmonically s-convexity.
iv) For $\varphi : I \subset (0,\infty) \to \mathbb{R}$, $\varphi(x) = x^p$, $p \in \mathbb{R}\backslash\{0\}$, then $M_\varphi A$-s-convexity deduce (p,s)-convexity.

3. Inequalities for $M_\varphi A$-s-convex functions in the first and second sense

Let I be a real interval, throughout this section we will take $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotonic function and $s \in (0,1]$.

Theorem 3.1. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a $M_\varphi A$-s-convex function in the first sense and $a,b \in I$ with $a<b$. If $f, \varphi' \in L[a,b]$ then the following inequalities hold*

$$f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right) \le \frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx \le \frac{f(a)+sf(b)}{s+1}. \tag{3.1}$$

The above inequalities are sharp.

Proof. Since $f : I \to \mathbb{R}$ is a $M_\varphi A$-s-convex function in the first sense, we have, for all $x,y \in I$ (with $t=\frac{1}{2}$ in the inequality (2.1))

$$f\left(\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)\right) \le \frac{1}{2^s}f(x) + \left(1-\frac{1}{2^s}\right)f(y).$$

Choosing $x = \varphi^{-1}\left(t\varphi(a)+(1-t)\varphi(b)\right)$, $y = \varphi^{-1}\left(t\varphi(b)+(1-t)\varphi(a)\right)$, we get

$$f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right) \le \frac{1}{2^s}f\left(\varphi^{-1}\left(t\varphi(a)+(1-t)\varphi(b)\right)\right) + \left(1-\frac{1}{2^s}\right)f\left(\varphi^{-1}\left(t\varphi(b)+(1-t)\varphi(a)\right)\right).$$

Further, integrating for $t \in [0,1]$, we have

$$\begin{aligned} & f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right) \\ & \le \frac{1}{2^s}\int_0^1 f\left(\varphi^{-1}\left(t\varphi(a)+(1-t)\varphi(b)\right)\right)dt \\ & + \left(1-\frac{1}{2^s}\right)\int_0^1 f\left(\varphi^{-1}\left(t\varphi(b)+(1-t)\varphi(a)\right)\right)dt. \end{aligned} \tag{3.2}$$

Since each of the integrals is equal to $\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx$, we obtain the left-hand side of the inequality (3.1) from (3.2).
Secondly, we observe that for all $t \in [0,1]$

$$f\left(\varphi^{-1}\left(t\varphi(a)+(1-t)\varphi(b)\right)\right) \le t^s f(a) + (1-t^s) f(b).$$

Integrating this inequality with respect to t over $[0,1]$, we obtain the right-hand side of the inequality (3.1).
Now, consider the function $f : (0,\infty) \to \mathbb{R}$, $f(x) = 1$. thus

$$\begin{aligned} 1 &= f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right) \\ &= tf(a)+(1-t)f(b) = 1 \end{aligned}$$

for all $x,y\in I$ and $t\in[0,1]$. Therefore f is $M_{\varphi}A$-convex on I. We also have

$$f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)=1,\ \frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx=1,$$

and

$$\frac{f(a)+sf(b)}{s+1}=1$$

which shows us the inequalities (3.1) are sharp. □

Similarly to Theorem 3.1, we will give the following theorem for $M_{\varphi}A$-s-convex function in the second sense:

Theorem 3.2. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a $M_{\varphi}A$-s-convex function in the second sense and $a,b\in I$ with $a<b$. If $f,\varphi'\in L[a,b]$, then the following inequalities hold*

$$2^{s-1}f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)\le\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx\le\frac{f(a)+f(b)}{s+1} \tag{3.3}$$

Proof. As f is $M_{\varphi}A$-s-convex function in the second sense, we have, for all $x,y\in I$

$$f\left(\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)\right)\le\frac{f(x)+f(y)}{2^s}. \tag{3.4}$$

Now, let $x=\varphi^{-1}\left(t\varphi(a)+(1-t)\varphi(b)\right),\ y=\varphi^{-1}\left(t\varphi(b)+(1-t)\varphi(a)\right)$ with $t\in[0,1]$. Then we get by (3.4) that:

$$f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)\le\frac{f(\varphi^{-1}\left(t\varphi(a)+(1-t)\varphi(b)\right))+f(\varphi^{-1}\left(t\varphi(b)+(1-t)\varphi(a)\right))}{2^s}$$

for all $t\in[0,1]$. Integrating this inequality on $[0,1]$, we deduce the first part of (3.3).
Secondly, we observe that for all $t\in[0,1]$

$$f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)\le t^sf(a)+(1-t)^sf(b).$$

Integrating this inequality on $[0,1]$, we get

$$\int_0^1 f\left(\varphi^{-1}\left(t\varphi(a)+(1-t)\varphi(b)\right)\right)dt=\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx\le\frac{f(a)+f(b)}{s+1}.$$

the second inequality in (3.3) is proved. □

The following proposition is obvious.

Proposition 3.3. *Let $f:[a,b]\to\mathbb{R}$ and $\varphi:I\to\mathbb{R}$ be a continuous and strictly monotonic incresing (or stricly monotonic decreasing). If we consider the function $g:[\varphi(a),\varphi(b)]\to\mathbb{R}$,(or if $\varphi:I\to\mathbb{R}$ is strictly monotonic decreasing, then $g:[\varphi(b),\varphi(a)]\to\mathbb{R}$,) defined by $g(t)=f\left(\varphi^{-1}(t)\right)$, then f is $M_{\varphi}A$-s-convex in the first sense (or second sense) on $[a,b]$ if and only if g is s-convex in the first sense (or second sense) on $[\varphi(a),\varphi(b)]$.*

Remark 3.4. *According to Proposition 3.3, we can obtain the inequalities (3.1) and (3.3) in a different manner as follow:*
For example, If f is a $M_{\varphi}A$-s-convex in the second sense on $[a,b]$ then we write the Hermite-Hadamard inequality for the s-convex function in the second sense $g(t)=f\left(\varphi^{-1}(t)\right)$ on the closed interval $[\varphi(a),\varphi(b)]$ (or $[\varphi(b),\varphi(a)]$) as follows

$$2^{s-1}g\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\le\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}g(t)dt\le\frac{g(\varphi(a))+g(\varphi(b))}{s+1}$$

that is equivalent to

$$2^{s-1}f\left(\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right)\le\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}f\left(\varphi^{-1}(t)\right)dt\le\frac{f(a)+f(b)}{s+1}. \tag{3.5}$$

Using the change of variable $x=\varphi^{-1}(t)$, then

$$\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}f\left(\varphi^{-1}(t)\right)dt=\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx$$

and by (3.5) we get the inequality (3.3).

For finding some new inequalities of Hermite-Hadamard type for functions whose derivatives are $M_\varphi A$-s-convex, we need a simple lemma as follows.

Lemma 3.5. *Let $f: I \to \mathbb{R}$ be a differentiable function on I° and $a,b \in I$ with $a<b$ and $\varphi: I \to \mathbb{R}$ be a continuous and strictly monotonic function such that $\varphi^{-1}: \varphi(I^\circ) \to I^\circ$ is continuously differentiable. If $f' \in L[a,b]$ then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{\varphi(b)-\varphi(a)} \int_a^b f(x)\varphi'(x)dx = \frac{\varphi(b)-\varphi(a)}{2} \tag{3.6}$$

$$\int_0^1 (1-2t) . \left(\varphi^{-1}\right)' (t\varphi(a)+(1-t)\varphi(b)) f' \left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right) dt.$$

Proof. Let

$$J = \frac{\varphi(b)-\varphi(a)}{2} \int_0^1 (1-2t) . \left(\varphi^{-1}\right)' (t\varphi(a)+(1-t)\varphi(b)) f' \left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right) dt.$$

By integrating by part, we have

$$J = \frac{(2t-1)}{2} f\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right)\Big|_0^1 - \int_0^1 f\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right) dt.$$

Setting $x = \varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))$, $dt = \frac{-\varphi'(x)}{\varphi(b)-\varphi(a)}dx$, we obtain

$$J = \frac{f(a)+f(b)}{2} - \frac{1}{\varphi(b)-\varphi(a)} \int_a^b f(x)\varphi'(x)dx$$

which gives the desired representation (3.6). □

Remark 3.6. *In Lemma 3.5*

(i) If we take $\varphi(x) = mx+n$, then we have the equality in [8, Lemma A].

(ii) If we take $\varphi(x) = lnx$, then we have the equality in [12, Lemma 1] as follow:

$$\begin{aligned} & \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_0^1 f(x)dx \\ = & \frac{lnb-lna}{2}\int_0^1 (1-2t)a^t b^{1-t} f'(a^t b^{1-t})dt \\ = & \frac{lnb-lna}{2}\left[a\int_0^1 t\left(\frac{b}{a}\right)^t f'\left(a^{1-t}b^t\right)dt - b\int_0^1 t\left(\frac{a}{b}\right)^t f'\left(b^{1-t}a^t\right)dt\right]. \end{aligned} \tag{3.7}$$

(iii) If we take $\varphi(x) = \frac{1}{x}$, then we have the equality in [13, 2.5. Lemma].

(iv) If we take $\varphi(x) = x^p$, $p \in \mathbb{R}\setminus\{0\}$, then we have the equality [19, Lemma 3].

Theorem 3.7. *Let $f: I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on I°, and $a,b \in I^\circ$ with $a<b$, $\varphi: I \to \mathbb{R}$ be a continuous and strictly monotonic function such that $\varphi^{-1}: \varphi(I^\circ) \to I^\circ$ is continuously differentiable and $f' \in L[a,b]$.*
a) If $|f'|$ is $M_\varphi A$-s-convex function in the second sense on $[a,b]$ and $s \in (0,1]$, then

$$\begin{aligned} & \left|\frac{f(a)+f(b)}{2} - \frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx\right| \\ \leq & \frac{|\varphi(b)-\varphi(a)|}{2}\left\{A_\varphi(a,b)\left|f'(a)\right| + B_\varphi(a,b)\left|f'(b)\right|\right\}, \end{aligned} \tag{3.8}$$

where

$$A_\varphi(a,b) = \int_0^1 |1-2t|\, t^s \left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right| dt$$

and

$$B_\varphi(a,b) = \int_0^1 |1-2t|\,(1-t)^s \left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right| dt.$$

b) If $|f'|$ is $M_{\varphi}A$-s-convex function in the first sense on $[a,b]$ and $s\in(0,1]$, then

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx\right| \\ \leq\ &\frac{|\varphi(b)-\varphi(a)|}{2}\left\{A_{\varphi}(a,b)\left|f'(a)\right|+C_{\varphi}(a,b)\left|f'(b)\right|\right\} \end{aligned} \tag{3.9}$$

where

$$C_{\varphi}(a,b)=\int_0^1 |1-2t|\left(1-t^s\right)\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|dt.$$

Proof. a) Since $|f'|$ is $M_{\varphi}A$-s-convex function in the second sense on $[a,b]$, from Lemma (3.5), we have

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx\right| \\ \leq\ &\frac{|\varphi(b)-\varphi(a)|}{2}\int_0^1 |1-2t|\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|\left|f'\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right)\right|dt \\ \leq\ &\frac{|\varphi(b)-\varphi(a)|}{2}\int_0^1 |1-2t|\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|\left[t^s\left|f'(a)\right|+(1-t)^s\left|f'(b)\right|\right]dt \\ =\ &\frac{|\varphi(b)-\varphi(a)|}{2}\left\{\begin{array}{c} |f'(a)|\int_0^1|1-2t|t^s\left|(\varphi^{-1})'(t\varphi(a)+(1-t)\varphi(b))\right|dt \\ +|f'(b)|\int_0^1|1-2t|(1-t)^s\left|(\varphi^{-1})'(t\varphi(a)+(1-t)\varphi(b))\right|dt \end{array}\right\} \\ =\ &\frac{|\varphi(b)-\varphi(a)|}{2}\left\{A_{\varphi}(a,b)\left|f'(a)\right|+B_{\varphi}(a,b)\left|f'(b)\right|\right\}. \end{aligned}$$

b) Similarly to a), since $|f'|$ is $M_{\varphi}A$-s-convex function in the first sense on $[a,b]$, we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx\right|$$

$$\leq\frac{|\varphi(b)-\varphi(a)|}{2}\left\{A_{\varphi}(a,b)\left|f'(a)\right|+C_{\varphi}(a,b)\left|f'(b)\right|\right\}.$$

□

Remark 3.8.

(i) If we take $\varphi(x)=mx+n$ in $[Theorem(3.7),a)]$, then we have the inequality in [22, Theorem 1,q=1].

(ii) If we take $\varphi(x)=lnx$ in $[Theorem(3.7),a)]$, then we have the follows inequality

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{lnb-lna}\int_a^b\frac{f(x)}{x}dx\right|\leq\frac{lnb-lna}{2}\left\{A_{(lnx)}(a,b)\left|f'(a)\right|+B_{(lnx)}(a,b)\left|f'(b)\right|\right\}$$

(iii)If we take $\varphi(x)=lnx$ in $[Theorem(3.7),b)]$, then we have the follows inequality

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{lnb-lna}\int_a^b\frac{f(x)}{x}dx\right|\leq\frac{lnb-lna}{2}\left\{A_{(lnx)}(a,b)\left|f'(a)\right|+C_{(lnx)}(a,b)\left|f'(b)\right|\right\}$$

(iv) If we take $\varphi(x)=\frac{1}{x}$ in $[Theorem(3.7),a)]$, then we have the inequality in [17, Corollary 2.4 (3),q=1].

(v) If we take $\varphi(x)=x^p$, $p\in\mathbb{R}\setminus\{0\}$ in $[Theorem(3.7),a,s=1]$, then we have the inequality [19, Theorem 7,q=1].

Theorem 3.9. *Let $f:I\subseteq\mathbb{R}_+\to\mathbb{R}$ be differentiable on I°, and $a,b\in I^{\circ}$ with $a<b$, $\varphi:I\to\mathbb{R}$ be a continuous and strictly monotonic function such that $\varphi^{-1}:\varphi(I^{\circ})\to I^{\circ}$ is continuously differentiable, $f'\in L[a,b]$ and $q>1$, $\frac{1}{p}+\frac{1}{q}=1$. a) If $|f'|^q$ is $M_{\varphi}A$-s-convex function in the second sense on $[a,b]$ and $s\in(0,1]$, then*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx\right| \\ \leq\ &\frac{|\varphi(b)-\varphi(a)|}{2}D_{\varphi}^{1/p}(a,b;p)\left(\frac{|f'(a)|^q+|f'(b)|^q}{s+1}\right)^{1/q} \end{aligned} \tag{3.10}$$

where

$$D_{\varphi}(a,b;p)=\int_0^1|1-2t|^p\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|^p dt.$$

b) If $|f'|^q$ is $M_{\varphi}A$-s-convex function in the first sense on $[a,b]$ and $s\in(0,1]$, then

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx\right| \\ \leq\ & \frac{|\varphi(b)-\varphi(a)|}{2}D_{\varphi}^{1/p}(a,b;p)\left(\frac{|f'(a)|^q+s|f'(b)|^q}{s+1}\right)^{1/q}.\end{aligned}\tag{3.11}$$

Proof. a) Since $|f'|$ is $M_{\varphi}A$-s-convex function in the second sense on $[a,b]$, from Lemma 3.5 and Hölder inequality, we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx\right| \\ \leq\ & \frac{|\varphi(b)-\varphi(a)|}{2}\int_0^1|1-2t|\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|\left|f'\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right)\right|dt \\ \leq\ & \frac{|\varphi(b)-\varphi(a)|}{2}\left(\int_0^1|1-2t|^p\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|^p dt\right)^{1/p} \\ & \times\left(\int_0^1\left|f'\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right)\right|^q dt\right)^{1/q} \\ \leq\ & \frac{|\varphi(b)-\varphi(a)|}{2}\left(\int_0^1|1-2t|^p\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|^p dt\right)^{1/p}\left(\frac{|f'(a)|^q+|f'(b)|^q}{s+1}\right)^{1/q} \\ =\ & \frac{|\varphi(b)-\varphi(a)|}{2}D_{\varphi}^{1/p}(a,b;p)\left(\frac{|f'(a)|^q+|f'(b)|^q}{s+1}\right)^{1/q}.\end{aligned}$$

b) Similarly to the proof of a), we can get easily the inequality (3.11). □

Theorem 3.10. *Let $f:I\subseteq\mathbb{R}_+\to\mathbb{R}$ be differentiable on I°, and $a,b\in I^\circ$ with $a<b$, $\varphi:I\to\mathbb{R}$ be a continuous and strictly monotonic increasing function $f'\in L[a,b]$ and $q>1$, $\frac{1}{p}+\frac{1}{q}=1$. a) If $|f'|^q$ is $M_{\varphi}A$-s-convex function in the second sense on $[a,b]$, $s\in(0,1]$ and $\left(\varphi^{-1}\right)'\in L_p[\varphi(a),\varphi(b)]$ then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx\right| \\ \leq\ & \frac{[\varphi(b)-\varphi(a)]^{1/q}}{2}\left\|\left(\varphi^{-1}\right)'\right\|_p\left(|f'(a)|^q E(q,s)+|f'(b)|^q F(q,s)dt\right)^{1/q}\end{aligned}\tag{3.12}$$

where

$$\left\|\left(\varphi^{-1}\right)'\right\|_p=\left(\int_{\varphi(a)}^{\varphi(b)}\left|\left(\varphi^{-1}\right)'(x)\right|^p dx\right)^{1/p}$$

$$E(q,s)=\int_0^1|1-2t|^q t^s dt$$

and

$$F(q,s)=\int_0^1|1-2t|^q(1-t)^s dt.$$

b) If $|f'|^q$ is $M_{\varphi}A$-s-convex function in the first sense on $[a,b]$ and $s\in(0,1]$, then

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx\right| \\ \leq\ & \frac{[\varphi(b)-\varphi(a)]^{1/q}}{2}\left\|\left(\varphi^{-1}\right)'\right\|_p\left(|f'(a)|^q E(q,s)+|f'(b)|^q G(q,s)dt\right)^{1/q},\end{aligned}\tag{3.13}$$

where

$$G(q,s)=\int_0^1 |1-2t|^q (1-t^s)dt.$$

Proof. a) Since $|f'|$ is $M_\varphi A$-s-convex function in the second sense on $[a,b]$, from Lemma (3.5) and Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx\right| \\
\leq\ & \frac{\varphi(b)-\varphi(a)}{2}\int_0^1 |1-2t|\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|\left|f'\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right)\right|dt \\
\leq\ & \frac{\varphi(b)-\varphi(a)}{2}\left(\int_0^1 \left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|^p dt\right)^{1/p} \\
\times\ & \left(\int_0^1 |1-2t|^q\left|f'\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right)\right|^q dt\right)^{1/q} \\
\leq\ & \frac{\varphi(b)-\varphi(a)}{2}\left(\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)}\left|\left(\varphi^{-1}\right)'(x)\right|^p dx\right)^{1/p} \\
\times\ & \left(\left|f'(a)\right|^q\int_0^1 |1-2t|^q t^s dt+\left|f'(b)\right|^q\int_0^1 |1-2t|^q (1-t)^s dt\right)^{1/q} \\
=\ & \frac{[\varphi(b)-\varphi(a)]^{1/q}}{2}\left\|\left(\varphi^{-1}\right)'\right\|_p\left(\left|f'(a)\right|^q\int_0^1 |1-2t|^q t^s dt+\left|f'(b)\right|^q\int_0^1 |1-2t|^q (1-t)^s dt\right)^{1/q} \\
=\ & \frac{[\varphi(b)-\varphi(a)]^{1/q}}{2}\left\|\left(\varphi^{-1}\right)'\right\|_p\left(\left|f'(a)\right|^q E(q,s)+\left|f'(b)\right|^q F(q,s)dt\right)^{1/q}
\end{aligned}$$

b) Similarly to the proof of a), we can get easily the inequality (3.13). □

Theorem 3.11. *Let $f: I\subseteq\mathbb{R}_+\to\mathbb{R}$ be differentiable on I°, and $a,b\in I^\circ$ with $a<b$, $\varphi: I\to\mathbb{R}$ be a continuous and strictly monotonic function such that $\varphi^{-1}:\varphi(I^\circ)\to I^\circ$ is continuously differentiable, $f'\in L[a,b]$ and $q\geq 1$.*

a) If $|f'|^q$ is $M_\varphi A$-s-convex function in the second sense on $[a,b]$ and $s\in(0,1]$, then

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int_0^1 f(x)\varphi'(x)dx\right| \\
&\leq \frac{|\varphi(b)-\varphi(a)|}{2^{3-\frac{1}{q}}}\left[\left(M_{1,\varphi}(t;a,b)\left|f'(a)\right|^q+M_{2,\varphi}(t;a,b)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(M_{3,\varphi}(t;a,b)\left|f'(a)\right|^q+M_{4,\varphi}(t;a,b)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]
\end{aligned} \tag{3.14}$$

where

$$\left(M_{1,\varphi}\right)(t;a,b)=\int_0^{1/2}(1-2t)t^s\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|^q dt$$

$$\left(M_{2,\varphi}\right)(t;a,b)=\int_0^{1/2}(1-2t)(1-t)^s\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|^q dt$$

$$\left(M_{3,\varphi}\right)(t;a,b)=\int_{1/2}^1(2t-1)t^s\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|^q dt$$

$$\left(M_{4,\varphi}\right)(t;a,b)=\int_{1/2}^1(2t-1)(1-t)^s\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|^q dt$$

b)If $|f'|^q$ is $M_\varphi A$-s-convex function in the first sense on $[a,b]$ and $s\in(0,1]$, then

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int_0^1 f(x)\varphi'(x)dx\right| \\ &\frac{|\varphi(b)-\varphi(a)|}{2^{3-\frac{2}{q}}}\left[\left(N_{1,\varphi}(t;a,b)\left|f'(a)\right|^q+N_{2,\varphi}(t;a,b)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right. \\ &\left.+\left(N_{3,\varphi}(t;a,b)\left|f'(a)\right|^q+N_{4,\varphi}(t;a,b)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]\end{aligned} \tag{3.15}$$

where

$$\left(N_{1,\varphi}\right)(t;a,b)=\int_0^{1/2}(1-2t)t^s\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|^q dt$$

$$\left(N_{2,\varphi}\right)(t;a,b)=\int_0^{1/2}(1-2t)(1-t^s)\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|^q dt$$

$$\left(N_{3,\varphi}\right)(t;a,b)=\int_{1/2}^{1}(2t-1)t^s\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|^q dt$$

$$\left(N_{4,\varphi}\right)(t;a,b)=\int_{1/2}^{1}(2t-1)(1-t^s)\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|^q dt$$

Proof.

a) Since $|f'|^q$, $q\geq 1$ is $M_\varphi A$-s-convex function in the second sense on $[a,b]$, from Lemma (3.5) and Hölder inequality, we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int_0^1 f(x)\varphi'(x)dx\right| \\ \leq\ &\frac{|\varphi(b)-\varphi(a)|}{2}\int_0^1 |1-2t|\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right| \left|f'\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right)\right| dt \\ \leq\ &\frac{|\varphi(b)-\varphi(a)|}{2}\left[\int_0^{1/2}(1-2t)\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right| \left|f'\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right)\right| dt\right. \\ &\left.+\int_{1/2}^1(2t-1)\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right| \left|f'\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right)\right| dt\right]\end{aligned} \tag{3.16}$$

$$\begin{aligned}\leq\ &\frac{|\varphi(b)-\varphi(a)|}{2}\left[\left(\int_0^{1/2}(1-2t)dt\right)^{1-\frac{1}{q}}\left(\int_0^{1/2}(1-2t)\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|^q \left|f'\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\int_{1/2}^1(2t-1)dt\right)^{1-\frac{1}{q}}\left(\int_{1/2}^1(2t-1)\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|^q \left|f'\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\ \leq\ &\frac{|\varphi(b)-\varphi(a)|}{2^{3-\frac{2}{q}}}\left[\left(\int_0^{1/2}(1-2t)\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|^q \left(t^s\left|f'(a)\right|^q+(1-t)^s\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\int_{1/2}^1(2t-1)\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|^q \left(t^s\left|f'(a)\right|^q+(1-t)^{1-s}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right].\end{aligned}$$

This proof is completed.

b) Similarly to the proof of a), we can get easily the inequality (3.15) □

Remark 3.12.

(i) If we take $\varphi(x) = mx + n$ *in* $[(Theorem(3.11), a), q = 1]$*, then we have the inequality in [22, Theorem 1,q=1].*

(ii) If we take $\varphi(x) = lnx$ *in* $[Theorem(3.11), a]$*, then we have the follows inequality*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{lnb-lna}\int_a^b \frac{f(x)}{x}dx \right| \leq \frac{lnb-lna}{2^{3-\frac{1}{q}}} \left[\left(M_{1,lnt}(t;a,b) \left| f'(a) \right|^q + M_{2,lnt}(t;a,b) \left| f'(b) \right|^q \right)^{\frac{1}{q}} \right.$$
$$\left. + \left(M_{3,lnt}(t;a,b) \left| f'(a) \right|^q + M_{4,lnt}(t;a,b) \left| f'(b) \right|^q \right)^{\frac{1}{q}} \right]$$

(iii)If we take $\varphi(x) = lnx$ *in* $[Theorem(3.11), b]$*, then we have the follows inequality*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{lnb-lna}\int_a^b \frac{f(x)}{x}dx \right| \leq \frac{lnb-lna}{2^{3-\frac{1}{q}}} \left[\left(N_{1,lnt}(t;a,b) \left| f'(a) \right|^q + N_{2,lnt}(t;a,b) \left| f'(b) \right|^q \right)^{\frac{1}{q}} \right.$$
$$\left. + \left(N_{3,lnt}(t;a,b) \left| f'(a) \right|^q + N_{4,lnt}(t;a,b) \left| f'(b) \right|^q \right)^{\frac{1}{q}} \right]$$

(iv) If we take $\varphi(x) = \frac{1}{x}$ *in* $[Theorem(3.11), a]$*, then we have the inequality in [17, Corollary 2.4 (3)].*

References

[1] J. Aczél , The notion of mean values, Norske Vid. Selsk. Forhdl., Trondhjem 19 (1947), 83–86.
[2] J. Aczél , A generalization of the notion of convex functions, Norske Vid. Selsk. Forhd., Trondhjem 19 (24) (1947), 87–90.
[3] G. Aumann , Konvexe Funktionen und Induktion bei Ungleichungen zwischen Mittelverten, Bayer. Akad. Wiss.Math.-Natur. Kl. Abh., Math. Ann. 109 (1933), 405–413.
[4] M. Avcı,H. Kavurmacıand M. E. Özdemir , New inequalities of Hermite-Hadamard type via s-convex functions in the second sense with applications, Appl. Math. Comput., vol. 217 (2011), pp. 5171–5176.
[5] G.D. Anderson, M.K. Vamanamurthy and M. Vuorinen , Generalized convexity and inequalities, Journal of Mathematical Analysis and Applications 335 (2) (2007), 1294–1308.
[6] Y.-M. Chu , M. Adil Khan , T. U. Khan , and J. Khan, Some new inequalities of Hermite-Hadamard type for s-convex functions with applications, Open Math., 15 (2017) 1414-1430.
[7] S.S. Dragomir , R.P. Agarwal , Two Inequalities for Differentiable Mappings and Applications to Special Means of Real Numbers and to Trapezoidal Formula, Appl. Math. Lett. 11 (5) (1998), 91–95.
[8] S. S. Dragomir and S. Fitzpatrick, The Hadamard's inequality for s-convex functions in the second sense, Demonstr. Math., 32 (4) (1999), 687–696.
[9] H. Hudzik and L. Maligranda, Some remarks on s-convex functions, Aequationes Math., 48 (1994), 100–111.
[10] İ. İşcan , A new generalization of some integral inequalities for -convex functions, Mathematical Sciences 2013, 7:22,1–8.
[11] İ. İşcan , New estimates on generalization of some integral inequalities for s-convex functions and their applications, International Journal of Pure and Applied Mathematics, 86 (4) (2013), 727–746.
[12] İ. İşcan , Some New Hermite-Hadamard Type Inequalities for Geometrically Convex Functions, Mathematics and Statistics 1(2) (2013), 86–91.
[13] İ. İşcan, Hermite-Hadamard type inequalities for harmonically convex functions, Hacettepe Journal of Mathematics and Statistics, 43 (6) (2014), 935–942.
[14] İ. İşcan , Some new general integral inequalities for h-convex and h-concave functions, Adv. Pure Appl. Math. 5 (1) (2014), 21–29.
[15] İ. İşcan , Hermite-Hadamard type inequalities for $GA-s$-convex functions, Le Matematiche, Vol. LXIX (2014) – Fasc. II, pp. 129—146.
[16] İ. İşcan , Hermite-Hadamard-Fejer type inequalities for convex functions via fractional integrals, Studia Universitatis Babeş-Bolyai Mathematica, 60(2015), no.3, 355–366.
[17] İ. İşcan , M. Kunt, Hermite-Hadamard-Fejér type inequalities for harmonically s-convex functions via fractional integrals, The Australian Journal of Mathematical Analysis and Applications, Volume 12, Issue 1, Article 10, (2015), 1–16.
[18] İ. İşcan , Ostrowski type inequalities for p-convex functions, New Trends in Mathematical Sciences, NTMSCI 4 No. 3 (2016), 140–150.
[19] İ. İşcan , Hermite-Hadamard type inequalities for p-convex functions, International Journal of Analysis and Applications, Volume 11, Number 2 (2016), 137–145.
[20] U.S. Kirmaci, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, *Appl. Math. Comput.* **147** (2004), 137-146.
[21] A.A. Kilbas ,H.M. Srivastava and J.J. Trujillo , *Theory and applications of fractional differential equations*, Amsterdam, Elsevier, 2006.
[22] U. S. Kirmaci ,M. K. Bakula ,M. E. Özdemir ,J. Peçarić , Hadamard-type inequalities for s-convex functions, Applied Mathematics and Computation 193 (2007) 26—35.
[23] M. Adil Khan ,T. Ali and T. U. Khan, Hermite-Hadamard Type Inequalities with Applications, Fasciculi Mathematici, 59 (2017), 57-74.
[24] Khan M. Adil Khan, T. Ali, M. Z. Sarikaya , and Q. Din, New bounds forHermite-Hadamard type inequalities with applications, Electronic Journal of Mathematical Analysis and Applications, to appear (2018).
[25] M. Adil Khan, Y. Khurshid , S. S. Dragomir and R. Ullah , Inequalities of the Hermite-Hadamard type with applications,Punjab Univ. J. Math., 50(3)(2018) 1-12.
[26] J. Matkowski , Convex functions with respect to a mean and a characterization of quasi-arithmetic means, Real Anal. Exchange 29 (2003/2004), 229–246.
[27] C. P. Niculescu , Convexity according to the geometric mean, Math. Inequal. Appl., vol. 3, no. 2 (2000), pp. 155–167.
[28] C.P. Niculescu, Convexity according to means, Math. Inequal. Appl. 6 (2003) 571–579.
[29] W. Orlicz, A note on modular spaces I, Bull. Acad. Polon. Sci. Ser. Math. Astronom. Phys., 9 (1961), 157-162.