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Research Article

Numerical investigation of dynamic Euler-Bernoulli equation via 3-Scale Haar wavelet collocation method

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Abstract

In this study, we analyze the performance of a numerical scheme based on 3-scale Haar wavelets for dynamic Euler-Bernoulli equation, which is a fourth order time dependent partial differential equation. This type of equations governs the behaviour of a vibrating beam and have many applications in elasticity. For its solution, we first rewrite the fourth order time dependent partial differential equation as a system of partial differential equations by introducing a new variable, and then use finite difference approximations to discretize in time, as well as 3-scale Haar wavelets to discretize in space. By doing so, we obtain a system of algebraic equations whose solution gives wavelet coefficients for constructing the numerical solution of the partial differential equation. To test the accuracy and reliability of the numerical scheme based on 3-scale Haar wavelets, we apply it to five test problems including variable and constant coefficient, as well as homogeneous and non-homogeneous partial differential equations. The obtained results are compared wherever possible with those from previous studies. Numerical results are tabulated and depicted graphically. In the applications of the proposed method, we achieve high accuracy even with small number of collocation points.

Mathematics Subject Classification (2020). 65M70, 65T99

Keywords. 3-Scale Haar wavelets, vibrating beam, dynamic Euler-Bernoulli equation

1. Introduction

The fourth-order problem considered in this paper is

$$\mu(x)\frac{\partial^2 u}{\partial t^2} + \mathrm{EI}(x)\frac{\partial^4 u}{\partial x^4} = F(x,t), \quad a \le x \le b, \quad 0 \le t \le T, \tag{1.1}$$

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subject to the initial conditions

$$u(x,0) = \xi(x),$$
$$u_t(x,0) = \eta(x), \quad a \le x \le b,$$

and the boundary conditions of the form

$$u(a,t) = f_1(t),\ u(b,t) = f_2(t),$$
$$u_{xx}(a,t) = f_3(t),\ u_{xx}(b,t) = f_4(t), \quad 0 \le t \le T.$$

Such problems occur in the study of the transverse displacements of a flexible beam hinged at both ends. Here $u = u(x,t)$ is the transverse displacement of the beam, t and x are time and spatial variables, $\mu(x) > 0$ is the density of the beam, $\mathrm{EI}(x) > 0$ is the beam bending stiffness and $F(x,t)$ is dynamic driving force per unit mass. Such an equation is also called dynamic Euler-Bernoulli equation, and its solution is important in many applications such as control of large flexible space structures or the development of robotics designs [3, 28, 41, 50].

The analytic solutions of variable coefficient nonhomogeneous Euler–Bernoulli equation are obtained by Wazwaz [52] using the Adomian decomposition method. Some exact solutions of variable coefficient homogeneous and nonhomogeneous Euler–Bernoulli equation are obtained by Adomian method in [14]. Analytical solutions of partial differential equations are very useful. However, it is not always possible to obtain the analytical solutions or it is possible only for limited initial and boundary conditions. So it is crucial to develop efficient numerical methods. For obtaining numerical solutions of Eq. (1.1), finite difference methods are employed in [1, 7–13, 20, 25, 47, 51]. A fully Sinc-Galerkin method is used in [49] by Smith et al. for solving fourth-order partial differential equations. A three level scheme based on parametric quintic spline is proposed by Aziz et al. [2] for the solution of fourth-order parabolic partial differential equations with constant coefficients. Khan et al. used sextic splines for solving a fourth-order parabolic partial differential equation in [26].

Caglar and Caglar [4] have developed a fifth degree B-spline method to obtain the numerical solution of constant coefficient fourth-order parabolic partial differential equations. Free vibration of an Euler–Bernoulli beam is obtained by Liu and Gurram [32] using He's variational iteration method. For variable coefficient fourth order parabolic partial differential equations a new three level implicit method based on sextic spline is proposed by Rashidinia and Mohammadi [46]. Mittal and Jain [36] used cubic and quintic B-spline method with redefined basis functions for obtaining numerical solutions of fourth-order parabolic partial differential equations with constant coefficients. Recently, Mohammadi [41] developed a numerical method based on sextic B-splines to solve the fourth-order time dependent partial differential equations subjected to fixed and cantilever boundary conditions.

Due to attractiveness of Haar wavelets for their simplicity, accuracy, computational cost, and so on, in recent years they have got much attention in numerical solutions of differential equations. A brief review of the literature can be given as follows. Chen and Hsiao[5] used Haar wavelet method for solving lumped and distributed parameter systems. In [6], they also discussed an optimal control problem. Hsiao and Wang [16, 17] used Haar wavelets for solving singular bilinear and nonlinear systems and [18] investigated nonlinear stiff systems. Hsiao [15] showed that the Haar wavelet approach is also effective for solving variational problems. Lepik applied this method to some well known problems [29–31]. Zhi Shi et al. [48] applied Haar wavelets to solve 2D and 3D Poisson equations and biharmonic equations.

Jiwari [21] used a hybrid numerical scheme based on implicit Euler method, quasilinearization and uniform Haar wavelets for the numerical solutions of Burgers' equation. Kaur et al. [24] solved Lane-Emden equations arising in astrophysics with Haar

Wavelets. Pandit et al. [45] solved second-order hyperbolic telegraph type equations by Haar wavelets. Majak et al. [33–35] studied functionally graded material (FGM) beams by means of Haar wavelet discretization method and convergence of Haar wavelet method. An efficient numerical scheme based on uniform Haar wavelets and the quasilinearization process is proposed for the numerical simulation of time dependent nonlinear Burgers' equation by Jiwari [22].

Oruç et al. [42–44] solved modified Burgers' equation, coupled Schrödinger-KdV equations and regularized long wave equation with the help of a Haar wavelet based method. Vibration analysis of nanobeams is investigated by Haar wavelets in [27]. A new type of solutions was obtained for the MHD Falkner–Skan boundary layer flow problem using the Haar wavelet quasilinearization approach via Lie symmetric analysis by Jiwari et. al. [23]. Mittal and Pandit [38] used Haar wavelet operational matrix along with quasi-linearization to detect the spin flow of fractional Bloch equations. Mittal and Pandit [40] developed a novel algorithm based on Scale-3 Haar wavelets and quasilinearization for numerical solution of a dynamical system of ordinary differential equations. Recently, Scale-3 Haar wavelet-based algorithm has been extended to find numerical approximations of second order initial and boundary value problems by Mittal and Pandit [39]. Most of the papers mentioned above are based on classical Haar wavelets (2-scale Haar wavelets).

In this study our aim is to analyze the performance of the 3-scale Haar wavelet collocation method (HWCM), recently introduced by Mittal and Pandit in their paper [37], for fourth order partial differential equations with variable and constant coefficients. As far as we know, the 3-scale Haar wavelets have not been employed to solve high order partial differential equations such as Euler-Bernoulli problems, which motivates us for conducting this study. This paper is organized as follows. In Section 2, 3-scale Haar wavelets and their integrals are introduced. In Section 3, a method based on discretization of time and space variables is described. Numerical results and discussion are given in Section 4. Finally, we summarize our findings in Section 5.

2. 3-Scale Haar wavelets and their integrals

The 3-scale Haar wavelets are constructed from two wavelet functions, namely symmetric and antisymmetric wavelet functions. This is the main difference with the 2-scale Haar wavelets, which employ only one wavelet function. The 3-scale Haar wavelets have advantages over the 2-scale ones: they converge rapidly, they can be represented by sparse matrices, in numerical applications solutions can be found at any point in the range, and they can easily detect singularity and discontinuity [37].

Using the orthogonality properties of 3-scale Haar wavelets, one can express any square integrable function $f(x)$ on the interval $[0,1)$ as an infinite series in the following form [37, 39]:

$$f(x) \approx c_1\phi_1(x) + \sum_{\text{even index}\, i,\, i\geq 2}^{\infty} c_i\psi_i^{(1)}(x) + \sum_{\text{odd index}\, i,\, i\geq 3}^{\infty} c_i\psi_i^{(2)}(x). \tag{2.1}$$

Herein, ϕ_1, $\psi_i^{(1)}$and $\psi_i^{(2)}$ are given by

$$\phi_1(x) = \begin{cases} 1 & a \leq x \leq b, \\ 0 & \text{elsewhere}, \end{cases} \tag{2.2}$$

$$\psi_i^{(1)}(x) = \frac{1}{\sqrt{2}} \begin{cases} -1 & \alpha(i) \leq x < \beta(i), \\ 2 & \beta(i) \leq x < \gamma(i), \\ -1 & \gamma(i) \leq x < \delta(i), \end{cases} \tag{2.3}$$

$$\psi_i^{(2)}(x) = \sqrt{\frac{3}{2}} \begin{cases} 1 & \alpha(i) \le x < \beta(i), \\ 0 & \beta(i) \le x < \gamma(i), \\ -1 & \gamma(i) \le x < \delta(i), \end{cases} \tag{2.4}$$

and

$$\alpha(i) = a + (b-a)\frac{k}{m},$$

$$\beta(i) = a + (b-a)\frac{k+1/3}{m},$$

$$\gamma(i) = a + (b-a)\frac{k+2/3}{m},$$

$$\delta(i) = a + (b-a)\frac{k+1}{m},$$

where m is defined as 3^j $(j = 0, 1, ...)$, and integer $k = 0, 1, .., m-1$ is the translation parameter. The index i in $\alpha(i)$, $\beta(i)$, $\gamma(i)$ and $\delta(i)$ shows the relation between wavelet level m and translation parameter k. If $i = 1$, then we get scaling function $\phi_1(x)$ which is defined in (2.2) and shown in Fig. 1 for $[a, b] = [0, 1]$. In case of $i > 1$, the index i is calculated according to formulae $i = m + 2k$ or $i = m + 2k + 1$. If i is even then consider $\psi_i^{(1)}$, if i is odd then consider $\psi_i^{(2)}$. In Figs. 2 and 3, first wavelets $\psi_i^{(1)}$ and $\psi_i^{(2)}$ are plotted for $[a, b] = [0, 1]$.

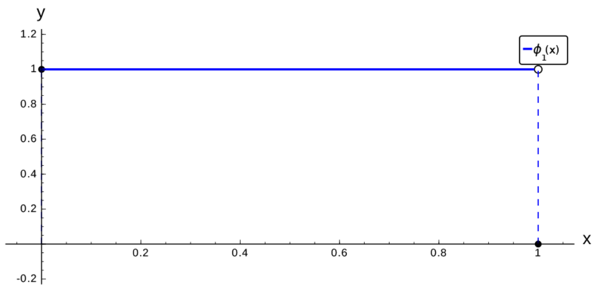

Figure 1. 3-scale Haar wavelet scaling function $\phi_1(x)$

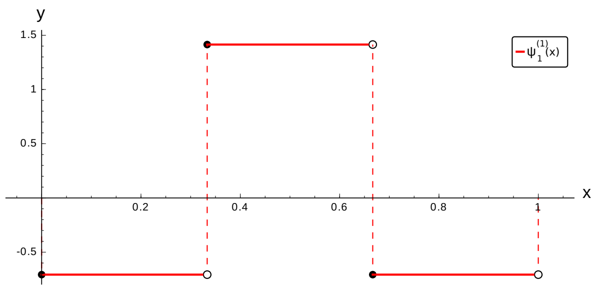

Figure 2. First symmetric wavelet $\psi_1^{(1)}(x)$

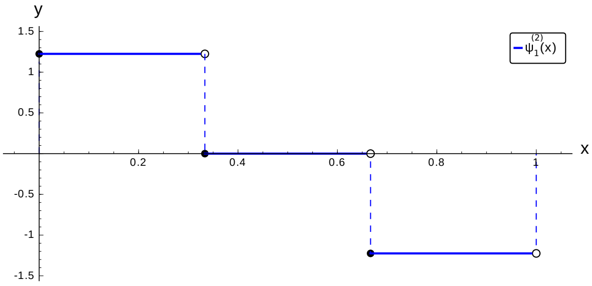

Figure 3. First anti-symmetric wavelet $\psi_1^{(2)}(x)$

Eq. (2.1) is an infinite series. We truncate this series to 3-scale Haar wavelets as [37]:

$$f(x) \approx c_1\phi_1(x) + \sum_{\text{even index } i,\, i\geq 2}^{3m} c_i\psi_i^{(1)}(x) + \sum_{\text{odd index } i,\, i\geq 3}^{3m} c_i\psi_i^{(2)}(x) = \boldsymbol{c}^T H_{3m}.$$

where $\boldsymbol{c}^T = [c_1, ..., c_{3m}]$ and $H_{3m} = [\phi_1(x), \psi_2^{(1)}(x), \psi_3^{(2)}(x), ..., \psi_{3m-1}^{(1)}(x), \psi_{3m}^{(2)}(x)]^T$ are in size of $1 \times 3m$.

In the solution process of a differential equation of any order, we need to integrate 3-scale Haar wavelets, that is we employ the integrals

$$\phi_{1,1}(x) = \int_0^x \phi_1(t)dt = \begin{cases} x & [a,b), \\ 0 & elsewhere, \end{cases}$$

$$\psi_{i,1}^{(1)}(x) = \int_0^x \psi_i^{(1)}(t)dt = \frac{1}{\sqrt{2}} \begin{cases} \alpha(i) - x & \alpha(i) \leq x < \beta(i), \\ 2x - 3\beta(i) + \alpha(i) & \beta(i) \leq x < \gamma(i), \\ \alpha(i) + 3\gamma(i) - 3\beta(i) - x & \gamma(i) \leq x < \delta(i), \end{cases}$$

$$\psi_{i,1}^{(2)}(x) = \int_0^x \psi_i^{(2)}(t)dt = \sqrt{\frac{3}{2}} \begin{cases} x - \alpha(i) & \alpha(i) \leq x < \beta(i), \\ \beta(i) - \alpha(i) & \beta(i) \leq x < \gamma(i), \\ \gamma(i) + \beta(i) - \alpha(i) - x & \gamma(i) \leq x < \delta(i). \end{cases}$$

Moreover, we introduce

$$\phi_{1,n+1}(x) = \int_0^x \phi_{1,n}(t)dt, \quad \psi_{1,n+1}^{(1)} = \int_0^x \psi_{1,n}^{(1)}(t)dt, \quad \psi_{1,n+1}^{(2)} = \int_0^x \psi_{1,n}^{(2)}(t)dt$$

which can explicitly be written as

$$\phi_{1,n+1}(x) = \begin{cases} \frac{x^{n+1}}{(n+1)!} & [a,b), \\ 0 & elsewhere, \end{cases}$$

$$\psi_{i,n+1}^{(1)}(x) = \frac{1}{\sqrt{2}} \begin{cases} \frac{-(x-\alpha(i))^{n+1}}{(n+1)!} & \alpha(i) \leq x < \beta(i), \\ \frac{3(x-\beta(i))^{n+1} - (x-\alpha(i))^{n+1}}{(n+1)!} & \beta(i) \leq x < \gamma(i), \\ \frac{3(x-\beta(i))^{n+1} - 3(x-\gamma(i))^{n+1} - (x-\alpha(i))^{n+1}}{(n+1)!} & \gamma(i) \leq x < \delta(i), \\ \frac{3(x-\beta(i))^{n+1} - 3(x-\gamma(i))^{n+1} - (x-\alpha(i))^{n+1} + (x-\delta(i))^{n+1}}{(n+1)!} & \delta(i) \leq x < 1, \end{cases}$$

$$\psi_{i,n+1}^{(2)}(x)=\sqrt{\frac{3}{2}}\begin{cases}\frac{(x-\alpha(i))^{n+1}}{(n+1)!} & \alpha(i)\le x<\beta(i),\\ \frac{(x-\alpha(i))^{n+1}-(x-\beta(i))^{n+1}}{(n+1)!} & \beta(i)\le x<\gamma(i),\\ \frac{(x-\alpha(i))^{n+1}-(x-\beta(i))^{n+1}-(x-\gamma(i))^{n+1}}{(n+1)!} & \gamma(i)\le x<\delta(i),\\ \frac{(x-\alpha(i))^{n+1}-(x-\beta(i))^{n+1}-(x-\gamma(i))^{n+1}+(x-\delta(i))^{n+1}}{(n+1)!} & \delta(i)\le x<1.\end{cases}$$

3. Discretization scheme for fourth order partial differential equations

To solve Eq. (1.1) we introduce a new variable, namely

$$v=\frac{\partial u}{\partial t}.$$

Now Eq. (1.1) can be rewritten as the system of partial differential equations that is first order in time given below.

$$\begin{aligned} u_t-v&=0,\\ \mu(x)v_t+\mathrm{EI}(x)u_{xxxx}&=F(x,t). \end{aligned}\tag{3.1}$$

We describe the discretization process of the equations above in the subsequent sections.

3.1. Time discretization

We use explicit finite difference schemes for time derivatives, as well as the time average for v and u_{xxxx} in Eq. (3.1). By doing so, we get

$$\begin{aligned} \frac{u^{j+1}-u^j}{\Delta t}-\frac{v^{j+1}+v^j}{2}&=0,\\ \mu(x)\frac{v^{j+1}-v^j}{\Delta t}+\mathrm{EI}(x)\frac{u_{xxxx}^{j+1}+u_{xxxx}^j}{2}&=F(x,t^{j+1}). \end{aligned}$$

The equations above can be rearranged as

$$\begin{aligned} u^{j+1}-\frac{\Delta t}{2}v^{j+1}&=u^j+\frac{\Delta t}{2}v^j,\\ \mu(x)v^{j+1}+\frac{\Delta t\mathrm{EI}(x)}{2}u_{xxxx}^{j+1}&=\mu(x)v^j-\frac{\Delta t\cdot\mathrm{EI}(x)}{2}u_{xxxx}^j+\Delta tF(x,t^{j+1}), \end{aligned}\tag{3.2}$$

with initial conditions

$$\begin{aligned} &u^0(x)=\xi(x),\\ &v^0(x)=\eta(x),\quad a\le x\le b \end{aligned}\tag{3.3}$$

and with the boundary conditions

$$\begin{aligned} &u^{j+1}(a)=f_1(t^{j+1}),\ u^{j+1}(b)=f_2(t^{j+1}),\\ &u_{xx}^{j+1}(a)=f_3(t^{j+1}),\ u_{xx}^{j+1}(b)=f_4(t^{j+1}), \end{aligned}\tag{3.4}$$

where u^{j+1} and v^{j+1} are the solutions of Eq. (3.2) at the $(j+1)$th time step and $t^{j+1}=\Delta t(j+1)$, $j=0,1,...,N-1$, $\Delta t\cdot N=T$.

3.2. Space discretization by Haar wavelets

Since Haar wavelets are generally defined for $[0,1]$. We have to transform the domain into unit interval. By introducing $y=(x-a)/L$, $L=b-a$, the interval $a\leq x\leq b$ can be transformed into the unit interval $0\leq y\leq 1$. Using this transformation, we can reduce a problem defined on $[a,b]$ to a problem defined on $[0,1]$. Hence, without loss of generality, the PDE we have at hand is defined over $[0,1]$ in space.

For the description of space discretization, we introduce notations

$$\sum_{i=1}^{3m} c_i h_i(x) := c_1\phi_1(x) + \sum_{\text{even index}\, i,\, i\geq 2}^{3m} c_i\psi_i^{(1)}(x) + \sum_{\text{odd index}\, i,\, i\geq 3}^{3m} c_i\psi_i^{(2)}(x)$$

$$\sum_{i=1}^{3m} c_i p_{i,j}(x) := c_1\phi_{1,j}(x) + \sum_{\text{even index}\, i,\, i\geq 2}^{3m} c_i\psi_{i,j}^{(1)}(x) + \sum_{\text{odd index}\, i,\, i\geq 3}^{3m} c_i\psi_{i,j}^{(2)}(x)$$

for $j=1,2,3,4$. Now we expand $u_{xxxx}^{j+1}(x)$ term in (3.2) into Haar wavelets, that is

$$u_{xxxx}^{j+1}(x) = \sum_{i=1}^{3m} c_i h_i(x). \tag{3.5}$$

By integrating the equation above from 0 to x, we get

$$u_{xxx}^{j+1}(x) = u_{xxx}^{j+1}(0) + \sum_{i=1}^{3m} c_i p_{i,1}(x) \tag{3.6}$$

We do not know the value of $u_{xxx}^{j+1}(0)$ term in Eq. (3.6), but we can calculate it by integrating Eq. (3.6) from 0 to 1 and using boundary conditions from Eq. (3.4) as follows:

$$u_{xxx}^{j+1}(0) = f_4(t^{j+1}) - f_3(t^{j+1}) - \sum_{i=1}^{3m} c_i p_{i,2}(1).$$

Now by integrating Eq. (3.6) from 0 to x we obtain the second derivative $u_{xx}^{j+1}(x)$ as

$$u_{xx}^{j+1}(x) = \sum_{i=1}^{3m} c_i p_{i,2}(x) + f_3(t^{j+1}) + \left[f_4(t^{j+1}) - f_3(t^{j+1})\right] x - x\sum_{i=1}^{3m} c_i p_{i,2}(1). \tag{3.7}$$

By integrating Eq. (3.7) once again from 0 to x, we deduce

$$\begin{aligned} u_x^{j+1}(x) - u_x^{j+1}(0) = &\sum_{i=1}^{3m} c_i p_{i,3}(x) + x f_3(t^{j+1}) \\ &+ \left[f_4(t^{j+1}) - f_3(t^{j+1})\right]\frac{x^2}{2} - \frac{x^2}{2}\sum_{i=1}^{3m} c_i p_{i,2}(1), \end{aligned} \tag{3.8}$$

which we integrate again from 0 to 1 to obtain

$$\begin{aligned} u^{j+1}(1) - u^{j+1}(0) - u_x^{j+1}(0) = &\sum_{i=1}^{3m} c_i p_{i,4}(1) + \frac{1}{2} f_3(t^{j+1}) \\ &+ \left[f_4(t^{j+1}) - f_3(t^{j+1})\right]\frac{1}{6} - \frac{1}{6}\sum_{i=1}^{3m} c_i p_{i,2}(1). \end{aligned} \tag{3.9}$$

By exploiting the boundary conditions $u^{j+1}(1) = f_2(t^{j+1})$ and $u^{j+1}(0) = f_1(t^{j+1})$ in the equation above, we retrieve

$$\begin{aligned}u_x^{j+1}(0) =& f_2(t^{j+1}) - f_1(t^{j+1}) - \sum_{i=1}^{3m} c_i p_{i,4}(1)\\ &- \frac{1}{2} f_3(t^{j+1}) - \left[f_4(t^{j+1}) - f_3(t^{j+1})\right]\frac{1}{6} + \frac{1}{6}\sum_{i=1}^{3m} c_i p_{i,2}(1).\end{aligned}$$

Plugging the right-hand side of the equation above for $u_x^{j+1}(0)$ in Eq.(3.8), we have

$$\begin{aligned}u_x^{j+1}(x) =& \sum_{i=1}^{3m} c_i p_{i,3}(x) + f_2(t^{j+1}) - f_1(t^{j+1}) - \frac{1}{3} f_3(t^{j+1})\\ &- \frac{1}{6} f_4(t^{j+1}) - \sum_{i=1}^{3m} c_i \left[p_{i,4}(1) - \frac{1}{6} p_{i,2}(1)\right] &(3.10)\\ &+ f_3(t^{j+1})x + \frac{x^2}{2}\left[f_4(t^{j+1}) - f_3(t^{j+1})\right] - \frac{x^2}{2}\sum_{i=1}^{3m} c_i p_{i,2}(1), &(3.11)\end{aligned}$$

which in turn yields

$$\begin{aligned}u^{j+1}(x) =& \sum_{i=1}^{3m} c_i p_{i,4}(x) + f_1(t^{j+1}) + \left[f_2(t^{j+1}) - f_1(t^{j+1}) - \frac{1}{3} f_3(t^{j+1}) - \frac{1}{6} f_4(t^{j+1})\right] x\\ &- x\sum_{i=1}^{3m} c_i \left[p_{i,4}(1) - \frac{1}{6} p_{i,2}(1)\right]\\ &+ f_3(t^{j+1})\frac{x^2}{2} + \frac{x^3}{6}\left[f_4(t^{j+1}) - f_3(t^{j+1})\right] - \frac{x^3}{6}\sum_{i=1}^{3m} c_i p_{i,2}(1). &(3.12)\end{aligned}$$

Additionally we express $v^{j+1}(x)$ in terms of Haar wavelets in the form

$$v^{j+1}(x) = \sum_{i=1}^{3m} d_i h_i(x). \tag{3.13}$$

By plugging Eqs. (3.5), (3.12) and (3.13) into Eq. (3.2) and discretizing at collocation points $x_l = \frac{l-0.5}{3m}$, $l = 1, 2, ..., 3m$ yields a system of linear equations whose solution gives the wavelet coefficients c_i and d_i. Then by plugging these wavelet coefficients into Eqs. (3.12) and (3.13) we can obtain the numerical solutions $u^{j+1}(x)$ and $v^{j+1}(x)$.

3.3. Convergence analysis of Haar wavelets

Let

$$u(x) = c_1\phi_1(x) + \sum_{\text{even index } i,\, i\geq 2}^{\infty} c_i\psi_i^{(1)}(x) + \sum_{\text{odd index } i,\, i\geq 3}^{\infty} c_i\psi_i^{(2)}(x)$$

and

$$u_{3m}(x) = c_1\phi_1(x) + \sum_{\text{even index } i,\, i\geq 2}^{3m} c_i\psi_i^{(1)}(x) + \sum_{\text{odd index } i,\, i\geq 3}^{3m} c_i\psi_i^{(2)}(x)$$

be exact and numerical solutions of Eq. (1.1) with $a = 0$ and $b = 1$. Furthermore, $E_J = u(x) - u_{3m}(x)$ with $J = 3m$ and $\left\|u(x)\right\| = \left(\int_0^1 |u(x)|^2 dx\right)^{1/2}$.

Theorem 3.1. [37] *Let the exact solution $u(x)$ be square integrable on $[0,1]$ with bounded derivatives on $(0,1)$. Then the error E_J satisfies*

$$\|E_J\| \le \frac{M}{\sqrt{24}}\frac{1}{3^J}$$

for some constant M independent of J.

Proof. See [37]. □

Theorem 3.1 implies that the error bound is inverse proportional to the level of resolution of scale-3 Haar wavelets. Therefore the error decreases as we increase J.

4. Numerical examples

Numerical computations have been done with python programming language and graphical outputs were generated by Matplotlib package [19].

In problem 1, we calculate the maximal absolute relative errors which are defined as follows:

$$E = \max_{i=1,\dots,3m}\left|\frac{u_i^{\text{exact}} - u_i^{\text{num}}}{u_i^{\text{exact}}}\right|.$$

In problems 2, 3, 4 and 5, for the sake of comparison with earlier studies, we calculate the absolute errors $|u(x) - u^{\text{num}}(x)|$ at the points $x = 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5$, where $u(x)$ and $u^{\text{num}}(x)$ denote the exact and numerical solutions at x. Here we should note that, u_i^{exact} and u_i^{num} denote exact and numerical solutions at collocation points x_i at a certain final time t. Since in the solution process we took the collocation points as $x_i = \frac{i-0.5}{3m}$, $i = 1,2,\dots,3m$, for calculating numerical results at the points $x = 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5$ we have used interpolation techniques.

Also for every problem, at the bottom of the tables, we provide the error norm L_∞ which is defined by

$$L_\infty(u,.) = \max_i\left|u_i^{\text{exact}} - u_i^{\text{num}}\right|,\ i = 1,2,\dots,3m.$$

Convergence rates are calculated according to the formula

$$\text{Rate} = \frac{\log\left(\frac{L_\infty(u,3\Delta x)}{L_\infty(u,\Delta x)}\right)}{\log\left(\frac{3\Delta x}{\Delta x}\right)} \tag{4.1}$$

where $\Delta x = \frac{1}{3m}$ is the step size of spatial variable x.

4.1. Problem 1

We consider

$$120x\frac{\partial^2 u}{\partial t^2} + \left(120 + x^5\right)\frac{\partial^4 u}{\partial x^4} = 0$$

subject to the initial conditions

$$u(x,0) = 0,\quad u_t(x,0) = 1 + \frac{x^5}{120},\quad \frac{1}{2} \le x \le 1$$

and with the boundary conditions at $x = 1/2$ and $x = 1$ of the form

$$u(\frac{1}{2},t) = \frac{3841}{3840}\sin t,\quad u(1,t) = \frac{121}{120}\sin t,$$
$$u_{xx}(\frac{1}{2},t) = \frac{1}{48}\sin t,\quad u_{xx}(1,t) = \frac{1}{6}\sin t,\quad t \ge 0.$$

This equation is also studied by [46], [1] and [25]. The exact solution of this problem is

$$u(x,t) = \left(1 + \frac{x^5}{120}\right)\sin t.$$

In Table 1, to see convergence in time variable we set $3m = 27$ and compute the errors at $t = 0.01$ for decreasing values of Δt. From Table 1, it is obvious that as the values of Δt are diminished, the error also decreases. Also to see convergence in space variable we fix $\Delta t = 0.00025$ and compute the errors at $t = 0.01$ for increasing values of collocation points in Table 2. It is clearly seen from Table 2 that the errors get smaller by increasing the number of collocation points. Using various values of Δt and $t = 0.01$ we compared the maximum absolute relative errors of the present method with the results from existing methods in the literature in Table 3. We choose the number of collocation points as $3m = 9$ for the present method for comparison. Table 3 shows that the obtained results from the present method, are more accurate in comparison to the sextic spline method [46], A.D.I methods [1] and difference scheme method [25] for this problem. Numerical and exact solutions are plotted for $3m = 9$, $\Delta t = 0.0025$ at $t = 1$ in Fig. 4.

Table 1. Maximum absolute relative errors for different values of Δt and $3m = 27$ at $t = 0.01$ for Problem 1

	Δt	E
	0.001	4.5780e-09
	0.0005	1.2246e-09
$3m = 27$	0.00025	2.8163e-10
	0.000125	6.7723e-11
	6.25e-05	1.9763e-11
	3.125e-05	3.5833e-13

Table 2. Maximum absolute relative errors for different values of $3m$ and $\Delta t = 0.00025$ at $t = 0.01$ for Problem 1

	$3m$	E
	3	3.4277e-09
	9	8.8387e-10
$\Delta t = 0.00025$	27	2.8162e-10
	81	9.4109e-11
	243	3.1088e-11
	729	1.0436e-11

Table 3. Maximum absolute relative errors at $t = 0.01$ in Problem 1

		Methods			
		HWCM	Rashidinia and Mohammadi [46]	Andrade and Mckee [1]	Khaliq and Twizell [25]
Parameters		$3m = 9$	$h = 0.05$	$h = 0.05$	$h = 0.05$
	$\Delta t = 0.000625$	5.8883e-009	3.51e-08	4.10e-07	3.30e-07
E	$\Delta t = 0.00025$	8.8387e-010	9.97e-08	7.20e-07	3.30e-07
	$\Delta t = 0.000125$	2.2098e-010	5.33e-08	1.90e-06	3.30e-07

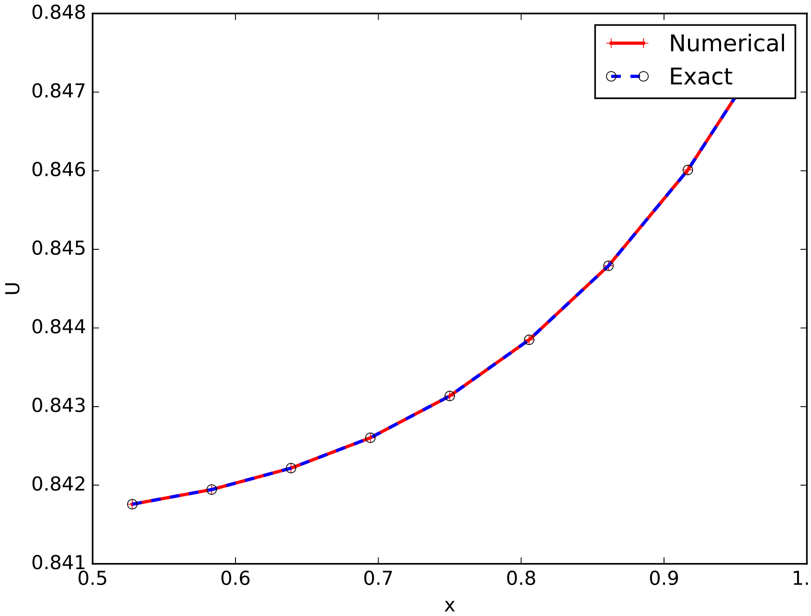

Figure 4. Exact solution versus numerical solution for $3m = 9$, $\Delta t = 0.0025$ at $t = 1$ in Problem 1

4.2. Problem 2

We consider

$$\sin x \frac{\partial^2 u}{\partial t^2} + (x - \sin x)\frac{\partial^4 u}{\partial x^4} = 0$$

subject to the initial conditions

$$u(x,0) = x - \sin x, \quad u_t(x,0) = -(x - \sin x), \quad 0 \leq x \leq 1$$

and with the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = e^{-t}(1 - \sin 1),$$
$$u_{xx}(0,t) = 0, \quad u_{xx}(1,t) = e^{-t}\sin 1, \qquad t \geq 0.$$

This problem is also also studied in [46]. The exact solution for this problem is

$$u(x,t) = (x - \sin x)\, e^{-t}.$$

We solve the problem for $3m = 27$ and $\Delta t = 0.05$ with 10 and 16 time steps. We compared the approximate solutions obtained by the present method with exact solutions and tabulated the absolute errors for the present method and for the sextic spline method by Rashidinia and Mohammadi [46] at the points $x = 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5$ and at times $t = 0.5$ and $t = 0.8$ in Table 4. It can be seen from the Table 4 that the present method gives more accurate results in comparison to [46] for all points. We plot the error with respect to Δt in Fig. 5 for $3m = 27$ at $t = 1$. Also a plot of the error with respect to the number of collocation points is given in Fig. 6 for $\Delta t = 0.0025$ at $t = 1$. From Figs. 5-6 we can deduce that, for fixed $3m$, lowering the value of Δt also reduces the error, and, for fixed Δt, increasing $3m$ decreases the error. Finally graphical representation of the exact solution and numerical solution are illustrated in Fig. 7 for $3m = 27$, $\Delta t = 0.005$ at $t = 0.08$. In Table 5 we tabulated the convergence rates in view of the errors calculated according to Eq. (4.1).

Table 4. L_∞ and Absolute errors for Problem 2

Methods	Time Steps	Parameters	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$	$x = 0.5$
HWCM	10	$3m = 27$	6.17e-11	3.55e-11	1.12e-09	8.03e-10	2.81e-10
HWCM	16	$3m = 27$	5.77e-11	1.41e-10	1.31e-09	1.85e-09	6.58e-10
[46]	10	$h = 0.05$	8.35e-08	4.51e-08	8.25e-08	2.33e-08	4.52e-08
[46]	16	$h = 0.05$	8.42e-08	2.62e-08	5.32e-08	1.45e-08	2.89e-08
HWCM	10	$3m = 27$	$L_\infty = 3.0466e-09$				
HWCM	16	$3m = 27$	$L_\infty = 4.2367e-09$				

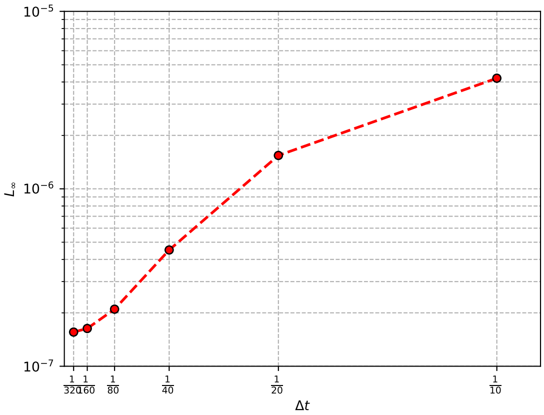

Figure 5. Error versus Δt for $3m = 27$ at $t = 1$ in Problem 2

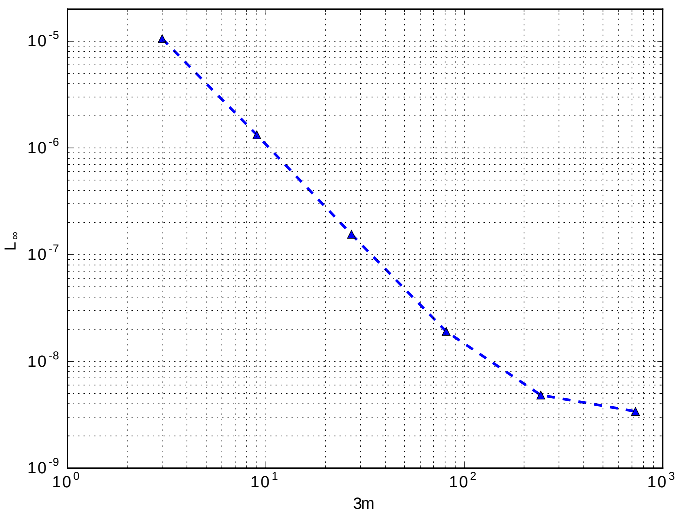

Figure 6. Error versus collocation points for $\Delta t = 0.0025$ at $t = 1$ in Problem 2

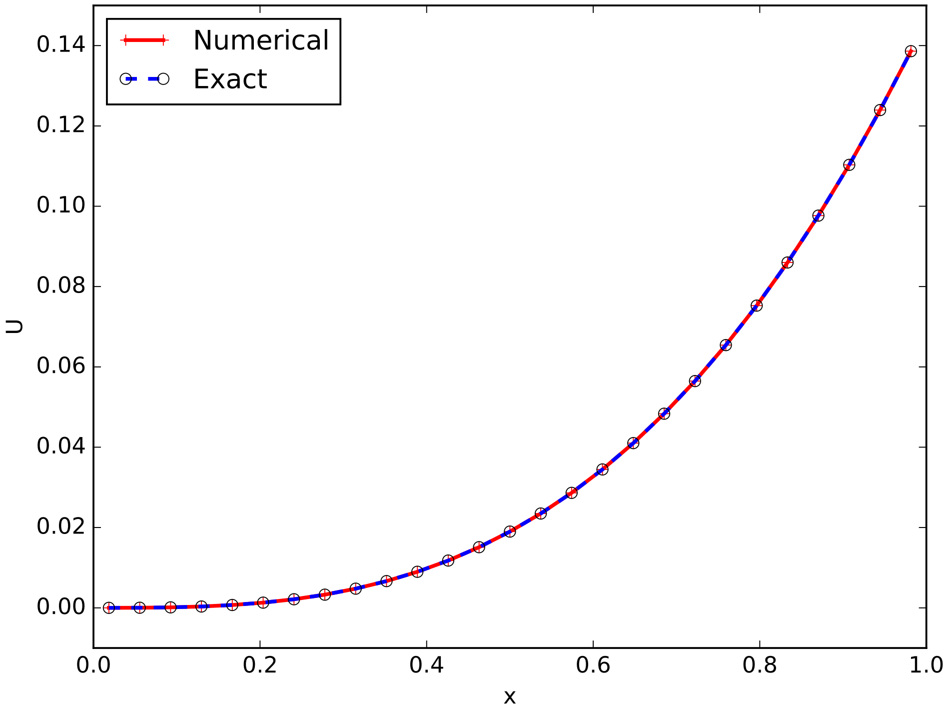

Figure 7. Exact solution versus numerical solution for $3m = 27$, $\Delta t = 0.005$ at $t = 0.08$ in Problem 2

Table 5. Convergence rates for $\Delta t = 0.005$ at time $t = 1$ in Problem 2

	L_∞	Rate
$3m = 3$	1.0535e-05	-
$3m = 9$	1.3257e-06	1.887
$3m = 27$	1.5933e-07	1.928
$3m = 81$	2.7831e-08	1.588

4.3. Problem 3

We consider a constant coefficient ($\mu(x) = \mathrm{EI}(x) = 1$) fourth order non-homogeneous parabolic partial differential equation given by

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = \left(\pi^4 - 1\right)\sin(\pi x)\cos t$$

subject to the initial conditions

$$u(x,0) = \sin(\pi x), \quad u_t(x,0) = 0, \quad 0 \le x \le 1$$

and with the boundary conditions

$$u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad t \ge 0.$$

The exact solution for this problem is [12]

$$u(x,t) = \sin(\pi x)\cos t.$$

In Table 6, we give absolute errors at the points $x = 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5$ using $3m = 27,\ 81$ and $\Delta t = 0.00125,\ 0.005$ at $t = 0.02,\ 0.05$. Also we give results from the previous studies for comparison. It can be seen from Table 6 that the present method gives more accurate results than AGE method [12], Fifth degree B-spline method [4], B-spline methods with redefined basis functions [36] and gives comparable results with other methods studied in [2, 26, 41, 46]. Note that n stands for the number of collocation points in Table 6. Figure 8 shows the evolution of numerical solution in time during simulation for $3m = 81$ and $\Delta t = 0.05$.

Table 6. L_∞ and Absolute errors for Problem 3

Methods	Time	Parameters	$x=0.1$	$x=0.2$	$x=0.3$	$x=0.4$	$x=0.5$
HWCM	$t=0.02$	$3m=81,\ \Delta t=0.00125$	3.80e-07	7.22e-07	9.92e-07	1.16e-06	1.22e-06
	$t=0.05$	$3m=81,\ \Delta t=0.005$	3.63e-06	6.91e-06	9.51e-06	1.12e-05	1.18e-05
	$t=0.02$	$3m=27,\ \Delta t=0.00125$	3.23e-06	6.13e-05	8.75e-06	1.02e-05	1.04e-05
	$t=0.05$	$3m=27,\ \Delta t=0.005$	2.04e-05	3.88e-05	5.37e-05	6.31e-05	6.60e-05
Evans and Yousif [12]	$t=0.02$	$h=0.05,\ \Delta t=0.00125$	2.50e- 05	4.70e- 05	6.60e- 05	7.80e- 05	8.20e- 05
	$t=0.05$	$h=0.05,\ \Delta t=0.005$	2.20e- 04	4.10e- 04	5.40e- 04	6.20e- 04	6.50e- 04
Caglar and Caglar [4]	$t=0.02$	$n=121,\ \Delta t=0.005$	4.80e-06	9.70e-06	1.40e-05	1.90e-05	2.40e-05
	$t=0.02$	$n=191,\ \Delta t=0.005$	5.20e-06	2.10e-06	3.10e-06	4.20e-06	5.20e-06
Mittal and Jain [36] Method 1	$t=0.02$	$n=181,\ \Delta t=0.005$	8.00e-06	1.52e-05	2.09e-05	2.46e-05	2.59e-05
	$t=0.05$	$n=181,\ \Delta t=0.005$	8.97e-06	1.71e-05	2.35e-05	2.76e-05	2.90e-05
Mittal and Jain [36] Method 2	$t=0.02$	$n=181,\ \Delta t=0.005$	1.50e-07	2.90e-07	3.90e-07	4.60e-07	4.90e-07
	$t=0.05$	$n=181,\ \Delta t=0.005$	1.10e-06	2.09e-06	2.88e-06	3.38e-06	3.56e-06
Khan et al [26]	$t=0.02$	$h=0.05,\ \Delta t=0.00125$	9.07e-06	7.79e-06	2.75e-06	1.01e-06	2.59e-06
	$t=0.05$	$h=0.05,\ \Delta t=0.005$	1.87e-06	2.13e-05	1.49e-05	8.60e-06	5.96e-06
Rashidinia and Mohammadi [46]	$t=0.02$	$h=0.05,\ \Delta t=0.00125$	4.47e-07	2.66e-07	1.39e-07	1.55e-07	1.57e-07
	$t=0.05$	$h=0.05,\ \Delta t=0.005$	2.91e-06	1.73e-06	1.60e-06	2.23e-06	2.60e-07
Aziz et al. [2]	$t=0.02$	$h=0.05,\ \Delta t=0.00125$	9.20e-06	7.90e-06	2.80e-06	9.80e-07	2.50e-06
	$t=0.05$	$h=0.05,\ \Delta t=0.005$	9.30e-06	8.00e-06	2.80e-06	1.00e-06	2.70e-06
Mohammadi [41]	$t=0.02$	$h=0.05,\ \Delta t=0.00125$	4.29e-07	2.51e-07	1.24e-07	1.38e-07	1.40e-07
	$t=0.05$	$h=0.05,\ \Delta t=0.005$	2.96e-06	1.77e-06	1.64e-06	2.28e-06	2.65e-07
HWCM	$t=0.02$	$3m=81,\ \Delta t=0.00125$	$L_\infty=1.2239e-06$				
	$t=0.05$	$3m=81,\ \Delta t=0.005$	$L_\infty=1.1752e-05$				

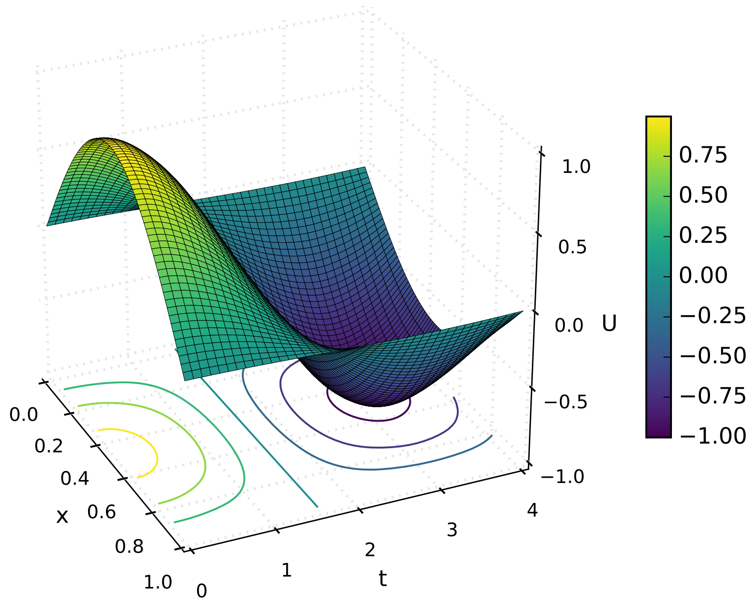

Figure 8. Evolution of numerical solution for $3m = 81$ and $\Delta t = 0.05$ from $t = 0$ to $t = 4$ in Problem 3

4.4. Problem 4

We consider a constant coefficient ($\mu(x) = \mathrm{EI}(x) = 1$) fourth order homogeneous parabolic partial differential equation given by

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = 0$$

subject to the initial conditions

$$u(x,0) = \frac{x}{12}\left(2x^2 - x^3 - 1\right), \quad u_t(x,0) = 0, \quad 0 \le x \le 1$$

and boundary conditions

$$u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad t \ge 0.$$

The exact solution of this problem [11] is

$$u(x,t) = \sum_{s=1}^{\infty} a_s \sin(s\pi x)\cos(s^2\pi^2 t)$$

where

$$a_s = \frac{4}{s^5\pi^5}(\cos(s\pi) - 1).$$

For the sake of comparing our results with existing results, we choose the number of collocation points as $3m = 27$ and $3m = 81$. We observe from the Table 7 that for $3m = 27$ the present method gives more accurate results in comparison to existing methods except H.O.C.M. [13] at $t = 0.02$, and while at $t = 1$ the present method gives the best results among other methods. When we increase the number of collocation points to $3m = 81$, we see from the Table 7 that none of the existing methods can reach to the performance of the present method in terms of accuracy. In Fig. 9, evolution of numerical solution for $3m = 81$ and $\Delta t = 0.01$ from $t = 0$ to $t = 1$ is given. In Table 8 we tabulated the convergence rates in view of the errors calculated according to Eq. (4.1).

Table 7. L_∞ and Absolute errors for Problem 4

Methods	Time	Parameters	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$	$x = 0.5$
HWCM	$t = 0.02$	$3m = 27,\ \Delta t = 0.00125$	3.33e-07	4.58e-07	1.45e-07	3.84e-07	1.97e-07
	$t = 1$	$3m = 27,\ \Delta t = 0.005$	2.04e-05	3.76e-05	2.16e-05	1.22e-05	2.45e-05
	$t = 0.02$	$3m = 81,\ \Delta t = 0.00125$	1.78e-07	1.35e-08	4.27e-07	4.07e-07	1.41e-07
	$t = 1$	$3m = 81,\ \Delta t = 0.005$	1.54e-05	1.06e-05	1.17e-05	3.13e-05	3.85e-05
H.O.C.M. [13]	$t = 0.02$	$h = 0.05,\ \Delta t = 0.00125$	1.40e-07	2.90e-07	5.60e-07	3.40e-07	1.70e-07
	$t = 1$	$h = 0.05,\ \Delta t = 0.005$	2.59e-03	1.91e-03	7.17e-04	2.20e-03	6.65e-04
Danea and Evans [10]	$t = 0.02$	$h = 0.05,\ \Delta t = 0.00125$	2.50e-06	3.90e-06	1.37e-05	2.60e-06	9.80e-06
	$t = 1$	$h = 0.05,\ \Delta t = 0.005$	3.19e-03	2.73e-03	9.80e-03	1.25e-02	1.40e-02
Evans [11]	$t = 0.02$	$h = 0.05,\ \Delta t = 0.00125$	8.44e-06	1.42e-05	1.74e-05	1.40e-06	1.20e-05
	$t = 1$	$h = 0.05,\ \Delta t = 0.005$	3.20e-03	2.73e-03	9.80e-03	1.25e-02	1.40e-02
Richtmyer [47]	$t = 0.02$	$h = 0.05,\ \Delta t = 0.00125$	2.24e-04	3.67e-04	4.03e-04	3.64e-04	3.35e-04
	$t = 1$	$h = 0.05,\ \Delta t = 0.005$	2.73e-03	9.48e-03	1.74e-02	2.30e-02	2.24e-02
Semi-explicit [13]	$t = 0.02$	$h = 0.05,\ \Delta t = 0.00125$	3.01e-05	6.19e-05	6.69e-05	5.10e-05	1.34e-05
	$t = 1$	$h = 0.05,\ \Delta t = 0.005$	2.74e-03	5.93e-03	4.48e-03	2.32e-03	6.51e-03
Mittal and Jain[36]	$t = 0.02$	$n = 181,\ \Delta t = 0.005$	1.14e-05	1.41e-05	9.70e-06	8.02e-06	1.92e-05
	$t = 1$	$n = 181,\ \Delta t = 0.005$	7.33e-04	1.44e-03	2.04e-03	2.47e-03	2.63e-03
HWCM	$t = 0.02$	$3m = 81,\ \Delta t = 0.00125$	$L_\infty = 4.4750\text{e-}07$				
	$t = 1$	$3m = 81,\ \Delta t = 0.005$	$L_\infty = 3.8503\text{e-}05$				

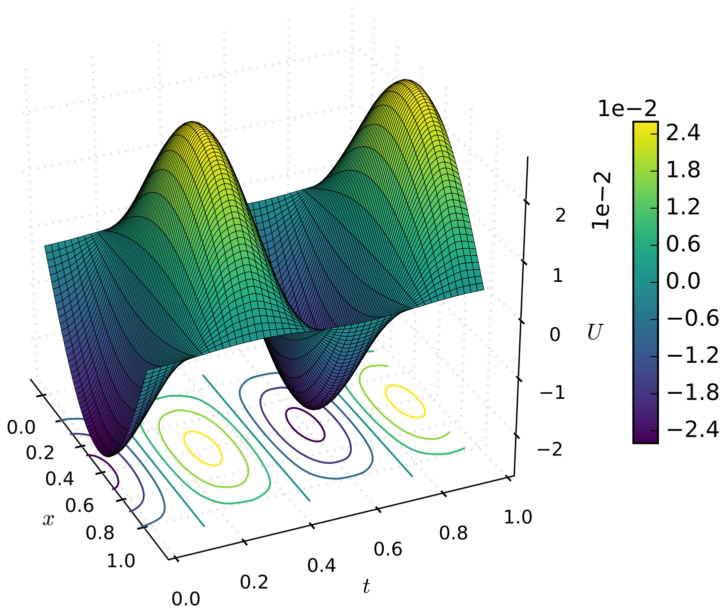

Figure 9. Evolution of numerical solution for $3m = 81$ and $\Delta t = 0.01$ from $t = 0$ to $t = 1$ in Problem 4

Table 8. Convergence rates for $\Delta t = 0.0001$ at final time $t = 1$ in Problem 4

	L_∞	Rate
$3m = 9$	4.693704e-04	-
$3m = 27$	4.495959e-05	2.135
$3m = 81$	4.810131e-06	2.034
$3m = 243$	6.716085e-07	1.792

4.5. Problem 5

We consider a constant coefficient ($\mu(x) = 1$, $\mathrm{EI}(x) = -1$) fourth order homogeneous parabolic partial differential equation which is also studied in [36]

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^4 u}{\partial x^4}$$

subject to the initial conditions

$$u(x, 0) = \sin(\pi x), \quad u_t(x, 0) = -\pi^2 \sin(\pi x), \quad 0 \leq x \leq 1$$

and with boundary conditions

$$u(0, t) = u(1, t) = u_{xx}(0, t) = u_{xx}(1, t) = 0, \quad t \geq 0.$$

The exact solution of the problem is given by

$$u(x,t) = \sin(\pi x)e^{-\pi^2 t}.$$

In Table 9, we give computed results by the present method for $3m = 27$ and $\Delta t = 0.005$ at $t = 0.02,\ 0.05$. We also give the results of [36] for comparison. We observe in Table 9 that the present method gives more accurate results than B-spline methods with redefined basis functions [36].

Table 9. L_∞ and Absolute errors for Problem 5

Methods	Time	Parameters	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$	$x = 0.5$
HWCM	$t = 0.02$	$3m = 27,\ \Delta t = 0.005$	7.74e-06	1.47e-05	2.05e-05	2.40e-05	2.50e-05
HWCM	$t = 0.05$	$3m = 27,\ \Delta t = 0.005$	5.99e-05	3.07e-05	2.89e-05	6.52e-05	8.15e-06
Mittal and Jain[36]	$t = 0.02$	$n = 31,\ \Delta t = 0.005$	2.80e-04	5.33e-04	7.33e-04	8.62e-04	9.06e-04
Method 1	$t = 0.05$	$n = 31,\ \Delta t = 0.005$	2.62e-04	4.98e-04	6.86e-04	8.07e-04	8.48e-04
Mittal and Jain [36]	$t = 0.02$	$n = 31,\ \Delta t = 0.005$	1.08e-04	2.06e-04	2.83e-04	3.33e-04	3.50e-04
Method 2	$t = 0.05$	$n = 31,\ \Delta t = 0.005$	6.13e-04	1.35e-03	1.95e-03	2.18e-03	2.20e-03
HWCM	$t = 0.02$	$3m = 27,\ \Delta t = 0.005$	$L_\infty = 2.4987e-05$				
HWCM	$t = 0.05$	$3m = 27,\ \Delta t = 0.005$	$L_\infty = 6.7356e-05$				

5. Conclusion

Our main goal in this study is to propose a new 3-scale Haar wavelet based method to high order partial differential equations and analyze the performance of the method. The comparisons of numerical solutions with exact solutions and the results from the previous studies that are based on numerical techniques such as finite differences, B-splines and high order spline methods indicate the power of the new 3-scale Haar wavelet based method in dealing with variable coefficient, constant coefficient, homogeneous and non-homogeneous partial differential equations. The implementation of the method is straight-forward and simpler than the existing methods. The advantages of the Haar wavelet based method can be listed as follows.

- High accuracy is attained even with small number of collocation points.
- Small computational costs are required, and the implementation of the method in computers is easy
- Coping with boundary conditions is very easy compared with other known methods.

We also note that the new 3-scale Haar wavelet based method introduced here with suitable modifications can be easily applied to similar problems.

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