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A Note on Generalized Relative Order (α,β) and Generalized Relative Type (α,β) of a Meromorphic Function with Respect to an Entire Function in the Unit Disc

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Abstract

In this paper we introduce the idea of generalized relative order (α,β) and generalized relative type (α,β) of a meromorphic function with respect to an analytic function in the unit disc D where α and β are continuous non-negative on $(-\infty,+\infty)$ functions. Hence we study some basic properties relating to the sum and product theorems of generalized relative order (α,β) and generalized relative type (α,β) of a meromorphic function with respect to an analytic function in the unit disc D.

Keywords: *Generalized relative order (α,β) ; generalized relative type (α,β); generalized relative weak type (α,β); meromorphic function; Property (D); unit disc.*
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Let us consider the functions which are meromorphic or analytic in the unit disc $D=\{z\in\mathbb{C}:|z|<1\}$ and have unbounded growth according to some specific growth indicator. Also we consider that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory in the unit disc D which are available in [6, 9, 12, 13]. Before starting the paper we just summarized the Nevanlinna theory for the reader's convenience. we denote by $n_f(r)$ the number of poles of f in $|z|\le r<1$ where each pole is counted according to its multiplicity. Similarly $\overline{n}_f(r)$ stands for the number of distinct poles of f in $|z|\le r<1$ disregarding the multiplicity. The Nevanlinna's Characteristic function of f is define as $T_f(r)=N_f(r)+m_f(r)$ where the function $N_f(r)$ and $m_f(r)$ are respectively known as counting function and proximity function which are as follows:

$$N_f(r)=\int_0^r\frac{n_f(t)-n_f(0)}{t}dt+n_f(0)\log r$$

$$\left(\overline{N}_f(r)=\int_0^r\frac{\overline{n}_f(t)-\overline{n}_f(0)}{t}dt+\overline{n}_f(0)\log r\right).$$

and

$$m_f(r)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|d\theta,\quad\text{where}$$

$\log^+x=\max(\log x,0)$ for all $x\geqslant 0$.

If f is an entire function, then the Nevanlinna's Characteristic function $T_f(r)$ of f is defined as

$T_f(r)=m_f(r)$.

An entire function f is said to have Property (D), if for any $\delta>1$, $\gamma>0$ and for all r, $0<r<1$, sufficiently close to 1

$$\left(M_f\left(\frac{1}{1-r}\right)^{\gamma}\right)^2\le M_f\left(\left(\left(\frac{1}{1-r}\right)^{\gamma}\right)^{\delta}\right)$$

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where $M_f(r) = \max_{|z|=r} |f(z)|$.We define $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ and $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $x \in [0,\infty)$ and $k \in \mathbb{N}$ where $\mathbb{N}$ be the set of all positive integers. We also denote $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$ and $\exp^{[-1]} x = \log x$.
In this connection we state the following definition which will be needed in the sequel:

Definition 0.1. *Let f be a meromorphic function in D. Then, the order $\rho(f)$ and lower order $\lambda(f)$ of f [12] are defined by*

$$\begin{matrix} \rho(f) \\ \lambda(f) \end{matrix} = \lim_{r\to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log T_f(r)}{\log(\frac{1}{1-r})}.$$

Further, if f is of order $\rho(f)$ $(0 < \rho(f) < \infty)$, one may introduced the definitions of type $\sigma(f)$ and lower type $\overline{\sigma}(f)$ of f which are as follows:

$$\begin{matrix} \sigma(f) \\ \overline{\sigma}(f) \end{matrix} = \lim_{r\to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{T_f(r)}{(\frac{1}{1-r})^{\rho(f)}}.$$

However during the last several years many authors have investigated different properties of meromorphic or analytic function in the unit disc D and derived so many great results e.g. [4, 5, 7, 8, 10, 11]. The notion of relative order was first introduced by Bernal [1, 2]. Considering this idea, one may give the definition of relative order and relative lower order of a meromorphic function f in the unit disc D with respect to an entire function in the following way:

Definition 0.2. *If f a meromorphic function f in the unit disc D and g be an entire function, then the relative order and relative lower order of f with respect to g, denoted by $\rho_g(f)$ and $\lambda_g(f)$ respectively are defined by*

$$\begin{matrix} \rho_g(f) \\ \lambda_g(f) \end{matrix} = \lim_{r\to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log T_g^{-1} T_f(r)}{\log(\frac{1}{1-r})}.$$

Now let L be a class of continuous non-negative on $(-\infty,+\infty)$ function α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$. For any $\alpha \in L$, we say that $\alpha \in L_1^0$, if $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$ and $\alpha \in L_2^0$, if $\alpha(\exp((1+o(1))x)) = (1+o(1))\alpha(\exp(x))$ as $x \to +\infty$. Finally for any $\alpha \in L$, we also say that $\alpha \in L_1$, if $\alpha(cx) = (1+o(1))\alpha(x)$ as $x_0 \leq x \to +\infty$ for each $c \in (0,+\infty)$ and $\alpha \in L_2$, if $\alpha(\exp(cx)) = (1+o(1))\alpha(\exp(x))$ as $x_0 \leq x \to +\infty$ for each $c \in (0,+\infty)$. Clearly, $L_1 \subset L_1^0$, $L_2 \subset L_2^0$ and $L_2 \subset L_1$.

Now considering this, one may introduce the definition of the generalized order (α,β) and generalized lower order (α,β) of a meromorphic function in the unit disc U which are as follows:

Definition 0.3. *Let $\alpha \in L_2$ and $\beta \in L_1$. The generalized order (α,β) denoted by $\rho_{(\alpha,\beta)}[f]$ and generalized lower order (α,β) denoted by $\lambda_{(\alpha,\beta)}[f]$ of a meromorphic function f in the unit disc U are defined as:*

$$\begin{matrix} \rho_{(\alpha,\beta)}[f] \\ \lambda_{(\alpha,\beta)}[f] \end{matrix} = \lim_{r\to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\exp(T_f(r)))}{\beta(\frac{1}{1-r})}.$$

Clearly if $\alpha(r) = \log(\log r)$ and $\beta(r) = \log r$, then Definition 0.3 reduces to Definition0.1.

In the case of relative order, it therefore seems reasonable to define suitably the generalized relative order (α,β) and generalized relative lower order (α,β) of a meromorphic function f in the unit disc D with respect to an entire function g respectively in the following way:

Definition 0.4. *Let α, $\beta \in L_1$. Let f be any meromorphic function in the unit disc D and g be any entire function. Then generalized relative order (α,β) denoted as $\rho_{(\alpha,\beta)}[f]_g$ and generalized relative lower order (α,β) denoted as $\lambda_{(\alpha,\beta)}[f]_g$ of a meromorphic function f with respect to an entire function g are define by*

$$\begin{matrix} \rho_{(\alpha,\beta)}[f]_g \\ \lambda_{(\alpha,\beta)}[f]_g \end{matrix} = \lim_{r\to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(T_g^{-1}(T_f(r)))}{\beta((1-r)^{-1})},$$

The previous definitions are easily generated as particular cases, e.g. if $\alpha(r) = \beta(r) = \log r$, then Definition 0.4 reduces Definition 0.2, and if $\alpha(r) = \beta(r) = \log r$ and $g(z) = \exp z$, then Definition 0.4 reduces to first part of Definition 0.1.A meromorphic function f in the unit disc D for which generalized relative order (α,β) and generalized relative lower order (α,β) with respect to an entire function g are the same is called a function of regular generalized relative growth (α,β) with respect to g. Otherwise, f is said to be irregular generalized relative growth (α,β) with respect to g.

Now in order to refine the above growth scale, one may give the definitions of an another growth indicators, such as generalized relative type (α,β) and generalized relative lower type (α,β) of meromorphic function in the unit disc D with respect to another entire function which are as follows:

Definition 0.5. *Let α, $\beta \in L_1$. Let f be meromorphic in the unit disc D and g be an entire function with $0 < \rho_{(\alpha,\beta)}[f]_g < \infty$, then the generalized relative type (α,β) and generalized relative lower type (α,β) denoted respectively by $\sigma_{(\alpha,\beta)}[f]_g$ and $\overline{\sigma}_{(\alpha,\beta)}[f]_g$ of f in the unit disc D with respect to g are respectively defined as follows:*

$$\begin{matrix} \sigma_{(\alpha,\beta)}[f]_g \\ \overline{\sigma}_{(\alpha,\beta)}[f]_g \end{matrix} = \lim_{r\to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{[\exp(\beta((1-r)^{-1})]^{\rho_{(\alpha,\beta)}[f]_g}}.$$

Analogously, to determine the relative growth of two meromorphic functions having same non-zero finite generalized relative lower order (α,β) in the unit disc D with respect to another entire function, one can introduced the definition of generalized relative weak type (α,β) of a meromorphic f in the unit disc D with respect to an entire g of finite positive generalized relative lower order (α,β), $\lambda_{(\alpha,\beta)}[f]_g$ in the following way:

Definition 0.6. *Let $\alpha, \beta \in L_1$. Let f be meromorphic in the unit disc D and g be an entire function having finite positive relative (α,β) the lower order $\lambda_{(\alpha,\beta)}[f]_g$ $(0<\lambda_{(\alpha,\beta)}[f]_g<\infty)$. Then the generalized relative weak type (α,β), $\tau_{(\alpha,\beta)}[f]_g$ and the growth indicator $\overline{\tau}_{(\alpha,\beta)}[f]_g$ of f with respect to g are defined as :*

$$\begin{matrix}\overline{\tau}_{(\alpha,\beta)}[f]_g \\ \tau_{(\alpha,\beta)}[f]_g\end{matrix} = \lim_{r\to 1} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{[\exp(\beta((1-r)^{-1})]^{\lambda_{(\alpha,\beta)}[f]_g}}.$$

Here, in this paper, we aim at investigating some basic properties of generalized relative order $\alpha,\beta)$, generalized relative type (α,β) and generalized relative weak type (α,β) of a meromorphic function in the unit disc D with respect to an entire function under somewhat different conditions which extend some earlier result (see, e.g., [3]). Henceforth throughout this paper,we assume that α, $\beta \in L_1$ and all the growth indicators are non-zero finite.

1. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1.1. *Let f be an entire function which satisfies the Property (D) then for any positive integer n and for all $\delta>1$,*

$$\left(M_f\left(\frac{1}{1-r}\right)\right)^n \le M_f\left(\left(\frac{1}{1-r}\right)^{\delta}\right)$$

holds for all r, $0<r<1$, sufficiently close to 1.

Lemma 1.1 follows from a result of Bernal [2].

Lemma 1.2. *Let f be an entire function. Then*

$$T_f\left(\frac{1}{1-r}\right) \le \log M_f\left(\frac{1}{1-r}\right) \le 3T_f\left(\frac{2}{1-r}\right).$$

for all r, $0<r<1$, sufficiently close to 1.

Lemma 1.2 follows from Theorem 1.6 of [6].

2. Main Results

In this section we present the main results of the paper.

Theorem 2.1. *Let f_1, f_2 be meromorphic functions in the unit disc D and g_1 be any entire function such that at least f_1 or f_2 is of regular generalized relative growth (α,β) with respect to g_1. Also let g_1 has the Property (D). Then*

$$\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \le \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\}.$$

The sign of equality holds when $\lambda_{(\alpha,\beta)}[f_i]_{g_1} > \lambda_{(\alpha,\beta)}[f_j]_{g_1}$ with at least f_j is of regular generalized relative growth (α,β) with respect to g_1 where $i=j=1,2$ and $i\neq j$.

Proof. The result is obvious when $\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1}=0$. So we suppose that $\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1}>0$. We can clearly assume that $\lambda_{(\alpha,\beta)}[f_k]_{g_1}$ is finite for $k=1,2$. Now let us consider that $\max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\}=\Delta$ and f_2 is of regular generalized relative growth (α,β) with respect to g_1.

Now for any arbitrary $\varepsilon>0$ from the definition of $\lambda_{(\alpha,\beta)}[f_1]_{g_1}$, we have for a sequence of r tending to 1 that

$$T_{f_1}(r) \le T_{g_1}[\alpha^{-1}[(\lambda_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon)\{\beta((1-r)^{-1})\}]]$$

$$i.e.,\ T_{f_1}(r) \le T_{g_1}[\alpha^{-1}[(\Delta+\varepsilon)\{\beta((1-r)^{-1})\}]]. \tag{2.1}$$

Also for any arbitrary $\varepsilon>0$ from the definition of $\rho_{(\alpha,\beta)}[f_2]_{g_1}(=\lambda_{(\alpha,\beta)}[f_2]_{g_1})$, we obtain for all r, $0<r<1$, sufficiently close to 1 that

$$T_{f_2}(r) \le T_{g_1}[\alpha^{-1}[(\lambda_{(\alpha,\beta)}[f_2]_{g_1}+\varepsilon)\{\beta((1-r)^{-1})\}]] \tag{2.2}$$

$$i.e.,\ T_{f_2}(r) \le T_{g_1}[\alpha^{-1}[(\Delta+\varepsilon)\{\beta((1-r)^{-1})\}]]. \tag{2.3}$$

Since $T_{f_1\pm f_2}(r) \le T_{f_1}(r)+T_{f_2}(r)+O(1)$, therefore there exists a sequence values of r tending to 1 for which we obtain in view of (2.1) , (2.3) and Lemma 1.2 that

$$T_{f_1\pm f_2}(r) \le 2\log M_{g_1}[\alpha^{-1}[(\Delta+\varepsilon)\{\beta((1-r)^{-1})\}]]+O(1)$$

$$i.e.,\ T_{f_1\pm f_2}(r) \le 3\log M_{g_1}[\alpha^{-1}[(\Delta+\varepsilon)\{\beta((1-r)^{-1})\}]]. \tag{2.4}$$

Therefore in view of Lemma 1.1 and Lemma 1.2, we obtain from (2.4) for a sequence values of r tending to 1 and $\sigma > 1$ that

$$T_{f_1 \pm f_2}(r) \leq \frac{1}{3} \log[M_{g_1}[\alpha^{-1}[(\Delta+\varepsilon)\{\beta((1-r)^{-1})\}]]]^9$$

$$i.e.,\ T_{f_1 \pm f_2}(r) \leq \frac{1}{3} \log M_{g_1}[\alpha^{-1}[(\Delta+\varepsilon)\{\beta((1-r)^{-1})\}]^{\sigma}]$$

$$i.e.,\ T_{f_1 \pm f_2}(r) \leq T_{g_1}[2\alpha^{-1}[(\Delta+\varepsilon)\{\beta((1-r)^{-1})\}]^{\sigma}].$$

Now we get from above by letting $\sigma \to 1^+$

$$i.e.,\ \underline{\lim}_{r \to 1} \frac{\alpha(T_{g_1}^{-1}(T_{f_1 \pm f_2}(r)))}{\{\beta((1-r)^{-1})\}} < (\Delta+\varepsilon).$$

Since $\varepsilon > 0$ is arbitrary,

$$\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \leq \Delta = \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\} .$$

Similarly, if we consider that f_1 is of regular generalized relative growth (α,β) with respect to g_1 or both f_1 and f_2 are of regular generalized relative growth (α,β) with respect to g_1, then one can easily verify that

$$\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \leq \Delta = \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\} . \tag{2.5}$$

Further without loss of any generality, let $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ and $f = f_1 \pm f_2$. Then in view of (2.5) we get that $\lambda_{(\alpha,\beta)}[f]_{g_1} \leq \lambda_{(\alpha,\beta)}[f_2]_{g_1}$. As, $f_2 = \pm(f - f_1)$ and in this case we obtain that $\lambda_{(\alpha,\beta)}[f_2]_{g_1} \leq \max\{\lambda_{(\alpha,\beta)}[f]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_1}\}$. As we assume that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_2]_{g_1}$, therefore we have $\lambda_{(\alpha,\beta)}[f_2]_{g_1} \leq \lambda_{(\alpha,\beta)}[f]_{g_1}$ and hence $\lambda_{(\alpha,\beta)}[f]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1} = \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\}$. Therefore, $\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_i]_{g_1} \mid i = 1,2$ provided $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \neq \lambda_{(\alpha,\beta)}[f_2]_{g_1}$. Thus the theorem is established. □

Theorem 2.2. *Let f_1 and f_2 be any two meromorphic functions in the unit disc D and g_1 be an entire function such that such that $\rho_{(\alpha,\beta)}[f_1]_{g_1}$ and $\rho_{(\alpha,\beta)}[f_1]_{g_2}$ exists and let g_1 has the Property (D). Then*

$$\rho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \leq \max\{\rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_2]_{g_1}\} .$$

The sign of equality holds when $\rho_{(\alpha,\beta)}[f_1]_{g_1} \neq \rho_{(\alpha,\beta)}[f_2]_{g_1}$.

We omit the proof of Theorem 2.2 as it can easily be carried out in the line of Theorem 2.1.

Theorem 2.3. *Let f_1 be a meromorphic function in the unit disc D and g_1, g_2 be any two entire functions such that $\lambda_{(\alpha,\beta)}[f_1]_{g_1}$ and $\lambda_{(\alpha,\beta)}[f_1]_{g_2}$ exists and let $g_1 \pm g_2$ has the Property (D). Then*

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \geq \min\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\} .$$

The sign of equality holds when $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \neq \lambda_{(\alpha,\beta)}[f_1]_{g_2}$.

Proof. The result is obvious when $\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \infty$. So we suppose that $\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} < \infty$. We can clearly assume that $\lambda_{(\alpha,\beta)}[f_1]_{g_k}$ is finite for $k = 1,2$. Further let $\Psi = \min\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\}$. Now for any arbitrary $\varepsilon > 0$ from the definition of $\lambda_{(\alpha,\beta)}[f_1]_{g_k}$ where $k = 1,2$, we have for all r, $0 < r < 1$, sufficiently close to 1 that

$$T_{g_k}[\alpha^{-1}[(\lambda_{(\alpha,\beta)}[f_1]_{g_k} - \varepsilon)\{\beta((1-r)^{-1})\}]]] \leq T_{f_1}(r) \tag{2.6}$$

$$i.e,\ T_{g_k}[\alpha^{-1}[(\Psi - \varepsilon)\{\beta((1-r)^{-1})\}]] \leq T_{f_1}(r)$$

Now we obtain from above and Lemma 1.2 for all r, $0 < r < 1$, sufficiently close to 1 that

$$T_{g_1 \pm g_2}[\alpha^{-1}[(\Psi - \varepsilon)\{\beta((1-r)^{-1})\}]] \leq 2T_{f_1}(r) + O(1)$$

$$i.e.,\ T_{g_1 \pm g_2}[\alpha^{-1}[(\Psi - \varepsilon)\{\beta((1-r)^{-1})\}]] < 3T_{f_1}(r) .$$

Therefore in view of Lemma 1.1 and Lemma 1.2, we obtain from above for all r, $0 < r < 1$, sufficiently close to 1 and any $\sigma > 1$ that

$$\frac{1}{9} \log M_{g_1 \pm g_2}[\frac{\alpha^{-1}[(\Psi - \varepsilon)\{\beta((1-r)^{-1})\}]}{2}] < T_{f_1}(r)$$

$$i.e.,\ \log M_{g_1 \pm g_2}[\frac{\alpha^{-1}[(\Psi - \varepsilon)[\beta(1-r)^{-1}]]}{2}]^{\frac{1}{9}} < T_{f_1}(r)$$

$$i.e.,\ \log M_{g_1 \pm g_2}[(\frac{\alpha^{-1}[(\Psi - \varepsilon)\{\beta((1-r)^{-1})\}]}{2})^{\frac{1}{\sigma}}] < T_{f_1}(r)$$

$$i.e.,\ T_{g_1 \pm g_2}[(\frac{\alpha^{-1}[(\Psi - \varepsilon)\{\beta((1-r)^{-1})\}]}{2})^{\frac{1}{\sigma}}] < T_{f_1}(r)$$

As $\varepsilon > 0$ is arbitrary, we get from above by letting $r \to 1$,

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \geq \Psi = \min\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\} . \tag{2.7}$$

Now without loss of any generality, we may consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and $g = g_1 \pm g_2$. Then in view of (2.7) we get that $\lambda_{(\alpha,\beta)}[f_1]_g \geq \lambda_{(\alpha,\beta)}[f_1]_{g_1}$. Further, $g_1 = (g \pm g_2)$ and in this case we obtain that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \geq \min\{\lambda_{(\alpha,\beta)}[f_1]_g, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\}$. As we assume that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$, therefore we have $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \geq \lambda_{(\alpha,\beta)}[f_1]_g$ and hence $\lambda_{(\alpha,\beta)}[f_1]_g = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \min\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\}$. Therefore, $\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_i} \mid i = 1,2$ provided $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \neq \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Thus the theorem follows. □

Theorem 2.4. *Let f_1 be a meromorphic function in the unit disc D and g_1, g_2 be any two entire functions such that f_1 is of regular generalized relative growth (α,β) with respect to at least any one of g_1 or g_2 .If $g_1 \pm g_2$ has the Property (D), then*

$$\rho_{(\alpha,\beta)}[f_1]_{g_1\pm g_2} \geq \min\{\rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_1]_{g_2}\} .$$

The sign of equality holds when $\rho_{(\alpha,\beta)}[f_1]_{g_i} < \rho_{(\alpha,\beta)}[f_1]_{g_j}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_j where $i = j = 1,2$ and $i \neq j$.

We omit the proof of Theorem 2.4 as it can easily be carried out in the line of Theorem 2.3.

Theorem 2.5. *Let f_1, f_2 be any two meromorphic functions in the unit disc D and g_1, g_2 be any two entire functions. Also let $g_1 \pm g_2$ has the Property (D). Then*

$$\rho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1\pm g_2}$$
$$\leq \max[\min\{\rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_1]_{g_2}\}, \min\{\rho_{(\alpha,\beta)}[f_2]_{g_1}, \rho_{(\alpha,\beta)}[f_2]_{g_2}\}]$$

when the following two conditions holds:
(i) $\rho_{(\alpha,\beta)}[f_1]_{g_i} < \rho_{(\alpha,\beta)}[f_1]_{g_j}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_j for $i = 1, 2$, $j = 1,2$ and $i \neq j$; and
(ii) $\rho_{g_i}^{(\alpha,\beta)}(f_2) < \rho_{g_j}^{(\alpha,\beta)}(f_2)$ with at least f_2 is of regular generalized relative growth (α,β) with respect to g_j for $i = 1, 2$, $j = 1,2$ and $i \neq j$.
The sign of equality holds when $\rho_{(\alpha,\beta)}[f_i]_{g_1} < \rho_{(\alpha,\beta)}[f_j]_{g_1}$ and $\rho_{(\alpha,\beta)}[f_i]_{g_2} < \rho_{(\alpha,\beta)}[f_j]_{g_2}$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$.

Proof. Let the conditions (i) and (ii) of the theorem hold. Therefore in view of Theorem 2.2 and Theorem 2.4 we get that

$$\begin{aligned} &\max[\min\{\rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_1]_{g_2}\}, \min\{\rho_{(\alpha,\beta)}[f_2]_{g_1}, \rho_{(\alpha,\beta)}[f_2]_{g_2}\}] \\ &= \max[\rho_{(\alpha,\beta)}[f_1]_{g_1\pm g_2}, \rho_{(\alpha,\beta)}[f_2]_{g_1\pm g_2}] \\ &\geq \rho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1\pm g_2} . \end{aligned} \tag{2.8}$$

Since $\rho_{(\alpha,\beta)}[f_i]_{g_1} < \rho_{(\alpha,\beta)}[f_j]_{g_1}$ and $\rho_{(\alpha,\beta)}[f_i]_{g_2} < \rho_{(\alpha,\beta)}[f_j]_{g_2}$ hold simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$, we obtain that

either $\min\{\rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_1]_{g_2}\} > \min\{\rho_{(\alpha,\beta)}[f_2]_{g_1}, \rho_{(\alpha,\beta)}[f_2]_{g_2}\}$ or

$\min\{\rho_{(\alpha,\beta)}[f_2]_{g_1}, \rho_{(\alpha,\beta)}[f_2]_{g_2}\} > \min\{\rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_1]_{g_2}\}$ holds.

Now in view of the conditions (i) and (ii) of the theorem, it follows from above that

either $\rho_{(\alpha,\beta)}[f_1]_{g_1\pm g_2} > \rho_{(\alpha,\beta)}[f_2]_{g_1\pm g_2}$ or $\rho_{(\alpha,\beta)}[f_2]_{g_1\pm g_2} > \rho_{(\alpha,\beta)}[f_1]_{g_1\pm g_2}$

which is the condition for holding equality in (2.8).

Hence the theorem follows. □

Theorem 2.6. *Let f_1, f_2 be any two meromorphic functions in the unit disc D and g_1, g_2 be any two entire functions. Also let g_1, g_2 and $g_1 \pm g_2$ satisfy the Property (D). Then we have*

$$\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1\pm g_2}$$
$$\geq \min[\max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\}, \{\max \lambda_{(\alpha,\beta)}[f_1]_{g_2}, \lambda_{(\alpha,\beta)}[f_2]_{g_2}\}]$$

when the following two conditions holds:
(i) Any one of $\lambda_{(\alpha,\beta)}[f_i]_{g_1} > \lambda_{(\alpha,\beta)}[f_j]_{g_1}$ hold and at least any one of f_j is of regular generalized relative growth (α,β) with respect to g_1 for $i = 1, 2$, $j = 1,2$ and $i \neq j$; and
(ii) Any one of $\lambda_{(\alpha,\beta)}[f_i]_{g_2} > \lambda_{(\alpha,\beta)}[f_j]_{g_2}$ hold and at least any one of f_j is of regular generalized relative growth (α,β) with respect to g_2 for $i = 1, 2$, $j = 1,2$ and $i \neq j$.
The equality holds when $\lambda_{(\alpha,\beta)}[f_1]_{g_i} < \lambda_{(\alpha,\beta)}[f_1]_{g_j}$ and $\lambda_{(\alpha,\beta)}[f_2]_{g_i} < \lambda_{(\alpha,\beta)}[f_2]_{g_j}$ hold simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$.

Proof. Suppose that the conditions (i) and (ii) of the theorem holds. Therefore in view of Theorem 2.1 and Theorem 2.3, we obtain that

$$\begin{aligned} &\min[\max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\}, \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_2}, \lambda_{(\alpha,\beta)}[f_2]_{g_2}\}] \\ &= \min[\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1}, \lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_2}] \\ &\geq \lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1\pm g_2} . \end{aligned} \tag{2.9}$$

Since $\lambda_{(\alpha,\beta)}[f_1]_{g_i} < \lambda_{(\alpha,\beta)}[f_1]_{g_j}$ and $\lambda_{(\alpha,\beta)}[f_2]_{g_i} < \lambda_{(\alpha,\beta)}[f_2]_{g_j}$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$, we get that

either $\max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\} < \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_2}, \lambda_{(\alpha,\beta)}[f_2]_{g_2}\}$ or

$\max\{\lambda_{(\alpha,\beta)}[f_1]_{g_2}, \lambda_{(\alpha,\beta)}[f_2]_{g_2}\} < \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\}$ holds.

Since condition (i) and (ii) of the theorem holds, it follows from above that

either $\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} < \lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_2}$ or $\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_2} < \lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1}$

which is the condition for holding equality in (2.9).

Hence the theorem follows. □

Theorem 2.7. *Let f_1, f_2 be any two meromorphic functions in the unit disc D and g_1 be any entire function such that at least f_1 or f_2 is of regular generalized relative growth (α,β) with respect to g_1. Also let g_1 satisfy the Property (D). Then we have*

$$\lambda_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1}\leq\max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1},\lambda_{(\alpha,\beta)}[f_2]_{g_1}\}.$$

The equality holds when any one of $\lambda_{(\alpha,\beta)}[f_i]_{g_1}>\lambda_{(\alpha,\beta)}[f_j]_{g_1}$ hold and at least any one of f_j is of regular generalized relative growth (α,β) with respect to g_1 where $i,j=1,2$ and $i\neq j$.

Proof. Since $T_{f_1\cdot f_2}(r)\leq T_{f_1}(r)+T_{f_2}(r)$,therefore applying the same procedure as adopted in Theorem 2.1 we get that

$$\lambda_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1}\leq\max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1},\lambda_{(\alpha,\beta)}[f_2]_{g_1}\}.$$

Now without loss of any generality, let $\lambda_{(\alpha,\beta)}[f_1]_{g_1}<\lambda_{(\alpha,\beta)}[f_2]_{g_1}$ and $f=f_1\cdot f_2$. Then $\lambda_{(\alpha,\beta)}[f]_{g_1}\leq\lambda_{(\alpha,\beta)}[f_2]_{g_1}$. Further, $f_2=\frac{f}{f_1}$ and $T_{f_1}(r)=T_{\frac{1}{f_1}}(r)+O(1)$. Therefore $T_{f_2}(r)\leq T_f(r)+T_{f_1}(r)+O(1)$ and in this case we obtain that $\lambda_{(\alpha,\beta)}[f_2]_{g_1}\leq\max\{\lambda_{(\alpha,\beta)}[f]_{g_1},\lambda_{(\alpha,\beta)}[f_1]_{g_1}\}$. As we assume that $\lambda_{(\alpha,\beta)}[f_1]_{g_1}<\lambda_{(\alpha,\beta)}[f_2]_{g_1}$, therefore we have $\lambda_{(\alpha,\beta)}[f_2]_{g_1}\leq\lambda_{(\alpha,\beta)}[f]_{g_1}$ and hence $\lambda_{(\alpha,\beta)}[f]_{g_1}=\lambda_{(\alpha,\beta)}[f_2]_{g_1}=\max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1},\lambda_{(\alpha,\beta)}[f_2]_{g_1}\}$. Therefore, $\lambda_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1}=\lambda_{(\alpha,\beta)}[f_i]_{g_1}\mid i=1,2$ provided $\lambda_{(\alpha,\beta)}[f_1]_{g_1}\neq\lambda_{(\alpha,\beta)}[f_2]_{g_1}$.

Hence the theorem follows. □

Next we prove the result for the quotient $\frac{f_1}{f_2}$, provided $\frac{f_1}{f_2}$ is meromorphic in the unit disc D.

Theorem 2.8. *Let f_1, f_2 be any two meromorphic functions in the unit disc D and g_1 be any entire function such that at least f_1 or f_2 is of regular generalized relative growth (α,β) with respect to g_1. Also let g_1 satisfy the Property (D). Then we have*

$$\lambda_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1}\leq\max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1},\lambda_{(\alpha,\beta)}[f_2]_{g_1}\},$$

provided $\frac{f_1}{f_2}$ is meromorphicin the unit disc D. The equality holds when at least f_2 is of regular generalized relative growth (α,β) with respect to g_1 and $\lambda_{(\alpha,\beta)}[f_1]_{g_1}\neq\lambda_{(\alpha,\beta)}[f_2]_{g_1}$.

Proof. Since $T_{f_2}(r)=T_{\frac{1}{f_2}}(r)+O(1)$ and $T_{\frac{f_1}{f_2}}(r)\leq T_{f_1}(r)+T_{\frac{1}{f_2}}(r)$, we get in view of Theorem 2.1 that

$$\lambda_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1}\leq\max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1},\lambda_{(\alpha,\beta)}[f_2]_{g_1}\}.\tag{2.10}$$

Now in order to prove the equality conditions, we discuss the following two cases:

Case I. Suppose $\frac{f_1}{f_2}(=h)$ satisfies the following condition

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1}<\lambda_{(\alpha,\beta)}[f_2]_{g_1},$$

and f_2 is of regular generalized relative growth (α,β) with respect to g_1.

Now if possible, let $\lambda_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1}<\lambda_{(\alpha,\beta)}[f_2]_{g_1}$. Therefore from $f_1=h\cdot f_2$ we get that $\lambda_{(\alpha,\beta)}[f_1]_{g_1}=\lambda_{(\alpha,\beta)}[f_2]_{g_1}$ which is a contradiction. Therefore $\lambda_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1}\geq\lambda_{(\alpha,\beta)}[f_2]_{g_1}$ and in view of (2.10), we get that

$$\lambda_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1}=\lambda_{(\alpha,\beta)}[f_2]_{g_1}.$$

Case II. Suppose $\frac{f_1}{f_2}(=h)$ satisfies the following condition

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1}>\lambda_{(\alpha,\beta)}[f_2]_{g_1},$$

and f_2 is of regular generalized relative growth (α,β) with respect to g_1.

Now from $f_1=h\cdot f_2$ we get that either $\lambda_{(\alpha,\beta)}[f_1]_{g_1}\leq\lambda_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1}$ or $\lambda_{(\alpha,\beta)}[f_1]_{g_1}\leq\lambda_{(\alpha,\beta)}[f_2]_{g_1}$. But according to our assumption $\lambda_{(\alpha,\beta)}[f_1]_{g_1}\nleq\lambda_{(\alpha,\beta)}[f_2]_{g_1}$. Therefore $\lambda_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1}\geq\lambda_{(\alpha,\beta)}[f_1]_{g_1}$ and in view of (2.10), we get that

$$\lambda_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1}=\lambda_{(\alpha,\beta)}[f_1]_{g_1}.$$

Hence the theorem follows. □

Now we state the following theorem which can easily be carried out in the line of Theorem 2.7 and Theorem 2.8 and therefore its proof is omitted.

Theorem 2.9. *Let f_1 and f_2 be any two meromorphic functions in the unit disc D and g_1 be any entire function such that such that $\rho_{(\alpha,\beta)}[f_1]_{g_1}$ and $\rho_{(\alpha,\beta)}[f_2]_{g_1}$ exist. Also let g_1 satisfy the Property (D). Then we have*

$$\rho_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1}\leq\max\{\rho_{(\alpha,\beta)}[f_1]_{g_1},\rho_{(\alpha,\beta)}[f_2]_{g_1}\}.$$

The equality holds when $\rho_{(\alpha,\beta)}[f_1]_{g_1}\neq\rho_{(\alpha,\beta)}[f_2]_{g_1}$. Similar results hold for the quotient $\frac{f_1}{f_2}$, provided $\frac{f_1}{f_2}$ is meromorphic in the unit disc D.

Theorem 2.10. *Let f_1 be a meromorphic function in the unit disc D and g_1, g_2 be any two entire functions such that $\lambda_{(\alpha,\beta)}[f_1]_{g_1}$ and $\lambda_{(\alpha,\beta)}[f_1]_{g_2}$ exist. Also let $g_1 \cdot g_2$ satisfy the Property (D). Then we have*

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} \geq \min\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\}.$$

The equality holds when any one of $\lambda_{(\alpha,\beta)}[f_1]_{g_i} < \lambda_{(\alpha,\beta)}[f_1]_{g_j}$ hold where $i, j = 1,2$ and $i \neq j$ and g_i satisfy the Property (D). Similar results hold for the quotient $\frac{g_1}{g_2}$, provided $\frac{g_1}{g_2}$ is entire and satisfies the Property (D). The equality holds when $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \neq \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and g_1 satisfy the Property (D).

Proof. Since $T_{g_1\cdot g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r)$, therefore applying the same procedure as adopted in Theorem 2.3 we get that

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} \geq \min\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\}.$$

Now without loss of any generality, we may consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and $g = g_1 \cdot g_2$. Then $\lambda_{(\alpha,\beta)}[f_1]_g \geq \lambda_{(\alpha,\beta)}[f_1]_{g_1}$. Further, $g_1 = \frac{g}{g_2}$ and and $T_{g_2}(r) = T_{\frac{1}{g_2}}(r) + O(1)$. Therefore $T_{g_1}(r) \leq T_g(r) + T_{g_2}(r) + O(1)$ and in this case we obtain that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \geq \min\{\lambda_{(\alpha,\beta)}[f_1]_g, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\}$. As we assume that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$, so we have $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \geq \lambda_{(\alpha,\beta)}[f_1]_g$ and hence $\lambda_{(\alpha,\beta)}[f_1]_g = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \min\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\}$. Therefore, $\lambda_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_i} \mid i = 1,2$ provided $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and g_1 satisfy the Property (D). Hence the first part of the theorem follows.

Now we prove our results for the quotient $\frac{g_1}{g_2}$, provided $\frac{g_1}{g_2}$ is entire and $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \neq \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Since $T_{g_2}(r) = T_{\frac{1}{g_2}}(r) + O(1)$ and $T_{\frac{g_1}{g_2}}(r) \leq T_{g_1}(r) + T_{\frac{1}{g_2}}(r)$, we get in view of Theorem 2.3 that

$$\lambda_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}} \geq \min\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\}. \tag{2.11}$$

Now in order to prove the equality conditions, we discuss the following two cases:

Case I. Suppose $\frac{g_1}{g_2}(=h)$ satisfies the following condition

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_1]_{g_2}.$$

Now if possible, let $\lambda_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}} > \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Therefore from $g_1 = h \cdot g_2$ we get that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}$, which is a contradiction. Therefore $\lambda_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}} \leq \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and in view of (2.11), we get that

$$\lambda_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}.$$

Case II. Suppose that $\frac{g_1}{g_2}(=h)$ satisfies the following condition

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}.$$

Therefore from $g_1 = h \cdot g_2$, we get that either $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \geq \lambda_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}}$ or $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \geq \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. But according to our assumption $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \not\geq \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Therefore $\lambda_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}} \leq \lambda_{(\alpha,\beta)}[f_1]_{g_1}$ and in view of (2.11), we get that

$$\lambda_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}} = \lambda_{(\alpha,\beta)}[f_1]_{g_1}.$$

Hence the theorem follows. □

Theorem 2.11. *Let f_1 be any meromorphic function in the unit disc D and g_1, g_2 be any two entire functions such that $\rho_{(\alpha,\beta)}[f_1]_{g_1}$ and $\rho_{(\alpha,\beta)}[f_1]_{g_2}$ exist. Further let f_1 be of regular generalized relative growth (α,β) with respect to at least any one of g_1 and g_2. Also let $g_1 \cdot g_2$ satisfies the Property (D). Then we have*

$$\rho_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} \geq \min\{\rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_1]_{g_2}\}.$$

The equality holds when any one of $\rho_{(\alpha,\beta)}[f_1]_{g_i} < \rho_{(\alpha,\beta)}[f_1]_{g_j}$ hold and at least f_1 is of regular generalized relative growth (α,β) with respect to any one of g_j where $i, j = 1,2$ and $i \neq j$ and g_i satisfies the Property (D).

Theorem 2.12. *Let f_1 be any meromorphic function in the unit disc D and g_1, g_2 be any two entire functions such that $\rho_{(\alpha,\beta)}[f_1]_{g_1}$ and $\rho_{(\alpha,\beta)}[f_1]_{g_2}$ exist. Further let f_1 be of regular generalized relative growth (α,β) with respect to at least any one of g_1 or g_2. Then we have*

$$\rho_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}} \geq \min\{\rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_1]_{g_2}\},$$

provided $\frac{g_1}{g_2}$ is entire and satisfies the Property (D). The equality holds when at least f_1 is of regular generalized relative growth (α,β) with respect to g_2, $\rho_{(\alpha,\beta)}[f_1]_{g_1} \neq \rho_{(\alpha,\beta)}[f_1]_{g_2}$ and g_1 satisfies the Property (D).

We omit the proof of Theorem 2.11 and Theorem 2.12 as those can easily be carried out in the line of Theorem 2.10.

Now we state the following four theorems without their proofs as those can easily be carried out in the line of Theorem 2.5 and Theorem 2.6 respectively.

Theorem 2.13. *Let f_1, f_2 be any two meromorphic functions in the unit disc D and g_1, g_2 be any two entire functions. Also let $g_1 \cdot g_2$ satisfy the Property (D). Then we have*

$$\begin{aligned} & \rho_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1 \cdot g_2} \\ \leq\ & \max[\min\{\rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_1]_{g_2}\}, \min\{\rho_{(\alpha,\beta)}[f_2]_{g_1}, \rho_{(\alpha,\beta)}[f_2]_{g_2}\}], \end{aligned}$$

when the following two conditions holds:
(i) Any one of $\rho_{(\alpha,\beta)}[f_1]_{g_i} < \rho_{(\alpha,\beta)}[f_1]_{g_j}$ hold and at least f_1 is of regular generalized relative growth (α,β) with respect to any one of g_j and g_i satisfy the Property (D) for $i = 1, 2$, $j = 1,2$ and $i \neq j$; and
(ii) Any one of $\rho_{(\alpha,\beta)}[f_2]_{g_i} < \rho_{(\alpha,\beta)}[f_2]_{g_j}$ hold and at least f_2 is of regular generalized relative growth (α,β) with respect to any one of g_j and g_i satisfy the Property (D) for $i = 1, 2$, $j = 1,2$ and $i \neq j$.
The quality holds when $\rho_{(\alpha,\beta)}[f_i]_{g_1} < \rho_{(\alpha,\beta)}[f_j]_{g_1}$ and $\rho_{(\alpha,\beta)}[f_i]_{g_2} < \rho_{(\alpha,\beta)}[f_j]_{g_2}$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$.

Theorem 2.14. *Let f_1, f_2 be any two meromorphic functions in the unit disc D and g_1, g_2 be any two entire functions. Also let $g_1 \cdot g_2$, g_1 and g_2 satisfy the Property (D). Then we have*

$$\lambda_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1 \cdot g_2} \geq \min[\max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\}, \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_2}, \lambda_{(\alpha,\beta)}[f_2]_{g_2}\}]$$

when the following two conditions holds:
(i) Any one of $\lambda_{(\alpha,\beta)}[f_i]_{g_1} > \lambda_{(\alpha,\beta)}[f_j]_{g_1}$ hold and at least any one of f_j is of regular generalized relative growth (α,β) with respect to g_1 for $i = 1, 2$, $j = 1,2$ and $i \neq j$; and
(ii) Any one of $\lambda_{(\alpha,\beta)}[f_i]_{g_2} > \lambda_{(\alpha,\beta)}[f_j]_{g_2}$ hold and at least any one of f_j is of regular generalized relative growth (α,β) with respect to g_2 for $i = 1, 2$, $j = 1,2$ and $i \neq j$.
The equality holds when $\lambda_{(\alpha,\beta)}[f_1]_{g_i} < \lambda_{(\alpha,\beta)}[f_1]_{g_j}$ and $\lambda_{(\alpha,\beta)}[f_2]_{g_i} < \lambda_{(\alpha,\beta)}[f_2]_{g_j}$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$.

Theorem 2.15. *Let f_1, f_2 be any two meromorphic functions in the unit disc D and g_1, g_2 be any two entire functions such that $\frac{f_1}{f_2}$ is meromorphic and $\frac{g_1}{g_2}$ is entire. Also let $\frac{g_1}{g_2}$ satisfy the Property (D). Then we have*

$$\rho_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{\frac{g_1}{g_2}} \leq \max[\min\{\rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_1]_{g_2}\}, \min\{\rho_{(\alpha,\beta)}[f_2]_{g_1}, \rho_{(\alpha,\beta)}[f_2]_{g_2}\}$$

when the following two conditions holds:
(i) At least f_1 is of regular generalized relative growth (α,β) with respect to g_2 and $\rho_{(\alpha,\beta)}[f_1]_{g_1} \neq \rho_{(\alpha,\beta)}[f_1]_{g_2}$; and
(ii) At least f_2 is of regular generalized relative growth (α,β) with respect to g_2 and $\rho_{(\alpha,\beta)}[f_2]_{g_1} \neq \rho_{(\alpha,\beta)}[f_2]_{g_2}$.
The equality holds when $\rho_{(\alpha,\beta)}[f_i]_{g_1} < \rho_{(\alpha,\beta)}[f_j]_{g_1}$ and $\rho_{(\alpha,\beta)}[f_i]_{g_2} < \rho_{(\alpha,\beta)}[f_j]_{g_2}$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$.

Theorem 2.16. *Let f_1, f_2 be any two meromorphic functions in the unit disc D and g_1, g_2 be any two entire functions such that $\frac{f_1}{f_2}$ is meromorphic and $\frac{g_1}{g_2}$ is entire. Also let $\frac{g_1}{g_2}$, g_1 and g_2 satisfy the Property (D). Then we have*

$$\lambda_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{\frac{g_1}{g_2}} \geq \min[\max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\}, \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_2}, \lambda_{(\alpha,\beta)}[f_2]_{g_2}\}]$$

when the following two conditions hold:
(i) At least f_2 is of regular generalized relative growth (α,β) with respect to g_1 and $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \neq \lambda_{(\alpha,\beta)}[f_2]_{g_1}$; and
(ii) At least f_2 is of regular generalized relative growth (α,β) with respect to g_2 and $\lambda_{(\alpha,\beta)}[f_1]_{g_2} \neq \lambda_{(\alpha,\beta)}[f_2]_{g_2}$.
The equality holds when $\lambda_{(\alpha,\beta)}[f_1]_{g_i} < \lambda_{(\alpha,\beta)}[f_1]_{g_j}$ and $\lambda_{(\alpha,\beta)}[f_2]_{g_i} < \lambda_{(\alpha,\beta)}[f_2]_{g_j}$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$.

Next we intend to find out the sum and product theorems of generalized relative type (α,β) (respectively generalized relative lower type (α,β)) and generalized relative weak type (α,β) of meromorphic function in the unit disc D with respect to an entire function taking into consideration of the above theorems.

Theorem 2.17. *Let f_1, f_2 be any two meromorphic functions in the unit disc D and g_1, g_2 be any two entire functions. Also let $\rho_{(\alpha,\beta)}[f_1]_{g_1}$, $\rho_{(\alpha,\beta)}[f_2]_{g_1}$, $\rho_{(\alpha,\beta)}[f_1]_{g_2}$ and$\rho_{(\alpha,\beta)}[f_2]_{g_2}$ be all non-zero and finite.*
***(A)** If any one of $\rho_{(\alpha,\beta)}[f_i]_{g_1} > \rho_{(\alpha,\beta)}[f_j]_{g_1}$ hold for i, $j = 1,2$; $i \neq j$, and g_1 has the Property (D), then*

$\sigma_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \sigma_{(\alpha,\beta)}[f_i]_{g_1}$ *and* $\overline{\sigma}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \overline{\sigma}_{(\alpha,\beta)}[f_i]_{g_1} \mid i = 1,2.$

***(B)** If any one of $\rho_{(\alpha,\beta)}[f_1]_{g_i} < \rho_{(\alpha,\beta)}[f_1]_{g_j}$ hold and at least f_1 is of regular generalized relative growth (α,β) with respect to any one of g_j for i, $j = 1,2$; $i \neq j$ and $g_1 \pm g_2$ has the Property (D), then*

$\sigma_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \sigma_{(\alpha,\beta)}[f_1]_{g_i}$ *and* $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_i} \mid i = 1,2.$

***(C)** Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:*
(i) Any one of $\rho_{(\alpha,\beta)}[f_1]_{g_i} < \rho_{(\alpha,\beta)}[f_1]_{g_j}$ hold and at least f_1 is of regular generalized relative growth (α,β) with respect to any one of g_j for $i = 1, 2$, $j = 1,2$ and $i \neq j$;
(ii) Any one of $\rho_{(\alpha,\beta)}[f_2]_{g_i} < \rho_{(\alpha,\beta)}[f_2]_{g_j}$ hold and at least f_2 is of regular generalized relative growth (α,β) with respect to any one of g_j for $i = 1, 2$, $j = 1,2$ and $i \neq j$;
(iii) $\rho_{(\alpha,\beta)}[f_i]_{g_1} > \rho_{(\alpha,\beta)}[f_j]_{g_1}$ and $\rho_{(\alpha,\beta)}[f_i]_{g_2} > \rho_{(\alpha,\beta)}[f_j]_{g_2}$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$;
(iv) $\rho_{(\alpha,\beta)}[f_l]_{g_m} =$

$\max[\min\{\rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_1]_{g_2}\}, \min\{\rho_{(\alpha,\beta)}[f_2]_{g_1}, \rho_{(\alpha,\beta)}[f_2]_{g_2}\}] \mid l,m = 1,2$, *and* $g_1 \pm g_2$ *has the Property (D); then*

$$\sigma_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1 \pm g_2} = \sigma_{(\alpha,\beta)}[f_l]_{g_m} \mid l,m = 1,2$$

and

$$\overline{\sigma}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1 \pm g_2} = \overline{\sigma}_{(\alpha,\beta)}[f_l]_{g_m} \mid l,m = 1,2.$$

Proof. From the definition of generalized relative type (α,β) and generalized relative lower type (α,β) of meromorphic function f_k in the unit disc D with respect to an entire function g_l, we have for all r, $0 < r < 1$, sufficiently close to 1 that

$$T_{f_k}(r) \le T_{g_l}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_k]_{g_l} + \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_k]_{g_l}}\})], \tag{2.12}$$

$$T_{f_k}(r) \ge T_{g_l}[\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[f_k]_{g_l} - \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_k]_{g_l}}\})], \tag{2.13}$$

and for a sequence of values of r tending to 1, we obtain that

$$T_{f_k}(r) \ge T_{g_l}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_k]_{g_l} - \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_k]_{g_l}}\})], \tag{2.14}$$

and

$$T_{f_k}(r) \le T_{g_l}[\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[f_k]_{g_l} + \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_k]_{g_l}}\})], \tag{2.15}$$

where $\varepsilon > 0$ is any arbitrary positive number $k = 1,2$ and $l = 1,2$.

Case I. Suppose that $\rho_{(\alpha,\beta)}[f_1]_{g_1} > \rho_{(\alpha,\beta)}[f_2]_{g_1}$ hold. Also let $\varepsilon(>0)$ be arbitrary. Since $T_{f_1 \pm f_2}(r) \le T_{f_1}(r) + T_{f_2}(r) + O(1)$, so in view of (2.12), we get for all r, $0 < r < 1$, sufficiently close to 1 that

$$T_{f_1 \pm f_2}(r) \le (1+A) \times T_{g_1}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_1]_{g_1} + \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\})]. \tag{2.16}$$

where $A = \dfrac{T_{g_1}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_2]_{g_1} + \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_2]_{g_1}}\})] + O(1)}{T_{g_1}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_1]_{g_1} + \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\})]}$,

and in view of $\rho_{(\alpha,\beta)}[f_1]_{g_1} > \rho_{(\alpha,\beta)}[f_2]_{g_1}$, and for all r, $0 < r < 1$, sufficiently close to 1, we can make the term A sufficiently small, i.e. $A < \varepsilon_1$. Hence for any $\delta = 1 + \varepsilon_1$, it follows from (2.16) for all r, $0 < r < 1$, sufficiently close to 1 that

$$T_{f_1 \pm f_2}(r) \le T_{g_1}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_1]_{g_1} + \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\})] \cdot (1+\varepsilon_1)$$

$$\text{i.e., } T_{f_1 \pm f_2}(r) \le T_{g_1}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_1]_{g_1} + \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\})] \cdot \delta.$$

Hence making $\delta \to 1+$, we get in view of Theorem 2.2, $\rho_{(\alpha,\beta)}[f_1]_{g_1} > \rho_{(\alpha,\beta)}[f_2]_{g_1}$ and above for all r, $0 < r < 1$, sufficiently close to 1 that

$$\limsup_{r \to \infty} \frac{\exp\left(\alpha\left(T_{g_1}^{-1}\left(T_{f_1 \pm f_2}(r)\right)\right)\right)}{\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1}}} \le \sigma_{(\alpha,\beta)}[f_1]_{g_1}$$

$$\text{i.e., } \sigma_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \le \sigma_{(\alpha,\beta)}[f_1]_{g_1}. \tag{2.17}$$

Now we may consider that $f = f_1 \pm f_2$. Since $\rho_{(\alpha,\beta)}[f_1]_{g_1} > \rho_{(\alpha,\beta)}[f_2]_{g_1}$ hold. Then $\sigma_{(\alpha,\beta)}[f]_{g_1} = \sigma_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \le \sigma_{(\alpha,\beta)}[f_1]_{g_1}$. Further, let $f_1 = (f \pm f_2)$. Therefore in view of Theorem 2.2 and $\rho_{(\alpha,\beta)}[f_1]_{g_1} > \rho_{(\alpha,\beta)}[f_2]_{g_1}$, we obtain that $\rho_{(\alpha,\beta)}[f]_{g_1} > \rho_{(\alpha,\beta)}[f_2]_{g_1}$ holds. Hence in view of (2.17) $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \le \sigma_{(\alpha,\beta)}[f]_{g_1} = \sigma_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1}$. Therefore $\sigma_{(\alpha,\beta)}[f]_{g_1} = \sigma_{(\alpha,\beta)}[f_1]_{g_1} \Rightarrow \sigma_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \sigma_{(\alpha,\beta)}[f_1]_{g_1}$.

Similarly, if we consider $\rho_{(\alpha,\beta)}[f_1]_{g_1} < \rho_{(\alpha,\beta)}[f_2]_{g_1}$, then one can easily verify that $\sigma_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \sigma_{(\alpha,\beta)}[f_2]_{g_1}$.

Case II. Let us consider that $\rho_{(\alpha,\beta)}[f_1]_{g_1} > \rho_{(\alpha,\beta)}[f_2]_{g_1}$ hold. Also let $\varepsilon(>0)$ are arbitrary. Since $T_{f_1 \pm f_2}(r) \le T_{f_1}(r) + T_{f_2}(r) + O(1)$ for all large r, from (2.12) and (2.15), we get for a sequence of values of r tending to 1, that

$$T_{f_1 \pm f_2}(r) \le T_{g_1}[\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} + \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\})](1+B). \tag{2.18}$$

where $B = \dfrac{T_{g_1}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_2]_{g_1} + \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_2]_{g_1}}\})] + O(1)}{T_{g_1}[\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} + \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\})]}$, and in view of $\rho_{(\alpha,\beta)}[f_1]_{g_1} > \rho_{(\alpha,\beta)}[f_2]_{g_1}$, we can make the term B sufficiently small for a sequence of values of r tending to 1 and therefore using the similar technique for as executed in the proof of Case I we get from (2.18) that $\overline{\sigma}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1}$ when $\rho_{(\alpha,\beta)}[f_1]_{g_1} > \rho_{(\alpha,\beta)}[f_2]_{g_1}$ hold. Likewise, if we consider $\rho_{(\alpha,\beta)}[f_1]_{g_1} < \rho_{(\alpha,\beta)}[f_2]_{g_1}$, then one can easily verify that $\overline{\sigma}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1}$.

Thus combining Case I and Case II, we obtain the first part of the theorem.

Case III. Let us consider that $\rho_{(\alpha,\beta)}[f_1]_{g_1} < \rho_{(\alpha,\beta)}[f_1]_{g_2}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_2. We can make the term

$$C = \frac{T_{g_2}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\})] + O(1)}{T_{g_2}[\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2} - \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_1]_{g_2}}\})]}$$

sufficiently small for all r,where $0 < r < 1$, sufficiently close to 1, since $\rho_{(\alpha,\beta)}[f_1]_{g_1} < \rho_{(\alpha,\beta)}[f_1]_{g_2}$. Hence $C < \varepsilon_1$.

As $T_{g_1 \pm g_2}(r) \le T_{g_1}(r) + T_{g_2}(r) + O(1)$, we get that

$$T_{g_1 \pm g_2}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\})] \le$$
$$T_{g_1}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\})] +$$
$$T_{g_2}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\})] + O(1).$$

Therefore for any $\delta = 1 + \varepsilon_1$, we obtain in view of $C < \varepsilon_1$, (2.13) and (2.14) for a sequence of values of r tending to 1 that

$$T_{g_1 \pm g_2}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\})] \le \delta T_{f_1}(r)$$

Now making $\delta \to 1+$, we obtain from above for a sequence of values of r tending to 1 that

$$(\sigma_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2}} < \exp\left(\alpha\left(T^{-1}_{g_1 \pm g_2}(T_{f_1}(r))\right)\right)$$

Since $\varepsilon > 0$ is arbitrary, we find that

$$\sigma_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \ge \sigma_{(\alpha,\beta)}[f_1]_{g_1}. \tag{2.19}$$

Now we may consider that $g = g_1 \pm g_2$. Also $\rho_{(\alpha,\beta)}[f_1]_{g_1} < \rho_{(\alpha,\beta)}[f_1]_{g_2}$ and at least f_1 is of regular generalized relative growth (α,β) with respect to g_2. Then $\sigma_{(\alpha,\beta)}[f_1]_{g} = \sigma_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \ge \sigma_{(\alpha,\beta)}[f_1]_{g_1}$. Further let $g_1 = (g \pm g_2)$. Therefore in view of Theorem 2.4 and $\rho_{(\alpha,\beta)}[f_1]_{g_1} < \rho_{(\alpha,\beta)}[f_1]_{g_2}$, we obtain that $\rho_{(\alpha,\beta)}[f_1]_{g} < \rho_{(\alpha,\beta)}[f_1]_{g_2}$ as at least f_1 is of regular generalized relative growth (α,β) with respect to g_2. Hence in view of (2.19), $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \ge \sigma_{(\alpha,\beta)}[f_1]_{g} = \sigma_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2}$. Therefore $\sigma_{(\alpha,\beta)}[f_1]_{g} = \sigma_{(\alpha,\beta)}[f_1]_{g_1} \Rightarrow \sigma_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \sigma_{(\alpha,\beta)}[f_1]_{g_1}$.

Similarly if we consider $\rho_{(\alpha,\beta)}[f_1]_{g_1} > \rho_{(\alpha,\beta)}[f_1]_{g_2}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_1, then $\sigma_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \sigma_{(\alpha,\beta)}[f_1]_{g_2}$.

Case IV. In this case suppose that $\rho_{(\alpha,\beta)}[f_1]_{g_1} < \rho_{(\alpha,\beta)}[f_1]_{g_2}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_2. we can also make the term $D = \frac{T_{g_2}[\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\})] + O(1)}{T_{g_2}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_1]_{g_2} - \varepsilon)[\exp(\beta(1-r)^{-1})]^{\rho_{(\alpha,\beta)}[f_1]_{g_2}}\})]}$ sufficiently small by taking r sufficiently close to 1 as $\rho_{(\alpha,\beta)}[f_1]_{g_1} < \rho_{(\alpha,\beta)}[f_1]_{g_2}$. So $D < \varepsilon_1$ for all r, $0 < r < 1$, sufficiently close to 1. As $T_{g_1 \pm g_2}(r) \le T_{g_1}(r) + T_{g_2}(r) + O(1)$ for all large r, therefore from (2.13), we get for all r, $0 < r < 1$, sufficiently close to 1 that

$$T_{g_1 \pm g_2}[\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\})] \le$$
$$T_{g_1}[\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\})] +$$
$$T_{g_2}[\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\})] + O(1)$$
$$i.e.,\ T_{g_1 \pm g_2}[\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\})]$$
$$\le (1 + \varepsilon_1) T_{f_1}(r), \tag{2.20}$$

and therefore using the similar technique for as executed in the proof of Case III we get from (2.20) that $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1}$ where $\rho_{(\alpha,\beta)}[f_1]_{g_1} < \rho_{(\alpha,\beta)}[f_1]_{g_2}$ and at least f_1 is of regular generalized relative growth (α,β) with respect to g_2.

Likewise if we consider $\rho_{(\alpha,\beta)}[f_1]_{g_1} > \rho_{(\alpha,\beta)}[f_1]_{g_2}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_1, then $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}$.

Thus combining Case III and Case IV, we obtain the second part of the theorem.

The third part of the theorem is a natural consequence of Theorem 2.5 and the first part and second part of the theorem. Hence its proof is omitted. □

Theorem 2.18. *Let f_1, f_2 be any two meromorphic functions in the unit disc D and g_1, g_2 be any two entire functions. Also let $\lambda_{(\alpha,\beta)}[f_1]_{g_1}$, $\lambda_{(\alpha,\beta)}[f_2]_{g_1}$, $\lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and $\lambda_{(\alpha,\beta)}[f_2]_{g_2}$ be all nonzero and finite.*
***(A)** Any one of $\lambda_{(\alpha,\beta)}[f_i]_{g_1} > \lambda_{(\alpha,\beta)}\left[f_j\right]_{g_1}$ hold and at least any one of f_j is of regular generalized relative growth (α,β) with respect to g_1 for $i,\ j = 1,2$; $i \neq j$, and g_1 has the Property (D), then*

$$\tau_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \tau_{(\alpha,\beta)}[f_i]_{g_1} \text{ and } \overline{\tau}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \overline{\tau}_{(\alpha,\beta)}[f_i]_{g_1} \mid i = 1,2.$$

***(B)** Any one of $\lambda_{(\alpha,\beta)}[f_1]_{g_i} < \lambda_{(\alpha,\beta)}[f_1]_{g_j}$ hold for $i,\ j = 1,2$; $i \neq j$ and $g_1 \pm g_2$ has the Property (D), then*

$$\tau_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \tau_{(\alpha,\beta)}[f_1]_{g_i} \text{ and } \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_i} \mid i = 1,2.$$

***(C)** Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:*
(i) Any one of $\rho_{(\alpha,\beta)}[f_i]_{g_1} > \rho_{(\alpha,\beta)}\left[f_j\right]_{g_1}$ hold and at least any one of f_j is of regular generalized relative growth (α,β) with respect to g_1 for $i,\ j = 1,2$ and $i \neq j$;
(ii) Any one of $\rho_{(\alpha,\beta)}[f_i]_{g_2} > \rho_{(\alpha,\beta)}\left[f_j\right]_{g_2}$ hold and at least any one of f_j is of regular generalized relative growth (α,β) with respect to g_2 for $i,\ j = 1,2$ and $i \neq j$;
(iii) $\rho_{(\alpha,\beta)}[f_1]_{g_i} < \rho_{(\alpha,\beta)}[f_1]_{g_j}$ and $\rho_{(\alpha,\beta)}[f_2]_{g_i} < \rho_{(\alpha,\beta)}[f_2]_{g_j}$ holds simultaneously for $i,\ j = 1,2$ and $i \neq j$;
(iv) $\lambda_{(\alpha,\beta)}[f_l]_{g_m} =$
$\min[\max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\}, \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_2}, \lambda_{(\alpha,\beta)}[f_2]_{g_2}\}] \mid l,m = 1,2$ *and $g_1 \pm g_2$ has the Property (D)*
then we have

$$\tau_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1 \pm g_2} = \tau_{(\alpha,\beta)}[f_l]_{g_m} \mid l,m = 1,2$$

and

$$\overline{\tau}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1 \pm g_2} = \overline{\tau}_{(\alpha,\beta)}[f_l]_{g_m} \mid l,m = 1,2.$$

Proof. For any arbitrary positive number $\varepsilon(>0)$, we have for all r, $0 < r < 1$, sufficiently close to 1 that

$$T_{f_k}(r) \leq T_{g_l}[\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[f_k]_{g_l} + \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\lambda_{(\alpha,\beta)}[f_k]_{g_l}}\})], \tag{2.21}$$

$$T_{f_k}(r) \geq T_{g_l}[\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[f_k]_{g_l} - \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\lambda_{(\alpha,\beta)}[f_k]_{g_l}}\})], \tag{2.22}$$

and for a sequence of values of r tending to 1, we obtain that

$$T_{f_k}(r) \geq T_{g_l}[\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[f_k]_{g_l} - \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\lambda_{(\alpha,\beta)}[f_k]_{g_l}}\})], \tag{2.23}$$

and

$$T_{f_k}(r) \leq T_{g_l}[\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[f_k]_{g_l} + \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\lambda_{(\alpha,\beta)}[f_k]_{g_l}}\})] \tag{2.24}$$

where $k = 1,2$ and $l = 1,2$.

Case I. Let $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ with at least f_2 is of regular generalized relative growth (α,β) with respect to g_1. Also let $\varepsilon(>0)$ be arbitrary. Since $T_{f_1 \pm f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r) + O(1)$, we get from (2.21) and (2.24), for a sequence of values of r tending to 1, that

$$T_{f_1 \pm f_2}(r) \leq$$

$$T_{g_1}[\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[f_1]_{g_1} + \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\})](1+E). \tag{2.25}$$

where $E = \dfrac{T_{g_1}[\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1} + \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\lambda_{(\alpha,\beta)}[f_2]_{g_1}}\})] + O(1)}{T_{g_1}[\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[f_1]_{g_1} + \varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\})]}$

and in view of $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$, we can make the term E sufficiently small by taking r sufficiently close to 1. Now with the help of Theorem 2.1 and using the similar technique of Case I of Theorem 2.17, we get from (2.25) that

$$\tau_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \leq \tau_{(\alpha,\beta)}[f_1]_{g_1}. \tag{2.26}$$

Further, we may consider that $f = f_1 \pm f_2$. Also suppose that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ and at least f_2 is of regular generalized relative growth (α,β) with respect to g_1. Then $\tau_{(\alpha,\beta)}[f]_{g_1} = \tau_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \leq \tau_{(\alpha,\beta)}[f_1]_{g_1}$. Now let $f_1 = (f \pm f_2)$. Therefore in view of Theorem 2.1, $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ and at least f_2 is of regular generalized relative growth (α,β) with respect to g_1, we obtain that $\lambda_{(\alpha,\beta)}[f]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ holds. Hence in view of (2.26), $\tau_{(\alpha,\beta)}[f_1]_{g_1} \leq \tau_{(\alpha,\beta)}[f]_{g_1} = \tau_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1}$. Therefore $\tau_{(\alpha,\beta)}[f]_{g_1} = \tau_{(\alpha,\beta)}[f_1]_{g_1} \Rightarrow \tau_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \tau_{(\alpha,\beta)}[f_1]_{g_1}$.

Similarly, if we consider $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_1 then one can easily verify that $\tau_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \tau_{(\alpha,\beta)}[f_2]_{g_1}$.

Case II. Let us consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ with at least f_2 is of regular generalized relative growth (α,β) with respect to g_1. Also let $\varepsilon\,(>0)$ be arbitrary. As $T_{f_1\pm f_2}(r) \le T_{f_1}(r) + T_{f_2}(r) + O(1)$, we obtain from (2.21) for all r, $0<r<1$, sufficiently close to 1 that

$$T_{f_1\pm f_2}(r) \le$$

$$T_{g_1}[\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\})](1+F). \tag{2.27}$$

where $F = \dfrac{T_{g_1}[\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}+\varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\lambda_{(\alpha,\beta)}[f_2]_{g_1}}\})]+O(1)}{T_{g_1}[\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\})]}$,

and in view of $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$, we can make the term F sufficiently small by taking r sufficiently close to 1 and therefore for similar reasoning of Case I we get from (2.27) that $\overline{\tau}_{(\alpha,\beta)}[f_1\pm f_2]_{g_1} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}$ when $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ and at least f_2 is of regular generalized relative growth (α,β) with respect to g_1.

Likewise, if we consider $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_1 then one can easily verify that $\overline{\tau}_{(\alpha,\beta)}[f_1\pm f_2]_{g_1} = \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}$.

Thus combining Case I and Case II, we obtain the first part of the theorem.

Case III. Let us consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Therefore we can make the term $G = \frac{T_{g_2}[\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\})]+O(1)}{T_{g_2}[\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[f_1]_{g_2}-\varepsilon)[\exp(\beta(1-r)^{-1})]^{\lambda_{(\alpha,\beta)}[f_1]_{g_2}}\})]}$ sufficiently small by taking r sufficiently close to 1, since $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. So $G<\varepsilon_1$. Since $T_{g_1\pm g_2}(r) \le T_{g_1}(r) + T_{g_2}(r) + O(1)$, we get from (2.22) for all r, $0<r<1$, sufficiently close to 1 that

$$T_{g_1\pm g_2}[\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\})] \le$$

$$T_{g_1}[\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\})]+$$

$$T_{g_2}[\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\})]+O(1)$$

$$i.e.,\ T_{g_1\pm g_2}[\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\})]$$

$$\le (1+\varepsilon_1)T_{f_1}(r). \tag{2.28}$$

Therefore in view of Theorem 2.3 and using the similar technique of Case III of Theorem 2.17, we get from (2.28) that

$$\tau_{(\alpha,\beta)}[f_1]_{g_1\pm g_2} \ge \tau_{(\alpha,\beta)}[f_1]_{g_1}. \tag{2.29}$$

Further, we may consider that $g = g_1 \pm g_2$. As $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$, so $\tau_{(\alpha,\beta)}[f_1]_{g} = \tau_{(\alpha,\beta)}[f_1]_{g_1\pm g_2} \ge \tau_{(\alpha,\beta)}[f_1]_{g_1}$. Further let $g_1 = (g \pm g_2)$. Therefore in view of Theorem 2.3 and $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ we obtain that $\lambda_{(\alpha,\beta)}[f_1]_{g} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ holds. Hence in view of (2.29) $\tau_{(\alpha,\beta)}[f_1]_{g_1} \ge \tau_{(\alpha,\beta)}[f_1]_{g} = \tau_{(\alpha,\beta)}[f_1]_{g_1\pm g_2}$. Therefore $\tau_{(\alpha,\beta)}[f_1]_{g} = \tau_{(\alpha,\beta)}[f_1]_{g_1} \Rightarrow \tau_{(\alpha,\beta)}[f_1]_{g_1\pm g_2} = \tau_{(\alpha,\beta)}[f_1]_{g_1}$.

Likewise, if we consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_1]_{g_2}$, then one can easily verify that $\tau_{(\alpha,\beta)}[f_1]_{g_1\pm g_2} = \tau_{(\alpha,\beta)}[f_1]_{g_2}$.

Case IV. In this case further we consider $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Further we can make the term $H = \frac{T_{g_2}\left[\alpha^{-1}\left(\log\left\{\left(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon\right)[\exp(\beta(1-r)^{-1})]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\right.\right.\right.}{T_{g_2}\left[\alpha^{-1}\left(\log\left\{\left(\tau_{(\alpha,\beta)}[f_1]_{g_2}-\varepsilon\right)[\exp(\beta(1-r)^{-1})]^{\lambda_{(\alpha,\beta)}[f_1}\right.\right.\right.}$ sufficiently small by taking r sufficiently close to 1, since $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Therefore $H<\varepsilon_1$ for all r, $0<r<1$, sufficiently close to 1. As $T_{g_1\pm g_2}(r) \le T_{g_1}(r) + T_{g_2}(r) + O(1)$, hence we obtain from (2.22) and (2.23), for a sequence of values of r tending to 1 that

$$T_{g_1\pm g_2}[\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\})] \le$$

$$T_{g_1}[\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\})]+$$

$$T_{g_2}[\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\})]+O(1)$$

$$i.e.,\ T_{g_1\pm g_2}[\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon)\left[\exp(\beta(1-r)^{-1})\right]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\})]$$

$$\le (1+\varepsilon_1)T_{f_1}(r), \tag{2.30}$$

and therefore using the similar technique for as executed in the proof of Case IV of Theorem 2.17, we get from (2.30) that $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1\pm g_2} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}$ when $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$.

Similarly, if we consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_1]_{g_2}$, then one can easily verify that $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1\pm g_2} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2}$.

Thus combining Case III and Case IV, we obtain the second part of the theorem.

The proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem 2.6 and the above cases. □

In the next two theorems we reconsider the equalities in Theorem 2.1 to Theorem 2.4 under somewhat different conditions.

Theorem 2.19. *Let f_1, f_2 be any two meromorphic functions in the unit disc D and g_1, g_2 be any two entire functions.*
***(A)** The following condition is assumed to be satisfied:*
(i) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_1}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1}$ holds and g_1 has the Property (D), then

$$\rho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1}.$$

***(B)** The following conditions are assumed to be satisfied:*
(i) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_1}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}$ holds and $g_1 \pm g_2$ has the Property (D);
(ii) f_1 is of regular generalized relative growth (α,β) with respect to at least any one of g_1 or g_2, then

$$\rho_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2}.$$

Proof. Let f_1, f_2, g_1 and g_2 be any four entire functions satisfying the conditions of the theorem.

Case I. Suppose that $\rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1}$ $(0 < \rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_2]_{g_1} < \infty)$. Now in view of Theorem 2.2 it is easy to see that $\rho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \leq \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1}$. If possible let

$$\rho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} < \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1}. \tag{2.31}$$

Let $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_1}$. Then in view of the first part of Theorem 2.17 and (2.31) we obtain that $\sigma_{(\alpha,\beta)}[f_1]_{g_1} = \sigma_{(\alpha,\beta)}[f_1 \pm f_2 \mp f_2]_{g_1} = \sigma_{(\alpha,\beta)}[f_2]_{g_1}$ which is a contradiction. Hence $\rho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1}$. Similarly with the help of the first part of Theorem 2.17, one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}$. This proves the first part of the theorem.

Case II. Let us consider that $\rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2}$ $(0 < \rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_1]_{g_2} < \infty)$, f_1 is of regular generalized relative growth (α,β) with respect to at least any one of g_1 or g_2 and $(g_1 \pm g_2)$ and $g_1 \pm g_2$ satisfy the Property (D). Therefore in view of Theorem 2.4, it follows that $\rho_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \geq \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2}$ and if possible let

$$\rho_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} > \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2}. \tag{2.32}$$

Let us consider that $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_1]_{g_2}$. Then. in view of the proof of the second part of Theorem 2.17 and (2.32) we obtain that $\sigma_{(\alpha,\beta)}[f_1]_{g_1} = \sigma_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2 \mp g_2} = \sigma_{(\alpha,\beta)}[f_1]_{g_2}$ which is a contradiction. Hence $\rho_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2}$. Also in view of the proof of second part of Theorem 2.17 one can derive the same conclusion for the condition $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}$ and therefore the second part of the theorem is established. □

Theorem 2.20. *Let f_1, f_2 be any two meromorphic functions in the unit disc D and g_1, g_2 be any two entire functions.*
***(A)** The following conditions are assumed to be satisfied:*
(i) $(f_1 \pm f_2)$ is of regular generalized relative growth (α,β) with respect to at least any one of g_1 and g_2; and g_1, g_2, $g_1 \pm g_2$ have the Property (D);
(ii) Either $\sigma_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_1 \pm f_2]_{g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_2}$;
(iii) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_1}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1}$;
(iv) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_2} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_2}$; then

$$\rho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1 \pm g_2} = \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2} = \rho_{(\alpha,\beta)}[f_2]_{g_2}.$$

***(B)** The following conditions are assumed to be satisfied:*
(i) f_1 and f_2 are of regular generalized relative growth (α,β) with respect to at least any one of g_1 or g_2, and $g_1 \pm g_2$ has the Property (D);
(ii) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_1 \pm g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1 \pm g_2}$;
(iii) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_1]_{g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}$;
(iv) Either $\sigma_{(\alpha,\beta)}[f_2]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_2}$; then

$$\rho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1 \pm g_2} = \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2} = \rho_{(\alpha,\beta)}[f_2]_{g_2}.$$

We omit the proof of Theorem 2.20 as it is a natural consequence of Theorem 2.19.

Theorem 2.21. *Let f_1, f_2 be any two meromorphic functions in the unit disc D and g_1, g_2 be any two entire functions.*
***(A)** The following conditions are assumed to be satisfied:*
(i) At least any one of f_1 or f_2 is of regular generalized relative growth (α,β) with respect to g_1;
(ii) Either $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_2]_{g_1}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}$ holds and g_1 has the Property (D), then

$$\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}.$$

***(B)** The following conditions are assumed to be satisfied:*
(i) f_1, g_1 and g_2 be any three entire functions such that $\lambda_{(\alpha,\beta)}[f_1]_{g_1}$ and $\lambda_{(\alpha,\beta)}[f_1]_{g_2}$ exists;
(ii) Either $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_1]_{g_2}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2}$ holds and $g_1 \pm g_2$ has the Property (D), then

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}.$$

Proof. Let f_1, f_2, g_1 and g_2 be any four entire functions satisfying the conditions of the theorem.

Case I. Let $\lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ $(0 < \lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1} < \infty)$ and at least f_1 or f_2 and $(f_1 \pm f_2)$ are of regular generalized relative growth (α,β) with respect to g_1. Now, in view of Theorem 2.1, it is easy to see that $\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \le \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}$. If possible let

$$\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}. \tag{2.33}$$

Let $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_2]_{g_1}$. Then in view of the proof of the first part of Theorem 2.18 and (2.33) we obtain that $\tau_{(\alpha,\beta)}[f_1]_{g_1} = \tau_{(\alpha,\beta)}[f_1 \pm f_2 \mp f_2]_{g_1} = \tau_{(\alpha,\beta)}[f_2]_{g_1}$ which is a contradiction. Hence $\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}$. Similarly in view of the proof of the first part of Theorem 2.18 , one can establish the same conclusion under the hypothesis $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}$. This proves the first part of the theorem.

Case II. Let us consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ $(0 < \lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2} < \infty$. Therefore in view of Theorem 2.3, it follows that $\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \ge \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and if possible let

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} > \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}. \tag{2.34}$$

Suppose $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_1]_{g_2}$. Then in view of the second part of Theorem 2.18 and (2.34), we obtain that $\tau_{(\alpha,\beta)}[f_1]_{g_1} = \tau_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2 \mp g_2} = \tau_{(\alpha,\beta)}[f_1]_{g_2}$ which is a contradiction. Hence $\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Analogously with the help of the second part of Theorem 2.18, the same conclusion can also be derived under the condition $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{g_2}^{(p,q)}(f_1)$ and therefore the second part of the theorem is established. ☐

Theorem 2.22. *Let f_1, f_2 be any two meromorphic functions in the unit disc D and g_1, g_2 be any two entire functions.*
***(A)** The following conditions are assumed to be satisfied:*
(i) At least any one of f_1 or f_2 is of regular generalized relative growth (α,β) with respect to g_1 and g_2. Also g_1, g_2, $g_1 \pm g_2$ have satisfy the Property (D);
(ii) Either $\tau_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \neq \tau_{(\alpha,\beta)}[f_1 \pm f_2]_{g_2}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_2}$;
(iii) Either $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_2]_{g_1}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}$;
(iv) Either $\tau_{(\alpha,\beta)}[f_1]_{g_2} \neq \tau_{(\alpha,\beta)}[f_2]_{g_2}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_2}$; then

$$\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1 \pm g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2} = \lambda_{(\alpha,\beta)}[f_2]_{g_2}.$$

***(B)** The following conditions are assumed to be satisfied:*
(i) At least any one of f_1 or f_2 are of regular generalized relative growth (α,β) with respect to $g_1 \pm g_2$, and $g_1 \pm g_2$ has satisfy the Property (D);
(ii) Either $\tau_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \neq \tau_{(\alpha,\beta)}[f_2]_{g_1 \pm g_2}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1 \pm g_2}$ holds;
(iii) Either $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_1]_{g_2}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2}$ holds;
(iv) Either $\tau_{(\alpha,\beta)}[f_2]_{g_1} \neq \tau_{(\alpha,\beta)}[f_2]_{g_2}$ or $\overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_2}$ holds, then

$$\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1 \pm g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2} = \lambda_{(\alpha,\beta)}[f_2]_{g_2}.$$

We omit the proof of Theorem 2.22 as it is a natural consequence of Theorem 2.21.

Theorem 2.23. *Let f_1, f_2 be any two meromorphic functions in the unit disc D and g_1, g_2 be any two entire functions. Also let $\rho_{(\alpha,\beta)}[f_1]_{g_1}$, $\rho_{(\alpha,\beta)}[f_2]_{g_1}$, $\rho_{(\alpha,\beta)}[f_1]_{g_2}$ and$\rho_{(\alpha,\beta)}[f_2]_{g_2}$ be all non-zero and finite.*
***(A)** Assume the functions f_1, f_2 and g_1 satisfy the following conditions:*
(i) Any one of $\rho_{(\alpha,\beta)}[f_i]_{g_1} > \rho_{(\alpha,\beta)}[f_j]_{g_1}$ hold for i, $j = 1,2$ and $i \neq j$;
(ii) g_1 satisfies the Property (D), then

$$\sigma_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \sigma_{(\alpha,\beta)}[f_i]_{g_1} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \overline{\sigma}_{(\alpha,\beta)}[f_i]_{g_1} \mid i = 1,2.$$

Similarly,

$$\sigma_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1} = \sigma_{(\alpha,\beta)}[f_i]_{g_1} \text{ and } \overline{\sigma}_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1} = \overline{\sigma}_{(\alpha,\beta)}[f_i]_{g_1} \mid i = 1,2$$

holds provided (i) $\frac{f_1}{f_2}$ is meromorphic in the unit disc D, (ii) $\rho_{(\alpha,\beta)}[f_i]_{g_1} > \rho_{(\alpha,\beta)}[f_j]_{g_1} \mid i,\ 1,2;\ j = 1,2;\ i \neq j$ and (iii) g_1 satisfy the Property (D).
***(B)** Assume the functions g_1, g_2 and f_1 satisfy the following conditions:*
(i) Any one of $\rho_{(\alpha,\beta)}[f_1]_{g_i} < \rho_{(\alpha,\beta)}[f_1]_{g_j}$ hold and at least f_1 is of regular generalized relative growth (α,β) with respect to any one of g_j for i, $j = 1,2$ and $i \neq j$, and g_i satisfies the Property (D);
(ii) $g_1 \cdot g_2$ satisfies the Property (D), then

$$\sigma_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \sigma_{(\alpha,\beta)}[f_1]_{g_i} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_i} \mid i = 1,2.$$

Similarly,

$$\sigma_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}} = \sigma_{(\alpha,\beta)}[f_1]_{g_i} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}} = \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_i} \mid i = 1,2$$

holds provided (i) $\frac{g_1}{g_2}$ is entire and satisfy the Property (D), (ii) At least f_1 is of regular generalized relative growth (α,β) with respect to g_2, (iii) $\rho_{(\alpha,\beta)}[f_1]_{g_i} < \rho_{(\alpha,\beta)}[f_1]_{g_j} \mid i = 1,2;\ j = 1,2;\ i \neq j$ and (iv) g_1 satisfy the Property (D).

(C) *Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:*
(i) $g_1 \cdot g_2$ satisfies the Property (D);
(ii) Any one of $\rho_{(\alpha,\beta)}[f_1]_{g_i} < \rho_{(\alpha,\beta)}[f_1]_{g_j}$ hold and at least f_1 is of regular generalized relative growth (α,β) with respect to g_j for $i = 1, 2$, $j = 1,2$ and $i \neq j$;
(iii) Any one of $\rho_{(\alpha,\beta)}[f_2]_{g_i} < \rho_{(\alpha,\beta)}[f_2]_{g_j}$ hold and at least f_2 is of regular generalized relative growth (α,β) with respect to g_j for $i = 1$, 2, $j = 1,2$ and $i \neq j$;
(iv) $\rho_{(\alpha,\beta)}[f_i]_{g_1} > \rho_{(\alpha,\beta)}[f_j]_{g_1}$ and $\rho_{(\alpha,\beta)}[f_i]_{g_2} > \rho_{(\alpha,\beta)}[f_j]_{g_2}$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$;
(v) $\rho_{(\alpha,\beta)}[f_l]_{g_m} =$
$\max[\min\{\rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_1]_{g_2}\}, \min\{\rho_{(\alpha,\beta)}[f_2]_{g_1}, \rho_{(\alpha,\beta)}[f_2]_{g_2}\}] \mid l,m = 1,2$; then

$$\sigma_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1 \cdot g_2} = \sigma_{(\alpha,\beta)}[f_l]_{g_m} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1 \cdot g_2} = \overline{\sigma}_{(\alpha,\beta)}[f_l]_{g_m} \mid l,m = 1,2 .$$

Similarly,

$$\sigma_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{\frac{g_1}{g_2}} = \sigma_{(\alpha,\beta)}[f_l]_{g_m} \text{ and } \overline{\sigma}_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{\frac{g_1}{g_2}} = \overline{\sigma}_{(\alpha,\beta)}[f_l]_{g_m} \mid l,m = 1,2.$$

holds provided $\frac{f_1}{f_2}$ is meromorphic function in the unit disc D and $\frac{g_1}{g_2}$ is entire function which satisfy the following conditions:
(i) $\frac{g_1}{g_2}$ satisfies the Property (D);
(ii) At least f_1 is of regular generalized relative growth (α,β) with respect to g_2 and $\rho_{(\alpha,\beta)}[f_1]_{g_1} \neq \rho_{(\alpha,\beta)}[f_1]_{g_2}$;
(iii) At least f_2 is of regular generalized relative growth (α,β) with respect to g_2 and $\rho_{(\alpha,\beta)}[f_2]_{g_1} \neq \rho_{(\alpha,\beta)}[f_2]_{g_2}$;
(iv) $\rho_{(\alpha,\beta)}[f_i]_{g_1} < \rho_{(\alpha,\beta)}[f_j]_{g_1}$ and $\rho_{(\alpha,\beta)}[f_i]_{g_2} < \rho_{(\alpha,\beta)}[f_j]_{g_2}$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$;
(v) $\rho_{(\alpha,\beta)}[f_l]_{g_m} =$
$\max[\min\{\rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_1]_{g_2}\}, \min\{\rho_{(\alpha,\beta)}[f_2]_{g_1}, \rho_{(\alpha,\beta)}[f_2]_{g_2}\}] \mid l,m = 1,2.$

Proof. Let us suppose that $\rho_{(\alpha,\beta)}[f_1]_{g_1}$, $\rho_{(\alpha,\beta)}[f_2]_{g_1}$, $\rho_{(\alpha,\beta)}[f_1]_{g_2}$ and$\rho_{(\alpha,\beta)}[f_2]_{g_2}$ are all non-zero and finite.

Case I. Suppose that $\rho_{(\alpha,\beta)}[f_1]_{g_1} > \rho_{(\alpha,\beta)}[f_2]_{g_1}$. Also let g_1 satisfy the Property (D). Since $T_{f_1 \cdot f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r)$, therefore applying the same procedure as adopted in Case I of Theorem 2.17 we get that

$$\sigma_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} \leq \sigma_{(\alpha,\beta)}[f_1]_{g_1} . \tag{2.35}$$

Further without loss of any generality, let $f = f_1 \cdot f_2$ and $\rho_{(\alpha,\beta)}[f_2]_{g_1} < \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f]_{g_1}$. Then in view of (2.35), we obtain that $\sigma_{(\alpha,\beta)}[f]_{g_1} = \sigma_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} \leq \sigma_{(\alpha,\beta)}[f_1]_{g_1}$. Also $f_1 = \frac{f}{f_2}$ and $T_{f_2}(r) = T_{\frac{1}{f_2}}(r) + O(1)$. Therefore $T_{f_1}(r) \leq T_f(r) + T_{f_2}(r) + O(1)$ and in this case also we obtain from (2.35) that $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \leq \sigma_{(\alpha,\beta)}[f]_{g_1} = \sigma_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1}$. Hence$\sigma_{(\alpha,\beta)}[f]_{g_1} = \sigma_{(\alpha,\beta)}[f_1]_{g_1} \Rightarrow \sigma_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \sigma_{(\alpha,\beta)}[f_1]_{g_1}$.

Similarly, if we consider $\rho_{(\alpha,\beta)}[f_1]_{g_1} < \rho_{(\alpha,\beta)}[f_2]_{g_1}$, then one can verify that $\sigma_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \sigma_{(\alpha,\beta)}[f_2]_{g_1}$.

Next we may suppose that $f = \frac{f_1}{f_2}$ with f_1, f_2 and f are all meromorphic functions.

Sub Case I$_{\mathbf{A}}$. Let $\rho_{(\alpha,\beta)}[f_2]_{g_1} < \rho_{(\alpha,\beta)}[f_1]_{g_1}$. Therefore in view of Theorem 2.9, $\rho_{(\alpha,\beta)}[f_2]_{g_1} < \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f]_{g_1}$. We have $f_1 = f \cdot f_2$. So, $\sigma_{(\alpha,\beta)}[f_1]_{g_1} = \sigma_{(\alpha,\beta)}[f]_{g_1} = \sigma_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1}$.

Sub Case I$_{\mathbf{B}}$. Let $\rho_{(\alpha,\beta)}[f_2]_{g_1} > \rho_{(\alpha,\beta)}[f_1]_{g_1}$. Therefore in view of Theorem 2.9, $\rho_{(\alpha,\beta)}[f_1]_{g_1} < \rho_{(\alpha,\beta)}[f_2]_{g_1} = \rho_{(\alpha,\beta)}[f]_{g_1}$. Since $T_f(r) = T_{\frac{1}{f}}(r) + O(1) = T_{\frac{f_2}{f_1}}(r) + O(1)$, So $\sigma_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1} = \sigma_{(\alpha,\beta)}[f_2]_{g_1}$.

Case II. Let $\rho_{(\alpha,\beta)}[f_1]_{g_1} > \rho_{(\alpha,\beta)}[f_2]_{g_1}$. Also let g_1 satisfy the Property (D). As $T_{f_1 \cdot f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r)$, therefore applying the same procedure as explored in Case II of Theorem 2.17, one can easily verify that $\overline{\sigma}_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1}$ and $\overline{\sigma}_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1} = \overline{\sigma}_{(\alpha,\beta)}[f_i]_{g_1} \mid i = 1,2$ under the conditions specified in the theorem.

Similarly, if we consider $\rho_{(\alpha,\beta)}[f_1]_{g_1} < \rho_{(\alpha,\beta)}[f_2]_{g_1}$, then one can verify that $\overline{\sigma}_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1}$ and $\overline{\sigma}_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1} = \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1}$.

Therefore the first part of theorem follows from Case I and Case II.

Case III. Let $g_1 \cdot g_2$ satisfy the Property (D) and $\rho_{(\alpha,\beta)}[f_1]_{g_1} < \rho_{(\alpha,\beta)}[f_1]_{g_2}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_2. Since $T_{g_1 \cdot g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r)$ for all large r, therefore applying the same procedure as adopted in Case III of Theorem 2.17 we get that

$$\sigma_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} \geq \sigma_{(\alpha,\beta)}[f_1]_{g_1} . \tag{2.36}$$

Further without loss of any generality, let $g = g_1 \cdot g_2$ and $\rho_{(\alpha,\beta)}[f_1]_g = \rho_{(\alpha,\beta)}[f_1]_{g_1} < \rho_{(\alpha,\beta)}[f_1]_{g_2}$. Then in view of (2.36), we obtain that $\sigma_{(\alpha,\beta)}[f_1]_g = \sigma_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} \geq \sigma_{(\alpha,\beta)}[f_1]_{g_1}$. Also $g_1 = \frac{g}{g_2}$ and $T_{g_2}(r) = T_{\frac{1}{g_2}}(r) + O(1)$. Therefore $T_{g_1}(r) \leq T_g(r) + T_{g_2}(r) + O(1)$ and in this case we obtain from (2.36) that $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \geq \sigma_{(\alpha,\beta)}[f_1]_g = \sigma_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2}$. Hence $\sigma_{(\alpha,\beta)}[f_1]_g = \sigma_{(\alpha,\beta)}[f_1]_{g_1} \Rightarrow \sigma_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \sigma_{(\alpha,\beta)}[f_1]_{g_1}$.

Similarly, if we consider $\rho_{(\alpha,\beta)}[f_1]_{g_1} > \rho_{(\alpha,\beta)}[f_1]_{g_2}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_1, then one can verify that $\sigma_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \sigma_{(\alpha,\beta)}[f_1]_{g_2}$.

Next we may suppose that $g = \frac{g_1}{g_2}$ with g_1, g_2, g are all entire functions satisfying the conditions specified in the theorem.

Sub Case $\mathbf{III_A}$. Let $\rho_{(\alpha,\beta)}[f_1]_{g_1} < \rho_{(\alpha,\beta)}[f_1]_{g_2}$. Therefore in view of Theorem 2.12, $\rho_{(\alpha,\beta)}[f_1]_g = \rho_{(\alpha,\beta)}[f_1]_{g_1} < \rho_{(\alpha,\beta)}[f_1]_{g_2}$. We have $g_1 = g\cdot g_2$. So $\sigma_{(\alpha,\beta)}[f_1]_{g_1} = \sigma_{(\alpha,\beta)}[f_1]_g = \sigma_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}}$.

Sub Case $\mathbf{III_B}$. Let $\rho_{(\alpha,\beta)}[f_1]_{g_1} > \rho_{(\alpha,\beta)}[f_1]_{g_2}$. Therefore in view of Theorem 2.12, $\rho_{(\alpha,\beta)}[f_1]_g = \rho_{(\alpha,\beta)}[f_1]_{g_2} < \rho_{(\alpha,\beta)}[f_1]_{g_1}$. Since $T_g(r) = T_{\frac{1}{g}}(r) + O(1) = T_{\frac{g_2}{g_1}}(r) + O(1)$, So $\sigma_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}} = \sigma_{(\alpha,\beta)}[f_1]_{g_2}$.

Case IV. Suppose $g_1\cdot g_2$ satisfy the Property (D). Also let $\rho_{(\alpha,\beta)}[f_1]_{g_1} < \rho_{(\alpha,\beta)}[f_1]_{g_2}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_2. As $T_{g_1\cdot g_2}(r) \le T_{g_1}(r) + T_{g_2}(r)$, the same procedure as explored in Case IV of Theorem 2.17, one can easily verify that $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} = \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1}$ and $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}} = \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_i} \mid i = 1,2$ under the conditions specified in the theorem.

Likewise, if we consider $\rho_{(\alpha,\beta)}[f_1]_{g_1} > \rho_{(\alpha,\beta)}[f_1]_{g_2}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_1, then one can verify that $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} = \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}$ and $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}} = \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}$. Therefore the second part of theorem follows from Case III and Case IV.

Proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem 2.13 and Theorem 2.15 and the above cases. □

Theorem 2.24. *Let f_1, f_2 be any two meromorphic functions in the unit disc D and g_1, g_2 be any two entire functions. Also let $\lambda_{(\alpha,\beta)}[f_1]_{g_1}$, $\lambda_{(\alpha,\beta)}[f_2]_{g_1}$, $\lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and $\lambda_{(\alpha,\beta)}[f_2]_{g_2}$ be all non-zero and finite.*
***(A)** Assume the functions f_1, f_2 and g_1 satisfy the following conditions:*
(i) Any one of $\lambda_{(\alpha,\beta)}[f_i]_{g_1} > \lambda_{(\alpha,\beta)}[f_j]_{g_1}$ hold and at least any one of f_j is of regular generalized relative growth (α,β) with respect to g_1 for i, $j = 1,2$ and $i \ne j$;
(ii) g_1 satisfies the Property (D), then

$$\tau_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1} = \tau_{(\alpha,\beta)}[f_i]_{g_1} \text{ and } \overline{\tau}_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1} = \overline{\tau}_{(\alpha,\beta)}[f_i]_{g_1} \mid i = 1,2.$$

Similarly,

$$\tau_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1} = \tau_{(\alpha,\beta)}[f_i]_{g_1} \text{ and } \overline{\tau}_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1} = \overline{\tau}_{(\alpha,\beta)}[f_i]_{g_1} \mid i = 1,2$$

holds provided $\frac{f_1}{f_2}$ is meromorphic in the unit disc D, at least f_2 is of regular generalized relative growth (α,β) with respect to g_1 where g_1 satisfy the Property (D) and $\lambda_{(\alpha,\beta)}[f_i]_{g_1} > \lambda_{(\alpha,\beta)}[f_j]_{g_1} \mid i = 1,2;\ j = 1,2;\ i \ne j$.
***(B)** Assume the functions g_1, g_2 and f_1 satisfy the following conditions:*
(i) Any one of $\lambda_{(\alpha,\beta)}[f_1]_{g_i} < \lambda_{(\alpha,\beta)}[f_1]_{g_j}$ hold for i, $j = 1,2$, $i \ne j$; and g_i satisfy the Property (D)
(ii) $g_1\cdot g_2$ satisfy the Property (D), then

$$\tau_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} = \tau_{(\alpha,\beta)}[f_1]_{g_i} \text{ and } \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_i} \mid i = 1,2.$$

Similarly,

$$\tau_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}} = \tau_{(\alpha,\beta)}[f_1]_{g_i} \text{ and } \overline{\tau}_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_i} \mid i = 1,2$$

holds provided $\frac{g_1}{g_2}$ is entire and satisfy the Property (D), g_1 satisfy the Property (D) and $\lambda_{(\alpha,\beta)}[f_1]_{g_i} < \lambda_{(\alpha,\beta)}[f_1]_{g_j} \mid i = 1,2;\ j = 1,2;\ i \ne j$.
***(C)** Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:*
(i) $g_1\cdot g_2$, g_1 and g_2 are satisfy the Property (D);
(ii) Any one of $\lambda_{(\alpha,\beta)}[f_i]_{g_1} > \lambda_{(\alpha,\beta)}[f_j]_{g_1}$ hold and at least any one of f_j is of regular generalized relative growth (α,β) with respect to g_1 for $i = 1, 2$, $j = 1,2$ and $i \ne j$;
(iii) Any one of $\lambda_{(\alpha,\beta)}[f_i]_{g_2} > \lambda_{(\alpha,\beta)}[f_j]_{g_2}$ hold and at least any one of f_j is of regular generalized relative growth (α,β) with respect to g_2 for $i = 1, 2$, $j = 1,2$ and $i \ne j$;
(iv) $\lambda_{(\alpha,\beta)}[f_1]_{g_i} < \lambda_{(\alpha,\beta)}[f_1]_{g_j}$ and $\lambda_{(\alpha,\beta)}[f_2]_{g_i} < \lambda_{(\alpha,\beta)}[f_2]_{g_j}$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \ne j$;
(v) $\lambda_{(\alpha,\beta)}[f_l]_{g_m} =$
$\min[\max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\}, \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_2}, \lambda_{(\alpha,\beta)}[f_2]_{g_2}\}] \mid l,m = 1,2$*; then*

$$\tau_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1\cdot g_2} = \tau_{(\alpha,\beta)}[f_l]_{g_m} \text{ and } \overline{\tau}_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1\cdot g_2} = \overline{\tau}_{(\alpha,\beta)}[f_l]_{g_m} \mid l,m = 1,2.$$

Similarly,

$$\tau_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{\frac{g_1}{g_2}} = \tau_{(\alpha,\beta)}[f_l]_{g_m} \text{ and } \overline{\tau}_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{\frac{g_1}{g_2}} = \overline{\tau}_{(\alpha,\beta)}[f_l]_{g_m} \mid l,m = 1,2.$$

holds provided $\frac{f_1}{f_2}$ is meromorphic in the unit disc D and $\frac{g_1}{g_2}$ is entire functions which satisfy the following conditions:
(i) $\frac{g_1}{g_2}$, g_1 and g_2 satisfy the Property (D);
(ii) At least f_2 is of regular generalized relative growth (α,β) with respect to g_1 and $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \ne \lambda_{(\alpha,\beta)}[f_2]_{g_1}$;
(iii) At least f_2 is of regular generalized relative growth (α,β) with respect to g_2 and $\lambda_{(\alpha,\beta)}[f_1]_{g_2} \ne \lambda_{(\alpha,\beta)}[f_2]_{g_2}$;
(iv) $\lambda_{(\alpha,\beta)}[f_1]_{g_i} < \lambda_{(\alpha,\beta)}[f_1]_{g_j}$ and $\lambda_{(\alpha,\beta)}[f_2]_{g_i} < \lambda_{(\alpha,\beta)}[f_2]_{g_j}$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \ne j$;
(v) $\lambda_{(\alpha,\beta)}[f_l]_{g_m} =$
$\min[\max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\}, \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_2}, \lambda_{(\alpha,\beta)}[f_2]_{g_2}\}] \mid l,m = 1,2.$

Proof. Let us consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1}$, $\lambda_{(\alpha,\beta)}[f_2]_{g_1}$, $\lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and $\lambda_{(\alpha,\beta)}[f_2]_{g_2}$ are all non-zero and finite.
Case I. Suppose $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ with at least f_2 is of regular generalized relative growth (α,β) with respect to g_1 and g_1 satisfy the Property (D). Since $T_{f_1\cdot f_2}(r) \le T_{f_1}(r) + T_{f_2}(r)$, therefore applying the same procedure as adopted in Case I of Theorem 2.18 we get that

$$\tau_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1} \le \tau_{(\alpha,\beta)}[f_1]_{g_1}. \tag{2.37}$$

Further without loss of any generality, let $f = f_1\cdot f_2$ and $\lambda_{(\alpha,\beta)}[f_2]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f]_{g_1}$. Then in view of (2.37), we obtain that $\tau_{(\alpha,\beta)}[f]_{g_1} = \tau_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1} \le \tau_{(\alpha,\beta)}[f_1]_{g_1}$. Also $f_1 = \frac{f}{f_2}$ and $T_{f_2}(r) = T_{\frac{1}{f_2}}(r) + O(1)$. Therefore $T_{f_1}(r) \le T_f(r) + T_{f_2}(r) + O(1)$ and in this case we obtain from the above arguments that $\tau_{(\alpha,\beta)}[f_1]_{g_1} \le \tau_{(\alpha,\beta)}[f]_{g_1} = \tau_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1}$. Hence $\tau_{(\alpha,\beta)}[f]_{g_1} = \tau_{(\alpha,\beta)}[f_1]_{g_1}$ $\Rightarrow \tau_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1} = \tau_{(\alpha,\beta)}[f_1]_{g_1}$.

Similarly, if we consider $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ with at least f_1 is of regular generalized relative growth with respect to g_1, then one can easily verify that $\tau_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1} = \tau_{(\alpha,\beta)}[f_2]_{g_1}$.

Next we may suppose that $f = \frac{f_1}{f_2}$ with f_1, f_2 and f are all meromorphic functions in the unit disc D satisfying the conditions specified in the theorem.

Sub Case I$_\mathbf{A}$. Let $\lambda_{(\alpha,\beta)}[f_2]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_1}$. Therefore in view of Theorem 2.8, $\lambda_{(\alpha,\beta)}[f_2]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f]_{g_1}$. We have $f_1 = f\cdot f_2$. So $\tau_{(\alpha,\beta)}[f_1]_{g_1} = \tau_{(\alpha,\beta)}[f]_{g_1} = \tau_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1}$.

Sub Case I$_\mathbf{B}$. Let $\lambda_{(\alpha,\beta)}[f_2]_{g_1} > \lambda_{(\alpha,\beta)}[f_1]_{g_1}$. Therefore in view of Theorem 2.8, $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f]_{g_1}$. Since $T_f(r) = T_{\frac{1}{f}}(r) + O(1) = T_{\frac{f_2}{f_1}}(r) + O(1)$, So $\tau_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1} = \tau_{(\alpha,\beta)}[f_2]_{g_1}$.

Case II. Let $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ with at least f_2 is of regular generalized relative growth (α,β) with respect to g_1 where g_1 satisfy the Property (D). As $T_{f_1\cdot f_2}(r) \le T_{f_1}(r) + T_{f_2}(r)$, so applying the same procedure as adopted in Case II of Theorem 2.18 we can easily verify that $\overline{\tau}_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}$ and $\tau_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}} = \tau_{(\alpha,\beta)}[f_1]_{g_i} \mid i = 1,2$ under the conditions specified in the theorem.

Similarly, if we consider $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_1, then one can easily verify that $\overline{\tau}_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1} = \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}$.

Therefore the first part of theorem follows Case I and Case II.
Case III. Let $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and $g_1\cdot g_2$ satisfy the Property (D). Since $T_{g_1\cdot g_2}(r) \le T_{g_1}(r) + T_{g_2}(r)$, therefore applying the same procedure as adopted in Case III of Theorem 2.18 we get that

$$\tau_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} \le \tau_{(\alpha,\beta)}[f_1]_{g_1}. \tag{2.38}$$

Further without loss of any generality, let $g = g_1\cdot g_2$ and $\lambda_{(\alpha,\beta)}[f_1]_g = \lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Then in view of (2.38), we obtain that $\tau_{(\alpha,\beta)}[f_1]_g = \tau_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} \ge \tau_{(\alpha,\beta)}[f_1]_{g_1}$. Also $g_1 = \frac{g}{g_2}$ and $T_{g_2}(r) = T_{\frac{1}{g_2}}(r) + O(1)$. Therefore $T_{g_1}(r) \le T_g(r) + T_{g_2}(r) + O(1)$ and in this case we obtain from above arguments that $\tau_{(\alpha,\beta)}[f_1]_{g_1} \ge \tau_{(\alpha,\beta)}[f_1]_g = \tau_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2}$. Hence $\tau_{(\alpha,\beta)}[f_1]_g = \tau_{(\alpha,\beta)}[f_1]_{g_1}$ $\Rightarrow \tau_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} = \tau_{(\alpha,\beta)}[f_1]_{g_1}$.

If $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_1]_{g_2}$, then one can easily verify that $\tau_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} = \tau_{(\alpha,\beta)}[f_1]_{g_2}$.

Next we may suppose that $g = \frac{g_1}{g_2}$ with g_1, g_2, g are all entire functions satisfying the conditions specified in the theorem.

Sub Case III$_\mathbf{A}$. Let $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Therefore in view of Theorem 2.10, $\lambda_{(\alpha,\beta)}[f_1]_g = \lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. We have $g_1 = g\cdot g_2$. So $\tau_{(\alpha,\beta)}[f_1]_{g_1} = \tau_{(\alpha,\beta)}[f_1]_g = \tau_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}}$.

Sub Case III$_\mathbf{B}$. Let $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Therefore in view of Theorem 2.10, $\lambda_{(\alpha,\beta)}[f_1]_g = \lambda_{(\alpha,\beta)}[f_1]_{g_2} < \lambda_{(\alpha,\beta)}[f_1]_{g_1}$. Since $T_g(r) = T_{\frac{1}{g}}(r) + O(1) = T_{\frac{g_2}{g_1}}(r) + O(1)$, So $\tau_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}} = \tau_{(\alpha,\beta)}[f_1]_{g_2}$.

Case IV. Suppose $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and $g_1\cdot g_2$ satisfy the Property (D). Since $T_{g_1\cdot g_2}(r) \le T_{g_1}(r) + T_{g_2}(r)$, then adopting the same procedure as of Case IV of Theorem 2.18, we obtain that $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}$ and $\overline{\tau}_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_i} \mid i = 1,2$.

Similarly if we consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_1]_{g_2}$, then one can easily verify that $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2}$.

Therefore the second part of the theorem follows from Case III and Case IV.

Proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem 2.14 , Theorem 2.16 and the above cases. □

Theorem 2.25. *Let f_1, f_2 be any two meromorphic functions in the unit disc D and g_1, g_2 be any two entire functions.*
***(A)** The following condition is assumed to be satisfied:*
(i) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \ne \sigma_{(\alpha,\beta)}[f_2]_{g_1}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \ne \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1}$ holds;
(ii) g_1 satisfies the Property (D), then

$$\rho_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1}.$$

***(B)** The following conditions are assumed to be satisfied:*
(i) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \ne \sigma_{(\alpha,\beta)}[f_1]_{g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \ne \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}$ holds;
(ii) f_1 is of regular generalized relative growth (α,β) with respect to at least any one of g_1 or g_2. Also $g_1\cdot g_2$ satisfy the Property (D). Then we have

$$\rho_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} = \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2}.$$

Proof. Let f_1, f_2 be any two meromorphic functions in the unit disc D and g_1, g_2 be any two entire functions satisfying the conditions of the theorem.

Case I. Suppose that $\rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1}$ $(0 < \rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_2]_{g_1} < \infty)$ and g_1 satisfy the Property (D). Now in view of Theorem 2.9, it is easy to see that $\rho_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} \leq \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1}$. If possible let

$$\rho_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} < \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1}. \tag{2.39}$$

Let $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_1}$. Now in view of the first part of Theorem 2.23 and (2.39) we obtain that $\sigma_{(\alpha,\beta)}[f_1]_{g_1} = \sigma_{(\alpha,\beta)}\left[\frac{f_1 \cdot f_2}{f_2}\right]_{g_1} = \sigma_{(\alpha,\beta)}[f_2]_{g_1}$ which is a contradiction. Hence $\rho_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1}$. Similarly with the help of the first part of Theorem 2.23, one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1}$. This prove the first part of the theorem.

Case II. Let us consider that $\rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2}$ $(0 < \rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_1]_{g_2} < \infty)$, f_1 is of regular generalized relative growth (α,β) with respect to at least any one of g_1 or g_2. Also $g_1 \cdot g_2$ satisfy the Property (D). Therefore in view of Theorem 2.11, it follows that $\rho_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} \geq \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2}$ and if possible let

$$\rho_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} > \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2}. \tag{2.40}$$

Further suppose that $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_1]_{g_2}$. Therefore in view of the proof of the second part of Theorem 2.23 and (2.40), we obtain that $\sigma_{(\alpha,\beta)}[f_1]_{g_1} = \sigma_{(\alpha,\beta)}[f_1]_{\frac{g_1 \cdot g_2}{g_2}} = \sigma_{(\alpha,\beta)}[f_1]_{g_2}$ which is a contradiction. Hence $\rho_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2}$. Likewise in view of the proof of second part of Theorem 2.23, one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}$. This proves the second part of the theorem. □

Theorem 2.26. *Let f_1, f_2 be any two meromorphic functions in the unit disc D and g_1, g_2 be any two entire functions.*
***(A)** The following conditions are assumed to be satisfied:*
(i) $(f_1 \cdot f_2)$ is of regular generalized relative growth (α,β) with respect to at least any one g_1 or g_2;
(ii) $(g_1 \cdot g_2)$, g_1 and g_2 all satisfy the Property (D);
(iii) Either $\sigma_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_2}$;
(iv) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_1}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1}$;
(v) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_2} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_2}$; then

$$\rho_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1 \cdot g_2} = \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2} = \rho_{(\alpha,\beta)}[f_2]_{g_2}.$$

***(B)** The following conditions are assumed to be satisfied:*
(i) $(g_1 \cdot g_2)$ satisfies the Property (D);
(ii) f_1 and f_2 are of regular generalized relative growth (α,β) with respect to at least any one g_1 or g_2;
(iii) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_1 \cdot g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1 \cdot g_2}$;
(iv) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_1]_{g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}$;
(v) Either $\sigma_{(\alpha,\beta)}[f_2]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_2}$; then

$$\rho_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1 \cdot g_2} = \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2} = \rho_{(\alpha,\beta)}[f_2]_{g_2}.$$

We omit the proof of Theorem 2.26 as it is a natural consequence of Theorem 2.25.

Theorem 2.27. *Let f_1, f_2 be any two meromorphic functions in the unit disc D and g_1, g_2 be any two entire functions.*
***(A)** The following conditions are assumed to be satisfied:*
(i) At least any one of f_1 or f_2 is of regular generalized relative growth (α,β) with respect to g_1;
(ii) If either $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_2]_{g_1}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}$ holds.
(iii) g_1 satisfies the Property (D), then

$$\lambda_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}.$$

***(B)** The following conditions are assumed to be satisfied:*
(i) f_1 is any meromorphic function in the unit disc D and g_1, g_2 are any two entire functions such that $\lambda_{(\alpha,\beta)}[f_1]_{g_1}$ and $\lambda_{(\alpha,\beta)}[f_1]_{g_2}$ exist and $g_1 \cdot g_2$ satisfy the Property (D);
(ii) If either $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_1]_{g_2}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2}$ holds, then

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}.$$

Proof. Let f_1, f_2 be any two meromorphic functions in the unit disc D and g_1, g_2 be any two entire functions satisfy the conditions of the theorem.

Case I. Let $\lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ $(0 < \lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1} < \infty)$, g_1 satisfies the Property (D) and at least f_1 or f_2 be of regular generalized relative growth (α,β) with respect to g_1. Now in view of Theorem 2.7 it is easy to see that $\lambda_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} \leq \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}$. If possible let

$$\lambda_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}. \tag{2.41}$$

Also let $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_2]_{g_1}$. Then in view of the proof of first part of Theorem 2.24 and (2.41), we obtain that $\tau_{(\alpha,\beta)}[f_1]_{g_1} = \tau_{(\alpha,\beta)}\left[\frac{f_1 \cdot f_2}{f_2}\right]_{g_1} = \tau_{(\alpha,\beta)}[f_2]_{g_1}$ which is a contradiction. Hence $\lambda_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}$. Analogously, in view of the proof of first part of Theorem 2.24 and using the same technique as above, one can easily derive the same conclusion under the hypothesis $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}$. Hence the first part of the theorem is established.

Case II. Let us consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ $(0 < \lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2} < \infty$ and $g_1 \cdot g_2$ satisfy the Property (D). Therefore in view of Theorem 2.10, it follows that $\lambda_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} \geq \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and if possible let

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} > \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}. \tag{2.42}$$

Further let $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_1]_{g_2}$. Then in view of second part of Theorem 2.24 and (2.42), we obtain that $\tau_{(\alpha,\beta)}[f_1]_{g_1} = \tau_{(\alpha,\beta)}[f_1]_{\frac{g_1\cdot g_2}{g_2}} = \tau_{(\alpha,\beta)}[f_1]_{g_2}$ which is a contradiction. Hence $\lambda_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Similarly by second part of Theorem 2.24, we get the same conclusion when $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2}$ and therefore the second part of the theorem follows. □

Theorem 2.28. *Let f_1, f_2 be any two meromorphic functions in the unit disc D and g_1, g_2 be any two entire functions.*
***(A)** The following conditions are assumed to be satisfied:*
(i) $g_1 \cdot g_2$, g_1 and g_2 satisfy the Property (D);
(ii) At least any one of f_1 or f_2 is of regular generalized relative growth (α,β) with respect to g_1 and g_2;
(iii)Either $\tau_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1} \neq \tau_{(\alpha,\beta)}[f_1\cdot f_2]_{g_2}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_1\cdot f_2]_{g_2}$;
(iv) Either $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_2]_{g_1}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}$;
(v) Either $\tau_{(\alpha,\beta)}[f_1]_{g_2} \neq \tau_{(\alpha,\beta)}[f_2]_{g_2}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_2}$; then

$$\lambda_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1\cdot g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2} = \lambda_{(\alpha,\beta)}[f_2]_{g_2}.$$

***(B)** The following conditions are assumed to be satisfied:*
(i) $g_1 \cdot g_2$ satisfies the Property (D);
(ii) At least any one of f_1 or f_2 is of regular generalized relative growth (α,β) with respect to $g_1 \cdot g_2$;
(iii) Either $\tau_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} \neq \tau_{(\alpha,\beta)}[f_2]_{g_1\cdot g_2}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1\cdot g_2}$ holds;
(iv) Either $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_1]_{g_2}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2}$ holds;
(v) If either $\tau_{(\alpha,\beta)}[f_2]_{g_1} \neq \tau_{(\alpha,\beta)}[f_2]_{g_2}$ or $\overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_2}$ holds, then

$$\lambda_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1\cdot g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2} = \lambda_{(\alpha,\beta)}[f_2]_{g_2}.$$

We omit the proof of Theorem 2.28 as it is a natural consequence of Theorem 2.27.

Remark 2.29. *If we take $\frac{f_1}{f_2}$ instead of $f_1 \cdot f_2$ and $\frac{g_1}{g_2}$ instead of $g_1 \cdot g_2$ where $\frac{f_1}{f_2}$ is meromorphic in the unit disc D and $\frac{g_1}{g_2}$ is entire function, and the other conditions of Theorem 2.25, Theorem 2.26, Theorem 2.27 and Theorem 2.28 remain the same, then conclusion of Theorem 2.25, Theorem 2.26, Theorem 2.27 and Theorem 2.28 remains valid.*

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