Communications in Advanced Mathematical Sciences
Vol. III, No. 2, 67-73, 2020
Research Article
e-ISSN: 2651-4001
DOI: 10.33434/cams.684102

Some New Results in Partial Cone b-Metric Space

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Abstract
In this paper, we introduce the concepts of the Ulam-Hyers-Rassias stability and the limit shadowing property of a fixed point problem and the P-property of a mapping in partial cone b-metric space. Also, we give such results by using the mapping which is studied by Fernandez et al. (Filomat **30**(10) (2016)) in partial cone b-metric space and provide some numerical examples to support our results. The results presented here extend and improve some recent results announced in the current literature.

Keywords: Fixed point, Limit shadowing property, P-property, Partial cone b-metric space, Ulam-Hyers-Rassias stability.
2010 AMS: Primary 47H09, Secondary 47H10

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Received: 4 February 2020, **Accepted:** 30 March 2020, **Available online:** 30 June 2020

1. Introduction

Fixed point theory plays an important role in applications of many branches of mathematics. There has been a number of generalizations of metric spaces. One of them is a b-metric space which is introduced by Czerwik [1]. After that a series of articles has been dedicated to the improvement of fixed point theory. In 2011, Hussain and Shah [2] introduced the concept of cone b-metric space and studied some topological properties. At the same year, Sönmez [3] introduced the concept of partial cone metric space and proved some important fixed point theorems in such spaces. In 2016, Fernandez et al. [4] introduced the concept of partial cone b-metric space which is a generalization of cone b-metric space and partial cone metric space. They also established the following fixed point result for asymptotically regular sequences in the setting of partial cone b-metric space.

Theorem 1.1. *(see [4, Theorem 5.1]) Let (X, p_b) be a complete partial cone b-metric space, P be a normal cone with the normal constant K and $T : X \to X$ be a mapping satisfying the inequality*

$$p_b(Tx,Ty) \leq a_1 p_b(x,Tx) + a_2 p_b(y,Ty) + a_3 p_b(x,Ty) + a_4 p_b(y,Tx) + a_5 p_b(x,y) \tag{1.1}$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4, a_5 are non-negative real numbers and satisfy the condition $a_3 + a_4 + a_5 < 1$. If there exists an asymptotically T-regular sequence in X, then T has a unique fixed point.

In this paper, we consider the mapping satisfying (1.1) in partial cone b-metric space. This paper contains four sections. In section 2, we give basic definitions and a detailed overview of the fundamental results. In section 3, we prove the Ulam-Hyers-Rassias stability and the limit shadowing property of the fixed point problem. In section 4, we present the P-property result of the mapping. Our results can be viewed as refinement and generalization of several well-known results in partial cone metric space and cone b-metric space.

2. Preliminaries

Let $(E,\|.\|)$ be a real Banach space. A subset P of E is called a cone if and only if

(1) P is closed, nonempty and $P \neq \{\theta\}$;

(2) $ax+by \in P$ for all $x,y \in P$ and $a,b \geq 0$;

(3) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subseteq E$, we define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y-x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y-x \in intP$ (the interior of P). A cone P is called normal if there is a number $K > 0$ such that for all $x,y \in E$, $\theta \leq x \leq y$ implies that

$$\|x\| \leq K\|y\|. \tag{2.1}$$

The least positive number satisfying (2.1) is called the normal constant of P. It is clear that $K \geq 1$.

Definition 2.1. (see [2]) Let X be a nonempty set, and let P be a cone in a real Banach space E. A vector-valued function $d : X \times X \to P$ is said to be cone b-metric with the constant $s \geq 1$ if the following conditions are satisfied:

(1) $\theta \leq d(x,y)$, for all $x,y \in X$, and $d(x,y) = \theta$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(3) $d(x,y) \leq s[d(x,z)+d(z,y)]$ for all $x,y,z \in X$.

Then the pair (X,d) is called a cone b-metric space.

Definition 2.2. (see [3]) Let X be a nonempty set, and let P be a cone in a real Banach space E. A partial cone metric on X is a function $p : X \times X \to P$ such that, for all $x,y,z \in X$:

(1) $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$;

(2) $\theta \leq p(x,x) \leq p(x,y)$;

(3) $p(x,y) = p(y,x)$;

(4) $p(x,y) \leq p(x,z)+p(z,y)-p(z,z)$.

In this case, the pair (X,p) is called a partial cone metric space.

Definition 2.3. (see [4, Definition 3.1]) Let X be a nonempty set, and let P be a cone in a real Banach space E. A partial cone b-metric on X is a function $p_b : X \times X \to P$ such that, for all $x,y,z \in X$:

(1) $x = y \iff p_b(x,x) = p_b(x,y) = p_b(y,y)$;

(2) $\theta \leq p_b(x,x) \leq p_b(x,y)$;

(3) $p_b(x,y) = p_b(y,x)$;

(4) $p_b(x,y) \leq s[p_b(x,z)+p_b(z,y)]-p_b(z,z)$.

Then the pair (X,p_b) is called a partial cone b-metric space. The number $s \geq 1$ is called the coefficient of (X,p_b).

In partial cone b-metric space (X,p_b), if $x,y \in X$ and $p_b(x,y) = \theta$, then $x = y$, but the converse may not be true. It is clear that every partial cone metric space is a partial cone b-metric space with the coefficient $s = 1$ and every cone b-metric space is a partial cone b-metric space with the same coefficient and zero self distance. However, the converse of these facts does not necessarily hold.

Example 2.4. (see [4]) (i) Let $E = \mathbb{R}^2$, $P = \{(x,y) \in E : x,y \geq 0\}$, $X = [0,\infty)$, $p > 1$ be a constant and $p_b : X \times X \to P$ be defined by

$$p_b(x,y) = ((\max\{x,y\})^p + |x-y|^p, \alpha(\max\{x,y\})^p) + |x-y|^p))$$

for all $x,y \in X$, where $\alpha \geq 0$ is a constant. Then (X,p_b) is a partial cone b-metric space with coefficient $s = 2^p > 1$. But it is not a partial cone metric space.

(ii) Let $E = \mathbb{R}^2$, $P = \{(x,y) \in E : x,y \geq 0\}$, $X = [0,\infty)$, $p > 1$ be a constant and $p_b : X \times X \to P$ be defined by

$$p_b(x,y) = ((\max\{x,y\})^p, \alpha(\max\{x,y\})^p)$$

for all $x,y \in X$, where $\alpha \geq 0$ is a constant. Then (X,p_b) is a partial cone b-metric space which is not a cone b-metric space.

Definition 2.5. (see [4]) Let (X,p_b) be a partial cone b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$ is:

(i) convergent to x and x is called a limit of $\{x_n\}$ if

$$\lim_{n\to\infty} p_b(x_n,x) = \lim_{n\to\infty} p_b(x_n,x_n) = p_b(x,x).$$

(ii) Cauchy sequence if there is $a \in P$ such that for every $\varepsilon > 0$ there is N such that for all $n,m > N$, $\|p_b(x_n,x_m)-a\| < \varepsilon$.

Definition 2.6. (see [4]) A partial cone b-metric space (X, p_b) is said to be complete if every Cauchy sequence in (X, p_b) is convergent in (X, p_b).

Theorem 2.7. *(see [4]) Let (X, p_b) be a partial cone b-metric space and P be a normal cone with a normal constant K. Let $x \in X$ and $\{x_n\}$ be a sequence in X. Then*
(i) $\{x_n\}$ converges to x if and only if $p_b(x_n, x) \to p_b(x, x)$ as $n \to \infty$.
(ii) $p_b(x_n, x_n) \to p_b(x, x)$ as $n \to \infty$ if $p_b(x_n, x) \to p_b(x, x)$ as $n \to \infty$.

Definition 2.8. (see [4, Definition 4.1]) Let (X, p_b) be a partial cone b-metric space. A sequence $\{x_n\}$ in X is said to be asymptotically T-regular if $\lim_{n\to\infty} p_b(x_n, Tx_n) = \theta$.

3. The Ulam-Hyers-Rassias stability and the limit shadowing property results

Speaking of the stability problem of functional equations, we follow a question raised in 1940 by Ulam, concerning approximate homomorphisms of groups (see [5]). Hyers [6] gave the first affirmative partial answer to the question of Ulam for Banach spaces in 1941 and after the fact, this type of stability is called the Ulam-Hyers stability. Hyers's theorem was generalized by Aoki [7] for additive mappings and by Rassias [8] for linear mappings by considering an unbounded Cauchy difference. Rassias [8] attempted to weaken the condition for the bound of the norm of the Cauchy difference as follows:

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

and derived Hyers's theorem. The work of Rassias has influenced a number of mathematicians to develop the notion what is now a days referred to as Ulam-Hyers-Rassias stability of linear mappings. Since then, stability of other functional equations, differential equations, and of various integral equations has been extensively investigated by many mathematicians (see [9, 10, 11, 12, 13]).

Now, we introduce the concept of Ulam-Hyers-Rassias stability of a fixed point problem in partial cone b-metric space.

Definition 3.1. Let (X, p_b) be a partial cone b-metric space and $T : X \to X$ be a mapping. A fixed point problem

$$Tx = x \tag{3.1}$$

has Ulam-Hyers-Rassias stability if and only if there exists the function $\sigma : [0, \infty) \to [0, \infty)$ which is increasing, continuous at 0 and $\sigma(0) = 0$ such that for $\varepsilon > 0$ and $y^* \in X$ which is an ε-solution of the fixed point equation (3.1), that is, y^* satisfied the inequality

$$\|p_b(y^*, Ty^*)\| \leq \sigma(t),$$

there exists a solution $x^* \in X$ of (3.1) such that

$$\|p_b(x^*, y^*)\| \leq c_1.\sigma(t)$$

for some $c_1 > 0$.

Remark 3.2. If the function σ is defined by $\sigma(t) = \varepsilon$ for all $t \geq 0$ where $\varepsilon > 0$, then the fixed point equation (3.1) has Ulam-Hyers stability.

Next, we prove that the fixed point equation (3.1) has the Ulam-Hyers-Rassias stability.

Theorem 3.3. *Let (X, p_b) be a complete partial cone b-metric space, P be a normal cone with the normal constant K and $T : X \to X$ be a mapping satisfying the inequality*

$$p_b(Tx, Ty) \leq a_1 p_b(x, Tx) + a_2 p_b(y, Ty) + a_3 p_b(x, Ty) + a_4 p_b(y, Tx) + a_5 p_b(x, y) \tag{3.2}$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4, a_5 are non-negative real numbers such that the condition $s(a_1 + a_3 s + a_4 + a_5) < 1$ holds. If there exists an asymptotically T-regular sequence in X, then the fixed point problem (3.1) has the Ulam-Hyers-Rassias stability.

Proof. Since $a_3 + a_4 + a_5 < s(a_1 + a_3 s + a_4 + a_5) < 1$, then all hypotheses of Theorem 1.1 are satisfied. Hence, we can say that the mapping T has a unique fixed point $x^* \in X$. Let $\varepsilon > 0$ and $y^* \in X$ be a ε-solution of (3.1), that is,

$$\|p_b(y^*, Ty^*)\| \leq \sigma(t).$$

Now we have

$$\begin{aligned} p_b(x^*,y^*) &= p_b(Tx^*,y^*) \\ &\leq s[p_b(Tx^*,Ty^*)+p_b(Ty^*,y^*)]-p_b(Ty^*,Ty^*) \\ &\leq sp_b(Tx^*,Ty^*)+sp_b(Ty^*,y^*). \end{aligned} \tag{3.3}$$

Also, we obtain

$$\begin{aligned} & sp_b(Tx^*,Ty^*) \\ \leq\ & s[a_1p_b(x^*,Tx^*)+a_2p_b(y^*,Ty^*)+a_3p_b(x^*,Ty^*)+a_4p_b(y^*,Tx^*)+a_5p_b(x^*,y^*)] \\ \leq\ & a_1sp_b(x^*,y^*)+a_2sp_b(y^*,Ty^*)+a_3s[s(p_b(x^*,y^*)+p_b(y^*,Ty^*))-p_b(y^*,y^*)]+a_4sp_b(y^*,x^*)+a_5sp_b(x^*,y^*) \\ \leq\ & a_1sp_b(x^*,y^*)+a_2sp_b(y^*,Ty^*)+a_3s^2p_b(x^*,y^*)+a_3s^2p_b(y^*,Ty^*)+a_4sp_b(y^*,x^*)+a_5sp_b(x^*,y^*). \end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4), we have

$$\left[1-(a_1s+a_3s^2+a_4s+a_5s)\right]p_b(x^*,y^*) \leq \left(a_2s+a_3s^2+s\right)p_b(y^*,Ty^*).$$

Hence, we get

$$\|p_b(x^*,y^*)\| \leq K.\frac{a_2s+a_3s^2+s}{1-s(a_1+a_3s+a_4+a_5)}\|p_b(y^*,Ty^*)\|.$$

Therefore, we obtain

$$\|p_b(x^*,y^*)\| \leq c_1\sigma(t)$$

where

$$c_1 = K.\frac{a_2s+a_3s^2+s}{1-s(a_1+a_3s+a_4+a_5)} > 0.$$

This completes the proof. □

The following example illustrates Theorem 3.3.

Example 3.4. Let (X,p_b) be a complete partial cone b-metric space which is defined as in Example 2.4 (i) such that $p=2$ and $s=4$. Let T be a self mapping of X such that $Tx=\frac{2x}{5}$ for all $x\in X$. Then, the mapping T satisfies the contractive condition (3.2) with $a_1=a_2=a_3=a_4=0$ and $a_5=\frac{1}{5}$. It is clearly seen that 0 is the unique fixed point of T. Assume that $\varepsilon>0$ and $y^*\in X$ is an ε-solution of the fixed point problem of T, that is,

$$\|p_b(y^*,Ty^*)\| \leq \sigma(t).$$

If we take $K=1$, we get

$$\|p_b(0,y^*)\| \leq 20.\sigma(t),$$

and so the fixed point problem (3.1) has the Ulam-Hyers-Rassias stability.

Corollary 3.5. *Under the assumptions of Theorem 3.3, the fixed point problem (3.1) has the Ulam-Hyers stability, that is, for every $y^*\in X$ and $\varepsilon>0$ with $\|p_b(y^*,Ty^*)\|\leq\varepsilon$, there exists a unique $x^*\in X$ such that*

$$Tx^*=x^* \quad \text{and} \quad \|p_b(x^*,y^*)\| \leq c_1\varepsilon$$

for some $c_1>0$.

The following example demonstrates Corollary 3.5.

Example 3.6. Let (X,p_b) be a complete partial cone b-metric space which is defined as in Example 2.4 (ii) such that $p=2$, and let T be a self mapping of X such that $Tx=\frac{x}{4}$ for all $x\in X$. Then, the mapping T satisfies the contractive condition (3.2) with $a_1=a_2=a_3=a_4=0$ and $a_5=\frac{1}{3}$. It is clearly seen that 0 is the unique fixed point of T. If we take $K=1$, we get

$$\|p_b(0,y^*)\| \leq 6.\varepsilon,$$

and so the fixed point problem (3.1) has the Ulam-Hyers stability.

The limit shadowing property of a fixed point problem have evoked much interest to many researchers, for example, Sintunavarat [12], Pilyugin [14].

In 2014, Sintunavarat [12] introduced the limit shadowing property of a fixed point problem in metric spaces.

Definition 3.7. (see [12]) Let (X,d) be a metric space and $T:X\to X$ be a mapping. We say that the fixed point problem of T has the limit shadowing property in X if for any sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} d(x_n,Tx_n)=0$, it follows that there exists $x^*\in X$ such that $\lim_{n\to\infty} d(T^n x^*,x_n)=0$.

Similarly, we define the limit shadowing property of a fixed point problem in partial cone b-metric space.

Definition 3.8. Let (X,p_b) be a partial cone b-metric space and $T:X\to X$ be a mapping. We say that the fixed point problem of T has the limit shadowing property in X if for any sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} p_b(x_n,Tx_n)=\theta$, it follows that there exists $x^*\in X$ such that $\lim_{n\to\infty} p_b(T^n x^*,x_n)=\theta$.

Now, we prove that the fixed point equation (3.1) has the limit shadowing property.

Theorem 3.9. *Let (X,p_b) be a complete partial cone b-metric space, P be a normal cone and $T:X\to X$ be a mapping satisfying (3.2) with $a_3+a_4+a_5<1$. If there exists an asymptotically T-regular sequence in X, then the fixed point problem of T has the limit shadowing property in X.*

Proof. Let $\{x_n\}$ is an asymptotically T-regular sequence in X. Then we say that

$$\lim_{n\to\infty} p_b(x_n,Tx_n)=\theta.$$

Also, from Theorem 1.1, the mapping T has a unique fixed point $x^*\in X$ and the sequence $\{x_n\}$ converges to x^*. Therefore, we can write

$$\lim_{n\to\infty} p_b(x_n,T^n x^*)=\lim_{n\to\infty} p_b(x_n,x^*)=\theta.$$

This completes the proof. □

The following example illustrates Theorem 3.9.

Example 3.10. Let (X,p_b) and T be defined as in Example 3.6. Choose a sequence $\{x_n\}$, $x_n\neq 0$ for any positive integer n, which converges to zero. Then $\{x_n\}$ is an asymptotically T-regular sequence in (X,p_b). We can see that there is $x^*=0\in X$ such that

$$\begin{aligned}\lim_{n\to\infty} p_b(T^n x^*,x_n) &= \lim_{n\to\infty} p_b(0,x_n)=\lim_{n\to\infty}(x_n^2,\alpha x_n^2)\\ &= (0,\alpha 0)\\ &= \theta.\end{aligned}$$

Hence the fixed point problem of T has the limit shadowing property.

4. The P-property result

Rhoades defined the P-property on metric spaces in his works [15], [16] and [17]. Denote, as usual, by $F(T)$ the set of fixed points of the mapping $T:X\to X$. We say that a self-mapping T has the P-property whenever $F(T)=F(T^n)$ for all $n\geq 1$, that is, it has no periodic points. Note that $F(T)\subseteq F(T^n)$ for all $n\geq 1$. It is clear that if T is a mapping which has a fixed point x^*, then x^* is also a fixed point of T^n for all $n\geq 1$. It is well known that the converse is not true. However if a mapping T satisfies $F(T^n)\subseteq F(T)$ for all $n\geq 1$, then it is said to have the P-property.

In 2018, Huang et al. [18] gave a characterization for the P-property in b-metric space.

Theorem 4.1. *(see [18]) Let (X,d) be a b-metric space with coefficient $s\geq 1$. Let $T:X\to X$ be a mapping such that $F(T)\neq\emptyset$ and*

$$d(Tx,T^2x)\leq \lambda d(x,Tx)$$

for all $x\in X$, where $0\leq\lambda<1$ is a constant. Then the mapping T has the P-property.

Now, we generalize Theorem 4.1 to partial cone b-metric space.

Theorem 4.2. *Let (X, p_b) be a partial cone b-metric space, P be a normal cone with the normal constant K and $T : X \to X$ be a mapping such that $F(T) \neq \emptyset$. Then T has the P-property if it is satisfied the following inequality*

$$p_b(Tx, T^2x) \leq \lambda p_b(x, Tx)$$

where $0 \leq \lambda < 1$.

Proof. We always assume that $n > 1$, since the statement for $n = 1$ is trivial. Let $x^* \in F(T^n)$. By the hypotheses, it is clear that

$$\begin{aligned} p_b(x^*, Tx^*) &= p_b(TT^{n-1}x^*, T^2T^{n-1}x^*) \leq \lambda p_b(T^{n-1}x^*, T^n x^*) \\ &= \lambda p_b(TT^{n-2}x^*, T^2T^{n-2}x^*) \\ &\leq \lambda^2 p_b(T^{n-2}x^*, T^{n-1}x^*) \leq \ldots \leq \lambda^n p_b(x^*, Tx^*). \end{aligned}$$

Since P is a normal cone with the normal constant K, then we have

$$\|p_b(x^*, Tx^*)\| \leq K\lambda^n \|p_b(x^*, Tx^*)\| \to 0 \text{ as } n \to \infty.$$

Hence, we get $p_b(x^*, Tx^*) = \theta$, that is, $x^* \in F(T)$. □

Next we prove that the mapping T has the P-property.

Theorem 4.3. *Let (X, p_b) be a complete partial cone b-metric space, P be a normal cone and $T : X \to X$ be a mapping satisfying the inequality (3.2) with $a_1 + a_2 + 2sa_3 + a_4 + a_5 < 1$. Then the mapping T has the P-property.*

Proof. Noting $a_3 + a_4 + a_5 < a_1 + a_2 + 2sa_3 + a_4 + a_5 < 1$, by Theorem 1.1, we get $x^* \in F(T)$. Using (3.2), we obtain

$$\begin{aligned} & p_b(Tx, T^2x) \\ = \; & p_b(Tx, TTx) \\ \leq \; & a_1 p_b(x, Tx) + a_2 p_b(Tx, T^2x) + a_3 p_b(x, T^2x) + a_4 p_b(Tx, Tx) + a_5 p_b(x, Tx) \\ \leq \; & a_1 p_b(x, Tx) + a_2 p_b(Tx, T^2x) + a_3[s(p_b(x, Tx) + p_b(Tx, T^2x)) - p_b(Tx, Tx)] + a_4 p_b(Tx, Tx) + a_5 p_b(x, Tx) \\ \leq \; & a_1 p_b(x, Tx) + a_2 p_b(Tx, T^2x) + a_3 s p_b(x, Tx) + a_3 s p_b(Tx, T^2x) + a_4 p_b(Tx, x) + a_5 p_b(x, Tx). \end{aligned}$$

Hence, we have

$$p_b(Tx, T^2x) \leq \frac{a_1 + sa_3 + a_4 + a_5}{1 - (a_2 + sa_3)} p_b(x, Tx).$$

Therefore, we obtain

$$p_b(Tx, T^2x) \leq \lambda . p_b(x, Tx)$$

where $\lambda = \frac{a_1 + sa_3 + a_4 + a_5}{1 - (a_2 + sa_3)} < 1$. Consequently, by Theorem 4.2, the mapping T has the P-property. □

Finally, we give an example to support Theorem 4.3.

Example 4.4. Let (X, p_b) and T be the same as in Example 3.6. If we take $a_1 = a_2 = a_3 = a_4 = 0$ and $a_5 = \frac{1}{16}$, then we get

$$p_b(Tx, T^2x) = p_b\left(\frac{x}{4}, \frac{x}{16}\right) = \left(\frac{x^2}{16}, \alpha\frac{x^2}{16}\right) = \frac{1}{16} p_b\left(x, \frac{x}{4}\right) = \frac{1}{16} p_b(x, Tx)$$

and so the mapping T has the P-property.

Conclusion

In this paper, based on the class of mappings studied by Fernandez et al. [4], we have proved the Ulam-Hyers-Rassias stability and the limit shadowing property results of a fixed point problem and the P-property of a mapping in partial cone b-metric space. If $P = [0, \infty)$ and $s = 1$ are taken in our results, the similar results are obtained in partial metric space.

Acknowledgement

The authors would like to thank Prof. Metin Başarır for his valuable suggestions to improve the content of the manuscript.

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