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Numerical Solutions Of Nonlinear Boundary Value Pantograph Type Delay Differential Equations

Doğrusal Olmayan Sınır Değerli Pantograf Tip Gecikmeli Diferansiyel Denklemlerin Nümerik Çözümleri

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Abstract
In this paper we compared the Daftardar-Jafari Method (DJM) with Adomian Decomposition Method (ADM) and Differential Transformation Method (DTM) in solving nonlinear boundary value delay differential equations of pantograph type. All these 3 methods provide series solutions to the problems. We analysed the first n-term approximate solutions of these 3 methods with 2 numerical examples to see if DJM is as good as ADM and DTM in solving nonlinear boundary value delay differential equations and we found DJM a reliable method in solving this kind of problems.
Keywords: Delay differential equations, Daftardar-Jafari, Adomian Decomposition, Differential Transformation, Boundary value

Öz
Bu çalışmada doğrusal olmayan sınır değerli pantograf tip gecikmeli diferansiyel denklemlerin çözümünde Daftardar-Jafari Metodunu (DJM), Adomian Ayrıştırma Metodu (ADM) ve Diferansiyel Transformasyon Metoduyla (DTM) karşılaştırdık. Bu 3 metot ta seri formunda çözümler oluştumaktadır. Bu 3 metodun ilk n-terimli yaklaşık çözümlerini 2 nümerik örnekle analiz ederek DJM nin sınır değerli gecikmeli diferansiyel denklemlerin çözümünde ADM ve DTM kadar iyi olup olmadığını araştırdık ve sonuç olarak DJM nin bu tip problemlerde güvenilir bir metot olduğunu gördük.
Anahtar Kelimeler: Gecikmeli diferansiyel denklemler, Daftardar-Jafari, Adomian Ayrıştırma, Diferansiyel Transformasyon, Sınır değer

I. INTRODUCTION

Delay Differential Equations (DDE) consider processes where rate of change depends not only on the present stage but also on the history. Linear and nonlinear delay differential equations are studied by many researchers using the ADM and DTM. Wazwaz et al [1], and Ogunfiditimi [2] used ADM, Cakir and Arslan [3] compared results of ADM and DTM on linear and nonlinear differential equations of pantograph type. In literature, to our knowledge, there isn‘t a research focused on DJM‘s application on nonlinear delay differential equations.

In this paper we investigate the efficiency of DJM compared with ADM and DTM on boundary value problems of ordinary nonlinear differential equations with proportional delay through 2 numerical examples.

We will briefly illustrate the theory of DJM, ADM and DTM, we will show applications of these 3 methods to 2 numerical problems, analyse the results and give a conclusion about the efficiency of DJM.

II. MATERIALS AND METHODS

2.1. Daftardar-Jafari Method (DJM)

DJM is a method that can be utilized to obtain solutions of nonlinear functional equations [4]. A general functional equation can be written as;

$$y = N(y) + g \tag{1}$$

where N might be a nonlinear operator and g is a known function. DJM decomposes the solution y into a series as;

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$$y = \sum_{n=0}^{\infty} y_n \tag{2}$$

By using the series expansion of y, nonlinear operator N in (1) can be decomposed as;

$$N\left(\sum_{n=0}^{\infty} y_n\right) = N(y_0) + \sum_{n=1}^{\infty}\left(N\left(\sum_{m=0}^{n} y_m\right) - N\left(\sum_{m=0}^{n-1} y_m\right)\right) \tag{3}$$

Then (1) can be re-written as;

$$y_0 + y_1 + \sum_{n=2}^{\infty} y_n = g + N(y_0) + \sum_{n=1}^{\infty}\left(N\left(\sum_{m=0}^{n} y_m\right) - N\left(\sum_{m=0}^{n-1} y_m\right)\right) \tag{4}$$

Taking $y_0 = g$ and $y_1 = N(y_0)$ we get the following recurrence relation;

$$\begin{aligned} &y_0 = g \\ &y_1 = N(y_0) \\ &\vdots \\ &y_{m+1} = N(y_0 + y_1 + \dots + y_m) \\ &-N(y_0 + y_1 + \dots + y_{m-1}) \qquad m = 1,2,\dots \end{aligned} \tag{5}$$

This yields;

$$\begin{aligned} &(y_1 + y_2 + \dots + y_{m+1}) \\ &= N(y_0 + y_1 + \dots + y_m) \qquad m = 1,2,\dots \end{aligned} \tag{6}$$

From this we can write (1) as;

$$y = g + \sum_{n=1}^{\infty} y_n \tag{7}$$

It is shown in [5] that if N is a contraction then $y = \sum_{n=0}^{\infty} y_n$ is absolutely and uniformly convergent and converges to the unique y in view of Banach fixed point theorem. k- term approximation to y is shown as follows;

$$y \approx \sum_{n=0}^{k} y_n \tag{8}$$

2.2. Adomian Decomposition Method (ADM)

A general form of a deterministic differential equation can be written as;

$$Ly + Ry + Ny = g \tag{9}$$

where L is the highest order linear differential operator $(L = \frac{d^n}{dt^n}(.)$; with invertibility assumption), R is the remainder linear operator and N is the nonlinear operator and g is any function [6], [7]. By applying the inverse operator L^{-1} we get the following;

$$L^{-1}Ly = L^{-1}g - L^{-1}Ry - L^{-1}Ny \tag{10}$$

where for initial value problems the inverse operator L^{-1} is defined as the n-fold definite integration operator from 0 to t. ADM decomposes the solution into a series;

$$y = \sum_{n=0}^{\infty} y_n \tag{11}$$

Then we obtain;

$$L^{-1}Ly = L^{-1}L\sum_{n=0}^{\infty} y_n = L^{-1}g - L^{-1}R\sum_{n=0}^{\infty} y_n - L^{-1}Ny \tag{12}$$

ADM replaces $N\ y$ with a series of so called Adomian polynomials (A_n s) which are generated for the particular nonlinearity of the operator N. Thus we have;

$$Ny = \sum_{n=0}^{\infty} A_n \tag{13}$$

With the definition of L^{-1} and taking y_0 (the first term of the series $< y_n >$) as the sum of $L^{-1}\ g$ and terms resulting from the initial conditions, (12) can be written as follows;

$$y = \sum_{n=0}^{\infty} y_n = y_0 - L^{-1}R\sum_{n=0}^{\infty} y_n - L^{-1}N\sum_{n=0}^{\infty} A_n \tag{14}$$

where A_n can be formulated as;

$$A_0 = f(y_0) \tag{15}$$

$$A_n = \sum_{v=1}^{n} c(v,n)\ f^{(v)}(y_0) \qquad n = 1,2,\dots \tag{16}$$

where f is the nonlinear term, $f^{(v)}$ is the v^{th} derivative of f, and $c(v, n)$ is the function as defined in [7].

Convergence of $\sum_{n=0}^{\infty} A_n$ is shown in [8] and [5] basing on the assumption that nonlinear operator N is a contraction in Banach space.

So that we reach at the following recursive relation;

$$\begin{aligned} y_1 &= -L^{-1}Ry_0 - L^{-1}A_0 \\ y_2 &= -L^{-1}Ry_1 - L^{-1}A_1 \\ &\vdots \\ y_n &= -L^{-1}Ry_{n-1} - L^{-1}A_{n-1} \end{aligned} \tag{17}$$

As y_0 is calculated from the initial conditions and A_n depends only on $(y_0, y_1 \cdots, y_n)$ we can find all y_n and A_n respectively.

In [9] it is shown that ADM and DJM generate series solution $y = \sum_{n=0}^{\infty} y_n$ converging to the same limit.

The practical solution will be the k-term approximation to y;

$$y \approx \sum_{n=0}^{k} y_n \tag{18}$$

2.3. Differential Transformation Method (DTM)

Any analytical function $y(x)$ can be expanded in Taylor series about a point x_0 as

$$y(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k y(x)}{dx^k}\bigg|_{x=x_0} (x - x_0)^k \tag{19}$$

Differential transformation of $y(x)$ is defined as;

$$Y(k) = \frac{1}{k!} \frac{d^k y(x)}{dx^k}\bigg|_{x=x_0} \qquad k = 0,1,2,\dots \tag{20}$$

Differential inverse transform of $Y(k)$ is defined as follows;

$$y(x) = \sum_{k=0}^{\infty} Y(k)\ (x - x_0)^k \tag{21}$$

with n-terms approximation we obtain;

$$y(x) \approx \sum_{k=0}^{n} \frac{1}{k!} \frac{d^k y(x)}{dx^k}\bigg|_{x=x_0} (x - x_0)^k \tag{22}$$

The fundamental operations of DTM performed at $x = 0$ is shown in Table 1.

Table 1. Most used differential transformation operators

Original Function	Transformed Function
$y(x) = u(x) \pm v(x)$	$Y(k) = U(k) \pm V(k)$
$y(x) = c\,u(x)$	$Y(k) = c\,U(k)$
$y(x) = u(x)\,v(x)$	$Y(k) = \sum_{l=0}^{k} U(l)\,V(k-l)$
$y(x) = \frac{d^n u(x)}{dx^n}$	$Y(k) = \frac{(k+n)!}{k!} U(k+n)$
$y(x) = x^n$	$Y(k) = \delta(k-n) = \begin{cases} 1, & if\ k = n \\ 0, & if\ k \neq n \end{cases}$
$y(x) = e^{\lambda x}$	$Y(k) = \frac{\lambda^k}{k!}$
$y(x) = u\left(\frac{x}{a}\right)$	$Y(k) = \frac{1}{a^k} U(k)$

III. NUMERICAL APPLICATIONS

In this section boundary value nonlinear delay differential equations have been solved by DJM, ADM and DTM and results analysed. We used Mathematica 10.4 in numerically solving of algebraic equations.

Example 1.

Nonlinear second order proportional delay differential equation

$$\frac{d^2 y(x)}{dx^2} = -y(x) + 5y^2\left(\frac{x}{2}\right), \quad x \in [0,1] \tag{23}$$

$$y(0) = 1, \qquad y(1) = e^{-2}$$

Exact solution is $y = e^{-2x}$

Solving with DJM:

By taking 2-fold integration of both sides of (23) we get the standard form for DJM;

$$y(x) = 1 + y'(0)x - \int_0^x\int_0^x y(x)\,dx\,dx + 5\int_0^x\int_0^x y^2\left(\frac{x}{2}\right) dx\,dx \tag{24}$$

Let`s call $y'(0) = b$.

Then in accordance with (5) we get

$$y_0(x) = 1 + bx$$

In (24) we have a linear term y as well. So we set

$$y_1 = Ly_0 + N y_0\left(\frac{x}{2}\right) = -\int_0^x\int_0^x y_0(x)\,dx\,dx + 5\int_0^x\int_0^x y_0^2\left(\frac{x}{2}\right) dx\,dx$$

and in accordance with (5) we have;

$$y_{m+1} = L(y_0 + y_1 + \dots + y_m) + N\left(y_0\left(\frac{x}{2}\right) + y_1\left(\frac{x}{2}\right) + \dots + y_m\left(\frac{x}{2}\right)\right) - N\left(y_0\left(\frac{x}{2}\right) + y_1\left(\frac{x}{2}\right) + \dots + y_{m-1}\left(\frac{x}{2}\right)\right) \qquad m = 1,2,\dots$$

which in this example is;

$$y_{m+1} = -\int_0^x\int_0^x y_0(x) + y_1(x) \dots + y_m(x)\ dx\,dx + 5\left[\int_0^x\int_0^x \left(y_0\left(\frac{x}{2}\right) + y_1\left(\frac{x}{2}\right) + \dots + y_m\left(\frac{x}{2}\right)\right)^2 dx\,dx - \int_0^x\int_0^x \left(y_0\left(\frac{x}{2}\right) + y_1\left(\frac{x}{2}\right) + \dots + y_{m-1}\left(\frac{x}{2}\right)\right)^2 dx\,dx\right]$$

So, we get;

$$y_1(x) = -\int_0^x\int_0^x (1+bx)\,dx\,dx + 5\int_0^x\int_0^x \left(1+b\left(\frac{x}{2}\right)\right)^2 dx\,dx = -\frac{x^2}{2} - \frac{b\,x^3}{6} + 5\left(\frac{x^2}{2} + \frac{b\,x^3}{6} + \frac{b^2x^4}{48}\right)$$

$$y_2(x) = -\frac{x^4}{6} - \frac{b\,x^5}{30} + \frac{b^2x^6}{288} + 5\left(\frac{x^4}{12} + \frac{b\,x^5}{30} + \frac{x^6}{120} + \frac{37b^2x^6}{11520} + \frac{b\,x^7}{504} + \frac{5b^3x^7}{32256} + \frac{31b^2x^8}{129024} + \frac{5b^3x^9}{331776} + \frac{5b^4x^{10}}{10616832}\right)$$

$$y_3(x) = -\int_0^x\int_0^x y_0(x) + y_1(x) + y_2(x)\; dx\,dx + 5\left[\int_0^x\int_0^x \left(y_0\left(\frac{x}{2}\right) + y_1\left(\frac{x}{2}\right) + y_2\left(\frac{x}{2}\right)\right)^2 dx\,dx - \int_0^x\int_0^x \left(y_0\left(\frac{x}{2}\right) + y_1\left(\frac{x}{2}\right)\right)^2 dx\,dx\right]$$

$\vdots$

By substituting the boundary condition $y(1) = e^{-2}$ in 4 terms approximate series solution ($y_0 + y_1 + y_2 + y_3$) we obtain $b = -2.0$ by numerically solving the algebraic equation, and hence we obtain the series solution of the problem.

Solving with ADM:

$L^{-1}: \int_0^x\int_0^x (.)\; dx\,dx$

Applying L^{-1} to both sides of (23) we get;

$y(x) = 1 + bx - L^{-1}y(x) + 5\,L^{-1}\,y^2\left(\frac{x}{2}\right)$

where b denotes $y'(0)$.

In accordance with (14) we have $y_0(x) = 1 + bx$ and the nonlinear term is $y^2\left(\frac{x}{2}\right)$.

In accordance with (15), (16) and (17) Adomian polynomials and y_n are calculated as follows;

$A_0 = y_0^2\left(\frac{x}{2}\right) = \left(1 + b\frac{x}{2}\right)^2$

$y_1(x) = -L^{-1}y_0(x) + 5\,L^{-1}A_0 = -\frac{x^2}{2} - b\frac{x^3}{6} + 5\left(\frac{x^2}{2} + b\frac{x^3}{6} + b^2\frac{x^4}{48}\right)$

$A_1 = y_1\left(\frac{x}{2}\right)2y_0\left(\frac{x}{2}\right) = 2\left(1 + b\frac{x}{2}\right)\left(-\frac{x^2}{8} - b\frac{x^3}{48} + 5\left(\frac{x^2}{8} + b\frac{x^3}{48} + b^2\frac{x^4}{768}\right)\right.$

$\vdots$

By substituting boundary condition at $x = 1$; starting from the 5th iteration b is found as -2.0. Then we obtain the series solution.

Solving with DTM:

Applying Differential Transformation to (23) at $x = 0$ in accordance with Table 1 we obtain the following;

$$(k+1)(k+2)\,Y(k+2) = -Y(k) + 5\sum_{l=0}^{k}\frac{1}{2^k}\,Y(l)Y(k-l)$$

From the initial condition of (23) we have $Y(0) = 1$. We also have $y'(0) = Y(1) = b$.

For
$k = 0;$ $\quad 1.2.Y(2) = -Y(0) + 5\left(Y(0)\right)^2 \xrightarrow{\text{yields}} Y(2) = 2$

$k = 1;$ $\quad Y(3) = \frac{2b}{3}$

$k = 2;$ $\quad Y(4) = \frac{1}{48}\left(12 + 5\,b^2\right)$

$k = 3;$ $\quad Y(5) = -\frac{2b}{15}$

$k = 4;$ $\quad Y(6) = \frac{444 + 145\,b^2}{11520}$

$\vdots \qquad \vdots$

By substituting boundary condition at $x = 1$; starting from 12th iteration b is found as -2.0.

With the definition of inverse transform, $n = 12$ term approximate solution is obtained as;

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^k \approx \sum_{k=0}^{12} Y(k)x^k$$

$$= 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 - \frac{4}{15}x^5 + \frac{4}{45}x^6 - \frac{8}{315}x^7 + \frac{2}{315}x^8 \cdots$$

The results of 3 methods applied to the problem compared with the exact solution is as follows;

Table 2. Error analysis

	DJM-Exact	ADM-Exact	DTM-Exact
$x = 0.0$	0	0	0
$x = 0.2$	-5.958613725 x 10^{-8}	3.338020682 x 10^{-8}	-1.746912571 x 10^{-7}
$x = 0.4$	-1.260420042 x 10^{-7}	7.063050189 x 10^{-8}	-3.696623786 x 10^{-7}
$x = 0.6$	-2.005382795 x 10^{-7}	1.131771987 x 10^{-7}	-5.944723387 x 10^{-7}
$x = 0.8$	-2.463858192 x 10^{-7}	1.450977594 x 10^{-7}	-7.92225276 x 10^{-7}
$x = 1.0$	0	0	0

Since boundary values are taken as input for all 3 methods there is no error at the boundaries.

Example 2.

Nonlinear second order proportional delay differential equation;

$$\frac{d^2y(x)}{dx^2} = \left(y^2(x) + y^3(x)\right)y\left(\frac{x}{2}\right), \quad x \in [0,1] \tag{25}$$

$y(0) = 1, \quad y(1) = \frac{1}{2}$

Exact solution is $y = \frac{1}{x+1}$

Solving with DJM:

By taking 2-fold integration of both sides of (25) we get the standard form for DJM.

$$y(x) = 1 + b\,x + \int_0^x\int_0^x (y^2(x) + y^3(x))\, y\left(\frac{x}{2}\right) dx\ dx$$

where b is for $y'(0)$. Then we have

$$y_0(x) = 1 + b\,x$$

Nonlinear term, $N\,y$ is $(y^2(x) + y^3(x))\, y\left(\frac{x}{2}\right)$.

y_n, $(n \geq 1)$ are as follows;

$$y_1(x) = x^2 + b\,x^3 + \frac{13\,b^2\,x^4}{24} + \frac{3\,b^3\,x^5}{20} + \frac{b^4\,x^6}{60}$$

$$\begin{aligned}
y_2(x) = {} & \frac{b^{16}x^{26}}{539136000000} + \frac{b^{15}x^{25}}{11059200000} + \frac{1913\,b^{14}x^{24}}{91570176000 0} + \frac{4261\,b^{13}x^{23}}{139898880000} + \frac{b^{13}x^{21}}{165888000} + \frac{14621\,b^{12}x^{22}}{139898880000} + \frac{13\,b^{12}x^{20}}{65664000} \\
& + \frac{75953\,b^{11}x^{21}}{30965760000} + \frac{607\,b^{11}x^{19}}{196992000} + \frac{7536349\,b^{10}x^{20}}{504299520000} + \frac{1579\,b^{10}x^{18}}{52224000} + \frac{11\,b^{10}x^{16}}{6144000} + \frac{203959\,b^{9}x^{19}}{2801664000} \\
& + \frac{1052329\,b^{9}x^{17}}{5013504000} + \frac{2137\,b^{9}x^{15}}{48384000} + \frac{11670373\,b^{8}x^{18}}{40609382400} + \frac{484303\,b^{8}x^{16}}{442368000} + \frac{4717\,b^{8}x^{14}}{9318400} + \frac{348059\,b^{7}x^{17}}{376012800} \\
& + \frac{1029967\,b^{7}x^{15}}{232243200} + \frac{8039\,b^{7}x^{13}}{2246400} + \frac{97\,b^{7}x^{11}}{422400} + \frac{3231751\,b^{6}x^{16}}{1327104000} + \frac{274529\,b^{6}x^{14}}{19353600} + \frac{354329\,b^{6}x^{12}}{20275200} \\
& + \frac{667\,b^{6}x^{10}}{172800} + \frac{16787\,b^{5}x^{15}}{3225600} + \frac{129307\,b^{5}x^{13}}{3594240} + \frac{315311\,b^{5}x^{11}}{5068800} + \frac{2677\,b^{5}x^{9}}{92160} + \frac{24917\,b^{4}x^{14}}{2795520} + \frac{9101\,b^{4}x^{12}}{126720} \\
& + \frac{227519\,b^{4}x^{10}}{1382400} + \frac{463\,b^{4}x^{8}}{3584} + \frac{119\,b^{4}x^{13}}{9984} + \frac{23311\,b^{3}x^{11}}{211200} + \frac{11111\,b^{3}x^{9}}{34560} + \frac{30463\,b^{3}x^{7}}{80640} + \frac{601\,b^{2}x^{12}}{50688} \\
& + \frac{359\,b^{2}x^{10}}{2880} + \frac{3207\,b^{2}x^{8}}{7168} + \frac{841\,b^{2}x^{6}}{1152} + \frac{7\,b\,x^{11}}{880} + \frac{3\,b\,x^{9}}{32} + \frac{45\,b\,x^{7}}{112} + \frac{17\,b\,x^{5}}{20} + \frac{x^{10}}{360} + \frac{x^{8}}{28} + \frac{7\,x^{6}}{40} + \frac{11\,x^{4}}{24}
\end{aligned}$$

$\vdots$

By substituting the boundary condition $y(1) = \frac{1}{2}$ in 3 terms approximate series solution $(y_0 + y_1 + y_2)$ we obtain $b = -0.9974$, and hence we obtain the series solution of the problem.

Solving with ADM:

$$L^{-1}\colon \int_0^x\int_0^x (.)\ dx\,dx$$

$$y_0(x) = 1 + b\,x$$

where b is for $y'(0)$.
There are 2 nonlinear terms; $y^2(x)\,y\left(\frac{x}{2}\right)$ and $y^3(x)\,y\left(\frac{x}{2}\right)$

We get;

$$y(x) = 1 + bx + L^{-1}y^2(x)\,y\left(\frac{x}{2}\right) + L^{-1}y^3(x)\,y\left(\frac{x}{2}\right)$$

Adomian polynomials (A_n for the 1st nonlinear term and B_n for the 2nd nonlinear term) and y_n, $(n \geq 1)$ are as follows;

$$A_0 = y_0^2(x)\,y_0\left(\frac{x}{2}\right) = (1 + b\,x)^2\left(1 + \frac{b\,x}{2}\right)$$

$$B_0 = y_0^3(x)\,y_0\left(\frac{x}{2}\right) = (1 + b\,x)^3\left(1 + \frac{b\,x}{2}\right)$$

$$y_1(x) = \frac{b^4\,x^6}{60} + \frac{3b^3\,x^5}{20} + \frac{13\,b^2\,x^4}{24} + b\,x^3 + x^2$$

$$A_1 = \left(\frac{b^4\,x^6}{60} + \frac{3b^3\,x^5}{20} + \frac{13\,b^2\,x^4}{24} + b\,x^3 + x^2\right) x\left(\frac{1}{2}(1 + b\,x)^2 + 2\left(1 + \frac{b\,x}{2}\right)(1 + b\,x)\right)$$

$$B_1 = \left(\frac{b^4\,x^6}{60} + \frac{3b^3\,x^5}{20} + \frac{13\,b^2\,x^4}{24} + b\,x^3 + x^2\right) x\left(\frac{1}{2}(1 + b\,x)^3 + 3\left(1 + \frac{b\,x}{2}\right)(1 + b\,x)^2\right)$$

$$y_2(x) = \frac{b^7\,x^{11}}{3300} + \frac{b^6\,x^{10}}{200} + \frac{53\,b^5\,x^9}{1440} + \frac{51\,b^4\,x^8}{320} + \frac{2273\,b^3\,x^7}{5040} + \frac{101\,b^2\,x^6}{120} + \frac{19\,b\,x^5}{20} + \frac{x^4}{2}$$

$\vdots$

By substituting the boundary condition $y(1) = \frac{1}{2}$ in 5 terms approximate series solution $(y_0 + y_1 + y_2 + y_3 + y_4)$ we obtain $b = -0.986677$, and hence we obtain the series solution of the problem.

Solving with DTM:
By applying differential transformation operations of Table1, transformed equation of (25) can be written a follows;

$$(k+1)(k+2)\,Y(k+2) = \sum_{z=0}^{k} \frac{\left(\sum_{m=0}^{z} Y(m) \sum_{l=0}^{z-m} Y(l)\,Y(z-m-l)\right) Y(k-z)}{2^{k-z}} + \sum_{r=0}^{k} \frac{\left(\sum_{l=0}^{r} Y(l)\,Y(r-l)\right) Y(k-r)}{2^{k-r}}$$

From the initial condition of (25) we have $Y(0) = 1$. We also have $y'(0) = Y(1) = b$.

For

$k=0;\quad Y(2)=1$

$k=1;\quad Y(3)=b$

$k=2;\quad Y(4)=\frac{1}{24}(13\,b^2+11)$

$k=3;\quad Y(5)=\frac{1}{20}(3\,b^2+17\,b)$

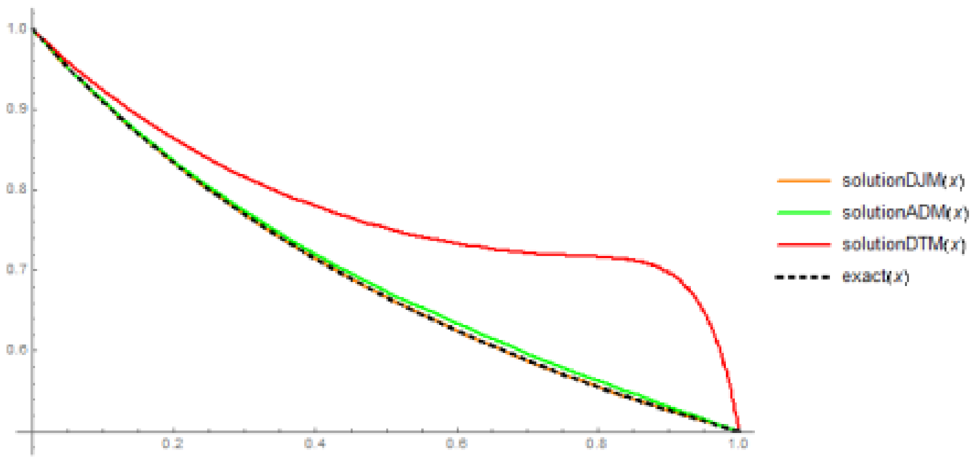
$\vdots \quad\quad \vdots$

By substituting the boundary condition $y(1)=\frac{1}{2}$ in 20 terms approximate series solution $(\sum_{k=0}^{20} Y(k)\,x^k)$ we obtain $b=-0.853709$, and with the definition of inverse transform, $n=20$ term approximate solution is obtained as follows;

$$y(x) = -0.470935\,x^{21} + 0.487609\,x^{20} - 0.504875\,x^{19} + 0.522751\,x^{18} - 0.541261\,x^{17} + 0.560426\,x^{16} - 0.580269\,x^{15} + 0.600816\,x^{14} - 0.622089\,x^{13} + 0.644117\,x^{12} - 0.666919\,x^{11} + 0.690543\,x^{10} - 0.714954\,x^{9} + 0.740367\,x^{8} - 0.766252\,x^{7} + 0.794215\,x^{6} - 0.818982\,x^{5} + 0.85311\,x^{4} - 0.853709\,x^{3} + x^{2} - 0.853709\,x + 1$$

Table 3. Error analysis

	DJM-Exact	ADM-Exact	DTM-Exact
$x=0.0$	0	0	0
$x=0.2$	0.0000053721909530568546	0.0027873595113915295	0.030240751176283664
$x=0.4$	0.00011459706021721416	0.006049697944179644	0.06539434024993596
$x=0.6$	0.00017334321661466312	0.008754821716905448	0.1087273721
$x=0.8$	0.00017662364074289005	0.007939509477603246	0.16114119396876214
$x=1.0$	0	0	0



IV. RESULTS AND CONCLUSION

We have investigated the convergence rates of DJM, ADM and DTM on 2 numerical examples of boundary value nonlinear pantograph type delay differential equations. We see that all 3 methods converge to the exact solution for the first problem very fast. For the second problem ADM and DJM are again very successful with few iterations but DTM doesn't converge as fast as them. In DTM *b* is found -0.793523 with 6 iterations, -0.826196 with 12 iterations and -0.853709 with 20 iterations where the exact value of *b* is -1.0. It seems DTM solution tends to converge to exact solution as number of iterations increases but as the Taylor series expansion of the exact solution of the problem is convergent only in radius less than 1, DTM would require much higher number of iterations to approximate the exact solution as DJM and ADM do.

We see in these two numerical examples that DJM can be used as a reliable method for solving boundary value nonlinear pantograph type delay differential equations.

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