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Research Article

Degenerate Pochhammer symbol, degenerate Sumudu transform, and degenerate hypergeometric function with applications

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Abstract

In the paper, we first define a degenerate Pochhammer symbol by using the degenerate gamma function and investigate its properties. By using the degenerate Pochhammer symbol, we introduce and investigate a degenerate hypergeometric function. We also define a degenerate Sumudu transform and investigate its properties by using degenerate exponential function. Finally, we give certain the integral representations, derivative formulas, integral transforms, factional calculus applications, and generating functions of the degenerate hypergeometric function.

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1. Introduction

Throughout this paper, we use the notations

$$\mathbb{N}_0 = \{0, 1, 2, \dots\} \quad \text{and} \quad \mathbb{N} = \{1, 2, 3, \dots\}.$$

The classical gamma function can be defined [1, 15, 19] by

$$\Gamma(z) = \int_0^\infty t^{z-1} \exp(-t) \mathrm{d}t, \quad \Re(z) > 0.$$

For $\alpha \in \mathbb{C}$, the Pochhammer symbol $(\alpha)_n$ is defined by

$$(\alpha)_n = \begin{cases} \prod_{k=0}^{n-1} (\alpha + k) = \dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha)} & n \geq 1; \\ 1, & n \leq 0. \end{cases}$$

The Pochhammer symbol $(\alpha)_n$ is also known as the rising factorial, see [10–14] and closely related references therein.

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For $\lambda \in (0,\infty)$ and $t \in \mathbb{R}$, the degenerate exponential function e_λ^ξ was defined [8] as

$$e_\lambda^\xi = (1+\lambda\xi)^{1/\lambda}. \tag{1.1}$$

Using the degenerate exponential function e_λ^ξ defined in (1.1), Kim and Kim defined in [8] the degenerate gamma function $\Gamma_\lambda(z)$ as

$$\Gamma_\lambda(z) = \int_0^\infty (1+\lambda\xi)^{-1/\lambda}\xi^{z-1}\mathrm{d}\xi, \quad 0 < \Re(z) < \frac{1}{\lambda}. \tag{1.2}$$

In 1993, Watugala defined [24] the Sumudu transform by

$$S[f(\zeta)](u) = \frac{1}{u}\int_0^\infty \exp(-u\zeta) f(\zeta)\mathrm{d}\zeta$$

or, equivalently,

$$G(u) = S[f(\zeta)] = \int_0^\infty \exp(-\zeta) f(u\zeta)\mathrm{d}\zeta,$$

provided that the integrals involved are convergent. For more details, please refer to [25,26] and closely related references therein.

The main aim of this paper is to define a degenerate Pochhammer symbol by using the degenerate gamma function defined in (1.2). A great number of extensions of the Pochammer symbol are available in the literature [16,18,20,22,23]. The paper is organized as follows: In Sec. 2, we introduce a degenerate hypergeometric function. In Sec. 3, a degenerate hypergeometric function and its properties are given. In Sec. 4, a degenerate Sumudu transform is presented. In Sec. (5), we give the integral transforms of the degenerate hypergeometric function. In Sections 6 and 7, we derive certain fractional calculus operators for the degenerate hypergeometric function. Section 8 is devoted to the families of generating relations for the degenerate hypergeometric function.

2. Degenerate Pochhammer symbol

For $\lambda \in (0,\infty)$ and $\alpha, n \in \mathbb{C}$, we define degenerate Pochhammer symbol $(\alpha)_n^\lambda$ by

$$(\alpha)_n^\lambda = \frac{\Gamma_\lambda(\alpha+n)}{\Gamma(\alpha)}. \tag{2.1}$$

The degenerate Pochhammer symbol $(\alpha)_n^\lambda$ has the integral representation

$$(\alpha)_n^\lambda = \frac{1}{\Gamma(\alpha)}\int_0^\infty \xi^{\alpha+n-1}(1+\lambda\xi)^{-1/\lambda}\mathrm{d}\xi, \quad \Re(\alpha+n) > 0. \tag{2.2}$$

Theorem 2.1. *For $\lambda \in (0,\infty)$ and $\alpha, \delta, \rho \in \mathbb{C}$, we have*

$$(\alpha)_{\delta+\rho}^\lambda = (\alpha)_\delta(\alpha+\delta)_\rho^\lambda. \tag{2.3}$$

Proof. This follows from

$$(\alpha)_{\delta+\rho}^\lambda = \frac{\Gamma_\lambda(\alpha+\delta+\rho)}{\Gamma(\alpha)} = \frac{\Gamma_\lambda(\alpha+\delta+\rho)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+\delta)}{\Gamma(\alpha+\delta)} = (\alpha)_\delta(\alpha+\delta)_\rho^\lambda.$$

The proof of Theorem 2.1 is complete. □

Making use of the relation (2.3) and properties of the Pochhammer symbol $(\alpha)_n$, we can derive the following features of the degenerate Pochhammer symbol $(\alpha)_n^\lambda$.

Corollary 2.2. *For $\lambda \in (0,\infty)$, $\alpha,\delta,\rho,\mu,\nu \in \mathbb{N}_0$, and $k \in \mathbb{N}$, we have*

$$\begin{aligned}
(\alpha)^{\lambda}_{2\delta+\rho} &= 2^{2\delta}\left(\frac{\alpha}{2}\right)_{\delta}\left(\frac{\alpha+1}{2}\right)_{\delta}(\alpha+2\delta)^{\lambda}_{\rho},\\
(\alpha)^{\lambda}_{k\delta+\rho} &= k^{k\delta}\left(\frac{\alpha}{k}\right)_{\delta}\left(\frac{\alpha+1}{k}\right)_{\delta}\cdots\left(\frac{\alpha+k-1}{k}\right)_{\delta}(\alpha+k\delta)^{\lambda}_{\rho},\\
(\alpha+\mu)^{\lambda}_{\nu+\rho} &= \frac{(\alpha)_{\nu}(\alpha+\nu)_{\mu}}{(\alpha)_{\mu}}(\alpha+\mu+\nu)^{\lambda}_{\rho},\\
(\alpha+\delta\mu)^{\lambda}_{\delta\nu+\rho} &= \frac{(\alpha)_{\delta\mu+\delta\nu}}{(\alpha)_{\delta\mu}}(\alpha+\delta\mu+\delta\nu)^{\lambda}_{\rho},\\
(\alpha-\nu)^{\lambda}_{\nu+\rho} &= (-1)^{n}(1-\alpha)_{\nu}(\alpha)^{\lambda}_{\rho},\\
(\alpha-\mu)^{\lambda}_{\nu+\rho} &= \frac{(1-\alpha)_{\mu}(\alpha)_{\nu}}{(1-\alpha-\nu)_{\mu}}(\alpha+\nu-\mu)^{\lambda}_{\rho},\\
(\alpha-\delta\mu)^{\lambda}_{\delta\nu+\rho} &= (-1)^{\delta\mu}(\alpha)_{\delta\nu-\delta\mu}(1-\alpha)_{\delta\mu}(\alpha+\delta\nu-\delta\mu)^{\lambda}_{\rho},\\
(-\alpha)^{\lambda}_{\nu+\rho} &= (-1)^{\nu}(\alpha-\nu+1)_{\nu}(1-\alpha)_{\delta\mu}(-\alpha+\nu)^{\lambda}_{\rho},\\
(\alpha)^{\lambda}_{\nu+\mu+\rho} &= (\alpha)_{\nu}(\alpha+\nu)_{\mu}(\alpha+\nu+\mu)^{\lambda}_{\rho},\\
(\alpha)^{\lambda}_{\nu-\mu+\rho} &= \frac{(-1)^{\mu}(\alpha)_{\nu}}{(1-\alpha-\nu)_{\mu}}(\alpha+\nu-\mu)^{\lambda}_{\rho},\\
(\alpha+\mu)^{\lambda}_{\nu-\mu+\rho} &= \frac{(\alpha)_{\nu}}{(\alpha)_{\mu}}(\alpha+\nu)^{\lambda}_{\rho},\\
(\alpha-\mu)^{\lambda}_{\nu-\mu+\rho} &= \frac{(-1)^{\mu}(\alpha)_{\nu}(1-\alpha)_{\mu}}{(1-\alpha-\nu)_{2\mu}}(\alpha+\nu-2\mu)^{\lambda}_{\rho},\\
(\alpha+\nu)^{\lambda}_{\nu+\rho} &= (\alpha+\nu)_{\nu}(\alpha+2\nu)^{\lambda}_{\rho} = \frac{(\alpha)_{2\nu}}{(\alpha)_{\nu}}(\alpha+2\nu)^{\lambda}_{\rho}.
\end{aligned}$$

3. Degenerate hypergeometric function

For $\lambda \in (0,\infty)$, $\alpha_\tau \in \mathbb{C}$ for $\tau = 1,2,\ldots,p$, and $\beta_\kappa \in \mathbb{C}\setminus\mathbb{Z}_0$ for $\kappa = 1,2,\ldots,q$, we define

$$ {}_pF_q^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)^{\lambda}_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!} \tag{3.1}$$

on the condition that the series on the right hand side converges.

When taking $p,q=1$ or taking $p=2$ and $q=1$ in (3.1), we have

$$ {}_1F_1^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda};\\ \beta_1;\end{matrix}z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)^{\lambda}_n}{(\beta_1)_n}\frac{z^n}{n!},$$

$$ {}_2F_1^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2;\\ \beta_1;\end{matrix}z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)^{\lambda}_n(\alpha_2)_n}{(\beta_1)_n}\frac{z^n}{n!},$$

which are respectively called a degenerate Kummer (confluent) hypergeometric function and a degenerate Gauss hypergeometric function.

Theorem 3.1. *The degenerate hypergeometric function ${}_pF_q^{\lambda}$ has the integral representation*

$$\begin{aligned}
&{}_pF_q^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z\right]\\
&\qquad = \frac{1}{\Gamma(\alpha_1)}\int_0^{\infty}\xi^{\alpha_1-1}(1+\lambda\xi)^{-1/\lambda}{}_{p-1}F_q\left[\begin{matrix}\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z\xi\right]\mathrm{d}\xi
\end{aligned} \tag{3.2}$$

for $\lambda \in (0,\infty)$ *and* $\Re(\alpha_1) > 0$.

Proof. The desired result (3.2) follows from combining the equation (2.1) with the integral representation (2.2). □

Theorem 3.2. *The Euler–Beta type integral representation of the degenerate hypergeometric function is*

$$ {}_pF_q^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\dots,\alpha_p;\\ \beta_1,\beta_2,\dots,\beta_q;\end{matrix} z\right] = \frac{1}{B(\alpha_p,\beta_q-\alpha_p)}\int_0^1 \xi^{\alpha_p-1}(1-\xi)^{\beta_q-\alpha_p-1} \times {}_{p-1}F_{q-1}^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\dots,\alpha_{p-1};\\ \beta_1,\dots,\beta_{q-1};\end{matrix} z\xi\right]\mathrm{d}\xi \quad (3.3)$$

for $\lambda \in (0,\infty)$ *and* $\Re(\beta_q) > \Re(\alpha_p) > 0$.

Proof. From the equation (3.1), it follows that

$$ {}_pF_q^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\dots,\alpha_p;\\ \beta_1,\beta_2,\dots,\beta_q;\end{matrix} z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n^{\lambda}(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}.$$

Further making use of the definition of the beta function yields

$$\frac{(\alpha_p)_n}{(\beta_q)_n} = \frac{B(\alpha_p+n,\beta_q-\alpha_p)}{B(\alpha_p,\beta_q-\alpha_p)} = \frac{1}{B(\alpha_p,\beta_q-\alpha_p)}\int_0^1 \xi^{\alpha_p-1}(1-\xi)^{\beta_q-\alpha_p-1}\mathrm{d}\xi.$$

The desired result (3.3) is thus obtained. □

Corollary 3.3. *The degenerate Kummer (confluent) hypergeometric function* ${}_1F_1^{\lambda}$ *and the degenerate Gauss hypergeometric function* ${}_2F_1^{\lambda}$ *have the integral representations*

$$ {}_1F_1^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda};\\ \beta_1;\end{matrix} z\right] = \frac{1}{\Gamma(\alpha_1)}\int_0^{\infty}\xi^{\alpha_1-1}(1+\lambda\xi)^{-1/\lambda}{}_0F_1\left[\begin{matrix}-;\\ \beta_1;\end{matrix} z\xi\right]\mathrm{d}\xi,$$

$$ {}_2F_1^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2;\\ \beta_1;\end{matrix} z\right] = \frac{1}{\Gamma(\alpha_1)}\int_0^{\infty}\xi^{\alpha_1-1}(1+\lambda\xi)^{-1/\lambda}{}_1F_1^{\lambda}\left[\begin{matrix}\alpha_2;\\ \beta_1;\end{matrix} z\xi\right]\mathrm{d}\xi,$$

$$ {}_2F_1^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2;\\ \beta_1;\end{matrix} z\right] = \frac{1}{B(\alpha_2,\beta_1-\alpha_2)}\int_0^{1}\xi^{\alpha_2-1}(1-\xi)^{\beta_1-\alpha_2-1}{}_1F_0^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda};\\ -;\end{matrix} z\xi\right]\mathrm{d}\xi$$

on the condition that the integrals involved are convergent.

Theorem 3.4. *Let* $\mathfrak{N} = \mathfrak{N}_{(\zeta\pm i\infty)}$ *for* $\zeta \in \mathbb{R}$ *be a Mellin–Barnes contour starting at the point* $\zeta - i\infty$ *and closing at the point* $\zeta + i\infty$ *with the ordinary indents in order to distinguish one set of poles from the other set of poles of the integrand. Then*

$$ {}_pF_q^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\dots,\alpha_p;\\ \beta_1,\beta_2,\dots,\beta_q;\end{matrix} z\right] = \frac{1}{2\pi i}\frac{\Gamma(\beta_1)\cdots\Gamma(\beta_q)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_p)}\int_{\mathfrak{N}}\frac{\Gamma_{\lambda}(\alpha_1+\xi)\cdots\Gamma(\alpha_p+\xi)}{\Gamma(\beta_1+\xi)\cdots\Gamma(\beta_q+\xi)}\Gamma(-\xi)(-z)^{\xi}\mathrm{d}\xi \quad (3.4)$$

for $|\arg(-z)| < \pi$.

Proof. Taking the sum of residues at the pole of $\Gamma(\xi)$ at the point $\xi = n$ $(n \in \mathbb{N}_0)$ in the equation (3.9), we readily get the following series expansion:

$$ {}_pF_q^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\cdots,\alpha_p;\\ \beta_1,\beta_2,\cdots,\beta_q;\end{matrix} z\right] = \frac{\Gamma(\beta_1)\Gamma(\beta_2)\cdots\Gamma(\beta_q)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_p)}\sum_{n=0}^{\infty}\frac{\Gamma_{\lambda}(\alpha_1+n)\Gamma(\alpha_2+n)\cdots\Gamma(\alpha_p+n)}{\Gamma(\beta_1+n)\Gamma(\beta_2+n)\cdots\Gamma(\beta_q+n)}\frac{z^n}{n!},$$

which completes the proof. □

Remark 3.5. Taking $p-1=q=1$ in (3.4) leads to

$$ {}_2F_1^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2;\\ \beta_1;\end{matrix}z\right]=\frac{1}{2\pi i}\frac{\Gamma(\beta_1)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{\mathfrak{N}}\frac{\Gamma_{\lambda}(\alpha_1+\xi)\Gamma(\alpha_2+\xi)}{\Gamma(\beta_1+\xi)}\Gamma(-\xi)(-z)^{\xi}\mathrm{d}\xi $$

for $|\arg(-z)|<\pi$.

Theorem 3.6. *The derivative formula of the degenerate hypergeometric function* ${}_pF_q^{\lambda}$ *is*

$$ \frac{\mathrm{d}^n}{\mathrm{d}z^n}{}_pF_q^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z\right] = \frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}{}_pF_q^{\lambda}\left[\begin{matrix}(\alpha_1+n)^{\lambda},\alpha_2+n,\ldots,\alpha_p+n;\\ \beta_1+n,\beta_2+n,\ldots,\beta_q+n;\end{matrix}z\right]. \quad (3.5) $$

Proof. Differentiating on both sides of (3.1) gives

$$ \frac{\mathrm{d}}{\mathrm{d}z}{}_pF_q^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z\right]=\sum_{n=1}^{\infty}\frac{(\alpha)_n^{\lambda}(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^{n-1}}{(n-1)!} $$
$$ =\sum_{n=0}^{\infty}\frac{(\alpha)_{n+1}^{\lambda}(\alpha_2)_{n+1}\cdots(\alpha_p)_{n+1}}{(\beta_1)_{n+1}\cdots(\beta_q)_{n+1}}\frac{z^n}{n!}. $$

Further utilizing (2.3) yields

$$ \frac{\mathrm{d}}{\mathrm{d}z}{}_pF_q^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z\right]=\frac{\alpha_1\cdots\alpha_p}{\beta_1\cdots\beta_q}{}_pF_q^{\lambda}\left[\begin{matrix}(\alpha_1+1)^{\lambda},\alpha_2+1,\ldots,\alpha_p+1;\\ \beta_1+1,\ldots,\beta_q+1;\end{matrix}z\right]. $$

By induction on $n\in\mathbb{N}_0$, the desired result (3.5) follows straightforwardly. □

Corollary 3.7. *The degenerate Kummer (confluent) hypergeometric function* ${}_1F_1^{\lambda}$ *and the degenerate Gauss hypergeometric function* ${}_2F_1^{\lambda}$ *satisfy*

$$ \frac{\mathrm{d}^n}{\mathrm{d}z^n}{}_1F_1^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda};\\ \beta_1;\end{matrix}z\right]=\frac{(\alpha_1)_n}{(\beta_1)_n}{}_1F_1^{\lambda}\left[\begin{matrix}(\alpha_1+n)^{\lambda};\\ \beta_1+n;\end{matrix}z\right] $$

and

$$ \frac{\mathrm{d}^n}{\mathrm{d}z^n}{}_2F_1^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2;\\ \beta_1;\end{matrix}z\right]=\frac{(\alpha_1)_n(\alpha_2)_n}{(\beta_1)_n}{}_2F_1^{\lambda}\left[\begin{matrix}(\alpha_1+n)^{\lambda},\alpha_2+n;\\ \beta_1+n;\end{matrix}z\right] $$

on the condition that the integrals included are convergent.

4. Degenerate Sumudu transforms

In this section, we define a degenerate Sumudu transform and calculate a degenerate Sumudu transforms of certain degenerate functions, such as the degenerate trigonometric functions and the degenerate hyperbolic functions, and the power function.

Definition 4.1 (A degenerate Sumudu transform)**.** Let $\lambda\in(0,\infty)$ and $f(\zeta)$ be a function defined for $\zeta>0$. Then

$$ S_{\lambda}[f(\zeta)](u)=\frac{1}{u}\int_0^{\infty}(1+\lambda\zeta)^{-1/u\lambda}f(\zeta)\mathrm{d}\zeta \quad (4.1) $$

or, equivalently,

$$ G_{\lambda}(u)=S_{\lambda}[f(\zeta)]=\int_0^{\infty}(1+\lambda\zeta)^{-1/\lambda}f(u\zeta)\mathrm{d}\zeta $$

on the condition that the integrals included are convergent.

It is clear that

$$ S_{\lambda}[\mu f(\zeta)+\nu g(\zeta)](u)=\mu S_{\lambda}[f(\zeta)](u)+\nu S_{\lambda}[g(\zeta)](u), $$

where μ and ν are fixed real numbers.

Theorem 4.2. *For $\lambda \in (0,\infty)$ and $\kappa \in \mathbb{C}$, we have*

$$S_\lambda[1](u) = \frac{1}{1-u\lambda}, \quad \frac{1}{u} > \lambda, \tag{4.2}$$

and

$$S_\lambda\big[(1+\lambda\zeta)^{-\kappa/\lambda}\big](u) = \frac{1}{1+(\kappa-\lambda)u}, \quad \frac{1}{u} > \lambda - \kappa. \tag{4.3}$$

Proof. Substituting $f(\zeta) = 1$ into the equality (4.1) gives

$$S_\lambda[1](u) = \frac{1}{u}\int_0^\infty (1+\lambda\zeta)^{-1/u\zeta}\mathrm{d}\zeta = \frac{1}{u}\lim_{\delta\to\infty}\int_0^\delta (1+\lambda\zeta)^{-1/u\lambda}\mathrm{d}\zeta = \frac{1}{1-u\lambda}.$$

The required result (4.2) is thus proved.

Similarly, we can obtain the desired result (4.3). □

Definition 4.3 (The degenerate trigonometric and hyperbolic functions [8])**.** For $\lambda \in (0,\infty)$ and $\kappa \in \mathbb{C}$, we define

$$\begin{aligned} \cos_\lambda(\zeta) &= \frac{1}{2}\big[(1+\lambda\zeta)^{i/\lambda} + (1+\lambda\zeta)^{-i/\lambda}\big], \\ \sin_\lambda(\zeta) &= \frac{1}{2i}\big[(1+\lambda\zeta)^{i/\lambda} - (1+\lambda\zeta)^{-i/\lambda}\big], \\ \cosh_\lambda(\kappa\zeta) &= \frac{1}{2}\big[(1+\lambda\zeta)^{\kappa/\lambda} + (1+\lambda\zeta)^{-\kappa/\lambda}\big], \end{aligned} \tag{4.4}$$

and

$$\sinh_\lambda(\kappa\zeta) = \frac{1}{2}\big[(1+\lambda\zeta)^{\kappa/\lambda} - (1+\lambda\zeta)^{-\kappa/\lambda}\big].$$

Theorem 4.4. *For $\lambda \in (0,\infty)$ and $\kappa \in \mathbb{C}$, the degenerate Sumudu transforms of the degenerate trigonometric and the hyperbolic functions are*

$$\begin{aligned} S_\lambda[\cos_\lambda(\kappa\zeta)](u) &= \frac{u(1-u\lambda)}{(1-u\lambda)^2+u^2\kappa^2}, \\ S_\lambda[\sin_\lambda(\kappa\zeta)](u) &= \frac{u\kappa}{(1-u\lambda)^2+u^2\kappa^2}, \\ S_\lambda[\cosh_\lambda(\kappa\zeta)](u) &= \frac{u(1-u\lambda)}{(1-u\lambda)^2-u^2\kappa^2}, \end{aligned} \tag{4.5}$$

and

$$S_\lambda[\sinh_\lambda(\kappa\zeta)](u) = \frac{u\kappa}{(1-u\lambda)^2-u^2\kappa^2}. \tag{4.6}$$

Proof. Putting the first equality (4.4) into (4.1) results in

$$\begin{aligned} S_\lambda[\cos_\lambda(\kappa\zeta)](u) &= \frac{1}{u}\int_0^\infty (1+\lambda\zeta)^{-1/u\lambda}\cos_\lambda(\kappa\zeta)\mathrm{d}\zeta \\ &= \frac{1}{2u}\int_0^\infty \big[(1+\lambda\zeta)^{-(1/u-\kappa i)/\lambda} + (1+\lambda\zeta)^{-(1/u+\kappa i)/\lambda}\big]\mathrm{d}\zeta \\ &= \frac{1}{2u}\left(\frac{1}{1/u-\lambda-\kappa i} + \frac{1}{1/u-\lambda+\kappa i}\right) \\ &= \frac{u(1-u\lambda)}{(1-u\lambda)^2+u^2\kappa^2}. \end{aligned}$$

The first required result in (4.5) is thus proved.
Similarly, we can obtain the desired results from the second one in (4.5) to (4.6) readily. □

Theorem 4.5. *For $\lambda \in (0,\infty)$, $\frac{1}{u} > (n+1)\lambda$, and $n \in \mathbb{N}$, the degenerate Sumudu transform of the power function ζ^n is*

$$S_\lambda[\zeta^n](u) = u^n\Gamma_{u\lambda}(n+1). \tag{4.7}$$

Proof. Applying the degenerate Sumudu transform (4.1) to the power function ζ^n gives

$$S_\lambda[\zeta^n](u) = \frac{1}{u}\int_0^\infty (1+\lambda\zeta)^{-1/u\lambda}\zeta^n \mathrm{d}\zeta. \tag{4.8}$$

Integrating by part consecutively in the equation (4.8) reveals

$$\begin{aligned} S_\lambda[\zeta^n](u) &= \frac{1}{u}\int_0^\infty (1+\lambda\zeta)^{-1/u\lambda}\zeta^n \mathrm{d}\zeta \\ &= \frac{n}{(1-u\lambda)}\int_0^\infty (1+\lambda\zeta)^{-(1-u\lambda)/u\lambda}\zeta^{n-1}\mathrm{d}\zeta \\ &= \frac{un(n-1)}{(1-u\lambda)(1-2u\lambda)}\int_0^\infty (1+\lambda\zeta)^{-(1-2u\lambda)/u\lambda}\zeta^{n-2}\mathrm{d}\zeta \\ &= \cdots \\ &= \frac{n!u^n}{(1-u\lambda)(1-2u\lambda)\cdots(1-(n+1)u\lambda)} \\ &= u^n\Gamma_{u\lambda}(n+1). \end{aligned}$$

Thus, we obtain the desired result (4.7). □

5. Integral transforms

Theorem 5.1. *The Euler–Beta transform of the degenerate hypergeometric function* ${}_pF_q^\lambda$ *in* (3.1) *is*

$$B\left\{{}_pF_q^\lambda\left[\begin{matrix}(\alpha_1)^\lambda,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix} wz\right];\mu,\nu\right\} = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}\,{}_{p+1}F_{q+1}^\lambda\left[\begin{matrix}\mu,(\alpha_1)^\lambda,\alpha_2,\ldots,\alpha_p;\\ \mu+\nu,\beta_1,\beta_2,\ldots,\beta_q;\end{matrix} w\right] \tag{5.1}$$

for $\Re(\mu) > 0$ *and* $\Re(\nu) > 0$.

Proof. Substituting the degenerate hypergeometric function ${}_pF_q^\lambda$ in (3.1) into the Euler–Beta transform in [3,4,9] gives

$$B\left\{{}_pF_q^\lambda\left[\begin{matrix}(\alpha_1)^\lambda,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix} wz\right];\mu,\nu\right\} = \int_0^1 z^{\mu-1}(1-z)^{\nu-1}\sum_{n=0}^\infty \frac{(\alpha_1)_n^\lambda(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\cdots(\beta_q)_n}\frac{(wz)^n}{n!}\mathrm{d}z. \tag{5.2}$$

Interchanging the order of the sum (5.2) leads to the formula (5.1). □

Theorem 5.2. *The Laplace transform of the degenerate hypergeometric function* ${}_pF_q^\lambda$ *in* (3.1) *is*

$$L\left\{z^{\mu-1}{}_pF_q^\lambda\left[\begin{matrix}(\alpha_1)^\lambda,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix} wz\right];\delta\right\} = \frac{\Gamma(\mu)}{\delta^\mu}\,{}_{p+1}F_q\left[\begin{matrix}\mu,(\alpha_1)^\lambda,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix} \frac{w}{\delta}\right] \tag{5.3}$$

for $\mu\in\mathbb{C}$, $\Re(\mu) > 0$, *and* $\left|\frac{w}{\delta}\right| < 1$.

Proof. Applying the Laplace transform in [1] to the degenerate hypergeometric function ${}_pF_q^\lambda$ in (3.1) gives

$$L\left\{z^{\mu-1}{}_pF_q^\lambda\left[\begin{matrix}(\alpha_1)^\lambda,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix} wz\right];\delta\right\} = \int_0^\infty z^{\mu-1}\exp(-\delta z)\sum_{n=0}^\infty \frac{(\alpha_1)_n^\lambda(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{(wz)^n}{n!}\mathrm{d}z. \tag{5.4}$$

Interchanging the order of sum and integral in (5.4) and making use of the Laplace equality

$$L\{z^{\mu+n-1};\delta\} = \frac{\Gamma(\mu+n)}{\delta^{\mu+n}}$$

in [3, 4, 9], we can obtain the desired result (5.3) readily. □

Theorem 5.3. *The degenerate Laplace transform of the degenerate hypergeometric function* ${}_pF_q^\lambda$ *in* (3.1) *is*

$$L_\lambda\left\{z^{\mu-1}{}_pF_q^\lambda\left[\begin{matrix}(\alpha_1)^\lambda,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix} wz\right];\delta\right\} = \frac{\Gamma(\mu)}{\delta^\mu}{}_{p+1}F_q\left[\begin{matrix}(\mu)^{\lambda/\delta},(\alpha_1)^\lambda,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}\frac{w}{\delta}\right] \tag{5.5}$$

for $\mu,\delta\in\mathbb{C}$.

Proof. Using the degenerate Laplace transform in [8] to the degenerate hypergeometric function ${}_pF_q^\lambda$ in (3.1) gives

$$L_\lambda\left\{z^{\mu-1}{}_pF_q^\lambda\left[\begin{matrix}(\alpha_1)^\lambda,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix} wz\right];\delta\right\} = \int_0^\infty z^{\mu-1}(1+\lambda z)^{-\delta/\lambda}\sum_{n=0}^\infty\frac{(\alpha_1)_n^\lambda(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{(wz)^n}{n!}\mathrm{d}z. \tag{5.6}$$

Shifting the order between sum and integral in (5.6) and utilizing the degenerate Laplace equality

$$L_\lambda\{z^{\mu+n-1};\delta\} = \frac{\Gamma_{\lambda/\delta}(\mu+n)}{\delta^{\mu+n}}.$$

for the power function in [8] result in the desired result (5.5). □

Theorem 5.4. *The Sumudu transform of the degenerate hypergeometric function* ${}_pF_q^\lambda$ *in* (3.1) *is*

$$S\left\{z^{\mu-1}{}_pF_q^\lambda\left[\begin{matrix}(\alpha_1)^\lambda,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix} wz\right]\right\}(u) = \Gamma(\mu){}_{p+1}F_q\left[\begin{matrix}\mu,(\alpha_1)^\lambda,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix} uw\right] \tag{5.7}$$

for $\Re(\mu)\in\mathbb{C}$.

Proof. Applying the Sumudu transform in [5, 24] to the degenerate hypergeometric function ${}_pF_q^\lambda$ in (3.1) yields

$$S\left\{z^{\mu-1}{}_pF_q^\lambda\left[\begin{matrix}(\alpha_1)^\lambda,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix} wz\right]\right\}(u) = \int_0^\infty z^{\mu-1}\exp(-z)\sum_{n=0}^\infty\frac{(\alpha_1)_n^\lambda(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{(uwz)^n}{n!}\mathrm{d}z. \tag{5.8}$$

Interchanging the order of integral and sum in (5.8) and employing the well-known integral formula

$$\Gamma(\mu+n) = \int_0^\infty z^{\mu+n-1}\exp(-z)\mathrm{d}z$$

in [1, 15] for the gamma function conclude the required result (5.7). □

Theorem 5.5. *The degenerate Sumudu transform of the degenerate hypergeometric function ${}_pF_q^{\lambda}$ in (3.1) is*

$$S_\lambda\left\{z^{\mu-1}{}_pF_q^{\lambda}\left[\begin{matrix}(\alpha_1)^\lambda,\alpha_2,\dots,\alpha_p;\\ \beta_1,\beta_2,\dots,\beta_q;\end{matrix}wz\right]\right\}(u)$$
$$= u^{\mu-1}\Gamma(\mu)_{p+1}F_q\left[\begin{matrix}(\mu)^{u\lambda},(\alpha_1)^\lambda,\alpha_2,\dots,\alpha_p;\\ \beta_1,\beta_2,\dots,\beta_q;\end{matrix}uw\right] \quad (5.9)$$

for $\mu,\delta\in\mathbb{C}$.

Proof. Using the degenerate Sumudu transform in (4.1) to the degenerate hypergeometric function ${}_pF_q^{\lambda}$ in (3.1) gives

$$S_\lambda\left\{z^{\mu-1}{}_pF_q^{\lambda}\left[\begin{matrix}(\alpha_1)^\lambda,\alpha_2,\dots,\alpha_p;\\ \beta_1,\beta_2,\dots,\beta_q;\end{matrix}wz\right]\right\}(u)$$
$$=\int_0^\infty z^{\mu-1}(1+\lambda z)^{-1/u\lambda}\sum_{n=0}^{\infty}\frac{(\alpha_1)_n^\lambda(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{(uwz)^n}{n!}\mathrm{d}z. \quad (5.10)$$

Shifting the order of sum and integral in (5.10) and employing degenerate Sumudu equality

$$S_\lambda\{z^{\mu+n-1}\}(u)=u^{\mu+n-1}\Gamma_{u\lambda}(\mu+n)$$

in (4.7) for the power function conclude the desired result (5.9) immediately. □

Theorem 5.6. *The Stieltjes transform of the degenerate hypergeometric function ${}_pF_q^{\lambda}$ in (3.1) is*

$$\mathfrak{S}\left\{z^{\mu-1}{}_pF_q^{\lambda}\left[\begin{matrix}(\alpha_1)^\lambda,\alpha_2,\dots,\alpha_p;\\ \beta_1,\beta_2,\dots,\beta_q;\end{matrix}wz\right]\right\}(t)$$
$$= t^{\mu-1}B(\mu,1-\mu)_{p+2}F_q\left[\begin{matrix}\mu,1-\mu,(\alpha_1)^\lambda,\alpha_2,\dots,\alpha_p;\\ \beta_1,\beta_2,\dots,\beta_q;\end{matrix}wt\right] \quad (5.11)$$

for $\Re(\mu)\in\mathbb{C}$.

Proof. Using the Stieltjes transform in [4, 9] to the degenerate hypergeometric function ${}_pF_q^{\lambda}$ in (3.1) leads to

$$\mathfrak{S}\left\{z^{\mu-1}{}_pF_q^{\lambda}\left[\begin{matrix}(\alpha_1)^\lambda,\alpha_2,\dots,\alpha_p;\\ \beta_1,\beta_2,\dots,\beta_q;\end{matrix}wz\right]\right\}(t)$$
$$=\int_0^\infty \frac{z^{\mu-1}}{z+t}\sum_{n=0}^{\infty}\frac{(\alpha_1)_n^\lambda(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{(wz)^n}{n!}\mathrm{d}z. \quad (5.12)$$

Setting $\frac{z}{t}=\nu$ and $\nu=\frac{\delta}{1-\delta}$, interchanging the order of integration and summation in (5.12), and using the integral representation

$$t^{\mu+n-1}\int_0^1\delta^{\mu+n-1}(1-\delta)^{1-n+\mu-1}\mathrm{d}\delta=t^{\mu+n-1}B(\mu+n,1-\mu+n)$$

in [1, 15] conclude the desired result (5.11). □

Theorem 5.7. *The Laguerre transform of the degenerate hypergeometric function ${}_pF_q^{\lambda}$ in (3.1) is*

$$\mathfrak{L}^{(a)}\left\{z^{\mu-1}{}_pF_q^{\lambda}\left[\begin{matrix}(\alpha_1)^\lambda,\alpha_2,\dots,\alpha_p;\\ \beta_1,\beta_2,\dots,\beta_q;\end{matrix}wz\right]\right\}$$
$$=\frac{\Gamma(2-\mu)\Gamma(\mu+a)}{\Gamma(1-\mu)}{}_{p+1}F_q\left[\begin{matrix}\mu+a,(\alpha_1)^\lambda,\alpha_2,\dots,\alpha_p;\\ \beta_1,\beta_2,\dots,\beta_q;\end{matrix}w\right] \quad (5.13)$$

for $\Re(\mu)\in\mathbb{C}$.

Proof. Applying the Laguerre transform in [4,9] to the degenerate hypergeometric function ${}_pF_q^\lambda$ in (3.1) gives

$$\mathfrak{L}^{(a)}\left\{z^{\mu-1}{}_pF_q^\lambda\left[\begin{matrix}(\alpha_1)^\lambda,\alpha_2,\dots,\alpha_p;\\ \beta_1,\beta_2,\dots,\beta_q;\end{matrix}wz\right]\right\} = \int_0^\infty z^{\mu+a-1}\exp(-z)L_m^{(a)}(z)\sum_{n=0}^\infty \frac{(\alpha_1)_n^\lambda(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{(wz)^n}{n!}\mathrm{d}z. \tag{5.14}$$

Shifting the order of integration and summation in (5.14), utilizing the integral equation

$$\int_0^\infty z^{\mu+n+a-1}\exp(-z)L_m^{(a)}(z)\mathrm{d}z = \frac{\Gamma(\mu+a+n)\Gamma(m-\mu+1)}{m!\Gamma(1-\mu)}$$

in [4,9] conclude the required result (5.14). □

6. Fractional calculus approach

In this section, we infer several formulas for the Riemann–Liouville fractional integral I_{a+}^ν and the fractional derivative operators D_{a+}^ν for the degenerate hypergeometric function ${}_pF_q^\lambda(z)$ in (3.1).

Let us recall from [6,7,21] that

$$(I_{a+}^\nu\varphi)(y) = \frac{1}{\Gamma(\nu)}\int_a^y \frac{\varphi(t)}{(y-t)^{1-\nu}}\mathrm{d}t \tag{6.1}$$

and

$$(D_{a+}^\nu\varphi)(y) = \left(\frac{\mathrm{d}}{\mathrm{d}y}\right)^n\frac{1}{\Gamma(n-\nu)}\int_a^y\frac{\varphi(t)}{(y-t)^{1-n+\nu}}\mathrm{d}t = \left(\frac{\mathrm{d}}{\mathrm{d}y}\right)^n(I_{a+}^{n-\nu}\varphi)(y) \tag{6.2}$$

for $\nu\in\mathbb{C}$, $\Re(\nu)>0$, and $n=[\Re(\nu)]+1$, where $[\nu]$ means the greatest integer not exceeding $\Re(\upsilon)$.

An alternative generalization of the Riemann–Liouville fractional derivative operator D_{a+}^ν in (6.2) by introducing a right-sided Riemann–Liouville fractional derivative operator $D_{a+}^{\mu,\nu}$ of order $0<\mu<1$ and type $0<\nu<1$ with respect to y by Hilfer [6] is given by

$$(D_{a+}^{\mu,\nu}\varphi)(y) = \left(I_{a+}^{\nu(1-\mu)}\frac{\mathrm{d}}{\mathrm{d}y}\right)\left(I_{a+}^{(1-\nu)(1-\mu)}\varphi\right)(y) \tag{6.3}$$

for $\mu\in\mathbb{C}$, $\Re(\mu)>0$, and $n=[\Re(\mu)]+1$.

Remark 6.1. When $\mu=0$ in (6.3), we derive the classical Riemann–Liouville fractional derivative operator D_{a+}^ν in [7].

Theorem 6.2. *Let $a\in[0,\infty)$, $\beta_q,\upsilon,w,\mu\in\mathbb{C}$, and $\Re(\beta_q),\Re(\upsilon),\Re(w),\Re(\mu)>0$. Then, for $\kappa>a$, we have*

$$\left\{I_{a+}^\nu(w-a)^{\beta_q-1}{}_pF_q^\lambda\left[\begin{matrix}(\alpha_1)^\lambda,\alpha_2,\dots,\alpha_p;\\ \beta_1,\beta_2,\dots,\beta_q;\end{matrix}z(w-a)\right]\right\}(\kappa) = \frac{(\kappa-a)^{\beta_q+\nu-1}\Gamma(\beta_q)}{\Gamma(\beta_q+\nu)}{}_pF_q^\lambda\left[\begin{matrix}(\alpha_1)^\lambda,\alpha_2,\dots,\alpha_p;\\ \beta_1,\beta_2,\dots,\beta_q+\nu;\end{matrix}z(\kappa-a)\right], \tag{6.4}$$

$$\left(D_{a+}^\nu(w-a)^{\beta_q-1}{}_pF_q^\lambda\left[\begin{matrix}(\alpha_1)^\lambda,\alpha_2,\dots,\alpha_p;\\ \beta_1,\beta_2,\dots,\beta_q;\end{matrix}w(z-a)\right]\right)(\kappa) = \frac{(\kappa-a)^{\beta_q-\nu-1}\Gamma(c)}{\Gamma(\beta_q-\nu)}{}_pF_q^\lambda\left[\begin{matrix}(\alpha_1)^\lambda,\alpha_2,\dots,\alpha_p;\\ \beta_1,\beta_2,\dots,\beta_q-\nu;\end{matrix}z(\kappa-a)\right], \tag{6.5}$$

and

$$\left(D^{\mu,\nu}_{a+}(w-a)^{\beta_q-1}{}_pF^{\lambda}_q\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z(w-a)\right]\right)(\kappa)$$
$$=\frac{(\kappa-a)^{\beta_q-\mu-\nu-1}\Gamma(\beta_q)}{\Gamma(\beta_q-\mu-\nu)}{}_pF^{\lambda}_q\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q-\mu-\nu;\end{matrix}z(\kappa-a)\right]. \quad (6.6)$$

Proof. Applying (6.1) to the degenerate hypergeometric function ${}_pF^{\lambda}_q$ in (3.1) and making use of

$$I^{\nu}_{\alpha+}\{(w-a)^{\beta_q-1}\}(\kappa)=\frac{(\kappa-a)^{\beta_q+\nu-1}\Gamma(\beta_q)}{\Gamma(\beta_q+\nu)}$$

in [6, 7] to fractionally integrate term-by-term lead to

$$\left(I^{\nu}_{a+}(w-a)^{\beta_q-1}{}_pF^{\lambda}_q\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z(w-a)\right]\right)(\kappa)$$
$$=\sum_{n=0}^{\infty}\frac{(\alpha_1)^{\lambda}_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}I^{\nu}_{a+}\{(w-\alpha)^{\beta_q+n-1}\}$$
$$=\frac{(\kappa-a)^{\beta_q+\nu-1}\Gamma(\beta_q)}{\Gamma(\beta_q+\nu)}{}_pF^{\lambda}_q\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q+\nu;\end{matrix}z(\kappa-a)\right].$$

Making use of (6.2) and (3.1) and taking into account (6.4) with ν replaced by $n-\upsilon$ yield

$$\left(D^{\nu}_{a+}\{(w-a)^{c-1}\}{}_pF^{\lambda}_q\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z(w-a)\right]\right)(\kappa)$$
$$=\left(\frac{\mathrm{d}}{\mathrm{d}\kappa}\right)^n\left(I^{n-\nu}_{a+}\{(w-a)^{\beta_q-1}\}{}_pF^{\lambda}_q\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q-\nu;\end{matrix}z(w-a)\right]\right)(\kappa)$$
$$=\left(\frac{\mathrm{d}}{\mathrm{d}\kappa}\right)^n\left\{\frac{(\kappa-a)^{c+n-\nu-1}\Gamma(\beta_q)}{\Gamma(\beta_q+n-\nu)}{}_pF^{\lambda}_q\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q-\nu;\end{matrix}z(\kappa-a)\right]\right\}. \quad (6.7)$$

Differentiating (6.7) term-by-term and using (3.1) again lead to the formula (6.5).

Finally, by (6.3) and (3.1), we have

$$\left(D^{\mu,\nu}_{a+}\{(w-a)^{\beta_q-1}\}{}_pF^{\lambda}_q\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z(w-a)\right]\right)(\kappa)$$
$$=\left(D^{\mu,\nu}_{a+}\left[\sum_{n=0}^{\infty}\frac{(\alpha_1)^{\lambda}_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}(w-a)^{\beta_q+n-1}\right]\right)(\kappa)$$
$$=\sum_{n=0}^{\infty}\frac{(\alpha_1)^{\lambda}_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}(D^{\mu,\nu}_{a+}(w-a)^{\beta_q+n-1})(\kappa). \quad (6.8)$$

Substituting the relation

$$D^{\mu,\nu}_{a+}\{(w-a)^{\rho-1}\}(\kappa)=\frac{\Gamma(\rho)}{\Gamma(\rho-\mu-\nu)}(\kappa-\alpha)^{\rho-\mu-\nu-1}$$

for $\kappa>a$, $0<\mu<1$, $0\le\nu\le1$, and $\Re(\rho)>0$ in [6, 7] into (6.8) leads to the desired result (6.6). □

7. Fractional calculus operators

In this section, we derive certain fractional derivative operators for the degenerate hypergeometric function ${}_pF^{\lambda}_q(z)$ defined by (3.1). For attaining this purpose, we evoke the derivative operators $\{D^{\mu,\delta,\xi}_{0+}f\}(y)$ and $\{D^{\mu,\delta,\xi}_{\infty-}f\}(y)$ which are given in terms of integral operators $\{I_{0+}^{\mu,\delta,\xi}f\}(y)$ and $\{I_{\infty-}^{\mu,-\delta,\xi}f\}(y)$. In addition to these fractional derivative operators, we give the following definitions for the relations among the Riemann–Liouville, Erdėlyi–Kober, and Weyl fractional operators in [6, 7, 21].

Definition 7.1. For $y > 0$, $\mu, \delta, \xi \in \mathbb{C}$, and $\Re(\mu) > 0$, the left sided fractional integral and derivative operators $\{D_{0+}^{\mu,\delta,\xi}f\}(y)$ and $\{I_{0+}^{\mu,\delta,\xi}f\}(y)$, the right sided fractional integral and derivative operators $\{D_{\infty-}^{\mu,\delta,\xi}f\}(y)$ and $\{I_{\infty-}^{\mu,-\delta,\xi}f\}(y)$ are defined [21] respectively by

$$\begin{aligned}\{I_{0+}^{\mu,\delta,\xi}f\}(y) &= \frac{y^{-\mu-\delta}}{\Gamma(\mu)}\int_0^y {}_2F_1^{\lambda}\left(\mu+\delta,-\xi;\mu;1-\frac{t}{y}\right)f(t)\mathrm{d}t,\\ \{D_{0+}^{\mu,\delta,\xi}f\}(y) &= \{I_{0+}^{-\mu,-\delta,\mu+\xi}f\}(y) = \left(\frac{\mathrm{d}}{\mathrm{d}y}\right)^m\{I_{0+}^{-\mu+\xi,-\delta-\xi,\mu+\xi-m}f\}(y), \\ \{I_{\infty-}^{\mu,\delta,\xi}f\}(y) &= \frac{1}{\Gamma(\mu)}\int_0^y (t-y)^{-\mu}t^{-\mu-\xi}{}_2F_1^{\lambda}\left(\mu+\delta,-\xi;\mu;1-\frac{y}{t}\right)f(t)\mathrm{d}t,\end{aligned} \tag{7.1}$$

and

$$\{D_{\infty-}^{\mu,\delta,\xi}f\}(y) = \{I_{\infty-}^{-\mu,-\delta,\mu+\xi}f\}(y) = \left(-\frac{\mathrm{d}}{\mathrm{d}y}\right)^m\{I_{0+}^{-\mu+\xi,-\delta-\xi,\mu+\xi-m}f\}(y), \tag{7.2}$$

where $m = [\Re(\mu)] + 1$.

Together with above fractional derivative operators, we have

$$RL_{0+}^{\mu} = D_{0+}^{\mu,-\mu,\xi}, \quad EK_{0+}^{\mu,\xi} = D_{0+}^{\mu,0,\xi}, \quad W_{\infty-}^{\mu} = D_{\infty-}^{\mu,-\mu,\xi}, \quad EK_{\infty-}^{\mu,\xi} = D_{\infty-}^{\mu,0,\xi},$$

where RL, EK, and W denote the Riemann–Liouville, Erdėlyi–Kober, and Weyl fractional calculus operators respectively. See [21] and closely related references therein.

Theorem 7.2. *The fractional derivative operator of the degenerate hypergeometric function ${}_pF_q^{\lambda}$ in* (3.1) *is*

$$\begin{aligned}\left\{D_{0+}^{\mu,\delta,\xi}w^{\nu-1}{}_pF_q^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\dots,\alpha_p;\\ \beta_1,\beta_2,\dots,\beta_q;\end{matrix}wz\right]\right\}(\eta) &= \eta^{\nu+\delta-1}\frac{\Gamma(\nu)\Gamma(\nu+\mu+\delta+\xi)}{\Gamma(\nu+\delta)\Gamma(\nu+\xi)}\\ &\quad\times {}_{p+2}F_{q+2}\left[\begin{matrix}\nu,\nu+\mu+\delta+\xi,(\alpha_1)^{\lambda},\alpha_2,\dots,\alpha_p;\\ \nu+\delta,\nu+\xi,\beta_1,\beta_2,\dots,\beta_q;\end{matrix}\eta z\right]\end{aligned} \tag{7.3}$$

for $\eta > 0$, $\Re(\mu) \geq 0$, and $\Re(\nu) > -\min\{0, \Re(\mu+\delta+\xi)\}$.

Proof. Applying the left sided hypergeometric fractional transform (7.1) to the degenerate hypergeometric function ${}_pF_q^{\lambda}$ in (3.1) and interchanging the order of integration and summation reveal

$$\begin{aligned}&\left\{D_{0+}^{\mu,\delta,\xi}w^{\nu-1}{}_pF_q^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\dots,\alpha_p;\\ \beta_1,\beta_2,\dots,\beta_q;\end{matrix}wz\right]\right\}(\eta)\\ &\qquad= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n^{\lambda}(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}\{D_{0+}^{\mu,\delta,\xi}w^{\nu+n-1}\}(\eta).\end{aligned} \tag{7.4}$$

Further substituting the well-known fractional derivative operator equality

$$\{D_{0+}^{\mu,\delta,\xi}w^{\nu+n-1}\}(\eta) = \eta^{\nu+n+\delta-1}\frac{\Gamma(\nu+n)\Gamma(\nu+n+\mu+\delta+\xi)}{\Gamma(\nu+n+\delta)\Gamma(\nu+n+\xi)}$$

for the power functions in [21] into the equation (7.4) results in (7.3). □

Theorem 7.3. *The fractional derivative operator of the degenerate hypergeometric function ${}_pF_q^\lambda$ in* (3.1) *is*

$$\left\{D_{\infty-}^{\mu,\delta,\xi} w^{\nu-1} {}_pF_q^\lambda \left[\begin{matrix}(\alpha_1)^\lambda, \alpha_2, \dots, \alpha_p; \\ \beta_1, \beta_2, \dots, \beta_q;\end{matrix} wz\right]\right\}(\eta)$$

$$= \eta^{\nu+\delta-1} \frac{\Gamma(1-\nu-\delta)\Gamma(1-\nu+\mu+\xi)}{\Gamma(1-\nu)\Gamma(1-\nu+\xi-\delta)} \times {}_{p+2}F_{q+2}\left[\begin{matrix}1-\nu-\delta, 1-\nu+\mu+\xi, (\alpha_1)^\lambda, \alpha_2, \dots, \alpha_p; \\ 1-\nu+\xi-\delta, 1-\nu, \beta_1, \beta_2, \dots, \beta_q;\end{matrix} \frac{z}{\eta}\right] \quad (7.5)$$

for $\eta > 0$, $\Re(\mu) \geq 0$, *and* $\Re(\nu) > -\min\{0, \Re(\mu+\delta+\xi)\}$.

Proof. Applying the right sided hypergeometric fractional transform (7.2) to the degenerate hypergeometric function ${}_pF_q^\lambda$ in (3.1) and interchanging the order of integration and summation acquire

$$\left\{D_{\infty-}^{\mu,\delta,\xi} w^{\nu-1} {}_pF_q^\lambda \left[\begin{matrix}(\alpha_1)^\lambda, \alpha_2, \dots, \alpha_p; \\ \beta_1, \beta_2, \dots, \beta_q;\end{matrix} wz\right]\right\}(\eta)$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n^\lambda (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \{D_{\infty-}^{\mu,\delta,\xi} w^{\lambda-n-1}\}(\eta). \quad (7.6)$$

Further substituting the well-known fractional derivative operator equality

$$\{D_{0+}^{\mu,\delta,\xi} w^{\nu-n-1}\}\eta = \eta^{\nu-n+\delta-1} \frac{\Gamma(1-\nu+n-\delta)\Gamma(1-\nu+n+\mu+\xi)}{\Gamma(1-\nu+n)\Gamma(1-\nu+n+\xi-\delta)}$$

for the power functions in [21] into the equation (7.6) yields (7.5). ☐

Corollary 7.4. *The left sided Riemann–Liouville fractional derivative operator for* ${}_pF_q^\lambda(wz)$ (3.1) *is*

$$\left\{RL_{0+}^{\mu} w^{\nu-1} {}_pF_q^\lambda \left[\begin{matrix}(\alpha_1)^\lambda, \alpha_2, \dots, \alpha_p; \\ \beta_1, \beta_2, \dots, \beta_q;\end{matrix} wz\right]\right\}(\eta)$$

$$= \eta^{\nu-\mu-1} \frac{\Gamma(\nu)}{\Gamma(\nu-\mu)} {}_{p+1}F_{q+1}\left[\begin{matrix}\nu, (\alpha_1)^\lambda, \alpha_2, \dots, \alpha_p; \\ \nu-\mu, \beta_1, \beta_2, \dots, \beta_q;\end{matrix} \eta z\right]$$

for $\eta > 0$, $\Re(\mu) \geq 0$, *and* $\Re(\nu) > -\min\{0, \Re(\mu)\}$.

The left sided Erdélyi–Kober fractional derivative operator for ${}_pF_q^\lambda(wz)$ (3.1) *is*

$$\left\{EK_{0+}^{\mu,\xi} w^{\nu-1} {}_pF_q^\lambda \left[\begin{matrix}(\alpha_1)^\lambda, \alpha_2, \dots, \alpha_p; \\ \beta_1, \beta_2, \dots, \beta_q;\end{matrix} wz\right]\right\}(\eta)$$

$$= \eta^{\nu-1} \frac{\Gamma(\nu+\mu+\xi)}{\Gamma(\nu+\xi)} {}_{p+1}F_{q+1}\left[\begin{matrix}\nu+\mu+\xi, (\alpha_1)^\lambda, \alpha_2, \dots, \alpha_p; \\ \nu+\xi, \beta_1, \beta_2, \dots, \beta_q;\end{matrix} \eta z\right]$$

for $\eta > 0$, $\Re(\mu) \geq 0$, *and* $\Re(\nu) > -\min\{0, \Re(\mu+\xi)\}$.

Proof. These follow from putting $\delta = -\mu$ and $\delta = 0$ in (7.3) respectively. ☐

Corollary 7.5. *The right sided Weyl fractional derivative operator for* ${}_pF_q^\lambda(z)$ *in* (3.1) *is*

$$\left\{W_{\infty-}^{\mu} w^{\nu-1} {}_pF_q^\lambda \left[\begin{matrix}(\alpha_1)^\lambda, \alpha_2, \dots, \alpha_p; \\ \beta_1, \beta_2, \dots, \beta_q;\end{matrix} wz\right]\right\}(\eta)$$

$$= \eta^{\nu-\mu-1} \frac{\Gamma(1-\nu+\mu)}{\Gamma(1-\nu)} {}_{p+1}F_{q+1}\left[\begin{matrix}1-\nu+\mu+\xi, (\alpha_1)^\lambda, \alpha_2, \dots, \alpha_p; \\ 1-\nu, \beta_1, \beta_2, \dots, \beta_q;\end{matrix} \frac{z}{\eta}\right]$$

for $\eta > 0$, $\Re(\mu) \geq 0$, *and* $\Re(\nu) < 1 + \min\{-\Re(-\mu+\xi), \Re(\mu+\xi)\}$.

The right sided Erdėlyi–Kober fractional derivative operator for ${}_pF_q^{\lambda}(z)$ *in* (3.1) *is*

$$\left\{EK_{\infty-}^{\mu,\xi}w^{\nu-1}{}_pF_q^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}wz\right]\right\}(\eta)$$
$$=\eta^{\nu-1}\frac{\Gamma(1-\nu+\mu+\xi)}{\Gamma(1-\nu+\xi)}{}_{p+1}F_{q+1}\left[\begin{matrix}1-\nu+\mu+\xi,(\alpha_1)^{\lambda},\alpha_2,\ldots,\alpha_p;\\ 1-\nu+\xi,\beta_1,\beta_2,\ldots,\beta_q;\end{matrix}\frac{z}{\eta}\right]$$

for $\eta>0$, $\Re(\mu)\geq 0$, $\Re(\nu)<1+\min\{-\Re(\xi),\Re(\mu+\xi)\}$.

Proof. These follow from applying $\delta=-\mu$ and $\delta=0$ in (7.5) respectively. □

8. Generating functions

In this section, we obtain from the generating functions for the generalized degenerate hypergeometric function ${}_pF_q^{\lambda}(z)$ in (3.1). We find it to be proper to choose the shortened notation $\Delta(K;\xi)$ which represent the array of K-parameters

$$\frac{\xi}{K},\quad\frac{\xi+1}{K},\quad\frac{\xi+2}{K},\quad\ldots,\quad\frac{\xi+K-1}{K}$$

for $\xi\in\mathbb{C}$ and $K\in\mathbb{N}$. The array $\Delta(K;\xi)$ is understood to be empty when $K=0$.

We first establish the following generating function for the corresponding deduced results mentioned in [2, 17, 19].

Theorem 8.1. *For* $|w|<1$, $\xi\in\mathbb{C}$, *and* $K\in\mathbb{N}$, *we have*

$$\sum_{n=0}^{\infty}\frac{(\xi)_n}{n!}{}_{p+K}F_q\left[\begin{matrix}\Delta(K;\xi+n),(\alpha_1)^{\lambda},\alpha_2,\cdots\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}w^n\right]$$
$$=\frac{1}{(1-w)^{\xi}}{}_{r+K}F_s\left[\begin{matrix}\Delta(K;\xi),(\alpha_1)^{\lambda},\alpha_2,\cdots\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}\frac{z}{(1-w)^K}\right]\quad(8.1)$$

on the condition such that each member of (8.1) *exists.*

Proof. By virtue of (3.1), the left-hand side of (8.1) can be reformulated as

$$\sum_{n=0}^{\infty}\frac{(\xi)_n}{n!}\left[\sum_{m=0}^{\infty}\frac{(\xi+n)_{Km}(\alpha_1)_m^{\lambda}(\alpha_2)_m\cdots(\alpha_p)_m}{(\beta_1)_m(\beta_2)_m\cdots(\beta_q)_m}\frac{z^m}{m!}\right]w^n$$
$$=\sum_{m=0}^{\infty}\frac{(\xi)_{Km}(\alpha_1)_m^{\lambda}(\alpha_2)_m\cdots(\alpha_p)_m}{(\beta_1)_m(\beta_2)_m\cdots(\beta_q)_m}\frac{z^m}{m!}\left[\sum_{n=0}^{\infty}(\xi+Km)_n\frac{w^n}{n!}\right],\quad(8.2)$$

where we have changed the order of summations and used the identity

$$(\xi+n)_{Km}(\xi)_n=(\xi)_{Km}(\xi+Km)_n$$

in [1, 15]. Making use of the binomial summation

$$(1-w)^{-\xi-Km}=\sum_{n=0}^{\infty}(\xi+Km)_n\frac{w^n}{n!},\quad |w|<1$$

from [19] in (8.2) and using (3.1) again arrive at the desired result (8.1). □

Theorem 8.2. *If* $|w|<1$, $\xi\in\mathbb{C}$, *and* $K\in\mathbb{N}$, *then*

$$\sum_{n=0}^{\infty}\frac{(\xi)_n}{n!}{}_{p+K}F_q\left[\begin{matrix}\Delta(K;-n),(\alpha_1)^{\lambda},\alpha_2,\cdots\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z\right]w^n$$
$$=(1-w)^{-\xi}{}_{p+K}F_q\left[\begin{matrix}\Delta(K;\xi),(\alpha_1)^{\lambda},\alpha_2,\cdots\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z\left(-\tfrac{w}{1-w}\right)^K\right],\quad(8.3)$$

$$\sum_{n=0}^{\infty}\frac{(\xi)_n}{n!}\,{}_{p+2K}F_q\left[\begin{matrix}\Delta(K;-n),\Delta(K;\xi+n),(\alpha_1)^{\lambda},\alpha_2,\cdots\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z\right]w^n$$

$$=(1-w)^{-\xi}\,{}_{p+2K}F_q\left[\begin{matrix}\Delta(2K;\xi),(\alpha_1)^{\lambda},\alpha_2,\cdots\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z(-\tfrac{4w}{(1-w)^2})^K\right], \quad (8.4)$$

and

$$\sum_{n=0}^{\infty}\frac{(\xi)_n}{n!}\,{}_{p+K}F_{q+K}\left[\begin{matrix}\Delta(K;-n),(\alpha_1)^{\lambda},\alpha_2,\cdots\alpha_p;\\ \Delta(K;1-\xi-n),\beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z\right]w^n$$

$$=(1-w)^{-\xi}\,{}_pF_q^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\cdots\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}zw^K\right] \quad (8.5)$$

for $|w|<1$, $\xi\in\mathbb{C}$, *and* $K\in\mathbb{N}$ *on conditions such that each member of the arguments in* (8.3), (8.4), *and* (8.5) *exists.*

Proof. The generating functions (8.3) to (8.5) can be established by following the method of derivation of the generating function (8.1). □

Corollary 8.3. *Each of the following generating functions hold true:*

$$\sum_{n=0}^{\infty}\frac{(\xi)_n}{n!}\,{}_{p+1}F_q\left[\begin{matrix}\xi+n,(\alpha_1)^{\lambda},\alpha_2,\cdots\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z\right]w^n$$

$$=(1-w)^{-\xi}\,{}_{p+1}F_q\left[\begin{matrix}\xi,(\alpha_1)^{\lambda},\alpha_2,\cdots\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}\tfrac{z}{1-w}\right],$$

$$\sum_{n=0}^{\infty}\frac{(\xi)_n}{n!}\,{}_{p+2}F_q\left[\begin{matrix}-n,\xi+n,(\alpha_1)^{\lambda},\alpha_2,\cdots\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z\right]w^n$$

$$=(1-w)^{-\xi}\,{}_{p+2}F_q\left[\begin{matrix}\Delta(2;\xi),(\alpha_1)^{\lambda},\alpha_2,\cdots\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}-\tfrac{4zw}{(1-w)^2}\right],$$

and

$$\sum_{n=0}^{\infty}\frac{(\xi)_n}{n!}\,{}_{p+1}F_{q+1}\left[\begin{matrix}-n,(\alpha_1)^{\lambda},\alpha_2,\cdots\alpha_p;\\ 1-\xi-n,\beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z\right]w^n$$

$$=(1-w)^{-\xi}\,{}_pF_q^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\cdots\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}zw\right].$$

Proof. These follow from taking $K=1$ in (8.3), (8.4), and (8.5). □

Theorem 8.4. *The generating function*

$$\sum_{n=0}^{\infty}\binom{\zeta+n-1}{n}\,{}_pF_{q+1}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\ldots,\alpha_p;\\ 1-\zeta-n,\beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z\right]w^n$$

$$=(1-w)^{-\zeta}\,{}_pF_{q+1}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\ldots,\alpha_p;\\ 1-\zeta,\beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z(1-w)\right] \quad (8.6)$$

for ${}_pF_q^{\lambda}(z)$ *in* (3.1) *holds true.*

Proof. This follows from applying the identity

$$(1-\zeta-n)_m=(1-\zeta)_m\binom{\zeta+n-1}{n}\binom{\zeta+m-n+1}{n}^{-1}, \quad m,n\in\mathbb{N}_0$$

in [1,15,19] into the left hand side of (8.6) and simplifying. □

Theorem 8.5. *The generating function*

$$\sum_{m=0}^{\infty}\binom{\xi+m-1}{m}{}_pF_q^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda},\xi+m,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z\right]w^m$$
$$=(1-w)^{-\xi}{}_pF_q^{\lambda}\left[\begin{matrix}(\alpha_1)^{\lambda},\xi,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}\frac{z}{(1-w)}\right] \quad (8.7)$$

for ${}_pF_q^{\lambda}(z)$ *in* (3.1) *holds true.*

Proof. When putting $\alpha_2=\xi+m$ and $\zeta=1-n$ in the equation (8.1), the left side of the equality (8.7) becomes

$$\sum_{m=0}^{\infty}\frac{(\alpha_1)_n^{\lambda}(\xi+m)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\left[\sum_{m=0}^{\infty}\binom{\zeta+m+n-1}{m}w^m\right]\frac{z^n}{n!}. \quad (8.8)$$

Applying

$$\sum_{m=0}^{\infty}\binom{\zeta+n+m-1}{m}w^m=(1-w)^{-\zeta-n}$$

in [19] to (8.8) yields the required result (8.7). □

Theorem 8.6. *Let*

$$\Phi_m^{(\mu,\nu),\lambda}(z)=\Phi_m^{(\mu,\nu)}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z\right]$$
$$={}_pF_{q+\nu}\left[\begin{matrix}(\alpha_1)^{\lambda},\alpha_2,\ldots,\alpha_p;\\ \Delta(\nu;1-\mu-m),\beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z\right] \quad (8.9)$$

for $\nu\in\mathbb{N}$ *and* $\mu\in\mathbb{C}$, *where*

$$\Delta(\nu;\mu)=\frac{\mu}{\nu}\frac{\mu+1}{\nu}\cdots\frac{\mu+\nu-1}{\nu}.$$

Then

$$\sum_{n=0}^{\infty}\binom{\mu+n-1}{n}\Phi_{m+n}^{(\mu,\nu),\lambda}(z)w^n=\frac{\Phi_m^{(\mu,\nu),\lambda}((1-w)^{\nu}z)}{(1-w)^{\mu+m}} \quad (8.10)$$

for $m\in\mathbb{N}_0$, $\mu\in\mathbb{C}$, $\nu\in\mathbb{N}$, *and* $|w|<1$.

Proof. The left side of the equality (8.10) can be arranged as

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{\mu+m+n-1}{n}\frac{(\alpha_1)_{m+n}^{\lambda}(\alpha_2)_{m+n}\cdots(\alpha_p)_{m+n}}{(1-\mu-m-n)_{\nu m+b\nu n}(\beta_1)_{m+n}\cdots(\beta_q)_{m+n}}\frac{z^{m+n}}{(m+n)!}w^n. \quad (8.11)$$

Applying the identity

$$(1-\mu-m-n)_{\nu k}=(1-\mu)_{\nu k}\binom{\mu+m+n-1}{n}\binom{\mu+m-\nu k+n+1}{n}^{-1}$$

for $m,n\in\mathbb{N}_0$ and $k\in\mathbb{N}$ in [19] into (8.11) and employing (8.9) yield the required result (8.10). □

Theorem 8.7. *Let*

$$\Phi_m^{(\delta,\nu),\lambda}(z)=\Phi_m^{(\delta,\nu)}\left[\begin{matrix}(\alpha_1)^{\lambda},\delta,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z\right]$$
$$={}_{p+\nu}F_q\left[\begin{matrix}(\alpha_1)^{\lambda},\Delta(\nu;\delta+m),\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}z\right] \quad (8.12)$$

for $\nu \in \mathbb{N}$ and $\delta \in \mathbb{C}$, where

$$\Delta(\nu;\delta) = \frac{\delta}{\nu}\frac{\delta+1}{\nu}\cdots\frac{\delta+\nu-1}{\nu}.$$

Then

$$\sum_{n=0}^{\infty}\binom{\delta+m+n-1}{n}\Phi_{m+n}^{(\delta,\nu),\lambda}(z)w^n = \frac{1}{(1-w)^{\delta+m}}\Phi_m^{(\delta,\nu),\lambda}\left(\frac{z}{(1-w)^{\nu}}\right) \tag{8.13}$$

for $m \in \mathbb{N}_0$, $\delta \in \mathbb{C}$, $\nu \in \mathbb{N}$, and $|w| < 1$.

Proof. The left side of the equality (8.13) can be rewritten as

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{\delta+m+n-1}{n}\frac{(\alpha_1)_{m+n}^{\lambda}(1-\delta+m+n)_{\nu m+\nu n}\cdots(\alpha_p)_{m+n}}{(\beta_1)_{m+n}\cdots(\beta_q)_{m+n}}\frac{z^{m+n}}{(m+n)!}w^n. \tag{8.14}$$

Applying the identity

$$(1-\delta+m+n)_{bk} = (1-\delta)_{\nu k}\binom{\delta+m+n-1}{n}^{-1}\binom{\delta+m+\nu k+n+1}{n}$$

for $k, n \in \mathbb{N}_0$ and $k \in \mathbb{N}$ in [19] to the equation (8.14) and utilizing (8.12) yield the required result (8.13). □

9. Concluding remarks

In our present paper, we first demonstrated the degenerate Pochhammer symbol by considering the degenerate gamma function. Then by means of newly defined a Pochhammer symbol, we demonstrated a degenerate hypergeometric function. Moreover, we introduced a degenerate Sumudu transform by using the degenerate exponential function. Also, we presented integral transforms, fractional calculus operators and generating function for the degenerate hypergeometric function. We make an observation limitting $\lambda \to 0^+$, then the result derived in this paper will be reduced to the classical Pochhammer symbol, hypergeometric function and Sumudu transform. Using the (2.1) relation, a new degenerate-special functions can be obtained.

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