Canal Surface Whose Center Curve is a Hyperbolic Curve with Hyperbolic Frame

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(Dedicated to the memory of **Prof. Dr. Aurel BEJANCU (1946 - 2020)**)

ABSTRACT

In this paper, we obtain the parametrization of the canal surfaces whose center curves are the hyperbolic curves on the hyperbolic space H^2 in $\mathbb{E}_1^3$. The parametrization of the canal surface is expressed according to the hyperbolic frame given in [10]. Then, the parallel surface of this surface is studied. Also, we define the notion of the associated canal surface. Lastly, we give the geometric properties of these surfaces such that Weingarten surface, (X, Y)-Weingarten surface and linear Weingarten surface.

1. Introduction

Canal surfaces was firstly investigated by Monge in 1850. A canal surface is defined as a surface formed as the envelope of a family of spheres whose centers lie on a space curve $C(t)$ with radius $r(t)$. If the radius $r(t)$ is constant, then the canal surface is called as pipe surface or tubular surface. Canal surfaces play an essential role in descriptive geometry, because in case of an orthographic projection its contour curve can be drawn as the envelope of circles. In technical area canal surfaces can be used for blending surfaces smoothly. Canal surface is useful to represent various objects e.g. pipe, hose, rope or intestine of a body. Moreover, canal surface is an important instrument in surface modelling for CAD/CAM such as tubular surfaces, torus and Dupin cyclides [7].

Canal surfaces and tubular surfaces have been studied by many researchers. In [5], [6], [7], [8], the authors study canal surfaces and tubular surfaces in Euclidean 3-space, Minkowski 3-space, Galilean and Pseudo Galilean spaces. Lately, in [13], the authors consider the new approach to canal surfaces. Also in [2] and [9], the authors study canal surfaces with quaternions.

In [10], the author defines the hyperbolic frame of the curves on the sphere H^2. Also the curves with constant curvatures were studied in many papers such as [3],[4] and [11].

In this paper, we obtain the parametrization of the canal surfaces whose center curves are the hyperbolic curves on the hyperbolic space H^2 in $\mathbb{E}_1^3$. The parametrization of the canal surface is expressed according to the hyperbolic frame given in [10]. Then the parallel surface of this surface is studied. Also we define the notion of the associated canal surface. Lastly we give the geometric properties of these surfaces such that Weingarten surface, (X, Y)-Weingarten surface and linear Weingarten surface.

2. Preliminaries

The Minkowski space $\mathbb{E}_1^3$ is the Euclidean 3-space $\mathbb{E}^3$ equipped with indefinite flat metric given by

$$ds^2 = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}_1^3$. We denote by $\langle \, , \rangle$ the inner product on $\mathbb{E}_1^3$.

Let m be a fixed point and $r > 0$ be a constant. The pseudo-Riemannian sphere is defined by

$$S_1^2(m, r) = \{u \in \mathbb{E}_1^3 : \langle u - m, u - m \rangle = r^2\};$$

the pseudo-Riemannian hyperbolic space is defined by

$$H^2(m, r) = \{u \in \mathbb{E}_1^3 : \langle u - m, u - m \rangle = -r^2\};$$

the pseudo-Riemannian lightlike cone is defined by

$$Q^2(m) = \{u \in \mathbb{E}_1^3 : \langle u - m, u - m \rangle = 0\}.$$

Now, for a unit speed regular curve $x(s) \subset H^2 \subset \mathbb{E}_1^3$, we choose $\{x(s), \alpha(s), y(s)\}$ forming a standart Pseudo orthonormal basis of $\mathbb{E}_1^3$. Then the hyperbolic Frenet formulas of the hyperbolic curve $x(s)$ in H^2 can be written as

$$x'(s) = \alpha(s), \quad \alpha'(s) = x(s) + \kappa(s)y(s), \quad y'(s) = -\kappa(s)\alpha(s) \tag{2.1}$$

where $\langle x(s), x(s) \rangle = -1$ and $\langle \alpha(s), \alpha(s) \rangle = \langle y(s), y(s) \rangle = 1$.

Here, the function $\kappa(s)$ is called the hyperbolic curvature function (or curvature) of $x(s)$ and the frame $\{x(s), \alpha(s), y(s)\}$ is called the hyperbolic Frenet frame of the hyperbolic curve $x(s)$ ([10]).

We recall some well-known formulas for the surfaces in $\mathbb{E}_1^3$. Let M be a surface of $\mathbb{E}_1^3$, the standart connection D on $\mathbb{E}_1^3$ induces the Levi-Civita connection $\triangledown$ on M. We have the following Gauss formula

$$D_X Y = \nabla_X Y + h(X, Y),$$

and the Weingarten formula

$$D_X \xi = -A_\xi X + {}^\perp\nabla_X \xi,$$

where $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(TM^\perp)$. Then ∇ is the Levi-Civita connection of M, h is the second fundamental form, A_ξ is the shape operator, and ${}^\perp\nabla$ is the normal connection. We note that

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

The mean curvature vector field $\overrightarrow{H}$, the mean curvature H and the Gauss curvature of M are given respectively by

$$\overrightarrow{H} = \frac{1}{2}(h(e_1, e_1) + h(e_2, e_2)), \quad H = \left\|\overrightarrow{H}\right\| \quad \text{and} \quad K = \det A$$

where $\{e_1, e_2\}$ is an pseudo orthonormal basis on M ([1]).

Let U be the unit normal vector field on a surface $M(s, t)$ defined by

$$U = \frac{M_s \times M_t}{\|M_s \times M_t\|}.$$

The second fundamental form II of a surface $M(s, t)$ is given as

$$II = eds^2 + 2fdsdt + gdt^2$$

where

$$e = g(M_{ss}, U), f = g(M_{st}, U), g = g(M_{tt}, U).$$

([14]) Thus the second Gaussian curvature K_{II} of a surface is given as

$$K_{II} = \frac{1}{(eg - f^2)^2} \left\{ \begin{vmatrix} -\frac{1}{2}e_{tt} + f_{st} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_t \\ f_t - \frac{1}{2}g_s & e & f \\ \frac{1}{2}g_t & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_t & \frac{1}{2}g_s \\ \frac{1}{2}e_t & e & f \\ \frac{1}{2}g_s & f & g \end{vmatrix} \right\}.$$

3. Canal surfaces with the hyperbolic curve in H^2

In this section, we consider the canal surfaces whose center curve is the hyperbolic curves in H^2. The following theorem can be obtained by similar technique in [13]. Thus we omit the proof.

Theorem 3.1. *Let $x(s)$ be a hyperbolic curve with arc-length parameter s on H^2 and be the center curve of a canal surface*
(i) obtained from the sphere $S^2(r)$. Then the parametrization of the canal surface can be as following

$$
\begin{aligned}
M(s,t) &= \left(1 + m_1 r(s)\sqrt{1 - r_s^2(s)}\sinh t\right) x(s) - r(s) r_s(s)\alpha(s) \\
&\quad + \left(m_2 r(s)\sqrt{1 - r_s^2(s)}\cosh t\right) y(s)
\end{aligned}
$$

and the parametrization of the tubular surface can be as following

$$
M(s,t) = (1 + m_1 r \sinh t) x(s) + (m_2 r \cosh t) y(s)
$$

(ii) obtained from the hyperbolic space $H^2(r)$. Then the parametrization of the canal surface can be as following

$$
\begin{aligned}
M(s,t) &= \left(1 + m_1 r(s)\sqrt{1 + r_s^2(s)}\cosh t\right) x(s) + r(s) r_s(s)\alpha(s) \\
&\quad + \left(m_2 r(s)\sqrt{1 + r_s^2(s)}\sinh t\right) y(s)
\end{aligned}
$$

and the parametrization of the tubular surface can be as following

$$
M(s,t) = (1 + m_1 r \cosh t) x(s) + (m_2 r \sinh t) y(s)
$$

(iii) obtained from the ligthlike cone Q^2. Then the parametrization of the canal surface(tubular surface) can be as following

$$
M(s,t) = (1 + a(s,t)) x(s) + m_1 a(s,t) y(s)
$$

where $m_1, m_2 \in \{-1, 1\}$ and $a(s,t)$ is a function of s and t, but not only s.

Remark 3.1. It is clear that the surface

$$
M(s,t) = (1 + a(s,t)) x(s) + m_1 a(s,t) y(s)
$$

is a degenerate surface.

In the following theorem, we classify all hyperbolic curve on H^2 with constant curvature.

Theorem 3.2. *Let κ be a real number. Then $x(s)$ is a hyperbolic curve on H^2 with arc-length parameter s and curvature κ satisfying*
(i) $\kappa^2 < 1$ if and only if $x(s)$ can be parameterized by

$$
x = \cosh\left(\sqrt{1 - \kappa^2}s\right) V_1 + \sinh\left(\sqrt{1 - \kappa^2}s\right) V_2 + V_3
$$

where V_1, V_2, V_3 are mutually orthogonal vectors satisfying the following equations

$$
\langle V_1, V_1\rangle = -\langle V_2, V_2\rangle = -\frac{1}{1 - \kappa^2} \quad and \quad \langle V_3, V_3\rangle = \frac{\kappa^2}{1 - \kappa^2}.
$$

(ii) $\kappa^2 > 1$ if and only if $x(s)$ can be parameterized by

$$
x = \cos\left(\sqrt{\kappa^2 - 1}s\right) V_1 + \sin\left(\sqrt{\kappa^2 - 1}s\right) V_2 + V_3
$$

where V_1, V_2, V_3 are mutually orthogonal vectors satisfying the following equations

$$
\langle V_1, V_1\rangle = \langle V_2, V_2\rangle = \frac{1}{\kappa^2 - 1} \quad and \quad \langle V_3, V_3\rangle = -\frac{\kappa^2}{\kappa^2 - 1}.
$$

(iii) $\kappa^2 = 1$ *if and only if* $x(s)$ *can be parameterized by*

$$x = s^2 V_1 + s V_2 + V_3$$

where

$$\langle V_1, V_1 \rangle = \langle V_1, V_2 \rangle = \langle V_2, V_3 \rangle = 0,$$

$$\langle V_2, V_2 \rangle = -\langle V_3, V_3 \rangle = 1 \quad and \quad \langle V_1, V_3 \rangle = -\frac{1}{2}.$$

Proof. Let $x(s)$ be a hyperbolic curve on H^2 with arc-length parameter s and constant curvature κ. By using the hyperbolic Frenet equations (4.2), we obtain the following homogeneous differential equation with constant coefficients

$$x''' + \left(\kappa^2 - 1 \right) x' = 0.$$

The characteristic equation of the previous equation is follows

$$r \left(r^2 + \left(\kappa^2 - 1 \right) \right) = 0.$$

(i) Assume that $\kappa^2 < 1$. Then we get

$$x = \cosh\left(\sqrt{1 - \kappa^2}s\right) V_1 + \sinh\left(\sqrt{1 - \kappa^2}s\right) V_2 + V_3. \tag{3.1}$$

Differentiating (3.1) with respect to s, we get

$$\alpha = \sqrt{1 - \kappa^2} \sinh\left(\sqrt{1 - \kappa^2}s\right) V_1 + \sqrt{1 - \kappa^2} \cosh\left(\sqrt{1 - \kappa^2}s\right) V_2.$$

By using $\langle \alpha, \alpha \rangle = 1$, we get V_1, V_2, V_3 are mutually orthogonal vectors satisfying the following equations

$$\langle V_1, V_1 \rangle = -\langle V_2, V_2 \rangle = -\frac{1}{1 - \kappa^2} \quad \text{and} \quad \langle V_3, V_3 \rangle = \frac{\kappa^2}{1 - \kappa^2}.$$

(ii) Assume that $\kappa^2 > 1$. Then the proof is similar.

(iii) Assume that $\kappa^2 = 1$. Then we get

$$x = s^2 V_1 + s V_2 + V_3. \tag{3.2}$$

Differentiating (3.2) with respect to s, we get

$$\alpha = 2s V_1 + V_2.$$

By using $\langle x, x \rangle = -1$ and $\langle \alpha, \alpha \rangle = 1$, we get $\langle V_1, V_1 \rangle = \langle V_1, V_2 \rangle = \langle V_2, V_3 \rangle = 0$,

$$\langle V_2, V_2 \rangle = -\langle V_3, V_3 \rangle = 1 \quad \text{and} \quad \langle V_1, V_3 \rangle = -\frac{1}{2}.$$

Then the proof is complete. $\qquad \square$

Example 3.1. Let us take $\kappa = \frac{\sqrt{2}}{2}$ in (i) of Theorem 3.2. Then we obtain

$$\langle V_1, V_1 \rangle = -\langle V_2, V_2 \rangle = -2 \quad \text{and} \quad \langle V_3, V_3 \rangle = 1.$$

Then we can choose

$$V_1 = \left(\sqrt{2}, 0, 0 \right), \quad V_2 = \left(0, \sqrt{2}, 0 \right), \quad V_3 = (0, 0, 1),$$

which implies that

$$
\begin{aligned}
x &= \left(\sqrt{2} \cosh\left(\frac{s}{\sqrt{2}}\right), \sqrt{2} \sinh\left(\frac{s}{\sqrt{2}}\right), 1 \right), \\
\alpha &= \left(\sinh\left(\frac{s}{\sqrt{2}}\right), \cosh\left(\frac{s}{\sqrt{2}}\right), 0 \right), \\
y &= \left(-\cosh\left(\frac{s}{\sqrt{2}}\right), -\sinh\left(\frac{s}{\sqrt{2}}\right), -\sqrt{2} \right).
\end{aligned}
$$

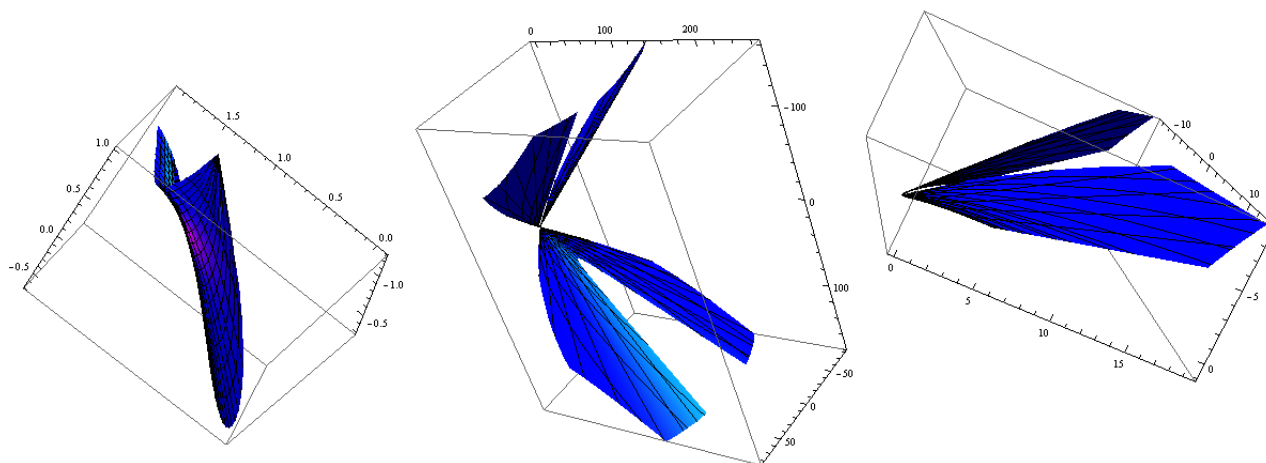Now let us take $m_1 = m_2 = 1$ in Theorem 3.1 and give the canal surfaces in Figure 2.

Figure 1. The canal surface for $r = e^s$ in (i) of Theorem 3.1 (left), $r = s$ in (ii) of Theorem 3.1(middle) and $a = s^2 \cosh t$ in (iii) of Theorem 3.1(right).

Example 3.2. Let us take $\kappa = \sqrt{2}$ in (ii) of Theorem 3.2. Then we choose

$$V_1 = (0, 1, 0) \,, \quad V_2 = (0, 0, 1) \,, \quad V_3 = \left(\sqrt{2}, 0, 0 \right),$$

which implies that

$$
\begin{aligned}
x &= \left(\sqrt{2}, \cos s, \sin s \right), \\
\alpha &= (0, -\sin s, \cos s) \,, \\
y &= \left(-1, -\sqrt{2} \cos s, -\sqrt{2} \sin s \right).
\end{aligned}
$$

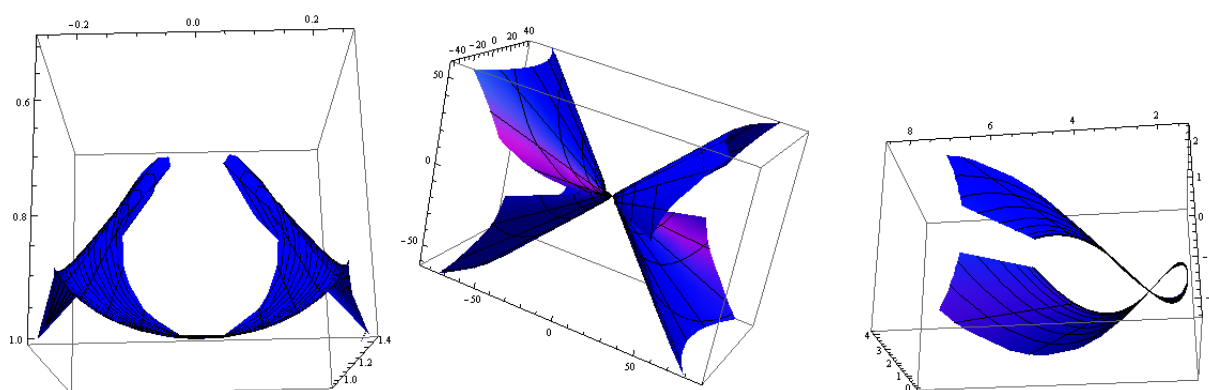Now let us take $m_1 = m_2 = 1$ in Theorem 3.1 and give the canal surfaces in Figure 3.

Figure 2. The canal surface for $r = s^2$ in (i) of Theorem 3.1 (left), $r = s$ in (ii) of Theorem 3.1(middle) and $a = s^2 \cosh t$ in (iii) of Theorem 3.1(right).

Example 3.3. Let us take $\kappa = 1$ in (ii) of Theorem 3.2. Then we choose

$$V_1 = \left(\frac{1}{2}, 0, \frac{1}{2} \right), \quad V_2 = (0, 1, 0) \,, \quad V_3 = (1, 0, 0) \,,$$

which implies that

$$
\begin{aligned}
x &= \left(\frac{s^2}{2} + 1, s, \frac{s^2}{2} \right), \\
\alpha &= (s, 1, s), \\
y &= \left(-\frac{s^2}{2}, -s, 1 - \frac{s^2}{2} \right).
\end{aligned}
$$

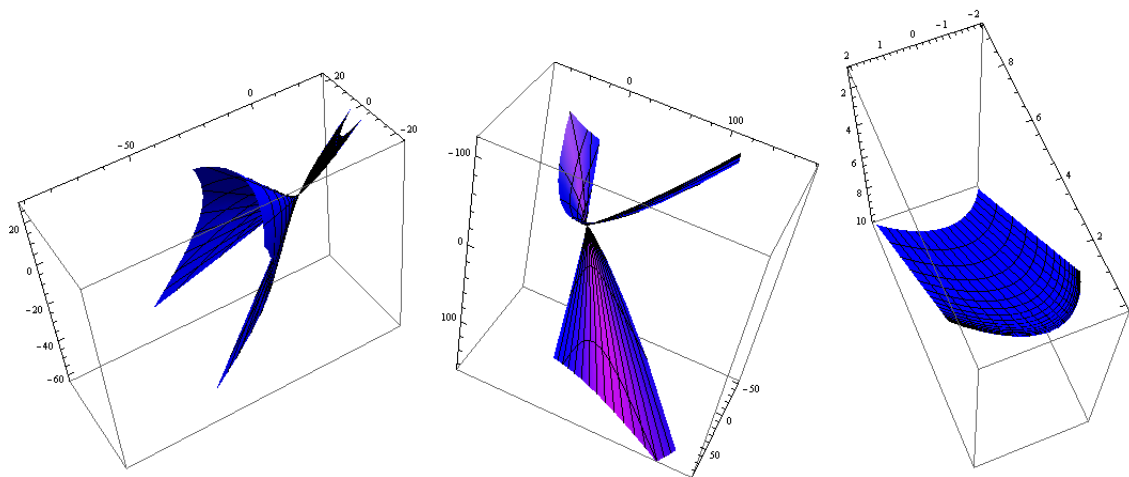Now let us take $m_1 = m_2 = 1$ in Theorem 3.1 and give the canal surfaces in Figure 4.

Figure 3. The canal surface for $r = 2$ in (i) of Theorem 3.1 (left), $r = s^3$ in (ii) of Theorem 3.1(middle) and $a = e^t$ in (iii) of Theorem 3.1(right).

4. Tubular surface whose center curve is the hyperbolic curve

In this section we consider the tubular surface whose center curve is the hyperbolic curve in H^2.

4.1. Tubular surface obtained from the sphere $S_1^2(r)$

In this subsection, we consider the tubular surface obtained from the sphere $S_1^2(r)$, given in theorem 3.1. We assume that $m_1 = m_2 = 1$. So we give the parametrization of the tubular surface as

$$
\psi(s, t) = (1 + r \sinh t) x + (r \cosh t) y(s) \tag{4.1}
$$

From (4.1), we find

$$
\begin{aligned}
\psi_s &= (1 + r \sinh t - r\kappa \cosh t) \alpha, \\
\psi_t &= (r \cosh t) x - (r \sinh t) y.
\end{aligned}
$$

We can find the components of first fundemental form as follows

$$
g_{11} = \langle \psi_s, \psi_s \rangle = (1 + r \sinh t - r\kappa \cosh t)^2, \quad g_{12} = \langle \psi_s, \psi_t \rangle = 0, \quad g_{22} = \langle \psi_t, \psi_t \rangle = -r^2.
$$

Then $g_{11}g_{22} - (g_{12})^2 = r^2 (1 + r \sinh t - r\kappa \cosh t)^2$. We assume that $1 + r \sinh t - r\kappa \cosh t > 0$ for the regularity of the surface ψ.

Now we will give an orthonormal basis on $\psi(s, t)$.

$$
\begin{aligned}
e_1 &= \frac{1}{\|\psi_s\|} \psi_s = \alpha, \\
e_2 &= \frac{1}{\|\psi_t\|} \psi_t = (\cosh t) x - (\sinh t) y,
\end{aligned}
$$

where $\{e_1, e_2\}$ is an orthonormal frame field on $\psi(s, t)$ with signature $(+, -)$. Set

$$e_3 = -(\sinh t)\, x - (\cosh t)\, y,$$

where e_3 is a normal vector field to $\psi(s, t)$. $\{e_1, e_2, e_3\}$ is an orthonormal basis on $\psi(s, t)$. Then we obtain

$$
\begin{aligned}
D_{e_1} e_1 &= \frac{1}{1 + r\sinh t - r\kappa \cosh t}\,(x + \kappa y), \\
D_{e_1} e_2 &= \frac{\cosh t - \kappa \sinh t}{1 + r\sinh t - r\kappa \cosh t}\alpha, \\
D_{e_2} e_2 &= \frac{1}{r}\left((\sinh t)\, x + (\cosh t)\, y\right).
\end{aligned}
$$

The components of the second fundamental form h are calculated as follows

$$
\begin{aligned}
h_{11} &= \langle D_{e_1} e_1, e_3 \rangle = \frac{\sinh t - \kappa \cosh t}{1 + r\sinh t - r\kappa \cosh t}, \\
h_{12} &= \langle D_{e_1} e_2, e_3 \rangle = 0 \quad \text{and} \quad h_{22} = \langle D_{e_2} e_2, e_3 \rangle = -\frac{1}{r}.
\end{aligned}
$$

Theorem 4.1. *The mean curvature H of $\psi(s, t)$ is obtained as*

$$H = \frac{1}{2}(h_{11} - h_{22}) = \frac{1 - 2r\kappa \cosh t + 2r\sinh t}{2r(1 + r\sinh t - r\kappa \cosh t)}. \tag{4.2}$$

Theorem 4.2. *The Gauss curvature K of $\psi(s, t)$ is obtained as*

$$K = -\left(h_{11} h_{22} - (h_{12})^2\right) = \frac{\sinh t - \kappa \cosh t}{r(1 + r\sinh t - r\kappa \cosh t)}. \tag{4.3}$$

A surface is called Weingarten surface if there exist a non-trivial function $\Psi(K, H)$ such that $\Psi(K, H) = K_s H_t - K_t H_s = 0$ for the Gauss curvature K and mean curvature H of the surface. Here subscripts denote partial derivatives. Also we a surface is called as a linear Weingarten surface if there exist real numbers a, b, $c \in \mathbb{R} \backslash \{0\}$ such that the linear combination $aK + bH = c$ is satisfied. For $(X, Y) \in \{(K, K_{II}), (H, K_{II})\}$, the surface is called as (X, Y)-Weingarten surface if $\Psi(X, Y) = 0$ ([12]).

From (4.2) and (4.3), we have

$$K_s = \frac{-\kappa' \cosh t}{r(1 + r\sinh t - r\kappa \cosh t)^2}, \quad K_t = \frac{\cosh t - \kappa \sinh t}{r(1 + r\sinh t - r\kappa \cosh t)^2}$$

and

$$H_s = \frac{-\kappa' \cosh t}{2(1 + r\sinh t - r\kappa \cosh t)^2}, \quad H_t = \frac{\cosh t - \kappa \sinh t}{2(1 + r\sinh t - r\kappa \cosh t)^2}.$$

Thus it can be easily seen that $\Psi(K, H) = K_s H_t - K_t H_s = 0$. So we can give the following theorem.

Theorem 4.3. *The surface $\psi(s, t)$ is a Weingarten surface.*

Now assume that there exist real numbers a, b, $c \in \mathbb{R} \backslash \{0\}$ such that the linear combination $aK + bH = c$ is satisfied.

$$aK + bH - c = \frac{b - 2cr + 2\left(a - cr^2 + br\right)\sinh t - 2\left(a - cr^2 + br\right)\kappa \cosh t}{2r(1 + r\sinh t - r\kappa \cosh t)} = 0$$

which implies that $b = 2cr$ and $a + cr^2 = 0$. So we can give the following theorem.

Theorem 4.4. *Let K and H be the Gauss curvature and mean curvature of the surface $\psi(s, t)$. Then there exists the following relation between K and H :*

$$-r^2 K + 2rH = 1$$

where r is a positive real number.

From above theorem, we get the following corollary.

Corollary 4.1. *The surface $\psi(s,t)$ is a linear Weingarten surface.*

Definition 4.1. The parallel surface of the surface $X(s,t)$ defined by

$$X^*(s,t) = X(s,t) + \mu U(s,t)$$

where

$$U(s,t) = \frac{X_s \times X_t}{\|X_s \times X_t\|}$$

is the unit normal vector of the surface $X(s,t)$ and $\mu \in \mathbb{R}$.

Now we will define the parallel surface $\psi^*(s,t)$ of the surface $\psi(s,t)$ as follows

$$
\begin{aligned}
\psi^*(s,t) &= \psi(s,t) + \mu e_3 \\
&= (1 + (r-\mu)\sinh t)\, x(s) + ((r-\mu)\cosh t)\, y(s)
\end{aligned}
\tag{4.4}
$$

From (4.4), we find

$$
\begin{aligned}
\psi_s^* &= (1 + (r-\mu)\sinh t - (r-\mu)\kappa \cosh t)\,\alpha, \\
\psi_t^* &= ((r-\mu)\cosh t)\, x - ((r-\mu)\sinh t)\, y.
\end{aligned}
$$

We can find the components of first fundamental form as follows

$$
\begin{aligned}
g_{11}^* &= \langle \psi_s^*, \psi_s^* \rangle = (1 + (r-\mu)\sinh t - (r-\mu)\kappa \cosh t)^2, \\
g_{12}^* &= \langle \psi_s^*, \psi_t^* \rangle = 0, \quad g_{22} = \langle \psi_t, \psi_t \rangle = -(r-\mu)^2.
\end{aligned}
$$

Then $g_{11}^* g_{22}^* - (g_{12}^*)^2 = (r-\mu)^2 (1 + (r-\mu)\sinh t - (r-\mu)\kappa \cosh t)^2$. We assume that $r - \mu > 0$ and $1 + (r-\mu)\sinh t - (r-\mu)\kappa \cosh t > 0$ for the regularity of the surface $\psi^*(s,t)$.

Now we will give an orthonormal basis on $\psi^*(s,t)$.

$$
\begin{aligned}
e_1^* &= \frac{1}{\|\psi_s^*\|}\psi_s^* = \alpha, \\
e_2^* &= \frac{1}{\|\psi_t^*\|}\psi_t^* = (\cosh t)\, x + (\sinh t)\, y,
\end{aligned}
$$

where $\{e_1^*, e_2^*\}$ is an orthonormal frame field on $\psi^*(s,t)$. Set

$$e_3^* = -(\sinh t)\, x - (\cosh t)\, y,$$

where e_3^* is a normal vector field to $\psi^*(s,t)$. $\{e_1^*, e_2^*, e_3^*\}$ is an orthonormal basis on $\psi^*(s,t)$. Then we obtain

$$
\begin{aligned}
D_{e_1^*}e_1^* &= \frac{1}{1 + (r-\mu)\sinh t - (r-\mu)\kappa \cosh t}\,(x + \kappa y), \\
D_{e_1^*}e_2^* &= \frac{\cosh t - \kappa \sinh t}{1 + (r-\mu)\sinh t - (r-\mu)\kappa \cosh t}\,\alpha, \\
D_{e_2^*}e_2^* &= \frac{1}{(r-\mu)}((\sinh t)\, x + (\cosh t)\, y).
\end{aligned}
$$

The components of the second fundamental form h^* are calculated as follows

$$
\begin{aligned}
h_{11}^* &= \langle D_{e_1^*}e_1^*, e_3^* \rangle = \frac{\sinh t - \kappa \cosh t}{1 + (r-\mu)\sinh t - (r-\mu)\kappa \cosh t}, \\
h_{12}^* &= \langle D_{e_1^*}e_2^*, e_3^* \rangle = 0 \quad \text{and} \quad h_{22} = \langle D_{e_2^*}e_2^*, e_3^* \rangle = -\frac{1}{(r-\mu)}.
\end{aligned}
$$

Similarly we can find the following results.

Theorem 4.5. *The mean curvature H^* of $\psi^*(s,t)$ is obtained as*

$$H^* = \frac{1 - 2(r-\mu)\kappa \cosh t + 2(r-\mu)\sinh t}{2(r-\mu)(1 + (r-\mu)\sinh t - (r-\mu)\kappa \cosh t)}.$$

Theorem 4.6. *The Gauss curvature K^* of $\psi^*(s,t)$ is obtained as*

$$K^* = \frac{\sinh t - \kappa \cosh t}{(r-\mu)\left(1 + (r-\mu)\sinh t - (r-\mu)\kappa \cosh t\right)}.$$

Theorem 4.7. *The surface $\psi^*(s,t)$ is a Weingarten surface.*

Theorem 4.8. *Let K^* and H^* be the Gauss curvature and mean curvature of the surface $\psi^*(s,t)$. Then there exists the following relation between K^* and H^* :*

$$-(r-\mu)^2 K^* + 2(r-\mu)H^* = 1$$

where r is a positive real number and μ is a real number.

From above theorem, we get the following corollary.

Corollary 4.2. *The surface $\psi^*(s,t)$ is a linear Weingarten surface.*

Definition 4.2. Let $x_1(\overline{s})$ be the associated curve of $x(s)$ such that $x_1(\overline{s}) = y(s)$ where there exists a diffeomorfism $\overline{s} = f_1(s)$. In this paper, we will call $x_1(\overline{s}) = y(s)$ as **the first associated curve** of the hyperbolic curve $x(s)$.
Let $x_2(s^*) = \alpha(s)$ where there exists a diffeomorfism $s^* = f_2(s)$ Then we will call $x_2(s^*) = \alpha(s)$ as **the second associated curve** of the hyperbolic curve $x(s)$.

Definition 4.3. Let $x_1(\overline{s})$ be the first associated curve of the hyperbolic curve $x(s)$ in H^2, $\psi(s,t)$ and $\psi_1(\overline{s},t)$ be canal surfaces (or tubular surfaces) whose center curves are $x(s)$ and $x_1(\overline{s})$, respectively. Then $\psi_1(\overline{s},t)$ is called as **"the first associated canal surface (or the first associated tubular surface)"** of $\psi(s,t)$.
Similarly, let $x_2(s^*)$ be the second associated curve of the hyperbolic curve $x(s)$ in H^2, $\psi(s,t)$ and $\psi_2(s^*,t)$ be canal surfaces (or tubular surfaces) whose center curves are $x(s)$ and $x_2(s^*)$, respectively. Then $\psi_2(s^*,t)$ is called as **"the second associated canal surface (or the second associated tubular surface)"** of $\psi(s,t)$.

4.1.1. The first associated curve and the first associated tubular surface Firstly we consider the first associated curve and the first associated tubular surface of the hyperbolic curve $x(s)$. So we can give the following corollary.

Corollary 4.3. *Let $x_1(\overline{s})$ be the first associated curve of the hyperbolic curve $x(s)$ in H^2 with the hyperbolic frame $\{x(s), \alpha(s), y(s)\}$ such that $x_1(\overline{s}) = y(s)$ where there exists a diffeomorfism $\overline{s} = f_1(s)$. Then we have*

$$x_1 = y, \quad \alpha_1 = -\varepsilon_1 \alpha, \quad y_1 = -\varepsilon_1 x, \quad \kappa_1 = \frac{\varepsilon_1}{\kappa}, \quad \frac{df_1}{ds} = \varepsilon_1 \kappa,$$

and

$$\frac{dx_1}{d\overline{s}} = \alpha_1, \quad \frac{d\alpha_1}{d\overline{s}} = -x_1 + \kappa_1 y_1, \quad \frac{dy_1}{d\overline{s}} = \kappa_1 \alpha_1$$

where $\varepsilon_1 = sgn(\kappa)$, $\{x_1(\overline{s}), \alpha_1(\overline{s}), y_1(\overline{s})\}$ is the spherical frame, $\kappa_1(\overline{s})$ is the spherical curvature of $x_1(\overline{s})$ and

$$\langle x_1(\overline{s}), x_1(\overline{s}) \rangle = \langle \alpha_1(\overline{s}), \alpha_1(\overline{s}) \rangle = 1, \quad \langle y_1(\overline{s}), y_1(\overline{s}) \rangle = -1.$$

Theorem 4.9. *Let $x_1(\overline{s})$ be the first associated curve of the hyperbolic curve $x(s)$ in H^2 with the hyperbolic frame $\{x(s), \alpha(s), y(s)\}$ such that $x_1(\overline{s}) = y(s)$. Also let $x_1(\overline{s})$ be the center curve of a canal surface*
(i) obtained from the sphere $S^2(r)$. Then the parametrization of the canal surface can be as following

$$
\begin{aligned}
M_1(\overline{s},t) &= \left(1 + m_1 r(\overline{s})\sqrt{1 - r_s^2(\overline{s})}\cosh t\right)x_1(\overline{s}) - r(\overline{s})r_s(\overline{s})\alpha_1(\overline{s}) \\
&\quad + \left(m_2 r(\overline{s})\sqrt{1 - r_s^2(\overline{s})}\sinh t\right)y_1(\overline{s})
\end{aligned}
$$

and the parametrization of the tubular surface can be as following

$$M_1(\overline{s},t) = (1 + m_1 r \cosh t)x_1(\overline{s}) + (m_2 r \sinh t)y_1(\overline{s})$$

(ii) *obtained from the hyperbolic space $H^2(r)$. Then the parametrization of the canal surface can be as following*

$$M_2(\bar{s}, t) = \left(1 + m_1 r(\bar{s}) \sqrt{1 + r_s^2(\bar{s})} \sinh t\right) x_1(\bar{s}) + r(\bar{s}) r_s(\bar{s}) \alpha_1(\bar{s})$$
$$+ \left(m_2 r(\bar{s}) \sqrt{1 + r_s^2(\bar{s})} \cosh t\right) y_1(\bar{s})$$

and the parametrization of the tubular surface can be as following

$$M_2(\bar{s}, t) = (1 + m_1 r \sinh t) x_1(\bar{s}) + (m_2 r \cosh t) y_1(\bar{s})$$

(iii) *obtained from the ligthlike cone Q^2. Then the parametrization of the canal surface(tubular surface) can be as following*

$$M_3(\bar{s}, t) = (1 + a(\bar{s}, t)) x_1(\bar{s}) + m_1 a(\bar{s}, t) y_1(\bar{s})$$

where $m_1, m_2 \in \{-1, 1\}$ and $a(\bar{s}, t)$ is a function of $\bar{s}$ and t, but not only $\bar{s}$.

Firstly we consider the tubular surface $M_1(\bar{s}, t)$ for $m_1 = m_2 = 1$ as follows

$$
\begin{aligned}
M_1(\bar{s}, t) &= (1 + r \cosh t) x_1(\bar{s}) + (r \sinh t) y_1(\bar{s}) \\
&= -\varepsilon_1 (r \sinh t) x(s) + (1 + r \cosh t) y(s).
\end{aligned}
\tag{4.5}
$$

From (4.5), we have

$$
\begin{aligned}
(M_1)_{\bar{s}} &= \frac{-\varepsilon_1 (\varepsilon_1 r \sinh t + \kappa (1 + r \cosh t))}{\kappa} \alpha, \\
(M_1)_t &= -(\varepsilon_1 r \cosh t) x + (r \sinh t) y,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\langle (M_1)_{\bar{s}}, (M_1)_{\bar{s}} \rangle &= \frac{(\varepsilon_1 r \sinh t + \kappa (1 + r \cosh t))^2}{\kappa^2}, \\
\langle (M_1)_{\bar{s}}, (M_1)_t \rangle &= 0, \quad \langle (M_1)_t, (M_1)_t \rangle = -r^2.
\end{aligned}
$$

Then

$$\langle (M_1)_{\bar{s}}, (M_1)_{\bar{s}} \rangle \langle (M_1)_t, (M_1)_t \rangle - \langle (M_1)_{\bar{s}}, (M_1)_t \rangle^2 = -r^2 \frac{(\varepsilon_1 r \sinh t + \kappa (1 + r \cosh t))^2}{\kappa^2}.$$

Theorem 4.10. *Let $M_1(\bar{s}, t)$ be the first associated tubular surfaces of $M(s, t)$, which is obtained by $S_1^2(r)$. Then $M_1(\bar{s}, t)$ has a singular point at $M(s_0, t_0)$ if and only if*

$$\varepsilon_1 r \sinh t_0 + \kappa(s_0)(1 + r \cosh t_0) = 0.$$

Now we assume that $\varepsilon_1 r \sinh t_0 + \kappa(s_0)(1 + r \cosh t_0) \neq 0$ for all (t, s). Then we will give an orthonormal basis on $M_1(\bar{s}, t)$.

$$
\begin{aligned}
\bar{e}_1 &= \frac{1}{\|(M_1)_{\bar{s}}\|} (M_1)_{\bar{s}} = -\varepsilon_2 \alpha, \\
\bar{e}_2 &= \frac{1}{\|(M_1)_t\|} (M_1)_t = -(\varepsilon_1 \cosh t) x + (\sinh t) y,
\end{aligned}
$$

where $\varepsilon_2 = sgn(\varepsilon_1 r \sinh t + \kappa (1 + r \cosh t))$ and $\{\bar{e}_1, \bar{e}_2\}$ is an orthonormal frame field on $M_1(\bar{s}, t)$. Set

$$\bar{e}_3 = -(\sinh t) x - (\varepsilon_1 \cosh t) y,$$

where $\bar{e}_3$ is a normal vector field to $M_1(\bar{s}, t)$. $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ is an orthonormal basis on $M_1(\bar{s}, t)$. Then we obtain

$$
\begin{aligned}
D_{\bar{e}_1} \bar{e}_1 &= \frac{-1}{\varepsilon_1 r \sinh t + \kappa (1 + r \cosh t)} (x + \kappa y), \\
D_{\bar{e}_1} \bar{e}_2 &= \frac{-\varepsilon_2 (\varepsilon_1 \cosh t + \kappa \sinh t)}{\varepsilon_1 r \sinh t + \kappa (1 + r \cosh t)} \alpha, \\
D_{\bar{e}_2} \bar{e}_2 &= \frac{1}{r} (-(\varepsilon_1 \sinh t) x + (\cosh t) y).
\end{aligned}
$$

The components of the second fundamental form $\overline{h}$ are calculated as follows

$$\overline{h}_{11} = \langle D_{\overline{e}_1}\overline{e}_1, \overline{e}_3 \rangle = \frac{\varepsilon_1 \kappa \cosh t - \sinh t}{\varepsilon_1 r \sinh t + \kappa (1 + r \cosh t)},$$

$$\overline{h}_{12} = \langle D_{\overline{e}_1}\overline{e}_2, \overline{e}_3 \rangle = 0 \quad \text{and} \quad \overline{h}_{22} = \langle D_{\overline{e}_2}\overline{e}_2, \overline{e}_3 \rangle = \frac{\varepsilon_1}{r}.$$

Theorem 4.11. *The mean curvature H_1 of $M_1(\overline{s}, t)$ is obtained as*

$$H_1 = \frac{1}{2}\left(\overline{h}_{11} - \overline{h}_{22}\right) = \frac{-\varepsilon_1 \left(2\varepsilon_1 r \sinh t + \kappa (1 + 2r \cosh t)\right)}{2r \left(\varepsilon_1 r \sinh t + \kappa (1 + r \cosh t)\right)}. \tag{4.6}$$

Theorem 4.12. *The Gauss curvature K_1 of $M_1(\overline{s}, t)$ is obtained as*

$$K_1 = -\left(\overline{h}_{11}\overline{h}_{22} - \left(\overline{h}_{12}\right)^2\right) = \frac{\varepsilon_1 \left(\sinh t + \varepsilon_1 \kappa \cosh t\right)}{r \left(\varepsilon_1 r \sinh t + \kappa (1 + r \cosh t)\right)}. \tag{4.7}$$

Thus it can be easily seen that $\Psi(K_1, H_1) = (K_1)_{\overline{s}} (H_1)_t - (K_1)_t (H_1)_{\overline{s}} = 0$. So we can give the following theorems.

Theorem 4.13. *The surface $M_1(\overline{s}, t)$ is a Weingarten surface.*

Theorem 4.14. *Let K_1 and H_1 be the Gauss curvature and mean curvature of the surface $M_1(\overline{s}, t)$. Then there exists the following relation between K_1 and H_1:*

$$-r^2 K_1 - 2\varepsilon_1 r H_1 = 1$$

where r is a positive real number.

From above theorem, we get the following corollary.

Corollary 4.4. *The surface $M_1(\overline{s}, t)$ is a linear Weingarten surface.*

Now we consider the tubular surface $M_2(\overline{s}, t)$ for $m_1 = m_2 = 1$ as follows

$$\begin{aligned} M_2(\overline{s}, t) &= (1 + r \sinh t) x_1(\overline{s}) + (r \cosh t) y_1(\overline{s}) \\ &= (-\varepsilon_1 r \cosh t) x(s) + (1 + r \sinh t) y(s). \end{aligned}$$

Similarly we can find the followings.

Theorem 4.15. *Let $M_2(\overline{s}, t)$ be the first associated tubular surfaces of $M(s, t)$, which is obtained by $H^2(r)$. Then $M_2(\overline{s}, t)$ has a singular point at $M(s_0, t_0)$ if and only if*

$$\varepsilon_1 r \cosh t_0 + \kappa(s_0)(1 + r \sinh t_0) = 0.$$

Theorem 4.16. *The mean curvature H_2 of $M_2(\overline{s}, t)$ is obtained as*

$$H_2 = \frac{\varepsilon_1 \left(2\varepsilon_1 r \cosh t + \kappa (1 + 2r \sinh t)\right)}{2r \left(\varepsilon_1 r \cosh t + \kappa (1 + r \sinh t)\right)}.$$

Theorem 4.17. *The Gauss curvature K_2 of $M_2(\overline{s}, t)$ is obtained as*

$$K_2 = \frac{-\varepsilon_1 \left(\cosh t + \varepsilon_1 \kappa \sinh t\right)}{r \left(\varepsilon_1 r \cosh t + \kappa (1 + r \sinh t)\right)}.$$

Thus it can be easily seen that $\Psi(K_2, H_2) = (K_2)_{\overline{s}} (H_2)_t - (K_2)_t (H_2)_{\overline{s}} = 0$. So we can give the following theorems.

Theorem 4.18. *The surface $M_2(\overline{s}, t)$ is a Weingarten surface.*

Theorem 4.19. *Let K_2 and H_2 be the Gauss curvature and mean curvature of the surface $M_2(\overline{s}, t)$. Then there exists the following relation between K_1 and H_1:*

$$r^2 K_2 + 2\varepsilon_1 r H_2 = 1$$

where r is a positive real number.

From above theorem, we get the following corollary.

Corollary 4.5. *The surface $M_2(\overline{s}, t)$ is a linear Weingarten surface.*

Remark 4.1. Let $M_3(\overline{s}, t)$ be the first associated tubular surfaces of $M(s, t)$, which is obtained by Q^2. Then $M_3(\overline{s}, t)$ is a degenerate surface.

*4.1.2. **The second associated curve and the second associated tubular surface*** Now, we consider the second associated curve and the second associated tubular surface of the hyperbolic curve $x(s)$. So we can give the following corollaries.

Corollary 4.6. *Let $x_2(s^*)$ be the second associated curve of the hyperbolic curve $x(s)$ with $\kappa^2 > 1$ in H^2 with the hyperbolic frame $\{x(s), \alpha(s), y(s)\}$ such that $x_2(s^*) = \alpha(s)$ where there exists a diffeomorfism $s^* = f_2(s)$. Then we have*

$$x_2 = \alpha, \quad \alpha_2 = \frac{1}{\sqrt{\kappa^2-1}}(x+\kappa y), \quad y_2 = \frac{1}{\sqrt{\kappa^2-1}}(-\kappa x - y),$$

$$\kappa_2 = \frac{\kappa'}{(\kappa^2-1)^{3/2}}, \quad \frac{df_2}{ds} = \sqrt{\kappa^2-1},$$

and

$$\frac{dx_2}{ds^*} = \alpha_2, \quad \frac{d\alpha_2}{ds^*} = -x_2 + \kappa_2 y_2, \quad \frac{dy_2}{ds^*} = \kappa_2 \alpha_2$$

where $\{x_2(s^*), \alpha_2(s^*), y_2(s^*)\}$ is the spherical frame, $\kappa_2(s^*)$ is the spherical curvature of $x_2(s^*)$ and

$$\langle x_2(s^*), x_2(s^*) \rangle = \langle \alpha_2(s^*), \alpha_2(s^*) \rangle = 1, \quad \langle y_2(s^*), y_2(s^*) \rangle = -1.$$

Theorem 4.20. *Let $x_2(s^*)$ be the second associated curve of the hyperbolic curve $x(s)$ with $\kappa^2 > 1$ in H^2 with the hyperbolic frame $\{x(s), \alpha(s), y(s)\}$ such that $x_2(s^*) = \alpha(s)$. Also let $x_2(s^*)$ be the center curve of a canal surface (i) obtained from the sphere $S^2(r)$. Then the parametrization of the canal surface can be as following*

$$M_4(s^*, t) = \left(1 + m_1 r(s^*) \sqrt{1-r_s^2(s^*)} \cosh t\right) x_2(s^*) - r(s^*) r_s(s^*) \alpha_2(s^*)$$
$$+ \left(m_2 r(s^*) \sqrt{1-r_s^2(s^*)} \sinh t\right) y_2(s^*)$$

and the parametrization of the tubular surface can be as following

$$M_4(s^*, t) = (1 + m_1 r \cosh t) x_2(s^*) + (m_2 r \sinh t) y_2(s^*)$$

(ii) obtained from the hyperbolic space $H^2(r)$. Then the parametrization of the canal surface can be as following

$$M_5(s^*, t) = \left(1 + m_1 r(s^*) \sqrt{1+r_s^2(s^*)} \sinh t\right) x_2(s^*) + r(s^*) r_s(s^*) \alpha_2(s^*)$$
$$+ \left(m_2 r(s^*) \sqrt{1+r_s^2(s^*)} \cosh t\right) y_2(s^*)$$

and the parametrization of the tubular surface can be as following

$$M_5(s^*, t) = (1 + m_1 r \sinh t) x_2(s^*) + (m_2 r \cosh t) y_2(s^*)$$

(iii) obtained from the ligthlike cone Q^2. Then the parametrization of the canal surface(tubular surface) can be as following

$$M_6(s^*, t) = (1 + a(s^*, t)) x_2(s^*) + m_1 a(s^*, t) y_2(s^*)$$

where $m_1, m_2 \in \{-1, 1\}$ and $a(s^*, t)$ is a function of s^* and t, but not only s^*.

Corollary 4.7. *Let $x_2(s^*)$ be the second associated curve of the hyperbolic curve $x(s)$ with $\kappa^2 < 1$ in H^2 with the hyperbolic frame $\{x(s), \alpha(s), y(s)\}$ such that $x_2(s^*) = \alpha(s)$ where there exists a diffeomorfism $s^* = f_2(s)$. Then we have*

$$x_2 = \alpha, \quad \alpha_2 = \frac{1}{\sqrt{1-\kappa^2}}(x+\kappa y), \quad y_2 = \frac{1}{\sqrt{1-\kappa^2}}(-\kappa x - y),$$

$$\kappa_2 = \frac{\kappa'}{(1-\kappa^2)^{3/2}}, \quad \frac{df_2}{ds} = \sqrt{1-\kappa^2},$$

and

$$\frac{dx_2}{ds^*} = \alpha_2, \quad \frac{d\alpha_2}{ds^*} = x_2 - \kappa_2 y_2, \quad \frac{dy_2}{ds^*} = -\kappa_2 \alpha_2$$

where $\{x_2(s^*), \alpha_2(s^*), y_2(s^*)\}$ is the spherical frame, $\kappa_2(s^*)$ is the spherical curvature of $x_2(s^*)$ and

$$\langle x_2(s^*), x_2(s^*) \rangle = \langle y_2(s^*), y_2(s^*) \rangle = 1, \quad \langle \alpha_2(s^*), \alpha_2(s^*) \rangle = -1.$$

Theorem 4.21. *Let $x_2(s^*)$ be the second associated curve of the hyperbolic curve $x(s)$ with $\kappa^2 < 1$ in H^2 with the hyperbolic frame $\{x(s), \alpha(s), y(s)\}$ such that $x_2(s^*) = \alpha(s)$. Also let $x_2(s^*)$ be the center curve of a canal surface (i) obtained from the sphere $S^2(r)$. Then the parametrization of the canal surface can be as following*

$$
\begin{aligned}
M_7(s^*, t) &= \left(1 + m_1 r(s^*) \sqrt{1 + r_s^2(s^*)} \cos t\right) x_2(s^*) + r(s^*) r_s(s^*) \alpha_2(s^*) \\
&\quad + \left(m_2 r(s^*) \sqrt{1 + r_s^2(s^*)} \sin t\right) y_2(s^*)
\end{aligned}
$$

and the parametrization of the tubular surface can be as following

$$
M_7(s^*, t) = (1 + m_1 r \cos t) x_2(s^*) + (m_2 r \sin t) y_2(s^*)
$$

(ii) obtained from the hyperbolic space $H^2(r)$. Then the parametrization of the canal surface can be as following

$$
\begin{aligned}
M_8(s^*, t) &= \left(1 + m_1 r(s^*) \sqrt{r_s^2(s^*) - 1} \cos t\right) x_2(s^*) - r(s^*) r_s(s^*) \alpha_2(s^*) \\
&\quad + \left(m_2 r(s^*) \sqrt{r_s^2(s^*) - 1} \sin t\right) y_2(s^*),
\end{aligned}
$$

where $m_1, m_2 \in \{-1, 1\}$.

Now, we consider the second associated tubular surface $M_4(s^*, t)$ of $M(s, t)$. Assume that $\kappa(s) = \kappa$ (constant). Let $M_4(s^*, t)$ be the second associated tubular surface of $M(s, t)$. Then we can write

$$
\begin{aligned}
M_4(s^*, t) &= (1 + r \cosh t) x_2(s^*) + (r \sinh t) y_2(s^*) \\
&= -\frac{\kappa r \sinh t}{\sqrt{\kappa^2 - 1}} x(s) + (1 + r \cosh t) \alpha(s) + \frac{r \sinh t}{\sqrt{\kappa^2 - 1}} y(s). \tag{4.8}
\end{aligned}
$$

From (4.8), we have

$$
\begin{aligned}
(M_4)_{s^*} &= \frac{1 + r \cosh t}{\sqrt{\kappa^2 - 1}} (-x + \kappa y), \\
(M_4)_t &= \frac{-\kappa r \cosh t}{\sqrt{\kappa^2 - 1}} x + (r \sinh t) \alpha - \frac{r \cosh t}{\sqrt{\kappa^2 - 1}} y,
\end{aligned}
$$

which implies that

$$
\langle (M_4)_{s^*}, (M_4)_{s^*} \rangle = (1 + r \cosh t)^2, \quad \langle (M_4)_{s^*}, (M_4)_t \rangle = 0, \quad \langle (M_4)_t, (M_4)_t \rangle = -r^2.
$$

Then

$$
\langle (M_4)_{s^*}, (M_4)_{s^*} \rangle \langle (M_4)_t, (M_4)_t \rangle - \langle (M_4)_{s^*}, (M_4)_t \rangle^2 = -r^2 (1 + r \cosh t)^2.
$$

Theorem 4.22. *Let $M_4(s^*, t)$ be the second associated tubular surfaces of $M(s, t)$. Then $M_4(s^*, t)$ has a singular point at $M(s, t_0)$ if and only if $1 + r \cosh t_0 = 0$.*

Now we assume that $1 + r \cosh t \neq 0$ for all (t, s). Then we will give an orthonormal basis on $M_4(s^*, t)$.

$$
\begin{aligned}
e_1^* &= \frac{1}{\|(M_4)_{s^*}\|} (M_4)_{s^*} = \frac{\varepsilon_2}{\sqrt{\kappa^2 - 1}} (x + \kappa y), \\
e_2^* &= \frac{1}{\|(M_4)_t\|} (M_4)_t = -\frac{\kappa \cosh t}{\sqrt{\kappa^2 - 1}} x + (\sinh t) \alpha - \frac{\cosh t}{\sqrt{\kappa^2 - 1}} y,
\end{aligned}
$$

where $\varepsilon_2 = sgn(1 + r \cosh t)$ and $\{e_1^*, e_2^*\}$ is an orthonormal frame field on $M_4(s^*, t)$. Set

$$
e_3^* = \frac{\varepsilon_2 \kappa \sinh t}{\sqrt{\kappa^2 - 1}} x - (\varepsilon_2 \cosh t) \alpha + \frac{\varepsilon_2 \sinh t}{\sqrt{\kappa^2 - 1}} y,
$$

where e_3^* is a normal vector field to $M_4(s^*, t)$. $\{e_1^*, e_2^*, e_3^*\}$ is an orthonormal basis on $M_4(s^*, t)$. Then we obtain

$$
\begin{aligned}
D_{e_1^*} e_1^* &= \frac{1}{1 + r \cosh t} \alpha \\
D_{e_1^*} e_2^* &= \frac{\varepsilon_2 \sinh t}{(1 + r \cosh t) \sqrt{\kappa^2 - 1}} (x + \kappa y), \\
D_{e_2^*} e_2^* &= -\frac{\kappa \sinh t}{r \sqrt{\kappa^2 - 1}} x + \frac{\cosh t}{r} \alpha - \frac{\sinh t}{r \sqrt{\kappa^2 - 1}} y.
\end{aligned}
$$

The components of the second fundamental form h^* are calculated as follows

$$
\begin{aligned}
h_{11}^* &= \left\langle D_{e_1^*} e_1^*, e_3^* \right\rangle = \frac{-\varepsilon_2 \cosh t}{1 + r \cosh t}, \\
h_{12}^* &= \left\langle D_{e_1^*} e_2^*, e_3^* \right\rangle = 0 \quad \text{and} \quad h_{22}^* = \left\langle D_{e_2^*} e_2^*, e_3^* \right\rangle = -\frac{\varepsilon_2}{r}.
\end{aligned}
$$

Theorem 4.23. *The mean curvature H_2 of $M_4(s^*, t)$ is obtained as*

$$
H_2 = \frac{\varepsilon_2}{2r(1 + r \cosh t)}. \tag{4.9}
$$

Theorem 4.24. *The Gauss curvature K_2 of $M_4(s^*, t)$ is obtained as*

$$
K_2 = \frac{-\cosh t}{r(1 + r \cosh t)}. \tag{4.10}
$$

From (4.9) and (4.10), we have

$$
(K_2)_{s^*} = 0, \quad (K_2)_t = \frac{-\sinh t}{r(1 + r \cosh t)^2}
$$

and

$$
(H_2)_{s^*} = 0, \quad (H_2)_t = \frac{-\sinh t}{2(1 + r \cosh t)^2}.
$$

Thus it can be easily seen that $\Psi(K_2, H_2) = 0$. So we can give the following theorem.

Theorem 4.25. *The surface $M_4(s^*, t)$ is a Weingarten surface.*

Also we can give the following theorem.

Theorem 4.26. *Let K_2 and H_2 be the Gauss curvature and mean curvature of the surface $M_4(s^*, t)$. Then there exists the following relation between K_2 and H_2:*

$$
-r^2 K_2 + 2\varepsilon_2 r H_2 = 1
$$

where r is a positive real number.

From above theorem, we get the following corollary.

Corollary 4.8. *The surface $M_4(s^*, t)$ is a linear Weingarten surface.*

The second Gaussian curvature K_{II} of the surface $M_4(s^*, t)$ is obtained that

$$
K_{II} = \frac{1 + 2r^2 + 6r \cosh t + 2r^2 \cosh 2t + \operatorname{sech}^2 t}{4\varepsilon_2 r(1 + r \cosh t)^2}.
$$

Then it can be easily seen that $\Psi(K_{II}, H_2) = 0$ and $\Psi(K_{II}, K_2) = 0$. So we can give the following theorem.

Theorem 4.27. *The surface $M_4(s^*, t)$ is a (X, Y)-Weingarten surface where $(X, Y) \in \{(K_2, K_{II}), (H_2, K_{II})\}$.*

Remark 4.2. Similar properties can be easily obtained for the tubular surfaces $M_5(s^*, t)$ and $M_7(s^*, t)$.

Remark 4.3. The tubular surface $M_6(s^*, t)$ is a degenerate surface.

4.2. Tubular surface obtained from the sphere $H^2(r)$

This subsection is omitted from the paper since the properties can be obtained similarly to previous subsection.

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