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Warped Translation Surfaces of Finite Type in Simply Isotropic 3-Spaces

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Abstract

In this paper, we classify warped translation surfaces being invariant surfaces of i-type, that is, the generating curve has formed by the intersection of the surface with the isotropic xz-plane in the three-dimensional simply isotropic space $\mathbb{I}^3$ under the condition

$$\Delta^J x_i = \lambda_i x_i, \quad \text{with} \quad J = I, II.$$

Here, Δ^J is the Laplace operator with respect to first and second fundamental form and λ_i, $i = 1,2,3$ are some real numbers. Also, as an application, we give some examples for these surfaces and also some explicit graphics of them. All graphics have been plotted with Maple14.

1. Introduction

Let $\mathbb{E}^m$ denotes the m-dimensional Euclidean space and M^n be a connected n-dimensional submanifold in this space. An isometric immersion $\mathbf{x} : M \to \mathbb{E}^m$ is said to be of *k-type* if it can be expressed as a sum of eigenvectors of the Laplace-Beltrami operator of the induced metric Δ, corresponding to k distinct eigenvalues of Δ:

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_1 + \cdots + \mathbf{x}_k, \quad \text{such that} \quad \Delta\mathbf{x}_i = \lambda_i \mathbf{x}_i, \quad i = 1, \ldots, k,$$

for a consant vector $\mathbf{x}_0$, smooth non-constant functions $\mathbf{x}_k$ and $\lambda_i \in \mathbb{R}$, [1]. If an isometric immersion $\mathbf{x}$ is of k-type, then the submanifold M is said to be of k-type [2, 3]. In [4], Chen gave a good survey related to finite type submanifolds. In [5], author proved that a submanifold M^n in $\mathbb{E}^m$ is of 1-type, that is, $\Delta\mathbf{x} = \lambda\mathbf{x}$, $\lambda \in \mathbb{R}^+$, if and only if it is either a minimal submanifold of $\mathbb{E}^m$ ($\lambda = 0$) or a minimal submanifold of hypersphere $\mathbb{S}^{m-1}$ in $\mathbb{E}^m$ ($\lambda \neq 0$). In [6, 7], by generalizing of this, authors showed that if a hypersurface M^n of $\mathbb{E}^{n+1}$ satisfies

$$\Delta\mathbf{x} = A\mathbf{x}, \tag{1.1}$$

where $A \in \mathbb{R}^{(n+1)\times(n+1)}$ is a diagonal matrix $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_{n+1})$. Moreover, Senoussi and Bekkar studied helicoidal surfaces in Euclidean 3-spaces satisfying the condition (1.1), [8]. Furthermore, in [9], Chen gave a detailed paper account of recent development about finite type submanifolds in Euclidean spaces.

On the other hand, very recently, the study on intrinsic (or extrinsic) properties of surfaces in (pseudo-) isotropic spaces has become a research subject for many researchers, see for examples [10]-[17]. Moreover, coordinate finite-type submanifolds

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have been studied in isotropic spaces [18, 19]. Moreover, the study of finite type submanifolds was studied in simply isotropic spaces. In particular, Karacan et. al. studied translation surfaces and surfaces of revolution satisfying

$$\Delta^J x_i = \lambda_i x_i, \tag{1.2}$$

where $J = I, II$ and $i = 1,2,3$, in these spaces in [20, 21] and [22], respectively. Also, in [23], [24] and [25], authors studied affine translation surfaces, helicoidal surfaces and ruled surfaces satisfying the same condition.

In this paper, we are going to study on warped translation surfaces of finite type in three dimensional Isotropic space $\mathbb{I}^3$ satisfying (1.2).

2. Preliminaries

The simply isotropic 3-space $\mathbb{I}^3$ is a Cayley-Klein space defined from the 3-dimensional real projective space $\mathbb{P}^3(\mathbb{R})$ with the absolute figure as given $\{\omega, d_1, d_2, f\}$. In this space, the homogeneous coordinates $[x_0 : x_1 : x_2 : x_3]$ are presented such that $\omega : x_0 = 0$ is a plane in $\mathbb{P}^3(\mathbb{R})$, $d_1 : x_0 = 0 = x_1 + \mathrm{i}x_2$ and $d_2 : x_0 = 0 = x_1 - \mathrm{i}x_2$ are two complex-conjugate straight lines in the plane, and also $f = [0:0:0:1]$ is a point in the intersection $d_1 \cap d_2$.

The group of motions of $\mathbb{I}^3$ is a six-parameter group given by [26]

$$\begin{aligned} \tilde{x} &= a_0 + x\cos\phi - y\sin\phi, \\ \tilde{y} &= a_1 + x\sin\phi + y\cos\phi, \\ \tilde{z} &= a_2 + c_1 x + c_2 y + z, \end{aligned} \tag{2.1}$$

where $\phi, a_0, a_1, a_2, c_1, c_2 \in \mathbb{R}$. Concerning this group of i-motions, it can be easily seen that these motions are indeed composed of an Euclidean motions onto the xy-plane and an affine shear transformation in z-direction. Thus, the projection of a point $S(x,y,z)$, in the z-direction onto $\mathbb{R}^2$, $\widetilde{S}(x,y,0)$ is called the *top view* of S. Let $\vec{A} = (x_1, x_2, x_3)$ be a vector in $\mathbb{I}^3$. If $x_1 = x_2 = 0$, then $\vec{A}$ is called as *isotropic*, otherwise *non-isotropic*. A plane having an isotropic line is said to be an *isotropic plane* and a line with an isotropic director is an *isotropic line*.

Given two vectors $\vec{A} = (x_1, x_2, x_3)$ and $\vec{B} = (y_1, y_2, y_3)$, the *isotropic inner product* is calculated by [26]

$$\left\langle \vec{A}, \vec{B} \right\rangle = x_1 y_1 + x_2 y_2.$$

Moreover, M^2 is called as an *admissible surface* when the metric in M^2 induced by the isotropic scalar product has rank 2. More precisely, M^2 parameterized by a C^2 map $\mathbf{x}(u^1, u^2) = \left(x^1(u^1,u^2), x^2(u^1,u^2), x^3(u^1,u^2)\right)$, is admissible if and only if $X_{12} = x_1^1 x_2^2 - x_2^1 x_1^2 \neq 0$, where $x_k^i = \partial x^i / \partial u^k$ and

$$X_{ij} = \left| x_1^i x_2^j - x_1^j x_2^i \right|, \tag{2.2}$$

[17, 26]. As a result, every admissible C^2 surface M^2 can be locally parameterized as $\mathbf{x}(u^1, u^2) = \left(u^1, u^2, f(u^1, u^2)\right)$: one can say that M is in its *normal form*.

Furthermore, the *isotropic first and second fundamental forms* I and II, and also their coefficients of the *isotropic metric* tensor g_{ij} and h_{ij} are given by, respectively, [17]

$$\begin{aligned} \mathrm{I} &= g_{ij} u^i u^j \text{ and } g_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle, \\ \mathrm{II} &= h_{ij} u^i u^j \text{ and } h_{ij} = \mathbf{x}_{ij} \cdot \mathbf{N}_m, \end{aligned} \tag{2.3}$$

where

$$\mathbf{N}_m = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{X_{12}} = \left(\frac{X_{23}}{X_{12}}, \frac{X_{31}}{X_{12}}, 1\right). \tag{2.4}$$

Here, we may call $\mathbf{N}_m$ the *minimal normal* because the trace of the Weingarten-like operator $-\mathbf{N}_m$ vanishes identically.

Also, the *isotropic Gaussian* and *mean curvatures* are given by [17]

$$K = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2} \text{ and } H = \frac{1}{2}\frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{g_{11}g_{22} - g_{12}^2}. \tag{2.5}$$

Note that, a surface in $\mathbb{I}^3$ is called as isotropic flat (resp. isotropic minimal) if K (resp. H) vanishes.

Consequently, according to a local coordinate system, the Laplacian Δ^J, $J = I, II$ in terms of the first and second fundamental forms are defined as usual by, ([20, 21],[26])

$$\Delta^I = -\frac{1}{\sqrt{g_{11}g_{22} - g_{12}^2}}\left[\partial_1\Big(\frac{g_{22}\partial_1 - g_{12}\partial_2}{\sqrt{g}}\Big) + \partial_2\Big(\frac{g_{11}\partial_2 - g_{12}\partial_1}{\sqrt{g}}\Big)\right], \tag{2.6}$$

and

$$\Delta^{II} = -\frac{1}{\sqrt{h_{11}h_{22} - h_{12}^2}}\left[\partial_1\Big(\frac{h_{22}\partial_1 - h_{12}\partial_2}{\sqrt{h_{11}h_{22} - h_{12}^2}}\Big) + \partial_2\Big(\frac{h_{11}\partial_2 - h_{12}\partial_1}{\sqrt{h_{11}h_{22} - h_{12}^2}}\Big)\right], \tag{2.7}$$

where $\partial_i = \partial/\partial u^i$ and g^{ij} is the inverse of the metric, that is, $g^{ik}g_{kj} = \delta^i_j$. Moreover, throughout paper we will take as $g_{11}g_{22} - g_{12}^2 \neq 0$ and $h_{11}h_{22} - h_{12}^2 \neq 0$.

2.1. Warped translation surface in Simply Isotropic 3-space

In this work, we will be working on warped translation surfaces being one of the types of invariant surfaces in $\mathbb{I}^3$ and some algebraic equations in terms of the Laplacian operator and the coordinate functions of these surfaces. So, in this section, we will work to explain how warped translation surfaces in $\mathbb{I}^3$ are parameterized, (For more details, see [17].)

Let M^2 be a *warped translation* surface being invariant. So, M^2 can be parametrized as

$$M^2_{(a_0,a_1,0,c_1,c_2)} : \mathbf{P}(u,v) = (a_0v + x(u), a_1v, c_1vx(u) + z(u)), \tag{2.8}$$

such that $a_2 = (a_0c_1 + a_1c_2) = 0$, (a_0,a_1), $(c_1,c_2) \neq (0,0)$. Also, $\phi, a_0, a_1, a_2, c_1, c_2$ are the real constants as in Eq. (2.1).

Notice that since all simply isotropic invariant surfaces are admissible, throughout paper we will assume that warped translation surfaces are admissible, (see for more details, [17].)

3. Warped translation surfaces of finite type

As mentioned in the previous section, the warped translation surfaces can be parametrized as in (2.8) in Isotropic 3-spaces. In this section, we calculate the Laplacian operator Δ^J for these surfaces in $\mathbb{I}^3$. And then, we examine the warped translation surfaces satisfying the condition (1.2). Finally, we give the complete classification of these surfaces of finite type in $\mathbb{I}^3$.

Now, let us consider on a warped translation surface $M^2_{(a_0,a_1,0,c_1,c_2)}$ defined as in (2.8) with the generating curve α, $\alpha(u) = (u,0,z(u))$, i.e.,

$$\mathbf{x}(u,v) = (a_0v + u, a_1v, c_1uv + z(u)),\ a_1 > 0. \tag{3.1}$$

Thus, we have

$$\mathbf{x}_u = (1,0,c_1v + z'),$$
$$\mathbf{x}_v = (a_0,a_1,c_1u).$$

Since $M^2_{(a_0,a_1,0,c_1,c_2)}$ is admissible, i.e., $a_1 \neq 0$ from (2.2), then $\mathbf{N}_m$ the minimal normal defined by (2.4) is computed as

$$\mathbf{N}_m = (-c_1v - z', \frac{a_0z' + c_1(a_0v - u)}{a_1}, 1).$$

By considering (2.3), we obtain the corresponding fundamental forms as [17]

$$\mathrm{I} = 1\mathrm{d}u^2 + 2a_0\mathrm{d}u\mathrm{d}v + ({a_0}^2 + {a_1}^2)\mathrm{d}v^2 \text{ and } \mathrm{II} = z''\mathrm{d}u^2 + 2c_1\mathrm{d}u\mathrm{d}v,$$

and from (2.5), the Gaussian and mean curvatures are

$$K = -\frac{c_1^2}{{a_1}^2} \text{ and } H = -\frac{a_0c_1}{{a_1}^2} + \frac{({a_0}^2 + {a_1}^2)}{2{a_1}^2}z''.$$

Finally, the Laplace-Beltrami operators defined as in (2.6) and (2.7) of a warped translation surface are obtained as, respectively,

$$\Delta^I = -\frac{{a_0}^2 + {a_1}^2}{{a_1}^2}\frac{\partial^2}{\partial u^2} + \frac{2a_0}{{a_1}^2}\frac{\partial^2}{\partial u\partial v} - \frac{1}{{a_1}^2}\frac{\partial^2}{\partial v^2}, \tag{3.2}$$

and

$$\Delta^{II} = -\frac{2}{c_1}\frac{\partial^2}{\partial u \partial v} + \frac{z''}{{c_1}^2}\frac{\partial^2}{\partial v^2},$$

where c_1 is a non-zero constant.

Now, firstly we would like to give the following theorem being the classification of parabolic revolution surfaces satisfying (3.2) in $\mathbb{I}^3$.

Theorem 3.1. *Let M^2 be a warped translation surface given by (3.1) in $\mathbb{I}^3$ such that it satisfies the condition $\Delta^I x_i = \lambda_i x_i$, where λ_i, $i = 1,2,3$ are some real constants. Then M^2 refers to one of the followings:*

1. *If $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 0$, then the function $z(u)$ is quadratic.*
2. *If $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 \neq 0$, then $(a_0, a_1, 0, c_1, c_2) = (a_0, a_1, 0, 0, c_2)$ and z is given by either*
   (a) $z(u) = z_1 \cosh(\sqrt{\Lambda_3}\, u) + z_2 \sinh(\sqrt{\Lambda_3}\, u)$, if $\lambda_3 > 0$, or
   (b) $z(u) = z_1 \cos(\sqrt{-\Lambda_3}\, u) + z_2 \sin(\sqrt{-\Lambda_3}\, u)$, if $\lambda_3 < 0$,

   where $\Lambda_3 = \frac{\lambda_3 {a_1}^2}{{a_0}^2 + {a_1}^2}$.

Proof. Assume that M^2 is a warped translation surface given by (3.1) and it satisfies the condition (1.2) for $J = I$. Let us take the expressions

$$\Delta^I x = \left(\Delta^I x_1, \Delta^I x_2, \Delta^I x_3\right),$$

(2.6) and (3.1) together. Thus by a straightforward computation, we get

$$\Delta^I x = \left(0, 0, -z''\frac{{a_0}^2 + {a_1}^2}{{a_1}^2} + \frac{2a_0 c_1}{{a_1}^2}\right).$$

So, as M^2 satisfies the condition $\Delta^I x_i = \lambda_i x_i$, where λ_i, $i = 1,2,3$ are some real constants, we have

$$0 = \lambda_1(a_0 v + u), \tag{3.3}$$

$$0 = \lambda_2 a_1 v, \tag{3.4}$$

$$-z''\frac{{a_0}^2 + {a_1}^2}{{a_1}^2} + \frac{2a_0 c_1}{{a_1}^2} = \lambda_3\left(c_1 uv + z\right). \tag{3.5}$$

So, from (3.3) and (3.4), we get directly $\lambda_1 = \lambda_2 = 0$. Now, we will consider on two possibilities coming from (3.5). First, if $\lambda_3 = 0$ then we get the following ODE

$$\frac{{a_0}^2 + {a_1}^2}{{a_1}^2} z'' - \frac{2a_0 c_1}{{a_1}^2} = 0, \tag{3.6}$$

whose solutions are given as in Case (1) in Theorem 3.1. Secondly, let $\lambda_3 \neq 0$. By considering $z = z(u)$ in (3.5), we obtain $c_1 = 0$ and

$$z'' + \frac{\lambda_3 {a_1}^2}{{a_0}^2 + {a_1}^2} z = 0.$$

By taking $\Lambda_3 = -\frac{\lambda_3 {a_1}^2}{{a_0}^2 + {a_1}^2}$, we can rewrite the last ODE as

$$z'' - \Lambda_3 z = 0,$$

whose solutions is given as in Case(2) in Theorem 3.1. □

Remark 3.2. *By comparing the second equality of (3.2) and (3.6), we conclude that the warped translation surface M^2 parametrized as in Case (1) in Theorem 3.1 is a isotropic minimal surface in $\mathbb{I}^3$.*

Remark 3.3. *By comparing the first equality of (3.2) and (3.6), we conclude that the warped translation surface M^2 parametrized as in Case (2) in Theorem 3.1 is a isotropic flat surface in $\mathbb{I}^3$.*

By considering the above Remark 3.2, we have the following:

Corollary 3.4. *A warped translation surface given by (3.1) in the three dimensional simply isotropic space $\mathbb{I}^3$ is harmonic if and only if the surface M^2 is isotropic minimal.*

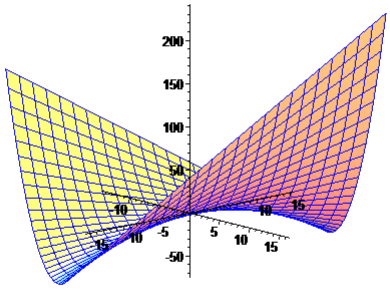

Figure 3.1: An isotropic minimal warped translation surface is parametrized as in Case (1) in Theorem 3.1.

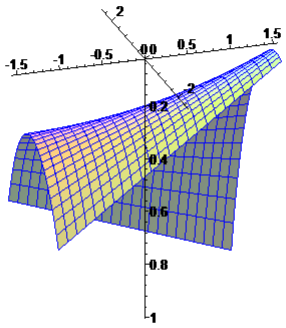

Figure 3.2: An isotropic flat warped translation surface is parametrized as in Case (2a) in Theorem 3.1.

Now, we would like to give some explicit examples of warped translation surfaces satisfing (1.2) for $J = I$ in $\mathbb{I}^3$:

Secondly, we would like to give the following theorem being the classification of warped translation surfaces satisfying (1.2) for $J = II$ in $\mathbb{I}^3$.

Theorem 3.5. *Let M^2 be a warped translation surface given by (3.1) in $\mathbb{I}^3$ such that it satisfies the condition $\Delta^{II}x_i = \lambda_i x_i$, where λ_i, $i = 1,2,3$ are some real constants. Then $M^2_{(a_0,a_1,0,0,c_2)}$ can be parametrized as*

$$\mathbf{x}(u,v) = (a_0 v + u, a_1 v, c_1 uv - \frac{2}{\lambda_3}),\ a_1 > 0.$$

Proof. Let M^2 be a warped translation surface given by (3.1) satisfying (1.2) and

$$\Delta^{II}x = \left(\Delta^{II}x_1, \Delta^{II}x_2, \Delta^{II}x_3\right).$$

By a straightforward computation, we get from (2.7)

$$\Delta^{II}x = \left(0,0,-2\right).$$

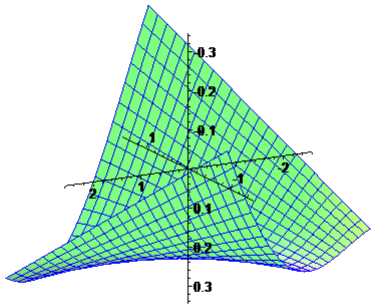

Figure 3.3: An isotropic flat warped translation surface is parametrized as in Case (2b) in Theorem 3.1.

So, as M^2 satisfies the condition $\Delta^{II}x_i = \lambda_i x_i$, where λ_i, $i = 1,2,3$ are some real constants, we have

$$0 = \lambda_1(a_0 v + u), \tag{3.7}$$
$$0 = \lambda_2 a_1 v, \tag{3.8}$$
$$-2 = \lambda_3\Big(c_1 uv + z(u)\Big). \tag{3.9}$$

From (3.7) and (3.8), we get directly, $\lambda_1 = \lambda_2 = 0$. Now by considering (3.9), we conclude directly that $\lambda_3 \neq 0$. And so we obtain $c_1 = 0$ and $z = -\frac{2}{\lambda_3}$. Thus M^2 can be parametrized as in Theorem 3.5. □

Definition 3.6. *A surface in a simply isotropic 3-space, $\mathbb{I}^3$ is called as $II-$harmonic if it satisfies the condition $\Delta^{II}x = 0$, [20].*

By considering the above Definition and the proof of Theorem 3.5, we have the following:

Corollary 3.7. *There are no $II-$harmonic warped translation surface in $\mathbb{I}^3$.*

Now, we would like to give some explicit examples of warped translation surfaces satisfying $\Delta^{II}x_i = \lambda_i x_i$ in $\mathbb{I}^3$:

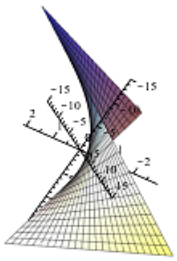

Figure 3.4: A warped translation surface is parametrized as in Theorem 3.5.

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