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Rough $\Delta\mathscr{I}-$Convergence

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Abstract

In this paper, we study the concept of rough $\mathscr{I}-$convergence for difference sequences in $(\mathbb{R}^n, \|.\|)$ where $\mathbb{R}^n$ denotes the real $n-$dimensional space with the norm $\|.\|$. At the same time, we examine some basic properties of the set $\mathscr{I}-\text{LIM}^r_{\Delta x_i} = \left\{x_* \in \mathbb{R}^n : \Delta x_i \xrightarrow{r} x_*\right\}$ which is called as $r-$ $\mathscr{I}-$ limit set of the difference sequence (Δx_i) and we give some properties of $\mathscr{I}-\liminf \Delta x_i$, $\mathscr{I}-\limsup \Delta x_i$ and $\mathscr{I}-\text{core}\{\Delta x_i\}$.

Keywords: *Statistical convergence, $\mathscr{I}-$Convergence, rough convergence, difference sequences, $\mathscr{I}-$ limit point set, $\mathscr{I}-$core*
2010 Mathematics Subject Classification: *40G15, 40A35.*

1. Introduction and Background

As can be seen from the title of the article, there are four important concepts that will form the basis of this article. These are;
- Statistical convergence,
- $\mathscr{I}-$convergence,
- Difference sequences,
- Rough convergence.

Now let us give the literature information and important definitions related to these concepts in order.
Statistical convergence was defined in 1951 by Fast ([16]) and Steinhaus ([29]), independently and later on, it found a wide application in many fields such as summability theory ([17]), number theory ([10]), measure theory ([24]) and trigonometric series ([30]). Therefore, it has become one of the most popular topics in the last seventy years. If we are talking about the concept of statistical convergence, it is necessary to know the concept of natural density because natural density is the basis of statistical convergence.

Definition 1.1. *Let $K \subseteq \mathbb{N}$ be a subset of $\mathbb{N}$, the set of all natural numbers.*

$$d(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$$

is said to be natural density of K where $K_n = \{k \in K : k \leq n\}$ and $|K_n|$ gives the number of elements in K_n. It is easy to see that if K is a finite set then, $d(K) = 0$.

Now we can give the definition of statistical convergence as follows:

Definition 1.2. *([16]) A real or complex sequence $x = (x_i)$ is statistically convergent to L provided that*

$$\lim_n \frac{1}{n} |\{i \leq n : |x_i - L| \geq \varepsilon\}| = 0$$

for each $\varepsilon > 0$. This is indicated by $st - \lim x = L$. So, it is obvious that each sequence that convergent is also statistically convergent.

Kostyrko et al. ([23]) defined the concept of ideal convergence, or shortly $\mathscr{I}-$convergence, in a metric space by using ideals and so they generalized many types of convergence including statistical convergence. In their study, they obtained that if $\mathscr{I} = \mathscr{I}_f = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$ then, $\mathscr{I}_f-$convergence coincides with the usual convergence and if $\mathscr{I} = \mathscr{I}_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$ then, $\mathscr{I}_d-$convergence (where $d(A)$ is natural density of A) coincides with the statistical convergence. Many examples about the concept of $\mathscr{I}-$convergence can be seen in Kostyrko and his friends' paper.
$\mathscr{I}-$convergence is based on the definition of an ideal $\mathscr{I}$ in $\mathbb{N}$. The concept of filter, which can be considered as the dual of the ideal, is also used in the conclusion of many proofs. Thus, before defining $\mathscr{I}-$convergence, the definitions of ideal and filter will be needed.

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Definition 1.3. *A family of sets $\mathscr{I} \subseteq 2^{\mathbb{N}}$ is an ideal if the following properties are provided:*

(i) $\emptyset \in \mathscr{I}$,
(ii) $A, B \in \mathscr{I}$ implies $A \cup B \in \mathscr{I}$,
(iii) For each $A \in \mathscr{I}$ and each $B \subseteq A$ implies $B \in \mathscr{I}$.
We say that $\mathscr{I}$ is non-trivial if $\mathbb{N} \notin \mathscr{I}$ and $\mathscr{I}$ is admissible if $\{n\} \in \mathscr{I}$ for each $n \in \mathbb{N}$.

Definition 1.4. *A family of sets $\mathscr{F} \subseteq 2^{\mathbb{N}}$ is a filter if the following properties are provided:*

(i) $\emptyset \notin \mathscr{F}$,
(ii) If $A, B \in \mathscr{F}$ then we have $A \cap B \in \mathscr{F}$,
(iii) For each $A \in \mathscr{F}$ and each $A \subseteq B$ we have $B \in \mathscr{F}$.

Proposition 1.5. *If $\mathscr{I}$ is an ideal in $\mathbb{N}$ then the collection,*

$$F(\mathscr{I}) = \{A \subset \mathbb{N} : \mathbb{N} \backslash A \in \mathscr{I}\}$$

forms a filter in $\mathbb{N}$ which is called the filter associated with $\mathscr{I}$.

Definition 1.6. *([23]) A sequence of reals $x = (x_i)$ is $\mathscr{I}-$convergent to $L \in \mathbb{R}$ if and only if the set*

$$A_\varepsilon = \{i \in \mathbb{N} : |x_i - L| \geq \varepsilon\} \in \mathscr{I}$$

for each $\varepsilon > 0$. In this case, we say that L is the $\mathscr{I}-$limit of the sequence x.

Definition 1.7. *A sequence $x = (x_i)$ is $\mathscr{I}-$bounded if there exists a positive real number M such that*

$$\{i \in \mathbb{N} : |x_i| \geq M\} \in \mathscr{I}.$$

In 1981, Kizmaz ([22]) defined difference sequences such that $\Delta x = (\Delta x_i) = (x_i - x_{i+1})$ where $x = (x_i)$ is a real number and $i \in \mathbb{N}$. In his paper, he also defined $c_0(\Delta) = \{x = (x_i) : \Delta x \in c_0\}$, $c(\Delta) = \{x = (x_i) : \Delta x \in c\}$ and $l_\infty(\Delta) = \{x = (x_i) : \Delta x \in l_\infty\}$ spaces where, l_∞, c and c_0 are bounded, convergent and null sequence spaces, respectively. He investigated relations between these spaces and he obtained $c_0(\Delta) \subseteq c(\Delta) \subseteq l_\infty(\Delta)$.
After Kızmaz's study, which can be considered as a base about difference sequences, Et ([11]), Et and Çolak ([12]), Başarır ([4]), Et and Başarır ([13]), Et and Nuray ([15]), Gümüş ([18]), Gümüş and Nuray ([19]), Aydın and Başar ([2]), Bektaş et al. ([5]), Et and Esi ([14]), Savaş ([28]) and many others searched various properties of this concept.
In 2011, Gümüş and Nuray ([18]) defined $\Delta\mathscr{I}-$convergence as follows:

Definition 1.8. *([18]) Let $x = (x_i)$ be a real sequence, $\Delta x = (\Delta x_i) = (x_i - x_{i+1})$ and $\mathscr{I}$ is an admissible ideal in $\mathbb{N}$. For each $\varepsilon > 0$ if the set*

$$\{i \in \mathbb{N} : |\Delta x_i - L| \geq \varepsilon\}$$

belongs to $\mathscr{I}$ then, the sequence x is called as $\Delta\mathscr{I}-$convergent to the real number L and it is denoted by $\Delta\mathscr{I} - \lim x_i = L$. The number L is said to be $\Delta\mathscr{I}-$limit of the sequence. The set of all $\Delta\mathscr{I}-$convergent sequences is denoted by $c_{\mathscr{I}}(\Delta)$. If studying with difference sequences and $\mathscr{I}-$convergence together, the relation between $c_{\mathscr{I}}$ and $c_{\mathscr{I}}(\Delta)$ is important. The author investigated this relation in her thesis. ([18])

Determining the place of sequences in that does not satisfy the convergence condition is as important as convergent ones. Although not convergent, the existence of this kind of sequences that show similar characteristics to the concept of convergent sequence under certain conditions, has led to the emergence of different types of convergence. One of these is the concept of rough convergence defined by Phu ([26]) in finite dimensional normed spaces. According to this idea, rough convergence of a sequence can be obtained by extending the range of convergence by a number $r > 0$. Here, it should be noted that rough convergence has quite interesting applications in numerical analysis**.** This concept was later extended by Phu ([27]) to infinite dimensional normed spaces. Accordingly, the definition of rough convergence in a finite dimensional normed space can be given as follows:

Definition 1.9. *([26]) Let $(X, \|.\|)$ be a normed linear space and r be a nonnegative real number. Then the sequence $x = (x_i)$ in X is said to be rough convergent (or $r-$convergent) to x_*, if for any $\varepsilon > 0$, there exists an $i_\varepsilon \in \mathbb{N}$ such that*

$$\|x_i - x_*\| < r + \varepsilon$$

for all $i \geq i_\varepsilon$ or equivalently

$$\limsup \|x_i - x_*\| < r.$$

In this definition, x_* is called as an $r-$limit point of (x_i), r is called by roughness degree and this situation denoted by $x_i \xrightarrow{r} x_*$.

Let (x_i) be a rough convergent sequence in a finite dimensional normed space $(X, \|.\|)$ and r be a non-negative real number. For each $r > 0$, we obtain a different x_* point. So, this point, which is called by the $r-$limit point of the sequence, may not be unique. Therefore, a set of these points can be mentioned. This set is called by $r-$limit set and it is indicated by $LIM^r_{x_i}$. As seen, the topological and analytical features of the set are very important. The $r-$limit set of the sequence (x_i) is defined by

$$LIM^r_{x_i} = \left\{x_* \in X : x_i \xrightarrow{r} x_*\right\}.$$

Following Phu ([26])'s definition, Aytar ([3]) and Dündar and Çakan ([9]) and Pal, Chandra and Dutta ([25]) described rough statistical convergent sequences and rough $\mathscr{I}-$convergent sequences, respectively.

Definition 1.10. *([3]) Let $(\mathbb{R}^n, \|.\|)$ be the real $n-$dimensional normed space and r be a non-negative real number. For every $\varepsilon > 0$, if the set*

$$\{i \in \mathbb{N} : \|x_i - x_*\| \geq r + \varepsilon\}$$

has natural density zero or

$$st - \limsup \|x_i - x_*\| \leq r$$

is satisfied then, the sequence $x = (x_i)$ is said to be rough statistically convergent (or $r-$statistically convergent) to $x_* \in \mathbb{R}^n$, and it is denoted by $x_i \overset{rst}{\to} x_*$.

Definition 1.11. *([9]) Let $(\mathbb{R}^n, \|.\|)$ be the real $n-$dimensional normed space, $\mathscr{I}$ is an admissible ideal and r be a non-negative real number. For every $\varepsilon > 0$, if the set*

$$\{i \in \mathbb{N} : \|x_i - x_*\| \geq r + \varepsilon\} \in \mathscr{I}$$

or equivalently

$$\mathscr{I} - \limsup \|x_i - x_*\| \leq r$$

then, $x = (x_i)$ is said to be rough $\mathscr{I}-$convergent to $x_* \in \mathbb{R}^n$ and it is denoted by $x_i \overset{r-\mathscr{I}}{\to} x_*$.

After these studies, Demir ([6],[7]) and Demir and Gümüş ([8]) studied the concept of rough convergence and rough statistical convergence for difference sequences and proved some basic theorems. Arslan and Dündar defined rough convergence in $2-$normed spaces ([1]). Kişi and Ünal, studied rough statistical and rough $\Delta\mathscr{I}_2-$statistical convergence of double sequences in normed linear spaces ([20]),([21]).

2. Main Results

In this part we define the concept of rough $\mathscr{I}-$convergence for difference sequences and we prove some important theorems. It should be noted here, throughout the paper, $\mathbb{R}^n$ denotes the real $n-$dimensional space with the norm $\|.\|$, $\Delta x = (\Delta x_i)$ is a difference sequence such that $\Delta x_i \in \mathbb{R}^n$, $\mathscr{I}$ is an admissible ideal and r is a nonnegative real number.

Definition 2.1. *A difference sequence $\Delta x = (\Delta x_i)$ in $\mathbb{R}^n$ is said to be rough $\mathscr{I}-$convergent to $x_* \in \mathbb{R}^n$, provided that the set*

$$\{i \in \mathbb{N} : \|\Delta x_i - x_*\| \geq r + \varepsilon\}$$

belongs to $\mathscr{I}$ for any $\varepsilon > 0$ or equivalently

$$\mathscr{I} - \limsup \|\Delta x_i - x_*\| \leq r.$$

In this case we write $\Delta x_i \overset{r-\mathscr{I}}{\to} x_*$.

The $r - \mathscr{I}-$limit set of the sequence (Δx_i) is defined by

$$\mathscr{I} - LIM^r_{\Delta x_i} = \left\{x_* \in \mathbb{R}^n : \Delta x_i \overset{r-\mathscr{I}}{\to} x_*\right\}.$$

In this notation, r denotes the degree of roughness and it is easy to see that if $r = 0$, $\Delta\mathscr{I}-$convergence is obtained.
If $\mathscr{I}$ is an admissible ideal, then usual rough convergence for a difference sequence (Δx_i) implies rough $\mathscr{I}-$convergence.
Similar to Phu ([26]), Aytar ([3]) and Dündar ([9])'s studies, the idea of rough $\mathscr{I}-$convergence for a difference sequence can be explained with following example.

Example 2.2. *Let $\Delta y = (\Delta y_i)$ be a difference sequence which is $\mathscr{I}-$convergent to x_* and cannot be measured or calculated exactly. Additionally, let $\Delta x = (\Delta x_i)$ be an approximated sequence that provides the property $\{i \in \mathbb{N} : \|\Delta x_i - \Delta y_i\| > r\} \in \mathscr{I}$. Then, $\mathscr{I}-$convergence of the sequence (Δx_i) is not assured, but as the inclusion*

$$\{i \in \mathbb{N} : \|\Delta y_i - x_*\| \geq \varepsilon\} \supseteq \{i \in \mathbb{N} : \|\Delta x_i - x_*\| \geq r + \varepsilon\}$$

and we get $\{i \in \mathbb{N} : \|\Delta x_i - x_*\| \geq r + \varepsilon\} \in \mathscr{I}$. It means that $\Delta x = (\Delta x_i)$ is rough $\mathscr{I}-$convergent to x_*.

Phu ([26]) observed that for a sequence $x = (x_i)$ of real numbers,

$$LIM^r_x = [\limsup x - r, \liminf x + r].$$

Similarly we have,

$$\mathscr{I} - LIM^r_{\Delta x_i} = [\mathscr{I} - \limsup \Delta x - r, \ \mathscr{I} - \liminf \Delta x + r].$$

As seen in the example below, there exists an unbounded difference sequence which is not rough convergent but it can be rough $\mathscr{I}-$convergent.

Example 2.3. *Let $\mathscr{I}$ be an admissible ideal and A be an infinite set such that $A \in \mathscr{I}$. Define a difference sequence*

$$\Delta x_i = \begin{cases} (-1)^i, & if\ i \notin A \\ i, & if\ i \in A \end{cases}.$$

It is obvious that Δx is unbounded and rough $\mathscr{I}-$convergent. Because,

$$\mathscr{I}-LIM^r_{\Delta x_i} = \begin{cases} \emptyset, & if\ r<1 \\ [1-r, r-1], & otherwise \end{cases}.$$

Corollary 2.4. *$\mathscr{I}-LIM^r_{\Delta x_i} \neq \emptyset$ does not imply $LIM^r_{\Delta x_i} \neq \emptyset$. Because $\mathscr{I}$ is an admissible ideal, $LIM^r_{\Delta x_i} \neq \emptyset$ implies $\mathscr{I}-LIM^r_{\Delta x_i} \neq \emptyset$. Therefore,*

$$LIM^r_{\Delta x_i} \subseteq \mathscr{I}-LIM^r_{\Delta x_i}$$

and

$$diam(LIM^r_{\Delta x_i}) \leq diam(\mathscr{I}-LIM^r_{\Delta x_i}).$$

Theorem 2.5. *Let $\mathscr{I}$ be an admissible ideal. For any difference sequence $\Delta x = (\Delta x_i)$, diameter of $\mathscr{I}-LIM^r_{\Delta x_i}$ is not greater than $2r$. Generally, there is no smaller bound.*

Proof. Suppose that $diam\left(\mathscr{I}-LIM^r_{\Delta x_i}\right) > 2r$. Then, there exists $y, z \in \mathscr{I}-LIM^r_{\Delta x_i}$ such that

$$d := \|y-z\| > 2r.$$

Take an arbitrary $\varepsilon \in \left(0, \frac{d}{2}-r\right)$. Define A_1 and A_2 sets such that

$$A_1 := \{i \in \mathbb{N} : \|\Delta x_i - y\| \geq r+\varepsilon\}$$

and

$$A_2 := \{i \in \mathbb{N} : \|\Delta x_i - z\| \geq r+\varepsilon\}.$$

Because $y, z \in \mathscr{I}-LIM^r_{\Delta x_i}$ we have $A_1 \in \mathscr{I}$ and $A_2 \in \mathscr{I}$ and hence $B = \mathbb{N}\backslash(A_1 \cup A_2) \in \mathscr{F}(\mathscr{I})$ and so $B \neq \emptyset$. Now,

$$\|y-z\| \leq \|\Delta x_i - y\| + \|\Delta x_i - z\| < 2(r+\varepsilon) < 2r + 2\left(\frac{d}{2}-r\right) = d = \|y-z\|$$

for all $i \in B$. As we can see this is a contradiction. Therefore, $diam\left(\mathscr{I}-LIM^r_{\Delta x_i}\right) \leq 2r$.
Now, let's show that there is generally no smaller bound. For this proof, we show that

$$\mathscr{I}-LIM^r_{\Delta x_i} = \bar{B}_r(x_*) := \{y \in X : \|x_* - y\| \leq r\}.$$

We know that $diam(\bar{B}_r(x_*)) = 2r$.
Choose a difference sequence (Δx_i) with $\mathscr{I}-\lim \Delta x = x_*$. For each $\forall \varepsilon > 0$ we have

$$K = \{i \in \mathbb{N} : \|\Delta x_i - x_*\| \geq \varepsilon\} \in \mathscr{I}.$$

Then,

$$\|\Delta x_i - y\| \leq \|\Delta x_i - x_*\| + \|x_* - y\| \leq \|\Delta x_i - x_*\| + r$$

for each $y \in \bar{B}_r(x_*)$. In this case,

$$\|\Delta x_i - y\| < r+\varepsilon$$

whenever $i \notin K$. Therefore, $y \in \mathscr{I}-LIM^r_{\Delta x_i}$ and we get $\mathscr{I}-LIM^r_{\Delta x_i} = \bar{B}_r(x_*)$. □

Theorem 2.6. *For a bounded sequence (Δx_i), there is a nonnegative real number r such that $\mathscr{I}-LIM^r_{\Delta x_i} \neq \emptyset$.*

The question of "whether the converse of the above theorem is also valid" is a question that can immediately come to mind. The answer is no. But if the difference sequence is $\mathscr{I}-$bounded, the converse is valid. The theorem that gives this case is below.

Theorem 2.7. *(Δx_i) is $\mathscr{I}-$bounded if and only if there exists a nonnegative real number r such that $\mathscr{I}-LIM^r_{\Delta x_i} \neq \emptyset$.*

Proof. First, let's show that $\mathscr{I}-LIM^r_{\Delta x_i} \neq \emptyset$ when Δx is $\mathscr{I}-$bounded. From the definition of the concept of $\mathscr{I}-$boundedness, there exists a positive real number M such that

$$A = \{i \in \mathbb{N} : \|\Delta x_i\| \geq M\} \in \mathscr{I}.$$

Let's define $r' := \sup\{\|\Delta x_i\| : i \in A^c\}$. Then, $\mathscr{I}-LIM^{r'}_{\Delta x_i}$ contains the origin of $\mathbb{R}^n$ and $\mathscr{I}-LIM^{r'}_{\Delta x_i} \neq \emptyset$.
Now, assume that $\mathscr{I}-LIM^{r'}_{\Delta x_i} \neq \emptyset$ for some $r \geq 0$. Then we have an x_* such that $x_* \in \mathscr{I}-LIM^{r'}_{\Delta x_i}$. In that case,

$$\{i \in \mathbb{N} : \|\Delta x_i - x_*\| \geq r+\varepsilon\} \in \mathscr{I}$$

for each $\varepsilon > 0$. So, we can say that almost all Δx_i 's are contained in some ball with any radius greater than r and Δx_i is $\mathscr{I}-$bounded. □

Theorem 2.8. *The set $\mathscr{I}-LIM^r_{\Delta x_i}$ is closed and convex.*

Proof. Let's first prove that $\mathscr{I}-LIM^r_{\Delta x_i}$ is closed. For this proof, we use one of the well-known theorems in Functional Analysis. According to this theorem, "Let $y=(y_i)$ be a convergent sequence and $y_i \to y_*$.When $y \in A$ at the same time $y_* \in A$, then the set A is closed".
If $\mathscr{I}-\text{LIM}^r_{\Delta x_i}=\emptyset$ then, the proof is trivial.
Suppose that $\mathscr{I}-\text{LIM}^r_{\Delta x_i}\neq\emptyset$. Then, we have a sequence $\Delta y_i \subseteq \mathscr{I}-\text{LIM}^r_{\Delta x_i}$ such that $\Delta y_i \to y_*$. From the definition of the concept of convergence, for each $\varepsilon>0$ there exists an $i_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that $\|\Delta y_i - y_*\| < \frac{\varepsilon}{2}$ for all $i > i_{\frac{\varepsilon}{2}}$. Choose an $i_0 \in \mathbb{N}$ such that $i_0 > i_{\frac{\varepsilon}{2}}$. Then, $\|\Delta y_{i_0} - y_*\| < \frac{\varepsilon}{2}$.
On the other hand, since $(\Delta y_i) \subseteq \mathscr{I}-\text{LIM}^r_{\Delta x_i}$, we have $y_{i_0} \in \mathscr{I}-\text{LIM}^r_{\Delta x_i}$, i.e.,

$$\left\{i \in \mathbb{N} : \left\|\Delta x_i - y_{i_0}\right\| \geq r + \frac{\varepsilon}{2}\right\} \in \mathscr{I}.$$

Let $k \in \left\{i \in \mathbb{N} : \|\Delta x_i - y_{i_0}\| < r + \frac{\varepsilon}{2}\right\}$ and choose $i_0 > i_{\frac{\varepsilon}{2}}$. Then, $\|\Delta x_k - y_{i_0}\| < r + \frac{\varepsilon}{2}$ and hence,

$$\|\Delta x_k - y_*\| \leq \left\|\Delta x_k - y_{i_0}\right\| + \left\|y_{i_0} - y_*\right\| < r + \varepsilon$$

therefore,

$$\{i \in \mathbb{N} : \|\Delta x_i - y_*\| < r + \varepsilon\} \supseteq \left\{i \in \mathbb{N} : \left\|\Delta x_i - y_{i_0}\right\| < r + \frac{\varepsilon}{2}\right\}$$

and so, $\{i \in \mathbb{N} : \|\Delta x_i - y_*\| < r + \varepsilon\} \in \mathscr{F}(\mathscr{I})$. Therefore, $\{i \in \mathbb{N} : \|\Delta x_i - y_*\| \geq r + \varepsilon\} \in \mathscr{I}$.
For the convexity of $\mathscr{I}-\text{LIM}^r_{\Delta x_i}$, let's show that when $y_0, y_1 \in \mathscr{I}-\text{LIM}^r_{\Delta x_i}$, $[(1-\lambda)y_0 + \lambda y_1] \in \mathscr{I}-\text{LIM}^r_{\Delta x_i}$ for each $\lambda \in [0,1]$. Suppose that $y_0, y_1 \in \mathscr{I}-\text{LIM}^r_{\Delta x_i}$ and let $\varepsilon > 0$ be given. Define the sets

$$K_1 := \{i \in \mathbb{N} : \|\Delta x_i - y_0\| \geq r + \varepsilon\}$$

and

$$K_2 := \{i \in \mathbb{N} : \|\Delta x_i - y_1\| \geq r + \varepsilon\}.$$

We know that $K_1, K_2 \in \mathscr{I}$ which implies $M = \mathbb{N}\backslash(K_1 \cup K_2) \in \mathscr{F}(\mathscr{I})$ and so M is not empty. Then, we have

$$\|\Delta x_i - [(1-\lambda)y_0 + \lambda y_1]\| = \|(1-\lambda)(\Delta x_i - y_0) + \lambda(\Delta x_i - y_1)\| < r + \varepsilon$$

for each $i \in M$ and each $\lambda \in [0,1]$. We get

$$\{i \in \mathbb{N} : \|\Delta x_i - [(1-\lambda)y_0 + \lambda y_1]\| \geq r + \varepsilon\} \in \mathscr{I},$$

this means $[(1-\lambda)y_0 + \lambda y_1] \in \mathscr{I}-\text{LIM}^r_{\Delta x_i}$ and so, $\mathscr{I}-\text{LIM}^r_{\Delta x_i}$ is convex. □

Theorem 2.9. *Let $r>0$. The sequence (Δx_i) is rough $\mathscr{I}-$convergent to x_* if and only if there exists a difference sequence $\Delta y = (\Delta y_i)$ such that $\mathscr{I}-\lim \Delta y = x_*$ and $\|\Delta x_i - \Delta y_i\| \leq r$ for each $i \in \mathbb{N}$.*

Proof. For the necessity part, suppose that (Δx_i) is rough $\mathscr{I}-$convergent to x_*. From the definition,

$$\mathscr{I}-\limsup \|\Delta x_i - x_*\| \leq r \tag{2.1}$$

Let's define the sequence (Δy_i) as follows:

$$\Delta y_i := \begin{cases} x_*, & if\ \|\Delta x_i - x_*\| \leq r \\ \Delta x_i + r\frac{x_* - \Delta x_i}{\|\Delta x_i - x_*\|}, & otherwise \end{cases}. \tag{2.2}$$

Then, it is easy to see that

$$\|\Delta y_i - x_*\| = \begin{cases} 0, & if\ \|\Delta x_i - x_*\| \leq r \\ \|\Delta x_i - x_*\| - r, & otherwise \end{cases}$$

thus, $\|\Delta x_i - \Delta y_i\| \leq r$ for each $i \in \mathbb{N}$. At the same time, from (2.1) and (2.2),

$$\mathscr{I}-\limsup \|\Delta y_i - x_*\| = 0$$

and we get $\mathscr{I}-\lim \Delta y = x_*$.
For the sufficiency, suppose that $\mathscr{I}-\lim \Delta y = x_*$ and $\|\Delta x_i - \Delta y_i\| \leq r$ for each $i \in \mathbb{N}$. From the definition of the concept of $\mathscr{I}-$convergence, for each $\varepsilon > 0$ we get

$$A = \{i \in \mathbb{N} : \|\Delta y_i - x_*\| \geq \varepsilon\} \in \mathscr{I}.$$

We know that,

$$\{i \in \mathbb{N} : \|\Delta y_i - x_*\| \geq \varepsilon\} \supseteq \{i \in \mathbb{N} : \|\Delta x_i - x_*\| \geq r + \varepsilon\}$$

if $i \in \mathbb{N}\backslash A$ and we obtain

$$\{i \in \mathbb{N} : \|\Delta x_i - x_*\| \geq r + \varepsilon\} \in \mathscr{I}.$$

□

In order to prove the next theorem, we will need the following lemma, which is related to $\mathscr{I}-$cluster points.

Definition 2.10. *Let X be a normed space with the norm $\|.\|$. A point $c\in X$ is called as an $\mathscr{I}-$cluster point of a difference sequence $x=(x_i)$ if for any $\varepsilon>0$,*

$\{i\in\mathbb{N}:\|x_i-c\|<\varepsilon\}\notin\mathscr{I}$.

Lemma 2.11. *Let $\mathscr{I}(\Gamma_{\Delta x})$ be the set of all $\mathscr{I}-$cluster points of Δx and c be an arbitrary element of this set. For all $x_*\in\mathscr{I}-LIM^r_{\Delta x_i}$, we have $\|x_*-c\|\le r$.*

Proof. Let's accept the contrary of the lemma and find the contradiction. Assume that there exist a point $c\in\mathscr{I}(\Gamma_{\Delta x})$ and $x_*\in\mathscr{I}-\mathrm{LIM}^r_{\Delta x_i}$ such that $\|x_*-c\|>r$. Define $\varepsilon=\frac{\|x_*-c\|-r}{2}$. From the fact that $c\in\mathscr{I}(\Gamma_{\Delta x})$, we have

$A=\{i\in\mathbb{N}:\|\Delta x_i-c\|<\varepsilon\}\notin\mathscr{I}$.

For $i\in A$,

$\|x_*-\Delta x_i\|\ge\|x_*-c\|-\|\Delta x_i-c\|>2\varepsilon+r-\varepsilon=r+\varepsilon$

and so

$\{i\in\mathbb{N}:\|\Delta x_i-c\|<\varepsilon\}\subseteq\{i\in\mathbb{N}:\|\Delta x_i-x_*\|\ge r+\varepsilon\}$.

Because $A\in\mathscr{F}(\mathscr{I})$, we obtain

$\{i\in\mathbb{N}:\|\Delta x_i-x_*\|\ge r+\varepsilon\}\in\mathscr{F}(\mathscr{I})$

which contradicts the fact $x_*\in\mathscr{I}-\mathrm{LIM}^r_{\Delta x_i}$. Thus, $\|x_*-c\|\le r$ for all $x_*\in\mathscr{I}-\mathrm{LIM}^r_{\Delta x_i}$. □

Theorem 2.12. *For a difference sequence $\Delta x=(\Delta x_i)$, $\Delta x_i\overset{r-\mathscr{I}}{\to}x_*$ if and only if $\mathscr{I}-LIM^r_{\Delta x_i}=\bar{B}_r(x_*)$.*

Proof. In Theorem 2.1, we proved the necessity part. So, we need to prove if $\mathscr{I}-\mathrm{LIM}^r_{\Delta x_i}=\bar{B}_r(x_*)$ then, $\Delta x_i\overset{r-\mathscr{I}}{\to}x_*$. Let $\mathscr{I}-\mathrm{LIM}^r_{\Delta x_i}=\bar{B}_r(x_*)\ne\emptyset$. Then, from the Theorem 2.3., we have that (Δx_i) is $\mathscr{I}-$bounded.
Let (Δx_i) sequence has two different $\mathscr{I}-$cluster points such as x_* and x'_*. Then, the point

$$\bar{x}_*:=x_*+\frac{r}{\|x_*-x'_*\|}\left(x_*-x'_*\right)$$

satisfies

$$\begin{aligned}\|\bar{x}_*-x'_*\| &= \left(\tfrac{r}{\|x_*-x'_*\|}+1\right)\|x_*-x'_*\|\\ &= r+\|x_*-x'_*\|>r.\end{aligned}$$

From the previous lemma, $\bar{x}_*\notin\mathscr{I}-\mathrm{LIM}^r_{\Delta x_i}$ but this contradicts with $\|\bar{x}_*-x_*\|=r$ and $\mathscr{I}-\mathrm{LIM}^r_{\Delta x_i}=\bar{B}_r(x_*)$. This means that x_* is the unique statistical cluster point of Δx. So, Δx is rough $\mathscr{I}-$convergent to x_*. □

Definition 2.13. *Let X be a normed space with the norm $\|.\|$. For the elements $z_0,z_1\in X$ which satisfy $\|z_0\|=\|z_1\|=1\,(z_0\ne z_1)$ and for the scalar $0<\lambda<1$, if $\|(1-\lambda)z_0+\lambda z_1\|<1$ then X is called by strictly convex space.*

According to previous theorems and results, we can say that if $\mathscr{I}-\mathrm{LIM}^r_{\Delta x_i}=x_*$ then there exist $y_1,y_2\in\mathscr{I}-\mathrm{LIM}^r_{\Delta x_i}$ such that $\|y_1-y_2\|=2r$. Next theorem proves that if the space is strictly convex, the inverse is also valid.

Theorem 2.14. *Let $(\mathbb{R}^n,\|.\|)$ be a strictly convex space and $\Delta x=(\Delta x_i)$ be a difference sequence in this space. If there exists $y_1,y_2\in\mathscr{I}-LIM^r_{\Delta x_i}$ such that $\|y_1-y_2\|=2r$ then, this sequence is $\mathscr{I}-$convergent to $\frac{y_1+y_2}{2}$.*

Proof. Choose a point $c\in\mathscr{I}(\Gamma_{\Delta x})$ and $y_1,y_2\in\mathscr{I}-\mathrm{LIM}^r_{\Delta x_i}$. From the Lemma 2.1, we have

$$\|y_1-c\|\le r \text{ and } \|y_2-c\|\le r. \tag{2.3}$$

From the assumption we know that

$$2r=\|y_1-y_2\|=\|y_1-c+c-y_2\|\le\|y_1-c\|+\|y_2-c\|. \tag{2.4}$$

From (2.3) and (2.4) we have

$\|y_1-c\|=\|y_2-c\|=r.$

Therefore,

$$\frac{1}{2}(y_2-y_1)=c-y_1=y_2-c$$

and so,

$$c=\frac{1}{2}(y_1+y_2).$$

It means that, c is the unique $\mathscr{I}-$cluster point of $\Delta x=(\Delta x_i)$.
On the other hand, since $y_1,y_2\in\mathscr{I}-\mathrm{LIM}^r_{\Delta x_i}$, $\mathscr{I}-\mathrm{LIM}^r_{\Delta x_i}$ is not empty and so, Δx is bounded from Theorem 2.3. Consequently, we have $\mathscr{I}-\mathrm{LIM}\ \Delta x=\frac{1}{2}(y_1+y_2)$. □

Theorem 2.15. *i) If $c \in \mathscr{I}(\Gamma_{\Delta x})$ then, $\mathscr{I}-LIM^r_{\Delta x_i} \subseteq \bar{B}_r(c)$.*

$$ii)\ \mathscr{I}-\text{LIM}^r_{\Delta x_i} = \bigcap_{c\in\mathscr{I}(\Gamma_{\Delta x})} \bar{B}_r(c) = \{x_* \in \mathbb{R}^n : \mathscr{I}(\Gamma_{\Delta x}) \subseteq \bar{B}_r(x_*)\}.$$

Proof. $i)$ Suppose that $c \in \mathscr{I}(\Gamma_{\Delta x})$. From Lemma 2.1, for all $x_* \in \mathscr{I}-\text{LIM}^r_{\Delta x_i}$ we have $\|x_* - c\| \leq r$. Otherwise we have

$$\{i \in \mathbb{N} : \|\Delta x_i - x_*\| \geq r+\varepsilon\} \notin \mathscr{I}$$

for $\varepsilon := \frac{\|x_*-c\|-r}{3}$. We know that c is an $\mathscr{I}-$cluster point of (Δx_i), this contradicts with the fact that $x_* \in \mathscr{I}-\text{LIM}^r_{\Delta x_i}$.
$ii)$ Because of the first part of the theorem, we have $\mathscr{I}-\text{LIM}^r_{\Delta x_i} \subseteq \bigcap_{c\in\mathscr{I}(\Gamma_{\Delta x})} \bar{B}_r(c)$. Now let's show that $\bigcap_{c\in\mathscr{I}(\Gamma_{\Delta x})} \bar{B}_r(c) \subseteq \mathscr{I}-\text{LIM}^r_{\Delta x_i}$. Let $y \in \bigcap_{c\in\mathscr{I}(\Gamma_{\Delta x})} \bar{B}_r(c)$. Then we have $\|y-c\| \leq r$ for all $c \in \mathscr{I}(\Gamma_{\Delta x})$, which is equivalent to $\mathscr{I}(\Gamma_{\Delta x}) \subseteq \bar{B}_r(y)$, i.e.,

$$\bigcap_{c\in\mathscr{I}(\Gamma_{\Delta x})} \bar{B}_r(c) \subseteq \{x_* \in \mathbb{R}^n : \mathscr{I}(\Gamma_{\Delta x}) \subseteq \bar{B}_r(x_*)\}.$$

Now, let $y \notin \mathscr{I}-\text{LIM}^r_{\Delta x_i}$. Then, there exists an $\varepsilon > 0$ such that

$$\{i \in \mathbb{N} : \|\Delta x_i - y\| \geq r+\varepsilon\} \notin \mathscr{I},$$

which implies the existence of an $\mathscr{I}-$cluster point c of the sequence Δx with $\|y-c\| \geq r+\varepsilon$, i.e.,

$$\mathscr{I}(\Gamma_{\Delta x}) \nsubseteq \bar{B}_r(y)$$

and $y \notin \{x_* \in \mathbb{R}^n : \mathscr{I}(\Gamma_{\Delta x}) \subseteq \bar{B}_r(x_*)\}$. Hence, $\mathscr{I}-\text{LIM}^r_{\Delta x_i}$ follows from

$$y \in \{x_* \in \mathbb{R}^n : \mathscr{I}(\Gamma_{\Delta x}) \subseteq \bar{B}_r(x_*)\}$$

i.e.,

$$\{x_* \in \mathbb{R}^n : \mathscr{I}(\Gamma_{\Delta x}) \subseteq \bar{B}_r(x_*)\} \subseteq \mathscr{I}-LIM^r_{\Delta x_i}.$$

Therefore,

$$\mathscr{I}-LIM^r_{\Delta x_i} = \bigcap_{c\in\mathscr{I}(\Gamma_{\Delta x})} \bar{B}_r(c).$$

□

Now, let's give an example about this theorem.

Example 2.16. *Let $\mathscr{I} = \mathscr{I}_d$ and consider the sequence $\Delta x = (\Delta x_i)$ in $\mathbb{R}^1$ defined as follows:*

$$\Delta x_i = \begin{cases} \cos i\pi, & if\ i \neq k^2\ (k \in \mathbb{N}) \\ i, & otherwise \end{cases}$$

Then, we have $\mathscr{I}(\Gamma_{\Delta x}) = \{-1, 1\}$ and

$$\mathscr{I}-LIM^r_{\Delta x_i} = \bar{B}_r(-1) \cap \bar{B}_r(1).$$

Theorem 2.17. *Let $\Delta x = (\Delta x_i)$ is $\mathscr{I}-$bounded difference sequence. If $r \geq diam(\mathscr{I}(\Gamma_{\Delta x}))$ then, we have $\mathscr{I}(\Gamma_{\Delta x}) \subseteq \mathscr{I}-LIM^r_{\Delta x_i}$.*

Proof. Assume that $r \geq diam(\mathscr{I}(\Gamma_{\Delta x}))$, $c \in \mathscr{I}(\Gamma_{\Delta x})$ but $c \notin \mathscr{I}-LIM^r_{\Delta x_i}$. Then, there exists an $\varepsilon > 0$ such that

$$\{i \in \mathbb{N} : \|\Delta x_i - c\| \geq r+\varepsilon\} \notin \mathscr{I}.$$

Since (Δx_i) is $\mathscr{I}-$bounded, we have an $\mathscr{I}-$cluster point c_1 such that $\|c-c_1\| > r+\varepsilon_1$ where $\varepsilon_1 := \frac{\varepsilon}{2}$. So, we get

$$diam(\mathscr{I}(\Gamma_{\Delta x})) > r+\varepsilon_1.$$

It means, our acceptance is not true and this proves the theorem.
The converse of the theorem is also true, i.e., if $\mathscr{I}(\Gamma_{\Delta x}) \subseteq \mathscr{I}-LIM^r_{\Delta x_i}$ then, $r \geq diam(\mathscr{I}(\Gamma_{\Delta x}))$. □

Now recall the definitions of $\mathscr{I}-\text{limsup}\Delta x$, $\mathscr{I}-\text{liminf}\Delta x$ and $\mathscr{I}-\text{core}\{\Delta x\}$ and give some results. Let $\Delta x = (\Delta x_i)$ is a real difference sequence, $t \in \mathbb{R}$, $M_t = \{i : \Delta x_i > t\}$, $M^t = \{i : \Delta x_i < t\}$.

$$a)\ \mathscr{I}-\limsup \Delta x = \begin{cases} \sup\{t \in \mathbb{R} : M_t \notin \mathscr{I}\}, & \text{if there is a } t \in \mathbb{R} \text{ such that } M_t \notin \mathscr{I} \\ -\infty & \text{if } M_t \in \mathscr{I} \text{ for each } t \in \mathbb{R} \end{cases}$$

$$b)\ \mathscr{I}-\liminf \Delta x = \begin{cases} \inf\{t \in \mathbb{R} : M^t \notin \mathscr{I}\}, & \text{if there is a } t \in \mathbb{R} \text{ such that } M^t \notin \mathscr{I} \\ +\infty & \text{if } M^t \in \mathscr{I} \text{ for each } t \in \mathbb{R} \end{cases}.$$

Definition 2.18. *For a real difference sequence $\Delta x = (\Delta x_i)$, $\mathscr{I}-core\{\Delta x\}$ is defined to be closed interval as follows:*

$$\mathscr{I}-core\{\Delta x\} = [\mathscr{I}-\liminf \Delta x,\ \mathscr{I}-\limsup \Delta x].$$

Theorem 2.19. *If $\mathscr{I}-LIM^{r}_{\Delta x_i} \neq \emptyset$, then, $\mathscr{I}-\limsup \Delta x$ and $\mathscr{I}-\liminf \Delta x$ belong to the set $\mathscr{I}-LIM^{2r}_{\Delta x_i}$.*

Proof. Since $\mathscr{I}-LIM^{r}_{\Delta x_i} \neq \emptyset$, $\Delta x = (\Delta x_i)$ difference sequence is $\mathscr{I}-$bounded. The number $\mathscr{I}-\liminf \Delta x$ is an $\mathscr{I}-$cluster point of Δx and consequently we have $\|x_* - (\mathscr{I}-\liminf \Delta x)\| \leq r$ for all $x_* \in \mathscr{I}-LIM^{r}_{\Delta x_i}$. Put

$$A = \{i \in \mathbb{N} : \|x_* - \Delta x_i\| \geq r + \varepsilon\}.$$

If $i \notin A$, then

$$\begin{aligned} \|x_i - (\mathscr{I}-\liminf \Delta x)\| &\leq \|x_i - x_*\| + \|x_* - (\mathscr{I}-\liminf \Delta x)\| \\ &< 2r + \varepsilon. \end{aligned}$$

Thus, $\mathscr{I}-\liminf \Delta x \in \mathscr{I}-LIM^{2r}_{\Delta x_i}$. Similarly it can be shown that $\mathscr{I}-\limsup \Delta x \in \mathscr{I}-LIM^{2r}_{\Delta x_i}$. □

Corollary 2.20. *If $\mathscr{I}-LIM^{r}_{\Delta x_i} \neq \emptyset$ then, $\mathscr{I}-core\{\Delta x\} \subseteq \mathscr{I}-LIM^{2r}_{\Delta x_i}$.*

Proposition 2.21. *$diam(\mathscr{I}-core\{\Delta x\}) = r$ if and only if $\mathscr{I}-core\{\Delta x\} = \mathscr{I}-LIM^{r}_{\Delta x_i}$.*

Proof. Assume that $diam(\mathscr{I}-core\{\Delta x\}) = r$. Then, we can easily write that,

$$diam(\mathscr{I}-core\{\Delta x\}) = r \iff (\mathscr{I}-\limsup \Delta x) - (\mathscr{I}-\liminf \Delta x) = r. \tag{2.5}$$

From the definition of $\mathscr{I}-core\{\Delta x\}$ and (2.5),

$$\begin{aligned} \mathscr{I}-core\{\Delta x\} &= [\mathscr{I}-\liminf \Delta x,\ \mathscr{I}-\limsup \Delta x] \\ &= [\mathscr{I}-\limsup \Delta x - r,\ \mathscr{I}-\liminf \Delta x + r] \\ &= \mathscr{I}-LIM^{r}_{\Delta x_i} \end{aligned}$$

At the same time, it is also possible to say the following relations:

$$r > diam(\mathscr{I}-core\{\Delta x\}) \iff \mathscr{I}-core\{\Delta x\} \subset \mathscr{I}-LIM^{r}_{\Delta x_i}$$

and

$$r < diam(\mathscr{I}-core\{\Delta x\}) \iff \mathscr{I}-core\{\Delta x\} \supset \mathscr{I}-LIM^{r}_{\Delta x_i}.$$

□

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References

[1] M. Arslan and E. Dündar, On Rough Convergence in 2-Normed Spaces and Some Properties, Filomat 33(6) (2019), 5077-5086.
[2] C. Aydın and F. Başar, Some new difference sequence spaces, Appl. Math.Comput. 157(3) (2004), 677-693.
[3] S. Aytar, The Rough Limit Set and the Core of a Real Sequence, Numer. Func. Anal. Optimiz. 29(3) (2008), 283-290.
[4] M. Başarır, On the $\Delta-$statistical convergence of sequences, Firat Uni., Jour. of Science and Engineering 7(2) (1995), 1-6.
[5] Ç.A. Bektaş, M. Et and R. Çolak, Generalized difference sequence spaces and their dual spaces, J. Math. Anal. Appl. 292 (2004), 423-432.
[6] N. Demir, Rough convergence and rough statistical convergence of difference sequences, Master Thesis in Necmettin Erbakan University, Institue of Natural and Applied Sciences, June 2019.
[7] N. Demir and H. Gümüş, Rough convergence for difference sequences, New Trends in Math.Sciences 8(2) (2020), 22-28.
[8] N. Demir and H. Gümüş, Rough statistical convergence for difference sequences, Kragujevac Journal of Mathematics 46(5) (2022), 733-742.
[9] E. Dündar and C. Çakan, Rough $\mathscr{I}-$convergence, Demonstratio Mathematica 47(3) (2014), 638-651.
[10] P. Erdös and G. Tenenbaum, Sur les densites de certaines suites d'entiers, Proceedings of the London Math. Soc. 59(3) (1989), 417-438.
[11] M. Et, On some difference sequence spaces, Doğa-Tr. J.of Mathematics 17 (1993), 18-24.
[12] M. Et and R. Çolak, On some generalized difference sequence spaces, Soochow Journal Of Mathematics 21(4) (1995), 377-386.
[13] M. Et and M. Başarır, On some new generalized difference sequence spaces, Periodica Mathematica Hungarica 35(3) (1997), 169-175.
[14] M. Et. and A. Eşi, On Köthe- Toeplitz duals of generalized difference sequence spaces, Bull. Malaysian Math. Sci. Soc. 23 (2000), 25-32.
[15] M. Et and F. Nuray, Δ^m-Statistical convergence, Indian J.Pure Appl. Math. 32(6) (2001), 961-969.
[16] H. Fast, Sur la convergence statistique, Colloquium Mathematicum 2 (1951), 241-244.
[17] A. R. Freedman, J. Sember and M. Raphael, Some Cesàro-type summability spaces, Proc. London Math. Soc. (3) 37 no. 3 (1978), 508–520.
[18] H. Gümüş, $\mathscr{I}-$convergence and asymptotic $\mathscr{I}-$equivalence of difference sequences, Phd Thesis in Afyon Kocatepe University, Institue of Natural and Applied Sciences, May 2011.
[19] H. Gümüş and F. Nuray, Δ^m-Ideal Convergence, Selçuk J. Appl. Math. 12(2) (2011), 101-110.
[20] Ö. Kişi and H. K. Ünal, Rough $\Delta\mathscr{I}_2-$statistical convergence of double difference sequences in normed linear spaces, Bull. Math. Anal. Appl. 12 (1) (2020), 1-11.
[21] Ö. Kişi and H. K. Ünal, Rough Statistical Convergence of Double Sequences in Normed Linear Spaces, Honam Math. J., in press.
[22] H. Kızmaz, On certain sequence spaces, Canad. Math. Bull. 24(2) (1981), 169-176.
[23] P. Kostyrko, T. Šalát, W. Wilezyński, $\mathscr{I}-$Convergence, Real Analysis Exchange, Vol. 26(2) (2000), 669-680.
[24] H. I. Miller, A measure theoretical subsequence characterization of statistical convergence, Trans. of the Amer. Math. Soc. 347(5) (1995), 1811-1819.
[25] S. K. Pal, D. Chandra and S. Dutta, Rough ideal convergence, Hacettepe Journal of Mathematics and Statistics, Vol 42 (6) (2013), 633-640.
[26] H. X. Phu, Rough convergence in normed lineer spaces, Numer. Funct. Anal. Optmiz., Vol. 22 (2001), 199-222.
[27] H. X. Phu, Rough Convergence infinite dimensional normed spaces, Numerical Functional Analysis and Optimization, Vol.24 (2003), 285-301.
[28] E. Savaş, Δ^m-strongly summable sequences spaces in 2-normed spaces defined by ideal convergence and an Orlicz function, Applied Mathematics and Computation 217(1) (2010), 271–276.
[29] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, Colloquium Athematicum 2 (1951), 73-74.
[30] A. Zygmund, Trigonometric Series, Cam. Uni. Press, Cambridge, UK., (1979).