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Differential Geometry of Curves in Euclidean 3-Space with Fractional Order

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(Dedicated to the memory of **Prof. Dr. Aurel BEJANCU (1946 - 2020)**)

Abstract

In this paper, for a given curve in the Euclidean 3-space $\mathbb{R}^3$ we introduce new invariants such as arc-length, curvature and torsion with fractional-order and provide certain relations between these and the standart invariants. We obtain the Frenet-Serret formulas in $\mathbb{R}^3$ and then construct the ways of determining a curve in $\mathbb{R}^2$ and $\mathbb{R}^3$ in terms of the new invariants. Several examples are also given by figures.

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1. Introduction

Fractional Calculus has been investigated and developed since seventeenth century by a larger group of mathematicians led by Leibniz, Bernoulli, L'Hôpital, Euler, Laplace among others. See [2, 34] for details of the process. This branch of Mathematics is an extension of the notion of ordinary differentiation and integration to non-integer orders.

When it comes to a fractional derivative operator, owing to the fact that the ordinary derivative possesses well-known physical and geometric responses, it is immediately questioned whether the fractional operator also presents a physical or geometric interpretation. In fact, it is supposed to have larger physical and geometric properties than that of the ordinary derivative, given that it generalizes operators of integer order. In order to satisfy the expectation, from the physical point of view, there have recently been published a great number of papers, see [1, 3, 6, 7, 11, 12, 14, 15, 18, 24, 31, 32, 40, 41]. Besides the physical capabilities, it is worth to mention that Fractional Calculus acknowledges a wide range of applications in science [10, 19, 21, 30, 35, 36, 38, 44].

Most recently there has been an ascending attention to bring in a geometric approach to a fractional derivative, mainly focusing on Differential Geometry, Fractal Geometry and Vector Calculus of which classic geometry are subclasses. See [3, 4, 13, 16, 20, 27, 28, 29, 42, 43]. Despite diversity of fractional derivative operators used in these approaches, they mostly fail to perform Leibniz rule and various composition rules [37]. Nevertheless, these are indispensable tools to establish a theory in differential geometry of curves, which is our main interest in the present study.

Before presenting how to tackle this obstruction, for a function $f(t)$, let us first introduce the Riemann-Liouville fractional integral and derivative and the Caputo fractional derivative of order α as ([5, 8, 34])

$$\left(\mathbf{I}_{0+}^{\alpha} f\right)(t)=\frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(u)}{(t-u)^{1-\alpha}} du$$

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and

$$\left(\mathbf{D}_{0+}^{\alpha}f\right)(t)=\frac{d\left(\mathbf{I}_{0+}^{1-\alpha}f\right)}{dt}(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{0}^{t}\frac{f(u)}{(t-u)^{\alpha}}du \tag{1.1}$$

and

$$\left({}^{C}\mathbf{D}_{0+}^{\alpha}f\right)(t)=\left(\mathbf{I}_{0+}^{1-\alpha}\frac{df}{du}\right)(t)=\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}\frac{1}{(t-u)^{\alpha}}\frac{df}{du}du,$$

respectively. We remark that f is supposed to be smooth in the case of Caputo derivative and $\Gamma(\alpha)$ denote the Euler gamma function of the parameter α given by ([23])

$$\Gamma(\alpha)=\int_{0}^{\infty}t^{\alpha-1}e^{-t}dt.$$

As clearly seen in (1.1), the fractional derivative of the function f is defined by the ordinary integration of f, presenting non-locality associated with the long range interactions in a space or a past history. Furthermore, this kind of 'fractional derivative' is non-local; that is, at a point t_0 the fractional derivative of $f(t)$ is determined by non-local values of $f(t)$.

Among the fractional derivatives, because the Caputo fractional derivative of a constant function vanishes ([3, 4]), it is the most preferred to construct a theory in the context of Differential Geometry, for example see [3, 4, 42, 43]. Besides, for a given curve in the Euclidean 3-space $\mathbb{R}^3$ Gozutok et al. [16] introduced curvature, torsion and Frenet-Serret formulas by using the conformable fractional derivative. In [27], local properties of curves and surfaces in $\mathbb{R}^3$ were considered via the Leibniz fractional derivative.

Aside from the mentioned advantage of the Caputo fractional derivative, it exhibits some difficulties as do other fractional derivative operators. More clearly, for the Caputo fractional derivative, the Leibniz rule and derivative of composite function appear in the form of infinite series given by respectively [6]

$$\left({}^{C}\mathbf{D}_{0+}^{\alpha}fg\right)(t)=\sum_{i=0}^{\infty}\binom{\alpha}{i}\frac{d^{i}f}{dt^{i}}\left(\mathbf{D}_{0+}^{\alpha-i}g\right)(t)-\frac{t^{-\alpha}}{\Gamma(1-\alpha)}f(0)\,g(0)$$

and

$$\left({}^{C}\mathbf{D}_{0+}^{\alpha}f\right)(g(t))=\frac{f(g(t))-f(g(0))}{\Gamma(1-\alpha)}t^{-\alpha}+\sum_{i=0}^{\infty}\binom{\alpha}{i}\frac{t^{i-\alpha}}{\Gamma(i-\alpha+1)}\frac{d^{i}f(g(t))}{dt^{i}}. \tag{1.2}$$

These infinite series make difficulties to introduce basic geometric objects of a curve as the tangent vector, curvature and etc. For this reason the authors in [43] were forced to impose a certain simplification for (1.2) as follows: for given smooth functions $f(t)$ and $t=g(s)$

$$\left({}^{C}\mathbf{D}_{0+}^{\alpha}f\right)(g(s))=\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\frac{df}{dt}\frac{dg}{ds}, \tag{1.3}$$

where and hereinafter we assume $0<\alpha\leq 1$.

In this paper, based on the simplification (1.3) we generalize for a given curve $\mathbf{x}(s)$ in $\mathbb{R}^3$ the arguments defined in $\mathbb{R}^2$ by Yajima et al. [43], introducing the arc-length paramater s, curvature κ_α and torsion τ_α with fractional-order α. We provide some relations between $\kappa_\alpha,\tau_\alpha$ and the standard curvatures of $\mathbf{x}(s)$. By showing that κ_α and τ_α are invariants under the Euclidean motions of $\mathbb{R}^3$, we prove the Fundamental Theorems of curves in $\mathbb{R}^2$ and $\mathbb{R}^3$ in terms of the curvatures with fractional-order. Several examples are also provided by figures.

2. Basics of differential geometry of curves

We briefly provide the differential geometric objects of curves in $\mathbb{R}^n$ from [9, 17, 33].

Let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the scalar product and induced norm in $\mathbb{R}^n$, respectively. Let $\mathbf{y}:\tilde{I}\rightarrow\mathbb{R}^n$ be a regular curve , i.e. $\|\dot{\mathbf{y}}\|\neq 0,$ for each $t\in\tilde{I}$ and $\dot{\mathbf{y}}=\frac{d\mathbf{y}}{dt}$. Remark that the derivative with respect to the parameter t is denoted by a dot throughout the paper. Then there exist a *unit speed reparameterization* $\mathbf{x}$ of the curve $\mathbf{y}$, $\mathbf{x}=\mathbf{y}\circ u^{-1}:I\rightarrow\mathbb{R}^n$, such that

$$u(t)=\int_{t_0}^{t}\left\|\frac{d\mathbf{y}}{d\sigma}\right\|d\sigma,\ t_0\in\tilde{I} \tag{2.1}$$

and $\left\|\frac{d\mathbf{x}}{du}\right\| = 1$. The parameter u is said to be *arc-length*. Point out that the arc-length parameter u is invariant under the Euclidean motions of $\mathbb{R}^n$. We seperate the formulas for local properties of the curves into two subsections:

2.1. Plane curves

Let $\mathbf{x} : I \to \mathbb{R}^2$ be a unit speed curve. Then the *curvature* κ of $\mathbf{x}$ at $u \in I$ is defined by $\kappa(u) = \left\|\frac{d^2\mathbf{x}}{du^2}\right\|$. The *Frenet-Serret frame* of $\mathbf{x}$ at u consists of $\mathbf{t}(u) = \frac{d\mathbf{x}}{du}\big|_u$ and $\mathbf{n}(u) = J\left(\frac{d\mathbf{x}}{du}\big|_u\right)$, where J is the complex structure of $\mathbb{R}^2$ given by $(v, w) \mapsto J(v, w) = (-w, v)$. For an arbitrary parameter t, κ is calculated by

$$\kappa(t) = \frac{\langle \ddot{\mathbf{x}}(t), J(\dot{\mathbf{x}}(t))\rangle}{\|\dot{\mathbf{x}}(t)\|^3}.$$

Thus the *Frenet-Serret formulas* follow

$$\begin{aligned} \frac{d\mathbf{t}}{du} &= \kappa\mathbf{n}, \\ \frac{d\mathbf{n}}{du} &= -\kappa\mathbf{t}. \end{aligned}$$

On the other hand, because $\frac{d\mathbf{x}}{du}\big|_u$ is a unit vector for each $u \in I$, there exists a smooth function $\theta(u)$, called *turning angle* of $\mathbf{x}$, such that

$$\frac{d\mathbf{x}}{du} = (\cos\theta(u), \sin\theta(u)).$$

The value $k(u) = |_u$ is referred to as *signed curvature* of $\mathbf{x}$ at u. Note that $k(u)$ is unchanged by the Euclidean motions of $\mathbb{R}^2$ and $\kappa = |k|$. The Fundamental Theorem of plane curves states that for a given smooth function $k(u)$, $u \in I$, there exist a unique plane curve $\mathbf{x}$ up to a Euclidean motion of $\mathbb{R}^2$ such that k and u are the signed curvature and arc-length parameter of $\mathbf{x}$, respectively.

2.2. Space curves

Let $\mathbf{x} : I \to \mathbb{R}^3$ be a unit speed curve. Then the *curvature* κ of $\mathbf{x}$ at $u \in I$ is defined by $\kappa(u) = \left\|\frac{d^2\mathbf{x}}{du^2}\right\|$. The *Frenet-Serret frame* of $\mathbf{x}$ at u consists of $\mathbf{t}(u) = \frac{d\mathbf{x}}{du}\big|_u$ and $\mathbf{n}(u) = \frac{1}{\kappa(u)}\frac{d^2\mathbf{x}}{du^2}\Big|_u$, $\kappa(u) \neq 0$, and $\mathbf{b}(u) = \mathbf{t}(u) \times \mathbf{n}(u)$, where $\times$ is the cross product in $\mathbb{R}^3$. Thus the *Frenet-Serret formulas* follow

$$\begin{aligned} \tfrac{d\mathbf{t}}{du} &= \kappa\mathbf{n}, \\ \tfrac{d\mathbf{n}}{du} &= -\kappa\mathbf{t} + \tau\mathbf{b}, \\ \tfrac{d\mathbf{b}}{du} &= -\tau\mathbf{n}, \end{aligned}$$

where the value $\tau(u) = \left\langle \frac{d\mathbf{n}}{du}\big|_u, \mathbf{b}(u)\right\rangle$ is called the *torsion* of $\mathbf{x}$ at u. Note that κ and $|\tau|$ are invariant under the Euclidean motions of $\mathbb{R}^3$. For an arbitrary parameter t, κ and τ are calculated by

$$\kappa(t) = \frac{\|\dot{\mathbf{x}}(t) \times \ddot{\mathbf{x}}(t)\|}{\|\dot{\mathbf{x}}(t)\|^3}, \ \tau(t) = \frac{\langle \dot{\mathbf{x}}(t) \times \ddot{\mathbf{x}}(t), \dddot{\mathbf{x}}(t)\rangle}{\|\dot{\mathbf{x}}(t) \times \ddot{\mathbf{x}}(t)\|^2}. \tag{2.2}$$

The curve $\mathbf{x}$ turns to a straight-line (resp. planar curve) when κ (resp. τ) vanishes identically. A curve $\mathbf{x}$ is called a *general helix* if there is a fixed unit vector $\mathbf{u}$ such that for each $u \in I$ the function $\langle \mathbf{t}(u), \mathbf{u}\rangle$ is constant. The Lancret Theorem [26] states that the ratio $\frac{\tau}{\kappa}$ is constant for each $u \in I$ if and only if $\mathbf{x}$ is a general helix. Furthermore, $\mathbf{x}$ turns to a circular helix provided both κ and τ are constant.

The Fundamental Theorem of space curves states that for given smooth functions $\kappa(u) > 0$ and $\tau(u)$, $u \in I$, there exist a unique space curve $\mathbf{x}$ up to a Euclidean motion of $\mathbb{R}^3$ such that κ, τ and u are the curvature, torsion and arc-length parameter of $\mathbf{x}$, respectively.

3. Curvatures of space curves with fractional-order

Let $\mathbf{x} : I \to \mathbb{R}^3$ be parameterized by the arc-length u. Consider another parameter s given by

$$u \to s = \left[\frac{\alpha^2}{\Gamma(2-\alpha)}u\right]^{\frac{1}{\alpha}}, \tag{3.1}$$

where Γ is the Euler gamma function and $0 < \alpha \leq 1$. Due to (2.1), s is also a function of t. Let us put $s = h(t)$. Then (3.1) gives

$$h(t) = \left(\frac{\alpha^2}{\Gamma(2-\alpha)} \int_{t_0}^{t} \left\| \frac{d\mathbf{x}}{d\sigma} \right\| d\sigma \right)^{\frac{1}{\alpha}}. \tag{3.2}$$

It follows from (3.2) that

$$\dot{h} = \frac{dh}{dt} = \frac{\alpha h^{1-\alpha}(t)}{\Gamma(2-\alpha)} \|\dot{\mathbf{x}}\|, \tag{3.3}$$

which implies that $\dot{h}$ is positive for each t and so its inverse function $t = h^{-1}(s)$ exists.

Remark 3.1. Throughout the paper we use the notation

$$\left({}^{C}\mathbf{D}_{0+}^{\alpha} f\right)(t) = \frac{d^{\alpha} f(t)}{dt^{\alpha}}.$$

Morever, the dash $'$ denotes the derivative with respect to s.

Following Yajima et al. [43], for the simplification of Caputo fractional derivative of composite function given by (1.3) we can take the monomial

$$\frac{d^{\alpha}\mathbf{x}(h(t))}{ds^{\alpha}} = \frac{\alpha h^{1-\alpha}(t)}{\Gamma(2-\alpha)} \left(h^{-1}\right)' \dot{\mathbf{x}}. \tag{3.4}$$

After considering (3.2) and (3.3) into (3.4), we conclude $\left\| \frac{d^{\alpha}\mathbf{x}}{ds^{\alpha}}\big|_s \right\| = 1$ for each s, namely s is the arc-length parameter of $\mathbf{x}$ by the virtue of (3.1) and (3.4). Hence we call the parameterized curve $\mathbf{x}$ by (3.1) with (3.4) *unit speed*. In the meanwhile, this is the justification why the parameter s is chosen as in (3.1).

In order to obtain the Frenet-Serret formulas with fractional-order α we firstly point out that the Frenet-Serret frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is independent of the choice of parametrization, namely

$$\texttt{Span}\left\{ \frac{d^{\alpha}\mathbf{x}}{ds^{\alpha}}, \left(\frac{d^{\alpha}\mathbf{x}}{ds^{\alpha}}\right)', \left(\frac{d^{\alpha}\mathbf{x}}{ds^{\alpha}}\right)'' \right\} = \texttt{Span}\left\{ \frac{d\mathbf{x}}{du}, \frac{d^2\mathbf{x}}{du^2}, \frac{d^3\mathbf{x}}{du^3} \right\}.$$

This means that $\mathbf{t}(s) = \frac{d^{\alpha}\mathbf{x}}{ds^{\alpha}}\big|_s$ is the unit tangent vector of $\mathbf{x}$ at s. Because $\langle \mathbf{t}, \mathbf{t} \rangle = 1$, we deduce that $\mathbf{t}' = \frac{d\mathbf{t}}{ds}$ is perpendicular to $\mathbf{t}$. We take $\mathbf{n}(s) = \frac{\mathbf{t}'(s)}{\|\mathbf{t}'(s)\|}$ as unit principal normal vector of $\mathbf{x}$ at s. We call $\kappa_{\alpha}(s) = \|\mathbf{t}'(s)\|$ *curvature* of $\mathbf{x}$ at s *with fractional-order* α. The vector $\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s)$ is the binormal vector of $\mathbf{x}$ at s.

For the Frenet-Serret formulas of $\mathbf{x}$ we have

Theorem 3.1. *Let $\mathbf{x}$ be a parameterized curve in $\mathbb{R}^3$ by (3.1) with (3.4). Then we have*

$$\begin{aligned} \mathbf{t}' &= \kappa_{\alpha}\mathbf{n}, \\ \mathbf{n}' &= -\kappa_{\alpha}\mathbf{t} + \tau_{\alpha}\mathbf{b}, \\ \mathbf{b}' &= -\tau_{\alpha}\mathbf{n}, \end{aligned} \tag{3.5}$$

where $\tau_{\alpha}(s) = \langle \mathbf{n}'(s), \mathbf{b}(s) \rangle$ is called torsion of $\mathbf{x}$ at s with fractional-order α.

Proof. The first equality in (3.5) is obvious. The proof of other equalities can be done by a similar way in usual case (i.e. the case $\alpha = 1$), for instance see [33, p. 50]. □

Remark 3.2. For the standard curvature κ and torsion τ of the curve $\mathbf{x}$, we have $\kappa_1 = \kappa$ and $\tau_1 = \tau$. We also call κ and τ the *curvature* and *torsion* of $\mathbf{x}$ *with integer-order*.

Theorem 3.2. *Let $\mathbf{x}$ be a parameterized curve in $\mathbb{R}^3$ by an arbitrary parameter t and let*

$$\phi(t) = \left(\frac{\Gamma(2-\alpha)}{\alpha} \right)^{\frac{1}{\alpha}} \left[\alpha \int_{t_0}^{t} \left\| \frac{d\mathbf{x}}{du} \right\| du \right]^{1-\frac{1}{\alpha}}.$$

Then the following occurs

$$\kappa_{\alpha}(t) = \phi(t)\,\kappa(t), \;\; \tau_{\alpha}(t) = \phi(t)\,\tau(t), \tag{3.6}$$

where κ_{α}, τ_{α} and κ, τ are respectively the curvatures and torsions of $\mathbf{x}$ with fractional-order α and integer-order.

Proof. Taking ordinary derivative of (3.4) with respect to s gives

$$\mathbf{t}' = \left[\frac{\alpha s^{-\alpha}}{\Gamma(2-\alpha)}\left((1-\alpha)\left(h^{-1}\right)' + s\left(h^{-1}\right)''\right)\right]\dot{\mathbf{x}} + \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\left(h^{-1}\right)'^{2}\ddot{\mathbf{x}}. \tag{3.7}$$

Substituting (3.4) and (3.7) into (2.2) leads to

$$\kappa_\alpha(s) = \frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}}\kappa(s). \tag{3.8}$$

By considering (3.1) into (3.8) we obtain the first equality of (3.6). Next we write

$$\tau_\alpha = \langle \mathbf{n}', \mathbf{t}\times\mathbf{n}\rangle = \left\langle \left(\frac{1}{\kappa_\alpha}\right)'\mathbf{t}' + \frac{1}{\kappa_\alpha}\mathbf{t}'', \mathbf{t}\times\frac{1}{\kappa_\alpha}\mathbf{t}'\right\rangle = \frac{1}{(\kappa_\alpha)^2}\langle \mathbf{t}'', \mathbf{t}\times\mathbf{t}'\rangle \tag{3.9}$$

Taking ordinary derivative of (3.7) with respect to s and substituting it into (3.9) yields

$$\tau_\alpha = \frac{1}{(\kappa_\alpha)^2}\left(\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\right)^3 (t')^6 \langle \dddot{\mathbf{x}}, \dot{\mathbf{x}}\times\ddot{\mathbf{x}}\rangle. \tag{3.10}$$

The proof is completed by considering (2.2) and (3.8) into (3.10). □

By (3.6) we have the following results.

Corollary 3.1. *Let $\mathbf{x}$ be a parameterized curve in $\mathbb{R}^3$ with nonzero curvatures κ, κ_α and torsions τ, τ_α. Then the following holds*

$$\frac{\kappa}{\kappa_\alpha} = \frac{\tau}{\tau_\alpha}.$$

Corollary 3.2. *Let $\mathbf{x}$ be a parameterized curve in $\mathbb{R}^3$ with the curvature κ_α and torsion τ_α of fractional-order α. Then $\mathbf{x}$ is a straight-line if $\kappa_\alpha = 0$, a planar curve if $\tau_\alpha = 0$ and a general helix if $\frac{\kappa_\alpha}{\tau_\alpha} = const. \neq 0$. The converses are true as well.*

Corollary 3.3. *Let $\mathbf{x}$ be a parameterized curve in $\mathbb{R}^3$ with constant curvature κ_α^0 of fractional-order α. Then the following ODE holds*

$$\frac{d\kappa}{ds} + \kappa_\alpha^0(\alpha-1)\kappa = 0.$$

In addition, if $\mathbf{x}$ has constant torsion τ_α^0 with fractional-order α then

$$\frac{d\tau}{ds} + \tau_\alpha^0(\alpha-1)\tau = 0,$$

where κ and τ are the curvature and torsion of $\mathbf{x}$ with integer-order.

Proof. It can be easily proved by using (3.6). □

4. Fundamental Theorems of plane and space curves

We first concern the case that $\mathbf{x}$ is a plane curve: let $\mathbf{x}$ be parameterized curve in $\mathbb{R}^2$ by the arc-length parameter u and κ the curvature of $\mathbf{x}$ with integer-order. Likewise $3-$dimensional case, we have the arc-length parameter s given by (3.1) with (3.4) and the relation

$$\kappa_\alpha(s) = \frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}}\kappa(s),$$

called *curvature* of $\mathbf{x}$ in $\mathbb{R}^2$ at s *with fractional-order* α (see [43, p. 1499]). For the Frenet-Serret formulas of $\mathbf{x}$ we have

$$\begin{aligned} \mathbf{t}' &= \kappa_\alpha \mathbf{n}, \\ \mathbf{n}' &= -\kappa_\alpha \mathbf{t}, \end{aligned}$$

where $\mathbf{t}(s) = \frac{d^\alpha \mathbf{x}}{ds^\alpha}\big|_s$ and $\mathbf{n}(s) = J(\mathbf{t}(s))$ are unit tangent and principal normal vectors of $\mathbf{x}$ at s. Because $\mathbf{t}'(s)$ is perpendicular to $\mathbf{t}(s)$ for each s, we conclude that $\left(\frac{d^\alpha \mathbf{x}}{ds^\alpha}\right)'$ and $\mathbf{n}$ are linearly dependent and so there exists a function $k_\alpha(s)$, called *signed curvature* of $\mathbf{x}$ *with fractional-order* α, such that

$$\left(\frac{d^\alpha \mathbf{x}}{ds^\alpha}\right)'(s) = k_\alpha(s)\,\mathbf{n}(s),$$

which implies $|k_\alpha| = \kappa_\alpha$. Because $\frac{d^\alpha \mathbf{x}}{ds^\alpha}\big|_s$ is a unit vector at s we write

$$\left.\frac{d^\alpha \mathbf{x}}{ds^\alpha}\right|_s = (\cos\theta(s), \sin\theta(s)),$$

for a smooth function $\theta(s)$ yielding $\theta'(s) = k_\alpha(s)$.

Theorem 4.1. *(Fundamental Theorem of Plane Curves) Let k_α be a real-valued smooth function on an open interval I which does not contain zero. Then there exists a unit speed curve $\mathbf{x}: I \to \mathbb{R}^2$ parameterized by (3.1) with (3.4) such that k_α is the signed curvature of $\mathbf{x}$ with fractional-order α. Further if $\mathbf{y}: I \to \mathbb{R}^2$ is another curve admitting k_α as the signed curvature with fractional-order α then $\mathbf{y}(s) = M(\mathbf{x}(s))$, for a Euclidean motion M of $\mathbb{R}^2$.*

Proof. The invariance of the derivative of the arc-length s and signed curvature k_α under a Euclidean motion can be easliy shown by a similar way in [17, p. 136]. This can be concluded by (3.1) and (3.6) as well. Let us introduce $\theta(s) = \int_{s_0}^s k_\alpha(t)\,dt$ for a fix $s_0 \in I$, and

$$\mathbf{x}(s) = \frac{\Gamma(2-\alpha)}{\alpha}\left(\int_{s_0}^s t^{\alpha-1}\cos\theta(t)\,dt, \int_{s_0}^s t^{\alpha-1}\sin\theta(t)\,dt\right). \tag{4.1}$$

If we differentiate (4.1) by using (3.4) then we easily achieve that k_α is the signed curvature of $\mathbf{x}$ with fractional-order α, implying the first part of the proof. For the second part, let

$$\mathbf{y}(s) = \frac{\Gamma(2-\alpha)}{\alpha}\left(\int_{s_0}^s t^{\alpha-1}\cos(\theta(t)+\theta_0)\,dt, \int_{s_0}^s t^{\alpha-1}\sin(\theta(t)+\theta_0)\,dt\right) + \mathbf{y}_0, \tag{4.2}$$

where $\mathbf{y}_0$ and θ_0 are a constant vector and scalar. By using the addition formulas for Sine and Cosine into (4.2), we conclude

$$\mathbf{y}(s) = T_{\mathbf{y}_0} R_{\theta_0}(\mathbf{x}(s)),$$

where R_{θ_0} denotes the anticlockwise rotation by the angle θ_0 about the origin and $T_{\mathbf{y}_0}$ the translation by the vector $\mathbf{y}_0$. This completes the proof. □

Theorem 4.2. *(Fundamental Theorem of Space Curves) Let $\kappa_\alpha > 0$ and τ_α be real-valued smooth functions on an open interval I which does not contain zero. Then there exists a unit speed curve $\mathbf{x}: I \to \mathbb{R}^3$ parameterized by (3.1) with (3.4) such that κ_α and τ_α are the curvature and torsion of $\mathbf{x}$ with fractional-order α. Further if $\mathbf{y}: I \to \mathbb{R}^2$ is another curve admitting κ_α and τ_α as the curvature and torsion with fractional-order α, then $\mathbf{y}(s) = M(\mathbf{x}(s))$, for a Euclidean motion M of $\mathbb{R}^3$.*

Remark 4.1. The proof performs similar techniques used in the usual case, namely the case $\alpha = 1$, for instance see [17, p. 219-222].

Proof. We divide the proof into two parts:
The uniqueness part. The invariance of the derivative of the arc-length s, the curvature κ_α and absolute value of the torsion τ_α under a Euclidean motion can be easliy shown by a similar way in [17, p. 218]. Furthermore, this can be concluded by (3.1) and (3.6). Let $\{\mathbf{t}_\mathbf{x}, \mathbf{n}_\mathbf{x}, \mathbf{b}_\mathbf{x}\}$ and $\{\mathbf{t}_\mathbf{y}, \mathbf{n}_\mathbf{y}, \mathbf{b}_\mathbf{y}\}$ denote the Frenet frames of $\mathbf{x}$ and $\mathbf{y}$, respectively, and fix $s_0 \in I$. By applying a translation and a rotation R of $\mathbb{R}^3$ we may assume that the following quadruples coincide

$$\{\mathbf{x}(s_0), \mathbf{t}_\mathbf{x}(s_0), \mathbf{n}_\mathbf{x}(s_0), \mathbf{b}_\mathbf{x}(s_0)\} \text{ and } \{\mathbf{y}(s_0), \mathbf{t}_\mathbf{y}(s_0), \mathbf{n}_\mathbf{y}(s_0), \mathbf{b}_\mathbf{y}(s_0)\}.$$

This means that there exists a Euclidean motion M of $\mathbb{R}^3$ such that $M(\mathbf{x}(s_0)) = \mathbf{y}(s_0)$ and $R(\mathbf{t}_\mathbf{x}(s_0)) = \mathbf{t}_\mathbf{y}(s_0)$, $R(\mathbf{n}_\mathbf{x}(s_0)) = \mathbf{n}_\mathbf{y}(s_0)$, $R(\mathbf{b}_\mathbf{x}(s_0)) = \mathbf{b}_\mathbf{y}(s_0)$, where R is the so-called *linear part* of M. For $s \in I$, let us introduce the following

$$\begin{aligned} f(s) \;=\; & \frac{1}{2}\left[\langle R(\mathbf{t}_\mathbf{x}(s)) - \mathbf{t}_\mathbf{y}(s), R(\mathbf{t}_\mathbf{x}(s)) - \mathbf{t}_\mathbf{y}(s)\rangle + \right. \\ & + \langle R(\mathbf{n}_\mathbf{x}(s)) - \mathbf{n}_\mathbf{y}(s), R(\mathbf{n}_\mathbf{x}(s)) - \mathbf{n}_\mathbf{y}(s)\rangle \\ & \left. + \langle R(\mathbf{b}_\mathbf{x}(s)) - \mathbf{b}_\mathbf{y}(s), R(\mathbf{b}_\mathbf{x}(s)) - \mathbf{b}_\mathbf{y}(s)\rangle\right]. \end{aligned}$$

By using (3.5) and the assumption that $\mathbf{x}$ and $\mathbf{y}$ have the same curvature and torsion with fractional-order α we obtain $\frac{df}{ds} = 0$. This follows that f is a zero function because $f(s_0) = 0$. Thereby we obtain

$$\frac{d^\alpha M(\mathbf{x}(s))}{ds^\alpha} = R(\mathbf{t}_\mathbf{x}(s)) = \mathbf{t}_\mathbf{y}(s) = \frac{d^\alpha \mathbf{y}(s)}{ds^\alpha}. \tag{4.3}$$

After considering (3.4) into (4.3) we get $M(\mathbf{x}(s))' = \mathbf{y}'(s)$, yielding

$$M(\mathbf{x}(s)) = \mathbf{y}(s) + \mathbf{z},$$

for each s and a constant vector $\mathbf{z}$. Notice also that $\mathbf{z} = 0$ because $M(\mathbf{x}(s_0)) = \mathbf{y}(s_0)$.
The existence part. We use the techniques from the theory of systems of ordinary differential equations. For $j = 1, 2, 3,$ let us consider the following system

$$\begin{array}{l} x_j' = \phi t_j, \\ t_j' = \kappa_\alpha n_j, \\ n_j' = -\kappa_\alpha t_j + \tau_\alpha b_j, \\ b_j' = -\tau_\alpha n_j \end{array} \tag{4.4}$$

with the initial conditions

$$x_j(s_0) = \lambda_j,\ t_j(s_0) = \mu_j,\ n_j(s_0) = \xi_j,\ b_j(s_0) = \mu_{\rho(j)}\xi_{\rho(j+1)} - \mu_{\rho(j+1)}\xi_{\rho(j)}, \tag{4.5}$$

where $\phi(s) = \frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}}$ and $\rho(j) = j + 1\,(mod 3)$. In addition let

$$\sum_{j=1}^{3} \mu_j^2 = \sum_{j=1}^{3} \xi_j^2 = 1 \text{ and } \sum_{j=1}^{3} \mu_j \xi_j = 0.$$

It is known that the system (4.4) with (4.5) admits a unique solution. Putting

$$\mathbf{x} = (x_1, x_2, x_3),\ \mathbf{t} = (t_1, t_2, t_3),\ \mathbf{n} = (n_1, n_2, n_3),\ \mathbf{b} = (b_1, b_2, b_3),$$

then the system (4.4) can be associated to into the equations

$$\begin{array}{rcl} \frac{d^\alpha \mathbf{x}}{ds^\alpha} &=& \mathbf{t}, \\ \mathbf{t}' &=& \kappa_\alpha \mathbf{n}, \\ \mathbf{n}' &=& -\kappa_\alpha \mathbf{t} + \tau_\alpha \mathbf{b}, \\ \mathbf{b}' &=& -\tau_\alpha \mathbf{n}. \end{array}$$

This implies from (3.5) that $\mathbf{x}$ is a unit speed curve parameterized by (3.1) with (3.4) admitting κ_α and τ_α as the curvature and torsion with fractional-order α. □

5. Examples

Example 5.1. Let $\mathbf{x}$ be a parameterized curve in $\mathbb{R}^2$ given by (3.1) with (3.4) and let $\kappa_\alpha = 1$ identically. By (4.1) we can get

$$\mathbf{x}(s) = \frac{\Gamma(2-\alpha)}{\alpha}\left(\int s^{\alpha-1}\cos s ds, \int s^{\alpha-1}\sin s ds\right).$$

For the different values of α, $\mathbf{x}$ can be plotted as in Figure 1.

Example 5.1 puts forward a construction method of a plane curve $\mathbf{x}$ from its curvature κ_α with fractional-order α. In addition, Figure 1 shows the change of the curve $\mathbf{x}$ with $\kappa_\alpha = 1$ about the value of α. As α goes to 1, $\mathbf{x}$ closes to a standard circle with radius 1, conforming to the fact that every plane curve having consant curvature with integer-order is a standard circle.

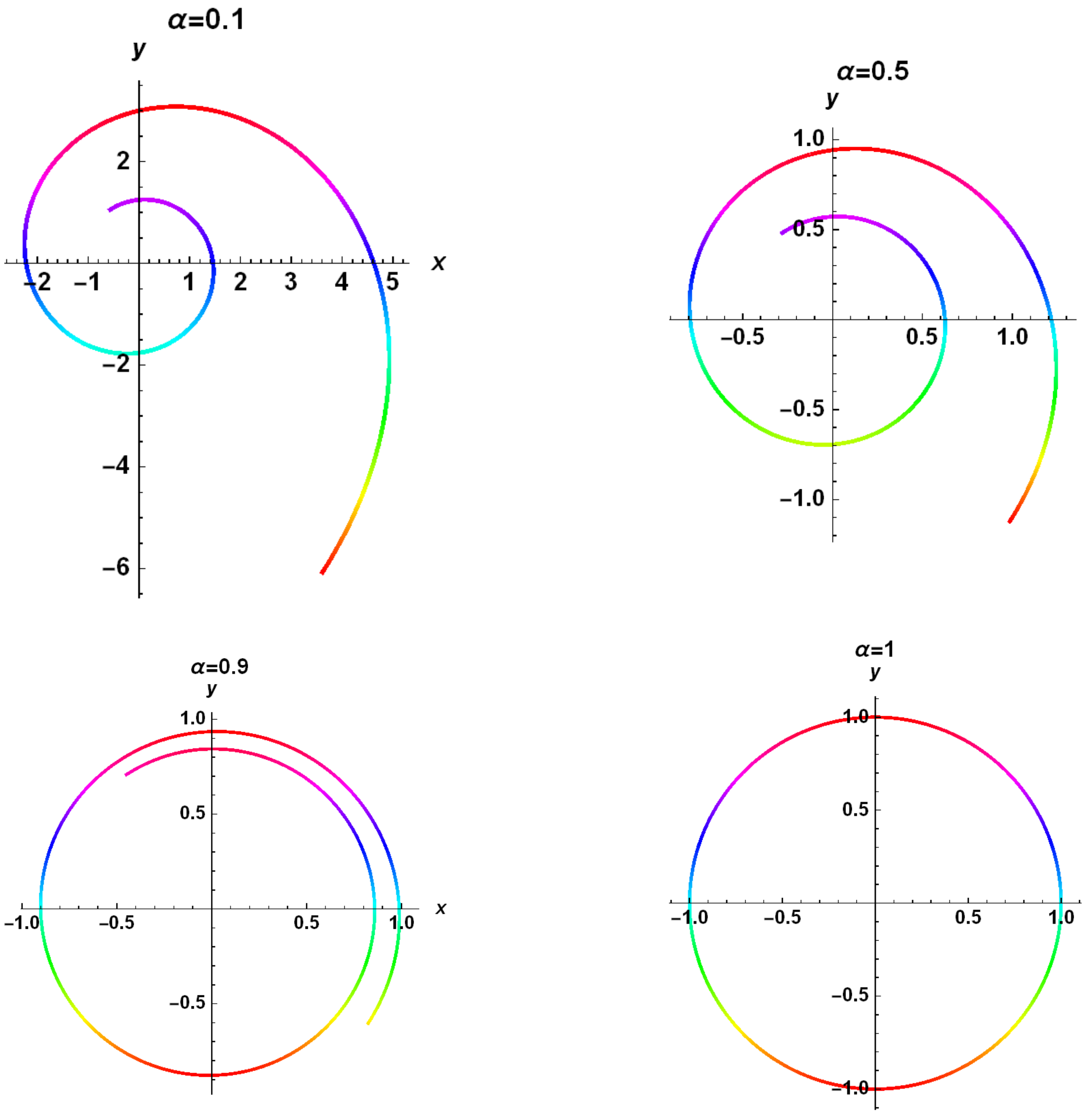


Figure 1. Plane curves with $\kappa_\alpha = 1$ for $\alpha \in \{0.1, 0.5, 0.9, 1\}$.

Example 5.2. Let $\kappa_\alpha = 1 = \tau_\alpha$ identically. (4.4) follows

$$\begin{array}{l} x_j' = \frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} t_j, \\ t_j' = n_j, \\ n_j' = -t_j + b_j \\ b_j = -n_j, \; j = 1, 2, 3. \end{array} \tag{5.1}$$

Put $\mathbf{t} = (t_1, t_2, t_3)$, $\mathbf{n} = (n_1, n_2, n_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ with the intial conditions

$$\mathbf{t}\left(\frac{\pi}{\sqrt{2}}\right) = (1, 0, 0), \; \mathbf{n}\left(\frac{\pi}{\sqrt{2}}\right) = (0, 1, 0), \; \mathbf{b}\left(\frac{\pi}{\sqrt{2}}\right) = (0, 1, 0). \tag{5.2}$$

After we solve (5.1) with (5.2) by considering (3.4) we derive

$$\begin{aligned}\mathbf{t}(s) &= \frac{1}{2}\left(-\cos\left(\sqrt{2}s\right)+1, -\sqrt{2}\sin\left(\sqrt{2}s\right), \cos\left(\sqrt{2}s\right)+1\right),\\ \mathbf{n}(s) &= \frac{1}{\sqrt{2}}\left(\sin\left(\sqrt{2}s\right), -\sqrt{2}\cos\left(\sqrt{2}s\right), -\sin\left(\sqrt{2}s\right)\right),\\ \mathbf{b}(s) &= \frac{1}{2}\left(\cos\left(\sqrt{2}s\right)+1, \sqrt{2}\sin\left(\sqrt{2}s\right), -\cos\left(\sqrt{2}s\right)+1\right),\end{aligned}$$

and

$$\begin{aligned}\mathbf{x}(s) &= \frac{\Gamma(2-\alpha)}{2\alpha}\left(\int s^{\alpha-1}\left[-\cos\left(\sqrt{2}s\right)+1\right]ds,\right.\\ &\left.\int s^{\alpha-1}\left[-\sqrt{2}\sin\left(\sqrt{2}s\right)\right]ds, \int s^{\alpha-1}\left[\cos\left(\sqrt{2}s\right)+1\right]ds\right).\end{aligned}$$

For different values of α, $\mathbf{x}$ can be plotted as in Figure 2.

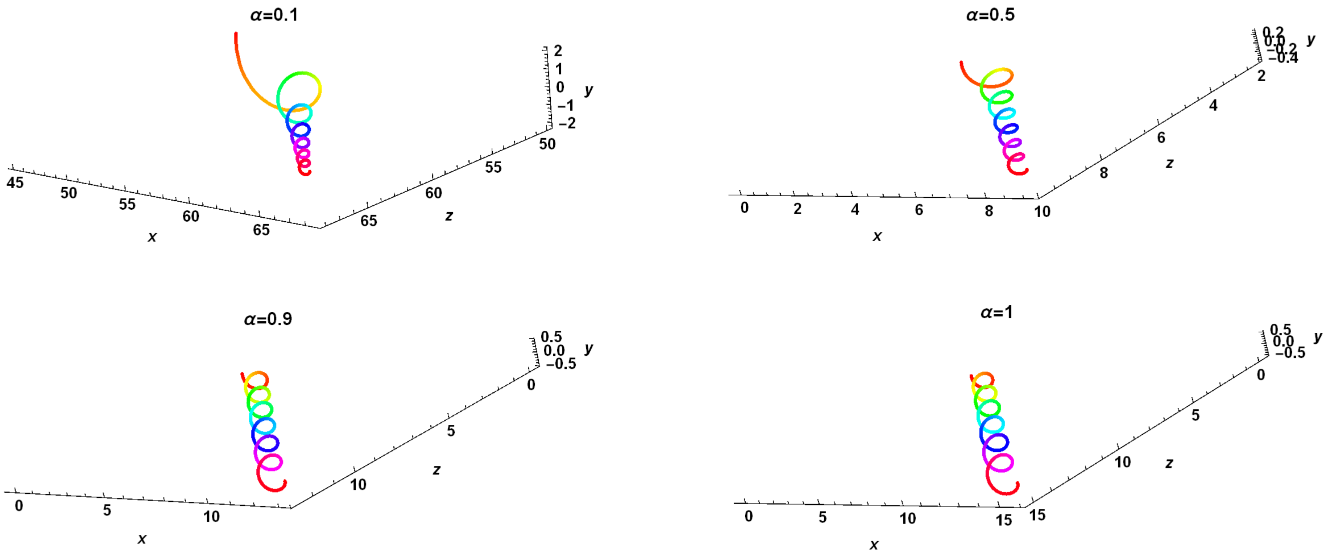


Figure 2. Space curves with $\kappa_\alpha = \tau_\alpha = 1$ for $\alpha \in \{0.1, 0.5, 0.9, 1\}$.

Example 5.2 puts forward a construction method of a space curve $\mathbf{x}$ from its curvature κ_α and torsion τ_α with fractional-order α. In addition, Figure 2 shows the change of the curve $\mathbf{x}$ with $\kappa_\alpha = \tau_\alpha = 1$ about the value of α. As α goes to 1, $\mathbf{x}$ closes to a standard circular helix, conforming to the fact that every space curve having consant curvature and torsion with integer-order is a standard circular helix.

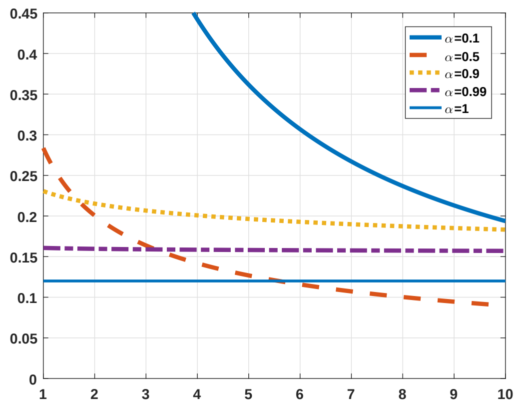
Example 5.3. Let x be a curve in $\mathbb{R}^3$ parameterized by

$$\mathbf{x}(s) = (3\cos\psi(s), 3\sin\psi(s), 4\psi(s)), \tag{5.3}$$

where $\psi(s) = \frac{\Gamma(2-\alpha)s^\alpha}{5\alpha^2}$. By the virtue of (3.4) we conclude from (5.3) that s is the arc-length parameter and

$$\begin{aligned}\mathbf{t}(s) &= \frac{d^\alpha \mathbf{x}(s)}{ds^\alpha} = \frac{1}{5}(-3\sin\psi(s), 3\cos\psi(s), 4),\\ \mathbf{n}(s) &= (-\cos\psi(s), -\sin\psi(s), 0),\\ \mathbf{b}(s) &= \frac{1}{5}(4\sin\psi(s), -4\cos\psi(s), 3),\\ \kappa_\alpha(s) &= \frac{3\Gamma(2-\alpha)s^{\alpha-1}}{25\alpha}, \ \tau_\alpha(s) = \frac{4\Gamma(2-\alpha)s^{\alpha-1}}{25\alpha},\\ \kappa(s) &= \frac{3}{25}, \ \tau(s) = \frac{4}{25}.\end{aligned}$$

For different values of α the graphs of the curvature κ_α and torsion τ_α with fractional-order and the curve $\mathbf{x}$ can be drawn as in Figures 3,4.



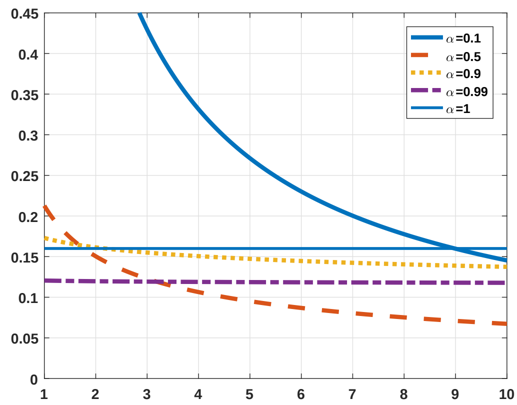


Figure 3. Graphs of $\kappa_\alpha(s) = \frac{3\Gamma(2-\alpha)s^{\alpha-1}}{25\alpha}$ (left-hand side) and $\tau_\alpha(s) = \frac{4\Gamma(2-\alpha)s^{\alpha-1}}{25\alpha}$ (right-hand side) for $\alpha \in \{0.1, 0.5, 0.9, 0.99, 1\}$.

As can be seen in Example 5.3, the part $s^{\alpha-1}$ of the curvature and torsion with fractional-order points to the effect of fractional derivative, intrinsically given by (3.4). The curvature and torsion takes a large value around an initial time and converges to zero for a long period of time, reflecting the memory effect of fractional derivative which is progressively declined for a long period of time.

Furthermore, Figure 4 shows the change of the curve $\mathbf{x}$ with $\tau_\alpha(s) = \frac{4\kappa_\alpha(s)}{3} = \frac{4\Gamma(2-\alpha)s^{\alpha-1}}{25\alpha}$ about the value of α. As α goes to 1, $\mathbf{x}$ closes to a standard circular helix.

6. Remarks and Discussions

By Figures 1,2,4 we can figure out that the curves given in Examples 1,2,3 close to the standard objects as α goes to 1, which implies that our approach conforms to the well-known geometric aspects. Figure 3 indicates the change of the curvature κ_α and torsion τ_α with fractional-order α about the value of α and these converge to zero for $s \to \infty$. It means that this intention of κ_α and τ_α points the memory effect of fractional derivative decreasing for a long period of time. In other words, our results and examples suggest that the property of fractional derivative associates with the differential geometry of plane and space curves based on the equation (3.4)

Furthermore, the approach carrying out in the present study puts forward a novelty from physical point of view, a *geodesic with fractional-order* α parameterized by

$$\mathbf{x}(s) = p + \frac{\Gamma(2-\alpha)s^\alpha}{\alpha^2}\mathbf{v},\ p, \mathbf{v} \in \mathbb{R}^n. \tag{6.1}$$

The following can be immediately checked by (3.1) and (3.4)

$$\nabla_{\mathbf{t}}\mathbf{t} = \frac{d}{ds}\left(\frac{d^\alpha \mathbf{x}}{ds^\alpha}\right) = 0,$$

which conforms with the usual geodesic equation in $\mathbb{R}^n$. Here $\mathbf{t}(s) = \frac{d^\alpha \mathbf{x}}{ds^\alpha}\big|_s$ and ∇ is the Levi-Civita connection in $\mathbb{R}^n$ [39]. Notice that (6.1) is geometrically nothing but a straight line.

7. Conclusions

The main goal of this paper was to make a conribution to the intersecting of two different disciplines such as Differential Geometry and Fractional Calculus. We remark that this contribution was mostly realized to the differential geometric parts of the intersecting. What we presented in our study can be summarized as follows:

We firstly introduced new invariants for a parameterized curve $\mathbf{x}(s)$ in $\mathbb{R}^3$, i.e. the curvature κ_α and torsion τ_α with fractional-order α, $0 < \alpha \leq 1$. Meanwhile, new geometric objects could be defined by using these invariants, for example

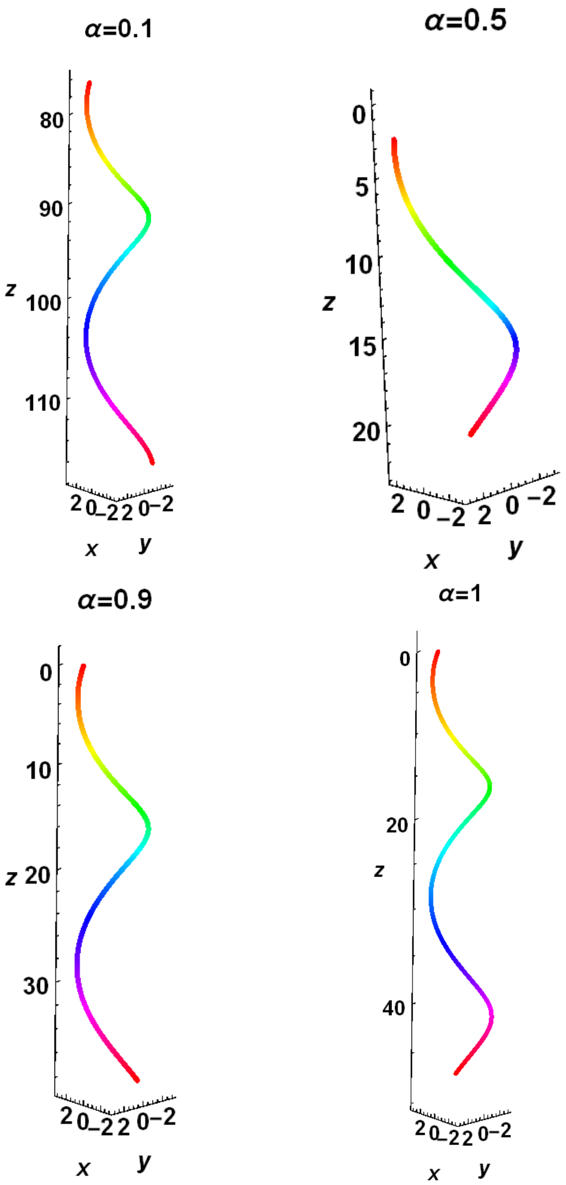


Figure 4. Space curves with $\tau_\alpha(s) = \frac{4\kappa_\alpha(s)}{3} = \frac{4\Gamma(2-\alpha)s^{\alpha-1}}{25\alpha}$ for $\alpha \in \{0.1, 0.5, 0.9, 1\}$.

- a circle with fractional-order α, i.e. a curve with $\kappa_\alpha = const.$ and $\tau_\alpha = 0$,
- a circular helix with fractional-order α, i.e. a curve with $\kappa_\alpha = const.$ and $\tau_\alpha = const.$,
- a slant helix (see [22, 25]) with fractional-order α, i.e. a curve with

$$\frac{(\kappa_\alpha)^2}{\left((\kappa_\alpha)^2 + (\tau_\alpha)^2\right)^{3/2}} \left(\frac{\tau_\alpha}{\kappa_\alpha}\right)' = const.$$

Investigating these objects could be an interesting problem.

We then provided the relations between the invariants with fractional-order α and standard invariants of $\mathbf{x}(s)$. We finally gave several examples illustrating the curves in both $\mathbb{R}^2$ and $\mathbb{R}^3$ with constant curvature and torsion with fractional-order α. Point out that we benefited from the Fundamental Theorems for plane and space curves stated in terms of the new invariants in order to illustrate these.

8. Conflict of Interests

The authors declare that they have no conflict of interests.

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