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On the Logarithmic Summability of Sequences in Intuitionistic Fuzzy Normed Spaces

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Article Info

Abstract

We introduce logarithmic summability in intuitionistic fuzzy normed spaces($IFNS$) and give some Tauberian conditions for which logarithmic summability yields convergence in $IFNS$. Besides, we define the concept of slow oscillation with respect to logarithmic summability in $IFNS$, investigate its relation with the concept of q-boundedness and give Tauberian theorems by means of q-boundedness and slow oscillation with respect to logarithmic summability. A comparison theorem between Cesàro summability method and logarithmic summability method in $IFNS$ is also proved in the paper.

1. Introduction and preliminaries

Fuzzy sets are put forward by Zadeh [1] in 1965 as a generalization of classical sets and have been studied by many mathematicians from varied branches. In classical sets, elements in the universal set are divided crisply into two groups as members and nonmembers, and partial membership is not allowed. Unlike the classical sets, fuzzy sets allow partial membership and take every elements in the universe into account by assigning degrees of membership between 1 and 0. Owing to the power in handling unclassifiable data, fuzzy sets are utilized in many real-world scenarios to cope with problems of uncertainty and indefiniteness. In 1983, inspired by fuzzy sets, Atanassov [2, 3] considered also partial non-membership and extended fuzzy sets to intuitionistic fuzzy sets. Following Atanassov's introduction, concepts of intuitionistic fuzzy metric [4] and intuitionistic fuzzy norm (IF-norm) [5,6] are defined and related topics are studied. In particular, convergence of sequences in $IFNS$ is investigated and different types of convergence(e.g., statistical convergence and ideal convergence) are applied to sequences in $IFNS$ to grasp the convergence [7–11].

Recently Talo and Yavuz [12] introduced Cesàro summability of sequences in $IFNS$ and gave Tauberian theorems for Cesàro summability method in $IFNS$, by which they initiated summability theory and Tauberian theory in $IFNS$. In their study, they also defined the concept of slow oscillation in $IFNS$ and gave related theorems. Following their study, we now define logarithmic summability of sequences in $IFNS$ and prove a Tauberian theorem for logarithmic summability method. In the sequel, we define the notion of slow oscillation with respect to logarithmic summability in $IFNS$ and give slowly oscillating type Tauberian conditions for which logarithmic summability yields convergence in $IFNS$. Besides, we compare Cesàro summability and logarithmic summability in $IFNS$. Before continuing with main results we now give some preliminaries.

Definition 1.1. *[6] The triplicate (N, μ, ν) is said to be an IFNS if N is a real vector space, and μ, ν are fuzzy sets on $N \times \mathbb{R}$ satisfying the following conditions for every $u, w \in N$ and $t, s \in \mathbb{R}$:*

(a) $\mu(u, t) = 0$ for $t \leq 0$,

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(b) $\mu(u,t) = 1$ *for all* $t \in \mathbb{R}^+$ *if and only if* $u = \theta$

(c) $\mu(cu,t) = \mu\left(u, \frac{t}{|c|}\right)$ *for all* $t \in \mathbb{R}^+$ *and* $c \neq 0$,

(d) $\mu(u+w, t+s) \geq \min\{\mu(u,t), \mu(w,s)\}$,

(e) $\lim_{t\to\infty} \mu(u,t) = 1$ *and* $\lim_{t\to 0} \mu(u,t) = 0$,

(f) $\nu(u,t) = 1$ *for* $t \leq 0$,

(g) $\nu(u,t) = 0$ *for all* $t \in \mathbb{R}^+$ *if and only if* $u = \theta$

(h) $\nu(cu,t) = \nu\left(u, \frac{t}{|c|}\right)$ *for all* $t \in \mathbb{R}^+$ *and* $c \neq 0$,

(i) $\max\{\nu(u,t), \nu(w,s)\} \geq \nu(u+w, t+s)$,

(j) $\lim_{t\to\infty} \nu(u,t) = 0$ *and* $\lim_{t\to 0} \nu(u,t) = 1$.

We call (μ, ν) *an IF$-$norm on N.*

Example 1.2. *Let* $(N, \|\cdot\|)$ *be a normed space and* μ_0, ν_0 *be fuzzy sets on* $N \times \mathbb{R}$ *defined by*

$$\mu_0(u,t) = \begin{cases} 0, & t \leq 0, \\ \frac{t}{t+\|u\|}, & t > 0, \end{cases} \qquad\qquad \nu_0(u,t) = \begin{cases} 1, & t \leq 0, \\ \frac{\|u\|}{t+\|u\|}, & t > 0. \end{cases}$$

Then (μ_0, ν_0) *is IF$-$norm on N.*

Throughout the paper (N, μ, ν) will denote an *IFNS*.

Definition 1.3. *[6] A sequence* (u_n) *in* (N, μ, ν) *is said to be convergent to* $a \in N$ *and denoted by* $u_n \to a$ *if for every* $\varepsilon > 0$ *and* $t > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that* $\mu(u_n - a, t) > 1 - \varepsilon$ *and* $\nu(u_n - a, t) < \varepsilon$ *for all* $n \geq n_0$.

Definition 1.4. *[6] A sequence* (u_n) *in* (N, μ, ν) *is said to be Cauchy if for every* $\varepsilon > 0$ *and* $t > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that* $\mu(u_k - u_n, t) > 1 - \varepsilon$ *and* $\nu(u_k - u_n, t) < \varepsilon$ *for all* $k, n \geq n_0$.

Every convergent sequence is Cauchy in *IFNS*.

Definition 1.5. *[13] A sequence* (u_n) *in* (N, μ, ν) *is called q-bounded if* $\lim_{t\to\infty} \inf_{n\in\mathbb{N}} \mu(u_n, t) = 1$ *and* $\lim_{t\to\infty} \sup_{n\in\mathbb{N}} \nu(u_n, t) = 0$.

2. Main results

Now we introduce logarithmic summability in *IFNS* and prove corresponding Tauberian theorems. For some other studies concerning logarithmic summability and convergence methods in fuzzy setting see [14–26].

Definition 2.1. *Let sequence* (u_n) *be in* (N, μ, ν). *Logarithmic mean* τ_n *of* (u_n) *is defined by*

$$\tau_n = \frac{1}{\ell_n} \sum_{k=1}^{n} \frac{u_k}{k} \quad \text{where} \quad \ell_n = \sum_{k=1}^{n} \frac{1}{k} .$$

(u_n) *is said to be logarithmic summable to* $a \in N$ *if*

$$\lim_{n\to\infty} \tau_n = a.$$

Following theorem shows that convergence yields logarithmic summability in *IFNS*.

Theorem 2.2. *Let sequence* (u_n) *be in* (N, μ, ν). *If* (u_n) *is convergent to* $a \in N$, *then* (u_n) *is logarithmic summable to a.*

Proof. Let sequence (u_n) converge to $a \in N$. Fix $t > 0$. For $\varepsilon > 0$

- There exists $n_0 \in \mathbb{N}$ such that $\mu\left(u_n - a, \frac{t}{2}\right) > 1 - \varepsilon$ and $\nu\left(u_n - a, \frac{t}{2}\right) < \varepsilon$ for $n > n_0$.
- There exists $n_1 \in \mathbb{N}$ such that

$$\mu\left(\sum_{k=1}^{n_0} \frac{u_k - a}{k}, \frac{\ell_n t}{2}\right) > 1 - \varepsilon \quad \text{and} \quad \nu\left(\sum_{k=1}^{n_0} \frac{u_k - a}{k}, \frac{\ell_n t}{2}\right) < \varepsilon$$

for $n > n_1$, since we have

$$\lim_{n\to\infty} \mu\left(\sum_{k=1}^{n_0} \frac{u_k - a}{k}, \frac{\ell_n t}{2}\right) = 1 \quad \text{and} \quad \lim_{n\to\infty} \nu\left(\sum_{k=1}^{n_0} \frac{u_k - a}{k}, \frac{\ell_n t}{2}\right) = 0.$$

Hence we get

$$
\begin{aligned}
\mu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{u_k}{k}-a,t\right) &= \mu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{u_k-a}{k},t\right)=\mu\left(\sum_{k=1}^{n}\frac{u_k-a}{k},\ell_n t\right)\\
&\geq \min\left\{\mu\left(\sum_{k=1}^{n_0}\frac{u_k-a}{k},\frac{\ell_n t}{2}\right),\mu\left(\sum_{k=n_0+1}^{n}\frac{u_k-a}{k},\frac{\ell_n t}{2}\right)\right\}\\
&\geq \min\left\{\mu\left(\sum_{k=1}^{n_0}\frac{u_k-a}{k},\frac{\ell_n t}{2}\right),\mu\left(\sum_{k=n_0+1}^{n}\frac{u_k-a}{k},\frac{(\ell_n-\ell_{n_0})t}{2}\right)\right\}\\
&\geq \min\left\{\mu\left(\sum_{k=1}^{n_0}\frac{u_k-a}{k},\frac{\ell_n t}{2}\right),\mu\left(\frac{u_{n_0+1}-a}{n_0+1},\frac{t}{2(n_0+1)}\right),\cdots,\mu\left(\frac{u_n-a}{n},\frac{t}{2n}\right)\right\}\\
&= \min\left\{\mu\left(\sum_{k=1}^{n_0}\frac{u_k-a}{k},\frac{\ell_n t}{2}\right),\mu\left(u_{n_0+1}-a,\frac{t}{2}\right),\cdots,\mu\left(u_n-a,\frac{t}{2}\right)\right\}\\
&> 1-\varepsilon
\end{aligned}
$$

and

$$
\begin{aligned}
\nu\left(\frac{1}{\ell_n}\sum_{k=1}^{n}\frac{u_k}{k}-a,t\right) &< \max\left\{\nu\left(\sum_{k=1}^{n_0}\frac{u_k-a}{k},\frac{\ell_n t}{2}\right),\nu\left(u_{n_0+1}-a,\frac{t}{2}\right),\cdots,\nu\left(u_n-a,\frac{t}{2}\right)\right\}\\
&< \varepsilon
\end{aligned}
$$

whenever $n>\max\{n_0,n_1\}$, which completes the proof. $\qquad\square$

Logarithmic summability does not imply convergence in *IFNS* by the next example.

Example 2.3. *Take* $(u_n)=((-1)^n)$ *in IF$-$normed space* $(\mathbb{R},\mu_0,\nu_0)$ *where* μ_0 *and* ν_0 *are as in Example 1.2. Sequence* (u_n) *is logarithmic summable to* 0 *in view of Theorem 2.13 and [12, Example 3.3], but it is not convergent.*

We now give some Tauberian conditions for which logarithmic summability yields convergence in *IFNS*.

Theorem 2.4. *Let sequence* (u_n) *be in* (N,μ,ν). *If* (u_n) *is logarithmic summable to* $a \in N$, *then it converges to* a *if and only if for each* $t>0$

$$
\sup_{\lambda>1}\liminf_{n\to\infty}\mu\left(\frac{1}{\ell_{\lfloor n^\lambda\rfloor}-\ell_n}\sum_{k=n+1}^{\lfloor n^\lambda\rfloor}\frac{u_k-u_n}{k},t\right)=1 \tag{2.1}
$$

and

$$
\inf_{\lambda>1}\limsup_{n\to\infty}\nu\left(\frac{1}{\ell_{\lfloor n^\lambda\rfloor}-\ell_n}\sum_{k=n+1}^{\lfloor n^\lambda\rfloor}\frac{u_k-u_n}{k},t\right)=0. \tag{2.2}
$$

Proof. Necessity. Let (u_n) converge to a. For all $\lambda>1$ and large enough n, that is when $\lfloor n^\lambda\rfloor>n$, we can write(see [27, Lemma 5.5(i)])

$$
u_n-\tau_n=\frac{\ell_{\lfloor n^\lambda\rfloor}}{\ell_{\lfloor n^\lambda\rfloor}-\ell_n}\left(\tau_{\lfloor n^\lambda\rfloor}-\tau_n\right)-\frac{1}{\ell_{\lfloor n^\lambda\rfloor}-\ell_n}\sum_{k=n+1}^{\lfloor n^\lambda\rfloor}\frac{u_k-u_n}{k}. \tag{2.3}
$$

Since (τ_n) is Cauchy, for each $t>0$ we have

$$
\lim_{n\to\infty}\mu\left(\tau_{\lfloor n^\lambda\rfloor}-\tau_n,t\right)=1 \quad\text{and}\quad \lim_{n\to\infty}\nu\left(\tau_{\lfloor n^\lambda\rfloor}-\tau_n,t\right)=0.
$$

Hence, for sufficiently large n such that $\frac{\ell_{\lfloor n^\lambda\rfloor}}{\ell_{\lfloor n^\lambda\rfloor}-\ell_n}\leq\frac{2\lambda}{\lambda-1}$ is satisfied, we have

$$
\mu\left(\frac{\ell_{\lfloor n^\lambda\rfloor}}{\ell_{\lfloor n^\lambda\rfloor}-\ell_n}\left(\tau_{\lfloor n^\lambda\rfloor}-\tau_n\right),t\right)=\mu\left(\tau_{\lfloor n^\lambda\rfloor}-\tau_n,\frac{t}{\frac{\ell_{\lfloor n^\lambda\rfloor}}{\ell_{\lfloor n^\lambda\rfloor}-\ell_n}}\right)\geq\mu\left(\tau_{\lfloor n^\lambda\rfloor}-\tau_n,\frac{t}{\frac{2\lambda}{\lambda-1}}\right)\to 1 \quad(n\to\infty)
$$

and

$$\nu\left(\frac{\ell_{\lfloor n^\lambda \rfloor}}{\ell_{\lfloor n^\lambda \rfloor} - \ell_n}\left(\tau_{\lfloor n^\lambda \rfloor} - \tau_n\right), t\right) = \nu\left(\tau_{\lfloor n^\lambda \rfloor} - \tau_n, \frac{t}{\frac{\ell_{\lfloor n^\lambda \rfloor}}{\ell_{\lfloor n^\lambda \rfloor} - \ell_n}}\right) \leq \mu\left(\tau_{\lfloor n^\lambda \rfloor} - \tau_n, \frac{t}{\frac{2\lambda}{\lambda-1}}\right) \to 0 \quad (n \to \infty)$$

revealing that $\frac{\ell_{\lfloor n^\lambda \rfloor}}{\ell_{\lfloor n^\lambda \rfloor} - \ell_n}\left(\tau_{\lfloor n^\lambda \rfloor} - \tau_n\right) \to 0$. So, by equation (2.3), we conclude

$$\lim_{n\to\infty}\mu\left(\frac{1}{\ell_{\lfloor n^\lambda \rfloor} - \ell_n}\sum_{k=n+1}^{\lfloor n^\lambda \rfloor}\frac{u_k - u_n}{k}, t\right) = 1 \quad \text{and} \quad \lim_{n\to\infty}\nu\left(\frac{1}{\ell_{\lfloor n^\lambda \rfloor} - \ell_n}\sum_{k=n+1}^{\lfloor n^\lambda \rfloor}\frac{u_k - u_n}{k}, t\right) = 0,$$

which means that (2.1) and (2.2) are satisfied.

Sufficiency. Let conditions (2.1) and (2.2) be satisfied. Let $t > 0$ be fixed. For $\varepsilon > 0$ we have:

- There exist $\lambda > 1$ and $n_0 \in \mathbb{N}$ such that

$$\mu\left(\frac{1}{\ell_{\lfloor n^\lambda \rfloor} - \ell_n}\sum_{k=n+1}^{\lfloor n^\lambda \rfloor}\frac{u_k - u_n}{k}, \frac{t}{3}\right) > 1 - \varepsilon \quad \text{and} \quad \mu\left(\frac{1}{\ell_{\lfloor n^\lambda \rfloor} - \ell_n}\sum_{k=n+1}^{\lfloor n^\lambda \rfloor}\frac{u_k - u_n}{k}, \frac{t}{3}\right) < \varepsilon$$

  for $n > n_0$.
- There exists $n_1 \in \mathbb{N}$ such that $\mu\left(\tau_n - a, \frac{t}{3}\right) > 1 - \varepsilon$ and $\nu\left(\tau_n - a, \frac{t}{3}\right) < \varepsilon$ for $n > n_1$.
- There exists $n_2 \in \mathbb{N}$ such that

$$\mu\left(\frac{\ell_{\lfloor n^\lambda \rfloor}}{\ell_{\lfloor n^\lambda \rfloor} - \ell_n}\left(\tau_{\lfloor n^\lambda \rfloor} - \tau_n\right), \frac{t}{3}\right) > 1 - \varepsilon \quad \text{and} \quad \nu\left(\frac{\ell_{\lfloor n^\lambda \rfloor}}{\ell_{\lfloor n^\lambda \rfloor} - \ell_n}\left(\tau_{\lfloor n^\lambda \rfloor} - \tau_n\right), \frac{t}{3}\right) < \varepsilon,$$

  for $n > n_2$, since $\frac{\ell_{\lfloor n^\lambda \rfloor}}{\ell_{\lfloor n^\lambda \rfloor} - \ell_n}\left(\tau_{\lfloor n^\lambda \rfloor} - \tau_n\right) \to 0$.

Hence, by equation (2.3), we get

$$\begin{aligned}
\mu(u_n - a, t) &= \mu(u_n - \tau_n + \tau_n - a, t) \\
&= \mu\left(\frac{\ell_{\lfloor n^\lambda \rfloor}}{\ell_{\lfloor n^\lambda \rfloor} - \ell_n}\left(\tau_{\lfloor n^\lambda \rfloor} - \tau_n\right) - \frac{1}{\ell_{\lfloor n^\lambda \rfloor} - \ell_n}\sum_{k=n+1}^{\lfloor n^\lambda \rfloor}\frac{u_k - u_n}{k} + \tau_n - a, t\right) \\
&\geq \min\left\{\mu\left(\frac{\ell_{\lfloor n^\lambda \rfloor}}{\ell_{\lfloor n^\lambda \rfloor} - \ell_n}\left(\tau_{\lfloor n^\lambda \rfloor} - \tau_n\right), \frac{t}{3}\right), \mu\left(\frac{1}{\ell_{\lfloor n^\lambda \rfloor} - \ell_n}\sum_{k=n+1}^{\lfloor n^\lambda \rfloor}\frac{u_k - u_n}{k}, \frac{t}{3}\right), \mu\left(\tau_n - a, \frac{t}{3}\right)\right\} \\
&> 1 - \varepsilon
\end{aligned}$$

and

$$\begin{aligned}
\nu(u_n - a, t) &< \max\left\{\nu\left(\frac{\ell_{\lfloor n^\lambda \rfloor}}{\ell_{\lfloor n^\lambda \rfloor} - \ell_n}\left(\tau_{\lfloor n^\lambda \rfloor} - \tau_n\right), \frac{t}{3}\right), \nu\left(\frac{1}{\ell_{\lfloor n^\lambda \rfloor} - \ell_n}\sum_{k=n+1}^{\lfloor n^\lambda \rfloor}\frac{u_k - u_n}{k}, \frac{t}{3}\right), \nu\left(\tau_n - a, \frac{t}{3}\right)\right\} \\
&< \varepsilon
\end{aligned}$$

for $n > \max\{n_0, n_1, n_2\}$, which completes the proof. $\quad\square$

Theorem 2.5. *Let sequence (u_n) be in (N, μ, ν). If (u_n) is logarithmic summable to $a \in N$, then it converges to a if and only if for each $t > 0$*

$$\sup_{0 < \lambda < 1}\liminf_{n\to\infty}\mu\left(\frac{1}{\ell_n - \ell_{\lfloor n^\lambda \rfloor}}\sum_{k=\lfloor n^\lambda \rfloor + 1}^{n}\frac{u_n - u_k}{k}, t\right) = 1$$

and

$$\inf_{0 < \lambda < 1}\limsup_{n\to\infty}\nu\left(\frac{1}{\ell_n - \ell_{\lfloor n^\lambda \rfloor}}\sum_{k=\lfloor n^\lambda \rfloor + 1}^{n}\frac{u_n - u_k}{k}, t\right) = 0.$$

Proof. The proof is done similarly to that of Theorem 2.4 by using equation(see [27, Lemma 5.5(ii)])

$$u_n - \tau_n = \frac{\ell_{\lfloor n^\lambda \rfloor}}{\ell_n - \ell_{\lfloor n^\lambda \rfloor}} \left(\tau_n - \tau_{\lfloor n^\lambda \rfloor} \right) + \frac{1}{\ell_n - \ell_{\lfloor n^\lambda \rfloor}} \sum_{k=\lfloor n^\lambda \rfloor + 1}^{n} \frac{u_n - u_k}{k} \quad (0 < \lambda < 1)$$

instead of (2.3). $\qquad \square$

Now we introduce the concept of slow oscillation with respect to logarithmic summability in *IFNS*.

Definition 2.6. (u_n) *in* (N, μ, ν) *is said to be slowly oscillating with respect to logarithmic summability if*

$$\sup_{\lambda > 1} \liminf_{n \to \infty} \min_{n < k \leq \lfloor n^\lambda \rfloor} \mu(u_k - u_n, t) = 1 \tag{2.4}$$

and

$$\inf_{\lambda > 1} \limsup_{n \to \infty} \max_{n < k \leq \lfloor n^\lambda \rfloor} \nu(u_k - u_n, t) = 0, \tag{2.5}$$

for each $t > 0$. "$\sup_{\lambda > 1}$" in (2.4) and "$\inf_{\lambda > 1}$" in (2.5) can be replaced by "$\lim_{\lambda \to 1^+}$".

A sequence (u_n) in (N, μ, ν) is slowly oscillating with respect to logarithmic summability if for each $t > 0$ and for all $\varepsilon > 0$ there exist $\lambda > 1$ and $n_0 \in \mathbb{N}$ such that

$$\mu(u_k - u_n, t) > 1 - \varepsilon \quad \text{and} \quad \nu(u_k - u_n, t) < \varepsilon$$

whenever $n_0 \leq n < k \leq \lfloor n^\lambda \rfloor$.

The proof of next theorem is analogous to that of Theorem 4.2 in [12] and hence omitted.

Theorem 2.7. *Let sequence (u_n) be in (N, μ, ν). For $t > 0$, conditions (2.4) and (2.5) are equivalent to*

$$\sup_{0 < \lambda < 1} \liminf_{n \to \infty} \min_{\lfloor n^\lambda \rfloor < k \leq n} \mu(u_k - u_n, t) = 1 \tag{2.6}$$

and

$$\inf_{0 < \lambda < 1} \limsup_{n \to \infty} \max_{\lfloor n^\lambda \rfloor < k \leq n} \nu(u_k - u_n, t) = 0, \tag{2.7}$$

respectively. "$\sup_{0 < \lambda < 1}$" in (2.6) and "$\inf_{0 < \lambda < 1}$" in (2.7) can be replaced by "$\lim_{\lambda \to 1^-}$".

Example 2.8. *Consider IF$-$normed space $(\mathbb{R}, \mu_0, \nu_0)$ where μ_0 and ν_0 are as in Example 1.2. $u_n = \sum_{j=1}^{n} \frac{1}{j \ln j}$ is slowly oscillating with respect to logarithmic summability by the calculations below:*

Fix $t > 0$. For $\varepsilon > 0$ take $\lambda = e^{\frac{t\varepsilon}{1-\varepsilon}}$. Then for $1 < n < k \leq \lfloor n^\lambda \rfloor$ we have

$$\mu_0(u_k - u_n, t) = \frac{t}{t + |u_k - u_n|} > \frac{t}{t + \frac{t\varepsilon}{1-\varepsilon}} = 1 - \varepsilon$$

and

$$\nu_0(u_k - u_n, t) = \frac{|u_k - u_n|}{|u_k - u_n| + t} < \frac{\frac{t\varepsilon}{1-\varepsilon}}{\frac{t\varepsilon}{1-\varepsilon} + t} = \varepsilon,$$

since $|u_k - u_n| = \sum_{j=n+1}^{k} \frac{1}{j \ln j} < \int\limits_{n}^{k} \frac{du}{u \ln u} \leq \ln\left(\frac{\ln k}{\ln n}\right) \leq \ln \lambda = \frac{t\varepsilon}{1-\varepsilon}$.

Theorem 2.9. *Let sequence (u_n) be in (N, μ, ν). If (u_n) is slowly oscillating with respect to logarithmic summability then (2.1) and (2.2) are satisfied.*

Proof. Suppose that (u_n) is slowly oscillating with respect to logarithmic summability. Fix $t > 0$. For $\varepsilon > 0$ there exist $\lambda > 1$ and $n_0 \in \mathbb{N}$ such that

$$\mu(u_k - u_n, t) > 1 - \varepsilon \quad \text{and} \quad \nu(u_k - u_n, t) < \varepsilon$$

whenever $n_0 \leq n < k \leq \lfloor n^\lambda \rfloor$. Hence, we have

$$
\begin{aligned}
\mu\left(\frac{1}{\ell_{\lfloor n^\lambda \rfloor} - \ell_n} \sum_{k=n+1}^{\lfloor n^\lambda \rfloor} \frac{u_k - u_n}{k}, t\right) &= \mu\left(\sum_{k=n+1}^{\lfloor n^\lambda \rfloor} \frac{u_k - u_n}{k}, (\ell_{\lfloor n^\lambda \rfloor} - \ell_n)t\right) \\
&\geq \min\left\{\mu\left(\frac{u_{n+1} - u_n}{n+1}, \frac{t}{n+1}\right), \ldots, \mu\left(\frac{u_{\lfloor n^\lambda \rfloor} - u_n}{\lfloor n^\lambda \rfloor}, \frac{t}{\lfloor n^\lambda \rfloor}\right)\right\} \\
&= \min\left\{\mu(u_{n+1} - u_n, t), \ldots, \mu(u_{\lfloor n^\lambda \rfloor} - u_n, t)\right\} \\
&> 1 - \varepsilon
\end{aligned}
$$

and

$$
\begin{aligned}
\nu\left(\frac{1}{\ell_{\lfloor n^\lambda \rfloor} - \ell_n} \sum_{k=n+1}^{\lfloor n^\lambda \rfloor} \frac{u_k - u_n}{k}, t\right) &\leq \max\left\{\nu(u_{n+1} - u_n, t), \ldots, \nu(u_{\lfloor n^\lambda \rfloor} - u_n, t)\right\} \\
&< \varepsilon
\end{aligned}
$$

for $n \geq n_0$ and this completes the proof. $\square$

In view of Theorem 2.4 and Theorem 2.9 we give the following Tauberian theorem.

Theorem 2.10. *Let sequence (u_n) be in (N, μ, ν). If (u_n) is logarithmic summable to $a \in N$ and slowly oscillating with respect to logarithmic summability, then (u_n) converges to a.*

Theorem 2.11. *Let sequence (u_n) be in (N, μ, ν). If $\{n \ln n(u_n - u_{n-1})\}$ is q-bounded, then (u_n) is slowly oscillating with respect to logarithmic summability.*

Proof. Let $\{n \ln n(u_n - u_{n-1})\}$ be q-bounded. In view of Definition 1.5, for given $\varepsilon > 0$ there exists $M_\varepsilon > 0$ so that

$$
t > M_\varepsilon \Rightarrow \inf_{n \in \mathbb{N}} \mu(n \ln n(u_n - u_{n-1}), t) > 1 - \varepsilon \quad \text{and} \quad \sup_{n \in \mathbb{N}} \nu(n \ln n(u_n - u_{n-1}), t) < \varepsilon.
$$

For every $t > 0$ choose $\lambda < 1 + \frac{t}{M_\varepsilon}$. Then for $n_0 < n < k \leq \lfloor n^\lambda \rfloor$ we have

$$
\begin{aligned}
\mu(u_k - u_n, t) &= \mu\left(\sum_{j=n+1}^{k}(u_j - u_{j-1}), t\right) \\
&\geq \min_{n+1 \leq j \leq k} \mu\left(u_j - u_{j-1}, \frac{t}{j(\ell_k - \ell_n)}\right) \\
&= \min_{n+1 \leq j \leq k} \mu\left(j \ln j(u_j - u_{j-1}), \frac{t \ln j}{\ell_k - \ell_n}\right) \\
&\geq \min_{n+1 \leq j \leq k} \mu\left(j \ln j(u_j - u_{j-1}), \frac{t \ln n}{\ell_k - \ell_n}\right) \\
&\geq \min_{n+1 \leq j \leq k} \mu\left(j \ln j(u_j - u_{j-1}), \frac{t}{\frac{\ln k}{\ln n} - 1}\right) \\
&\geq \min_{n+1 \leq j \leq k} \mu\left(j \ln j(u_j - u_{j-1}), \frac{t}{\lambda - 1}\right) \\
&\geq \inf_{n \in \mathbb{N}} \mu\left(n \ln n(u_n - u_{n-1}), \frac{t}{\lambda - 1}\right) \\
&> 1 - \varepsilon
\end{aligned}
$$

and

$$
\nu(u_k - u_n, t) < \sup_{n \in \mathbb{N}} \nu\left(n \ln n(u_n - u_{n-1}), \frac{t}{\lambda - 1}\right) < \varepsilon.
$$

Hence, (u_n) is slowly oscillating with respect to logarithmic summability. $\square$

By Theorem 2.10 and Theorem 2.11, we conclude following Tauberian theorem.

Theorem 2.12. *Let sequence (u_n) be in (N, μ, ν). If (u_n) is logarithmic summable to $a \in N$ and $\{n \ln n(u_n - u_{n-1})\}$ is q-bounded, then (u_n) converges to a.*

Now we prove a comparison theorem.

Theorem 2.13. *Let sequence (u_n) be in (N, μ, ν). If (u_n) is Cesàro summable to $a \in N$, then (u_n) is logarithmic summable to a.*

Proof. Let (u_n) be Cesàro summable to $a \in N$. Then, Cesàro means $\sigma_n = \frac{1}{n} \sum_{k=1}^{n} u_k$ converges to a and $\frac{1}{\ell_n} \sum_{k=1}^{n} \frac{\sigma_{k-1}}{k} \to a$ by Theorem 2.2 with the agreement $\sigma_0 = 0$.

Fix $t > 0$. For $\varepsilon > 0$

- There exists $n_0 \in \mathbb{N}$ such that $\mu\left(\sigma_n - a, \frac{t}{2}\right) > 1 - \varepsilon$ and $\nu\left(\sigma_n - a, \frac{t}{2}\right) < \varepsilon$ whenever $n > n_0$.
- There exists $n_1 \in \mathbb{N}$ such that

$$\mu\left(\frac{1}{\ell_n} \sum_{k=1}^{n} \frac{\sigma_{k-1}}{k} - a, \frac{t}{2}\right) > 1 - \varepsilon \quad \text{and} \quad \nu\left(\frac{1}{\ell_n} \sum_{k=1}^{n} \frac{\sigma_{k-1}}{k} - a, \frac{t}{2}\right) < \varepsilon$$

  whenever $n > n_1$.
- There exists $n_2 \in \mathbb{N}$ such that $\mu\left(a, \frac{(\ell_n - 1)t}{2}\right) > 1 - \varepsilon$ and $\nu\left(a, \frac{(\ell_n - 1)t}{2}\right) < \varepsilon$ whenever $n > n_2$, since $\lim_{n \to \infty} \mu\left(a, \frac{(\ell_n - 1)t}{2}\right) = 1$ and $\lim_{n \to \infty} \nu\left(a, \frac{(\ell_n - 1)t}{2}\right) = 0$.

Then, we have(see [28])

$$
\begin{aligned}
\mu(\tau_n - a, t) &= \mu\left(\frac{\sigma_n}{\ell_n} + \frac{1}{\ell_n} \sum_{k=1}^{n} \frac{\sigma_{k-1}}{k} - a, t\right) \\
&\geq \min\left\{\mu\left(\frac{\sigma_n}{\ell_n}, \frac{t}{2}\right), \mu\left(\frac{1}{\ell_n} \sum_{k=1}^{n} \frac{\sigma_{k-1}}{k} - a, \frac{t}{2}\right)\right\} \\
&= \min\left\{\mu\left(\sigma_n, \frac{\ell_n t}{2}\right), \mu\left(\frac{1}{\ell_n} \sum_{k=1}^{n} \frac{\sigma_{k-1}}{k} - a, \frac{t}{2}\right)\right\} \\
&\geq \min\left\{\mu\left(\sigma_n - a, \frac{t}{2}\right), \mu\left(a, \frac{(\ell_n - 1)t}{2}\right), \mu\left(\frac{1}{\ell_n} \sum_{k=1}^{n} \frac{\sigma_{k-1}}{k} - a, \frac{t}{2}\right)\right\} \\
&> 1 - \varepsilon
\end{aligned}
$$

and

$$\nu(\tau_n - a, t) \leq \max\left\{\nu\left(\sigma_n - a, \frac{t}{2}\right), \nu\left(a, \frac{(\ell_n - 1)t}{2}\right), \nu\left(\frac{1}{\ell_n} \sum_{k=1}^{n} \frac{\sigma_{k-1}}{k} - a, \frac{t}{2}\right)\right\} < \varepsilon$$

whenever $n > \max\{n_0, n_1, n_2\}$, which completes the proof. $\qquad\square$

Logarithmic summability does not imply Cesàro summability in *IFNS* by the next example.

Example 2.14. *Consider sequence $(u_n) = ((-1)^n n)$ in IF$-$normed space $(\mathbb{R}, \mu_0, \nu_0)$ where μ_0 and ν_0 are as in Example 1.2. Since*

$$\lim_{n \to \infty} \mu_0(\tau_{2n+1}, t) = \lim_{n \to \infty} \mu_0\left(-\frac{1}{\ell_{2n+1}}, t\right) = \lim_{n \to \infty} \frac{t}{t + \left|-\frac{1}{\ell_{2n+1}}\right|} = 1$$

$$\lim_{n \to \infty} \nu_0(\tau_{2n+1}, t) = \lim_{n \to \infty} \nu_0\left(-\frac{1}{\ell_{2n+1}}, t\right) = \lim_{n \to \infty} \frac{\left|-\frac{1}{\ell_{2n+1}}\right|}{\left|-\frac{1}{\ell_{2n+1}}\right| + t} = 0$$

we have $\tau_{2n+1} \to 0$, and since

$$\lim_{n \to \infty} \mu_0(\tau_{2n}, t) = \lim_{n \to \infty} \mu_0(0, t) = \lim_{n \to \infty} \frac{t}{t + 0} = 1, \quad \lim_{n \to \infty} \nu_0(\tau_{2n}, t) = \lim_{n \to \infty} \nu_0(0, t) = \lim_{n \to \infty} \frac{0}{0 + t} = 0$$

we have $\tau_{2n} \to 0$ which yields that $\lim_{n \to \infty} \tau_n = 0$. So, (u_n) is logarithmic summable to 0. But, sequence (u_n) is not Cesàro summable.

We note that converse of Theorem 2.13 is true under the condition $\ln n(\tau_n - a) \to 0$, which can be seen by the following:

$$
\begin{aligned}
\mu(\sigma_n - a, t) &= \mu\left(\ell_n(\tau_n - a) - \frac{1}{n}\sum_{k=1}^{n-1}\ell_k(\tau_k - a), t\right) \\
&\geq \min\left\{\mu\left(\ell_n(\tau_n - a), \frac{t}{2}\right), \mu\left(\frac{1}{n}\sum_{k=1}^{n-1}\ell_k(\tau_k - a), \frac{t}{2}\right)\right\} \to 1 \quad as \quad n \to \infty
\end{aligned}
$$

and

$$
\nu(\sigma_n - a, t) \leq \max\left\{\nu\left(\ell_n(\tau_n - a), \frac{t}{2}\right), \nu\left(\frac{1}{n}\sum_{k=1}^{n-1}\ell_k(\tau_k - a), \frac{t}{2}\right)\right\} \to 0 \quad as \quad n \to \infty.
$$

By Theorem 2.13 and Example 2.14, we see that logarithmic summability method is stronger than Cesàro summability method in summing up sequences in *IFNS*.

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