SIMILAR AND SELF-SIMILAR NULL CARTAN CURVES IN MINKOWSKI-LORENTZIAN SPACES

Hakan ŞİMŞEK

Department of Industrial Engineering, Antalya Bilim University, Antalya, TURKEY

ABSTRACT. In this paper, differential invariants of null Cartan curves are studied in (n+2) dimensional Lorentzian similarity geometry. The fundamental theorem for a null Cartan curve in similarity geometry is investigated and the characterization of all self-similar null Cartan curves parameterized by de Sitter parameter in Minkowski space-time is given.

1. INTRODUCTION

A similarity transformation of Euclidean space, which consists of a rotation, a translation and an isotropic scaling, is an automorphism preserving the angles and ratios between lengths. The structure consisting of unchanging geometric properties under the similarity transformation is called similarity geometry. The whole Euclidean geometry can be considered as a class of similarity geometry. The similarity transformations are studied in most areas of the pure and applied mathematics. For example, S. Li [23] presented a system for matching and pose estimation of 3D space curves under the similarity transformation. Brook et al. [5] discussed various problems of image processing and analysis by using the similarity transformation. Sahbi [26] investigated a method for shape description based on kernel principal component analysis (KPCA) in the similarity invariance of KPCA. On the other hand, the self-similar objects, whose images under the similarity map are themselves, have had a wide range of applications in areas such as fractal geometry, dynamical systems, computer networks, statistical physics and so on. The Cantor set, the von Koch snowflake curve and the Sierpinski gasket are some of most famous examples of such objects (see [14, 21]). Recently, the self-similarity started playing a role in algebra as well, first of all in group theory [17].

Bonnor [4] introduced the Cartan frame to study the behaviors of a null curve and proved the fundamental existence and congruence theorems in Minkowski space-time. Bejancu [2] represented a method for the general study of the geometry of null curves in Lorentz manifolds and, more generally, in semi-Riemannian manifolds (see also the book [11]). Ferrandez, Gimenez and Lucas [15] gave a reference along a null curve in an n-dimensional Lorentzian space. They showed the fundamental existence and uniqueness theorems and described the null helices in higher dimensions. Cöken and Ciftci [10] studied null curves in the Minkowski space-time and characterized pseudo-spherical null curves and Bertrand null curves.

The study of the geometry of null curves has a growing importance in the mathematical physics. The null curves are useful to find the solution of some equations in the classical relativistic string theory (see [6,19,20]) Moreover, there exists a geometric particle model associated with the geometry of null curves in the Minkowski space-time (see [16,24]).

Berger [3] represented the broad content of similarity transformations of Euclidean space. Encheva and Georgiev [12,13] studied the differential geometric invariants of curves according to a similarity in the Euclidean n-space. Chou and Qu [9] showed that the motions of curves in two, three and n-dimensional (n > 3) similarity geometries correspond to the Burgers hierarchy, Burgers-mKdV hierarchy and a multi-component generalization of these hierarchies. Şimşek and Özdemir [27] introduced the geometry of non-lightlike curves in the n-dimensional Lorentzian similarity geometry. Ateş et.al. [1] studied the similarity invariants of Frenet curves by considering the parametrization of any spherical indicatrix curve in Eucliedan space $\mathbb{E}^n$. Kamishima [22] examined the properties of compact Lorentzian similarity manifolds using developing maps and holonomy representations. The main idea of this paper is to study the differential geometry of a null curve under the similarity mapping.

The scope of paper is as follows. First, we give basic information about null Cartan curves. Then, we introduce a new parameter, which is called de Sitter parameter that is invariant under the similarity transformation. We represent the differential geometric invariants of a null Cartan curve, which are called shape Cartan curvatures, in (n+2)-dimensional Lorentzian similarity geometry. We prove the uniqueness theorem which states that two null Cartan curves are equivalent according to a similarity mapping. Furthermore, we show the existence theorem that is a process for constructing a null Cartan curve by the shape Cartan curvatures under some initial conditions. Lastly, we obtain the equations of all self-similar null Cartan curves parameterized by the de Sitter parameter in Minkowski spacetime.

2. Preliminaries

Let $\mathbf{u} = (u_1, u_2, \ldots, u_{n+2})$, $\mathbf{v} = (v_1, v_2, \ldots, v_{n+2})$ be two arbitrary vectors in Minkowski-Lorentzian space $\mathbb{M}^{n+2}$. The Lorentzian inner product of $\mathbf{u}$ and $\mathbf{v}$ can be stated as $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} I^* \mathbf{v}^T$ where $I^* = diag(-1, 1, \ldots, 1)$. We say that a vector

$\mathbf{u}$ in $\mathbb{M}^{n+2}$ is called spacelike, null (lightlike) or timelike if $\mathbf{u} \cdot \mathbf{u} > 0$, $\mathbf{u} \cdot \mathbf{u} = 0$ or $\mathbf{u} \cdot \mathbf{u} < 0$, respectively. The norm of the vector $\mathbf{u}$ is defined by $\|\mathbf{u}\| = \sqrt{|\mathbf{u} \cdot \mathbf{u}|}$. The *pseudohyperbolic space* (or anti-de Sitter space) is defined by

$$\mathbb{H}_0^{n+1}(r) = \left\{ \mathbf{u} \in \mathbb{M}^{n+2} : \mathbf{u} \cdot \mathbf{u} = -r^2 \right\}$$

and *pseudo-sphere* (or *de Sitter space)* is defined by

$$\mathbb{S}_1^{n+1}(r) = \left\{ \mathbf{u} \in \mathbb{M}^{n+2} : \mathbf{u} \cdot \mathbf{u} = r^2 \right\}, \quad ([25]).$$

A basis $\mathbf{B} = \{\mathbf{L}, \mathbf{N}, \mathbf{W}_1, \ldots, \mathbf{W}_n\}$ is said pseudo-orthonormal if it satisfies the following conditions:

$$\mathbf{L} \cdot \mathbf{L} = \mathbf{N} \cdot \mathbf{N} = 0, \quad \mathbf{L} \cdot \mathbf{N} = 1$$
$$\mathbf{L} \cdot \mathbf{W}_i = \mathbf{N} \cdot \mathbf{W}_i = \mathbf{W}_i \cdot \mathbf{W}_j = 0, \ i \neq j$$
$$\mathbf{W}_i \cdot \mathbf{W}_i = 1$$

where $i, j \in \{1, \ldots, n\}$ ([11]) .

Now, we consider the mapping $\mathcal{A} : \left(\bar{\mathbf{L}}, \bar{\mathbf{N}}, \bar{\mathbf{W}}_1, \cdots, \bar{\mathbf{W}}_n\right) \rightarrow (\mathbf{L}, \mathbf{N}, \mathbf{W}_1, \cdots, \mathbf{W}_n)$ of one pseudo-orthonormal basis onto another at any point P in $\mathbb{M}^{n+2}$, which is given as either

$$
\begin{bmatrix} \mathbf{L} \\ \mathbf{N} \\ \mathbf{W}_1 \\ \mathbf{W}_2 \\ \vdots \\ \mathbf{W}_{n-1} \\ \mathbf{W}_n \end{bmatrix} =
\begin{bmatrix}
\lambda & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-\frac{1}{2}\lambda\left(\varepsilon_1^2 + \varepsilon_2^2 + \ldots + \varepsilon_n^2\right) & \lambda^{-1} & -\epsilon_1 & \epsilon_2 & -\epsilon_3 & \epsilon_4 & \cdots & -\epsilon_{n-1} & \epsilon_n \\
\lambda\varepsilon_1\cos\theta + \lambda\varepsilon_2\sin\theta & 0 & \cos\theta & -\sin\theta & 0 & 0 & \cdots & 0 & 0 \\
\lambda\varepsilon_1\sin\theta - \lambda\varepsilon_2\cos\theta & 0 & \sin\theta & \cos\theta & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda\varepsilon_{n-1}\cos\theta + \lambda\varepsilon_n\sin\theta & 0 & 0 & 0 & 0 & \cdots & 0 & \cos\theta & -\sin\theta \\
\lambda\varepsilon_{n-1}\sin\theta - \lambda\varepsilon_n\cos\theta & 0 & 0 & 0 & 0 & \cdots & 0 & \sin\theta & \cos\theta
\end{bmatrix}
\begin{bmatrix} \bar{\mathbf{L}} \\ \bar{\mathbf{N}} \\ \bar{\mathbf{W}}_1 \\ \bar{\mathbf{W}}_2 \\ \vdots \\ \bar{\mathbf{W}}_{n-1} \\ \bar{\mathbf{W}}_n \end{bmatrix} \quad (1)
$$

when n is even, or the same matrix with the additional row (column) $\begin{matrix} 0 & 0 & \cdots & 1 \end{matrix}$ when n is odd, where λ, ε_i $(1 \leq i \leq n)$ and θ are real constants and $\lambda \neq 0$. The image of pseudo-orthonormal basis under the mapping $\mathcal{A}$ is a pseudo-orthonormal basis. Moreover, we have $\mathcal{A}^T \mathbf{J}^* \mathcal{A} = \mathbf{J}^*$, $\det \mathcal{A} = 1$ where

$$
\mathbf{J}^* = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1
\end{bmatrix}
$$

and the orientation is preserved by (1). Bonnor [4] defined the mapping $\mathcal{A}$ as a *null rotation*. A null rotation at P is equivalent to a Lorentzian transformation between two sets of natural coordinate functions whose values coincide at P.

A curve locally parameterized by $\gamma : J \subset \mathbb{R} \rightarrow \mathbb{M}^{n+2}$ is called a *null curve* if $\frac{d}{dt}\gamma(t) \neq 0$ is a null vector for all t. We know that a null curve $\gamma(t)$ satisfies $\frac{d^2}{dt^2}\gamma(t) \cdot \frac{d^2}{dt^2}\gamma(t) \geq 0$ (see [11]). If the acceleration vector of the null curve is a unit

vector, that is, $\frac{d^2}{dt^2}\gamma(t) \cdot \frac{d^2}{dt^2}\gamma(t) = 1$, then, null curve $\gamma(t)$ in $\mathbb{M}^{n+2}$ is said to be parameterized by *pseudo-arc*. If the acceleration vector of the null curve is not a unit vector, then the pseudo-arc parametrization becomes as the following

$$s = \int_{t_0}^{t} \left(\frac{d^2}{du^2}\gamma(u) \cdot \frac{d^2}{du^2}\gamma(u) \right)^{1/4} du \quad ([4, 10]). \tag{2}$$

A null curve $\gamma(t)$ in $\mathbb{M}^{n+2}$ with $\frac{d^2}{dt^2}\gamma(t) \cdot \frac{d^2}{dt^2}\gamma(t) \neq 0$ is a *Cartan curve* if

$$\left\{ \frac{d}{dt}\gamma(t), \frac{d^2}{dt^2}\gamma(t), \frac{d^3}{dt^3}\gamma(t), \cdots, \frac{d^{n+1}}{dt^{n+1}}\gamma(t) \right\}$$

is linearly independent for any t.

There exists a unique *Cartan frame* $C_\gamma := \{\mathbf{L}, \mathbf{N}, \mathbf{W}_1, \cdots, \mathbf{W}_n\}$ of the Cartan curve parameterized by a pseudo arc-parameter s such that the following equations are satisfied

$$\begin{aligned}
\gamma'(s) &= \mathbf{L}(s), \\
\mathbf{L}'(s) &= \mathbf{W}_1(s), \\
\mathbf{N}'(s) &= \kappa(s)\mathbf{W}_1(s) + \tau(s)\mathbf{W}_2(s), \\
\mathbf{W}_1'(s) &= -\kappa(s)\mathbf{L}(s) - \mathbf{N}(s), \\
\mathbf{W}_2'(s) &= -\tau(s)\mathbf{L}(s) + \kappa_3\mathbf{W}_3(s), \\
\mathbf{W}_i'(s) &= -\kappa_i(s)\mathbf{W}_{i-1}(s) + \kappa_{i+1}\mathbf{W}_{i+1}(s) \quad i \in \{3, \ldots, n-1\}, \\
\mathbf{W}_n'(s) &= -\kappa_n(s)\mathbf{W}_{n-1}(s)
\end{aligned} \tag{3}$$

where $\mathbf{N}$ is a null vector called *null transversal vector field*, and C_γ is pseudo-orthonormal, $\{\gamma', \gamma'', \ldots, \gamma^{(i+2)}\}$ and $\{\mathbf{L}, \mathbf{N}, \mathbf{W}_1, \ldots, \mathbf{W}_i\}$ have the same orientation for $2 \leq i \leq n-1$, C_γ is positively oriented and the differentiation with respect to s is denoted by prime "$'$". The functions κ, τ, κ_j $(3 \leq j \leq n)$ are called the *Cartan curvatures* of $\gamma(s)$ and are given as

$$\begin{aligned}
\kappa(s) &= \frac{1}{2}\left(\gamma^{(3)}(s) \cdot \gamma^{(3)}(s)\right) \\
\tau^2(s) &= \gamma^{(4)}(s) \cdot \gamma^{(4)}(s) - \left(\gamma^{(3)}(s) \cdot \gamma^{(3)}(s)\right)^2 \\
\kappa_j^2(s) &= \frac{D_j(s)\,D_{j+2}(s)}{D_{j+1}^2(s)}
\end{aligned} \tag{4}$$

for the pseudo-arc parameter s, where D_j denotes the j-th order main determinant of the matrix of the metric with respect to $\{\gamma', \gamma'', \ldots, \gamma^{(n+2)}\}$. We know that $\tau < 0$, $\kappa_i > 0$ $(3 \leq i \leq n-1)$, and $\kappa_n > 0$ or $\kappa_n < 0$ according to $\{\gamma', \gamma'', \ldots, \gamma^{(n+2)}\}$ is positively or negatively oriented, respectively. More information about the geometry of null curves can be found in the papers [2], [4], [11] and [15].

3. Geometric Invariants of Null Curves Under a Similarity Map

In this section, we introduce the similarity geometry of null curves in $\mathbb{M}^{n+2}$. A *null-similarity (n-similarity)* $f : \mathbb{M}^{n+2} \to \mathbb{M}^{n+2}$ is determined by

$$f(x) = \mu \mathcal{A}(x) + \mathbf{C}, \tag{5}$$

where $\mu > 0$ is a real constant, $\mathcal{A}$ is a null rotation and $\mathbf{C}$ is a translation vector. The n-similarity transformations form a group under the composition of maps and denoted by $\mathbf{Simn}(\mathbb{M}^{n+2})$. The n-similarity transformations in $\mathbb{M}^{n+2}$ preserve the orientation.

Let $\gamma(t) : J \subset \mathbb{R} \to \mathbb{M}^{n+2}$ be a null curve. The image of γ under $f \in \mathbf{Simn}(\mathbb{M}^{n+2})$ is denoted by β. Then, the null curve β can be stated as

$$\beta(t) = \mu \mathcal{A}(\gamma(t)) + \mathbf{b}, \qquad t \in J. \tag{6}$$

The pseudo-arc length function β starting at $t_0 \in J$ is

$$s^*(t) = \int_{t_0}^{t} \left(\frac{d^2}{du^2}\beta(u) \cdot \frac{d^2}{du^2}\beta(u) \right)^{1/4} du = \sqrt{\mu} s(t) \tag{7}$$

where $s \in I \subset \mathbb{R}$ is pseudo-arc parameter of $\gamma : I \to \mathbb{M}^{n+2}$. We can compute the Cartan curvatures $\kappa_\beta \left(\sqrt{\mu} s \right)$ and $\tau_\beta \left(\sqrt{\mu} s \right)$ of β by using (4) as

$$\kappa_\beta = \frac{1}{\mu}\kappa_\gamma, \quad \tau_\beta = \frac{1}{\mu}\tau_\gamma, \text{ and } \kappa_{i\beta} = \frac{1}{\sqrt{\mu}}\kappa_{i\alpha}, \quad 3 \leq i \leq n. \tag{8}$$

We define $\mathbf{W}_1$-indicatrix γ_{W_1} of the null curve γ parameterized by $\gamma_{W_1}(s) = \mathbf{W}_1(s)$. The $\mathbf{W}_1$-indicatrix is a pseudo-spherical non-null curve lies on the de Sitter (n+1)-space $S_1^{n+1}(1)$. If we state the arc-length parameter of γ_{W_1} as σ_γ, we can find $d\sigma_\gamma = \sqrt{2|\kappa_\gamma|}ds$. The arc-length element $d\sigma_\gamma$ is invariant under the n-similarity transformation since the equality $d\sigma_\beta = d\sigma_\gamma$ can be easily found, where σ_β is the de Sitter parameter of β. The parameter σ_γ is called *de Sitter parameter* of γ. Therefore, we reparametrize a null curve by the de Sitter parameter so that we can study the differential geometry of a null curve under the n-similarity transformation.

The derivative formulas of γ and C_γ with respect to σ_γ are given by

$$\frac{d\gamma}{d\sigma_\gamma} = \frac{1}{\sqrt{2|\kappa_\gamma|}}\mathbf{L}, \qquad \frac{d^2\gamma}{d\sigma_\gamma^2} = \frac{-d|\kappa_\gamma|}{2|\kappa_\gamma|\,d\sigma_\gamma}\frac{d\gamma}{d\sigma_\gamma} + \frac{1}{2|\kappa_\gamma|}\mathbf{W}_1 \tag{9}$$

and

$$\frac{d\mathbf{L}}{d\sigma_\gamma} = \frac{1}{\sqrt{2|\kappa_\gamma|}}\mathbf{W}_1$$

$$\frac{d\mathbf{N}}{d\sigma_\gamma} = \frac{\kappa_\gamma}{\sqrt{2|\kappa_\gamma|}}\mathbf{W}_1 + \frac{\tau_\gamma}{\sqrt{2|\kappa_\gamma|}}\mathbf{W}_2$$

$$\frac{d\mathbf{W}_1}{d\sigma_\gamma} = -\frac{\kappa_\gamma}{\sqrt{2\left|\kappa_\gamma\right|}}\mathbf{L} - \frac{1}{\sqrt{2\left|\kappa_\gamma\right|}}\mathbf{N}$$

$$\frac{d\mathbf{W}_2}{d\sigma_\gamma} = -\frac{\tau_\gamma}{\sqrt{2\left|\kappa_\gamma\right|}}\mathbf{L} + \frac{\kappa_{3\gamma}}{\sqrt{2\left|\kappa_\gamma\right|}}\mathbf{W}_3 \qquad (10)$$

$$\frac{d\mathbf{W}_3}{d\sigma_\gamma} = -\frac{\kappa_{3\gamma}}{\sqrt{2\left|\kappa_\gamma\right|}}\mathbf{W}_2 + \frac{\kappa_{4\gamma}}{\sqrt{2\left|\kappa_\gamma\right|}}\mathbf{W}_3$$

$$\vdots$$

$$\frac{d\mathbf{W}_n}{d\sigma_\gamma} = -\frac{\kappa_n}{\sqrt{2\left|\kappa_\gamma\right|}}\mathbf{W}_{n-1}$$

Similarly, we can find the same formulas (9) and (10) for the null curve β.

Now, we construct a new frame corresponding to n-similarity transformation for a null curve. Let's define the functions

$$\tilde{\kappa}_\gamma := \frac{-d\left|\kappa_\gamma\right|}{2\left|\kappa_\gamma\right|d\sigma_\gamma}, \quad \tilde{\tau}_\gamma := \frac{\tau_\gamma}{2\left|\kappa_\gamma\right|} \text{ and } \tilde{\kappa}_{i\gamma} = \frac{\kappa_{i\gamma}}{\sqrt{2\left|\kappa_\gamma\right|}}, \ 3 \le i \le n,$$

which are invariant under the n-similarity since we get the equalities

$$\tilde{\kappa}_\beta = \tilde{\kappa}_\gamma, \ \tilde{\tau}_\beta = \tilde{\tau}_\gamma \text{ and } \tilde{\kappa}_{i\beta} = \tilde{\kappa}_{i\gamma}.$$

If we set $\mathbf{L}^{sim} = \sqrt{2\left|\kappa_\gamma\right|}\mathbf{L}$, then we get a unit spacelike vector

$$\mathbf{W}_1^{sim} = \frac{d\mathbf{L}^{sim}}{d\sigma_\gamma} = \frac{d\left|\kappa_\gamma\right|}{\sqrt{2\left|\kappa_\gamma\right|}d\sigma_\gamma}\mathbf{L} + \mathbf{W}_1$$

such that $\mathbf{L}^{sim} \cdot \mathbf{W}_1^{sim} = 0$. From [2], we know that there exists a null vector $\mathbf{N}^{sim}$ satisfying

$$\mathbf{L}^{sim} \cdot \mathbf{N}^{sim} = 1, \qquad \mathbf{N}^{sim} \cdot \mathbf{W}_1^{sim} = 0,$$

in the space spanned by $\{\gamma', \ \gamma'', \ \gamma'''\}$ such that $\mathbf{N}^{sim}$ can be given in the form

$$\mathbf{N}^{sim} = \frac{1}{\mathbf{L}^{sim} \cdot \mathbf{V}}\left(\mathbf{V} - \frac{\mathbf{V} \cdot \mathbf{V}}{2\mathbf{L}^{sim} \cdot \mathbf{V}}\mathbf{L}^{sim}\right)$$

where $\mathbf{V} \in span\left\{\gamma', \ \gamma'', \ \gamma'''\right\}$. Choosing $\mathbf{V} = \frac{1}{\sqrt{2\left|\kappa_\gamma\right|}}\mathbf{N} + \tilde{\kappa}_\gamma\mathbf{W}_1$ bring about

$$\mathbf{N}^{sim} = \frac{1}{\sqrt{2\left|\kappa_\gamma\right|}}\mathbf{N} + \tilde{\kappa}_\gamma\mathbf{W}_1 - \frac{\tilde{\kappa}_\gamma^2\sqrt{2\left|\kappa_\gamma\right|}}{2}\mathbf{L},$$

which satisfies the relations $\mathbf{N}^{sim} \cdot \mathbf{N}^{sim} = 0$, $\mathbf{L}^{sim} \cdot \mathbf{N}^{sim} = 1$ and $\mathbf{N}^{sim} \cdot \mathbf{W}_1^{sim} = 0$. Moreover, If we choose $\mathbf{W}_i^{sim} = \mathbf{W}_i$, $2 \le i \le n$, then

$$C_\gamma^{sim} := \left\{\mathbf{L}^{sim}, \mathbf{N}^{sim}, \mathbf{W}_1^{sim}, \mathbf{W}_2^{sim}, \ldots, \mathbf{W}_n^{sim}\right\}$$

becomes a pseudo-orthonormal frame of γ under the n-similarity map. Then, the derivative formulas of C_γ^{sim} are computed as

$$\frac{d}{d\sigma_\gamma}\left(C_\gamma^{sim}\right)^T = P\left(C_\gamma^{sim}\right)^T \tag{11}$$

where

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \tilde{\xi}_\gamma & \tilde{\tau}_\gamma & 0 & 0 & \cdots & 0 \\ -\tilde{\xi}_\gamma & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -\tilde{\tau}_\gamma & 0 & 0 & 0 & \tilde{\kappa}_{3\gamma} & 0 & \cdots & 0 \\ 0 & 0 & 0 & -\tilde{\kappa}_{3\gamma} & 0 & \tilde{\kappa}_{4\gamma} & \cdots & 0 \\ 0 & 0 & 0 & 0 & -\tilde{\kappa}_{4\gamma} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -\tilde{\kappa}_{n\gamma} & 0 \end{bmatrix} \tag{12}$$

where $\tilde{\xi}_\gamma = \pm\dfrac{1}{2} - \dfrac{\tilde{\kappa}_\gamma^2}{2} + \dfrac{d\tilde{\kappa}_\gamma}{d\sigma_\gamma}$.

Let's consider the pseudo-orthogonal frame $C_\gamma^H := \left\{\mathbf{H}_1^\gamma, \mathbf{H}_2^\gamma, \mathbf{H}_3^\gamma, ..., \mathbf{H}_{n+2}^\gamma\right\}$ of γ where

$$\mathbf{H}_1^\gamma = \tfrac{1}{\sqrt{2|\kappa_\gamma|}}\mathbf{L}^{sim}, \ \mathbf{H}_2^\gamma = \tfrac{1}{\sqrt{2|\kappa_\gamma|}}\mathbf{N}^{sim}, \ \mathbf{H}_3^\gamma = \tfrac{1}{\sqrt{2|\kappa_\gamma|}}\mathbf{W}_1^{sim}, \ldots, \ \mathbf{H}_{n+2}^\gamma = \tfrac{1}{\sqrt{2|\kappa_\gamma|}}\mathbf{W}_n^{sim}.$$

Since, from (5), we can obtain $f(\mathbf{H}_i^\gamma) = \mathbf{H}_i^\beta$, $i = 1, \cdots n + 2$, the pseudo-orthogonal frame C_γ^H is invariant according to n-similarity map. Then, using (9) and (11), we get the derivative formulas of C_γ^H as the following

$$\frac{d}{d\sigma}\left(C_\gamma^H\right)^T = \tilde{P}\left(C_\gamma^H\right)^T \tag{13}$$

where

$$\tilde{P} = \begin{bmatrix} \tilde{\kappa}_\gamma & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \tilde{\kappa}_\gamma & \tilde{\xi}_\gamma & \tilde{\tau}_\gamma & 0 & 0 & \cdots & 0 \\ -\tilde{\xi}_\gamma & -1 & \tilde{\kappa}_\gamma & 0 & 0 & 0 & \cdots & 0 \\ -\tilde{\tau}_\gamma & 0 & 0 & \tilde{\kappa}_\gamma & \tilde{\kappa}_{3\gamma} & 0 & \cdots & 0 \\ 0 & 0 & 0 & -\tilde{\kappa}_{3\gamma} & \tilde{\kappa}_\gamma & \tilde{\kappa}_{4\gamma} & \cdots & 0 \\ 0 & 0 & 0 & 0 & -\tilde{\kappa}_{4\gamma} & \tilde{\kappa}_\gamma & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -\tilde{\kappa}_{n\gamma} & \tilde{\kappa}_\gamma \end{bmatrix}.$$

We can consider the equation (13) as the Frenet-Serret equation of a null Cartan curve γ according to the pseudo-orthogonal moving frame C_γ^H under the group $\mathbf{Simn}\left(\mathbb{M}^{n+2}\right)$. As a result, the following theorem is obtained.

Theorem 1. *Let $\gamma : I \to \mathbb{M}^{n+2}$ be a null Cartan curve with pseudo-de Sitter parameter σ_γ and $\{\kappa_\gamma, \tau_\gamma, \tilde{\kappa}_{i\gamma} \ (3 \leq i \leq n)\}$ be Cartan curvatures of γ with the Cartan frame C_γ. Then, the functions*

$$\tilde{\kappa}_\gamma := \frac{-d\,|\kappa_\gamma|}{2\,|\kappa_\gamma|\,d\sigma_\gamma}, \quad \tilde{\tau}_\gamma := \frac{\tau_\gamma}{2\,|\kappa_\gamma|} \ \ and \ \tilde{\kappa}_{i\gamma} = \frac{\kappa_{i\gamma}}{\sqrt{2\,|\kappa_\gamma|}}, \ 3 \leq i \leq n, \qquad (14)$$

and the pseudo-orthogonal frame C_γ^H are invariant under the n-similarity transformation in $\mathbb{M}^{n+2}$ and the derivative formulas of C_γ^H with respect to σ_γ are given by the equation (13).

Definition 1. *The functions $\tilde{\xi}_\gamma \ \tilde{\tau}_\gamma$ and $\tilde{\kappa}_{i\gamma} \ (3 \leq i \leq n)$ are called* shape Cartan curvatures *of a null Cartan curve γ and the pseudo-orthonormal frame C_γ^{sim} are called* shape Cartan frame *of γ.*

4. The Fundamental Theorem for a Null Curve

The existence and uniqueness theorems were shown by [2], [15] and [4] for a null Cartan curve under the Lorentz transformations. This notion can be extended with respect to $\mathbf{Simn}\big(\mathbb{M}^{n+2}\big)$ for the null Cartan curves parameterized by de Sitter parameter.

Theorem 2. *Let $\gamma, \beta : I \to \mathbb{M}^{n+2}$ be two null Cartan curves parameterized by the same de Sitter parameter σ, where $I \subset \mathbb{R}$ is an open interval. Suppose that γ and β have the same shape Cartan curvatures; namely, $\tilde{\kappa}_\gamma = \tilde{\kappa}_\beta$, $\tilde{\tau}_\gamma = \tilde{\tau}_\beta$ and $\tilde{\kappa}_{i\gamma} = \tilde{\kappa}_{i\beta} \ (3 \leq i \leq n)$ for all $\sigma \in I$. Then, there exists a $f \in \mathbf{Simn}\big(\mathbb{M}^{n+2}\big)$ such that $\beta = f \circ \gamma$.*

Proof. Let κ_γ, τ_γ, $\kappa_{i\gamma}$ and κ_β, τ_β, $\kappa_{i\beta}$ $(3 \leq i \leq n)$ be the Cartan curvatures and also s and s^* be the pseudo-arc length parameters of γ and β, respectively. Using the equality $\tilde{\kappa}_\gamma = \tilde{\kappa}_\beta$, we get $|\kappa_\gamma| = \mu\,|\kappa_\beta|$ for some real constant $\mu > 0$. Then, the equality $\tilde{\tau}_\gamma = \tilde{\tau}_\beta$ imply $\tau_\gamma = \mu\tau_\beta$. Therefore, we find $ds = \dfrac{1}{\sqrt{\mu}}ds^*$ from the definition of de Sitter parameter σ.

There exists a Lorentzian motion $\varphi = \mathcal{A} \circ T$ of $\mathbb{M}^{n+2}$ satisfying the equality $\varphi\,(\gamma\,(\sigma_0)) = \beta\,(\sigma_0)$ for any fixed $\sigma_0 \in I$, where $\mathcal{A}$ is a null rotation and T is a translation map, such that φ maps the pseudo-orthonormal frame C_γ^{sim} to pseudo-orthonormal frame C_β^{sim}. Therefore, the map $g = \mu\varphi : \mathbb{M}^{n+2} \to \mathbb{M}^{n+2}$ is a n-similarity transformation of $\mathbb{M}^{n+2}$. Let's define a function $\Phi : I \to \mathbb{R}$ as the following

$$\Phi\,(\sigma) = \left\|\frac{d}{d\sigma}g\,(\gamma\,(\sigma)) - \frac{d}{d\sigma}\beta\,(\sigma)\right\|^2 \quad \text{for } \forall \sigma \in I.$$

Taking derivative of this function with respect to σ, we conclude that

$$\frac{d\Phi}{d\sigma} = 2g\left(\frac{d^2\gamma}{d\sigma^2}\right) \cdot g\left(\frac{d\gamma}{d\sigma}\right) - 2\left[g\left(\frac{d^2\gamma}{d\sigma^2}\right) \cdot \frac{d\beta}{d\sigma}\right]$$

$$-2\frac{d^2\beta}{d\sigma^2}\cdot g\left(\frac{d\gamma}{d\sigma}\right)+2\left[\frac{d^2\beta}{d\sigma^2}\cdot\frac{d\beta}{d\sigma}\right].$$

Then, we obtain the following equation

$$\frac{d\Phi}{d\sigma}=0$$

since we have

$$\frac{d\gamma}{d\sigma}=\frac{1}{2\left|\kappa_\gamma\right|}\mathbf{L}_\gamma^{sim},\qquad\frac{d^2\gamma}{d\sigma^2}=\frac{\tilde{\kappa}_\gamma}{2\left|\kappa_\gamma\right|}\mathbf{L}_\gamma^{sim}+\frac{1}{2\left|\kappa_\gamma\right|}\mathbf{W}_{1\gamma}^{sim}$$

from (9). Also, we can find

$$\frac{d}{d\sigma}g\left(\gamma\left(\sigma_0\right)\right)=g\left(\frac{1}{2\left|\kappa_\gamma\right|}\mathbf{L}_\gamma^{sim}\left(\sigma_0\right)\right)=\frac{1}{2\left|\kappa_\beta\right|}\mathbf{L}_\beta^{sim}\left(\sigma_0\right)=\frac{d}{d\sigma}\beta\left(\sigma_0\right),$$

which implies $\Phi\left(\sigma_0\right)=0$. This means that

$$\frac{d}{d\sigma}g\left(\gamma\left(\sigma\right)\right)=\frac{d}{d\sigma}\beta\left(\sigma\right)$$

or equivalently $\beta\left(\sigma\right)=g\left(\gamma\left(\sigma\right)\right)+\mathbf{b}$ for all σ where $\mathbf{b}$ is a constant vector. Thus, the image of γ under the n-similarity $f=E\circ g$ is the null Cartan curve β, where $E:\mathbb{M}^{n+2}\to\mathbb{M}^{n+2}$ is a translation determined by $\mathbf{b}$. $\qquad\square$

The following theorem shows that every two functions determine a null Cartan curve according to a n-similarity under some initial conditions.

Theorem 3. *Let $z_i:I\to\mathbb{R}$, $i=1,2,\ldots n$ be smooth functions and $\mathbf{L}^{0sim}$, $\mathbf{N}^{0sim}$, $\mathbf{W}_1^{0sim}$, $\mathbf{W}_2^{0sim},\ldots,\mathbf{W}_n^{0sim}$ be a pseudo-orthonormal frame at a point x_0 in the Minkowski space $\mathbb{M}^{n+2}$. According to a n-similarity, there exists a unique null Cartan curve $\gamma:I\to\mathbb{M}^{n+2}$ parameterized by the de Sitter parameter σ such that γ satisfies the following conditions:*

(i) There exists $\sigma_0\in I$ such that $\gamma\left(\sigma_0\right)=x_0$ and the shape Cartan frame of γ at x_0 is $\mathbf{L}^{0sim}$, $\mathbf{N}^{0sim}$, $\mathbf{W}_1^{0sim}$, $\mathbf{W}_2^{0sim},\ldots,\mathbf{W}_n^{0sim}$.

(ii) $\tilde{\kappa}_\gamma\left(\sigma\right)=z_1\left(\sigma\right)$, $\tilde{\tau}_\gamma\left(\sigma\right)=z_2\left(\sigma\right)$ and $\tilde{\kappa}_{i\gamma}\left(\sigma\right)=z_i\left(\sigma\right)$ $(3\leq i\leq n)$ for all $\sigma\in I$.

Proof. Let us consider the following system of differential equations with respect to a matrix-valued function $\mathbf{K}\left(\sigma\right)=\left(\mathbf{L}^{sim},\mathbf{N}^{sim},\mathbf{W}_1^{sim},\mathbf{W}_2^{sim},\ldots,\mathbf{W}_n^{sim}\right)^T$

$$\frac{d\mathbf{K}}{d\sigma}\left(\sigma\right)=\mathbf{M}\left(\sigma\right)\mathbf{K}\left(\sigma\right)\qquad\qquad(15)$$

with a given matrix

$$\mathbf{M}\left(\sigma\right) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & z & z_2 & 0 & 0 & \ldots & 0 \\ -z & -1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ -z_2 & 0 & 0 & 0 & z_3 & 0 & \ldots & 0 \\ 0 & 0 & 0 & -z_3 & 0 & z_4 & \ldots & 0 \\ 0 & 0 & 0 & 0 & -z_4 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -z_n & 0 \end{bmatrix},$$

where $z\left(\sigma\right) = \pm\dfrac{1}{2} - \dfrac{z_1^2}{2} + \dfrac{dz_1}{d\sigma}$. The system (15) has a unique solution $\mathbf{K}\left(\sigma\right)$ which satisfies the initial conditions

$$\mathbf{K}\left(\sigma_0\right) = \left(\mathbf{L}^{0sim}, \mathbf{N}^{0sim}, \mathbf{W}_1^{0sim}, \mathbf{W}_2^{0sim}, \ldots, \mathbf{W}_n^{0sim}\right)^T$$

Then, we can write

$$\frac{d}{d\sigma}\left(\mathbf{J}^*\mathbf{K}^T\mathbf{J}^*\mathbf{K}\right) = \mathbf{J}^*\frac{d}{d\sigma}\mathbf{K}^T\mathbf{J}^*\mathbf{K} + \mathbf{J}^*\mathbf{K}^T\mathbf{J}^*\frac{d}{d\sigma}\mathbf{K}$$
$$= \mathbf{J}^*\mathbf{K}^T\left(\mathbf{M}^T\mathbf{J}^* + \mathbf{J}^*\mathbf{M}\right)\mathbf{K} = 0$$

since we have the equation $\mathbf{M}^T\mathbf{J}^* + \mathbf{J}^*\mathbf{M} = \left[0\right]_{(n+2)\times(n+2)}$. Also, we have

$$\mathbf{J}^*\mathbf{K}^T\left(\sigma_0\right)\mathbf{J}^*\mathbf{K}\left(\sigma_0\right) = \mathbf{I}$$

where $\mathbf{I}$ is the unit matrix since $\mathbf{L}^{0sim}$, $\mathbf{N}^{0sim}$, $\mathbf{W}_1^{0sim}$, $\mathbf{W}_2^{0sim}, \ldots, \mathbf{W}_n^{0sim}$ is the pseudo-orthonormal (n+2)-frame. As a result, we find $\mathbf{J}^*\mathbf{X}^T\left(\sigma\right)\mathbf{J}^*\mathbf{X}\left(\sigma\right) = \mathbf{I}$ for all $\sigma \in I$. This means that the vector fields $\mathbf{L}^{sim}$, $\mathbf{N}^{sim}$, $\mathbf{W}_1^{sim}$, $\mathbf{W}_2^{sim}, \ldots, \mathbf{W}_n^{sim}$ form a pseudo-orthonormal frame field in $\mathbb{M}^{n+2}$.

Let $\gamma : I \to \mathbb{M}^{n+2}$ be a null curve given by

$$\gamma\left(\sigma\right) = x_0 + \frac{1}{2}\int_{\sigma_0}^{\sigma} e^{2\int z_1(\sigma)d\sigma}\mathbf{L}^{sim}\left(\sigma\right)d\sigma, \qquad \sigma \in I. \tag{16}$$

By the equality (15), we get that $\gamma\left(\sigma\right)$ is a similar null Cartan curve with the curvatures $\tilde{\kappa}_\gamma\left(\sigma\right) = z_1\left(\sigma\right)$, $\tilde{\tau}_\gamma\left(\sigma\right) = z_2\left(\sigma\right)$ and $\tilde{\kappa}_{i\gamma}\left(\sigma\right) = z_i\left(\sigma\right)$ $(3 \leq i \leq n)$ in $\mathbb{M}^{n+2}$. Also, we find $\sqrt{\frac{1}{2}e^{2\int z_1(\sigma)d\sigma}}d\sigma = ds$ by using (2) and (15), where s is a pseudo-arc parameter; thus, σ is the de Sitter parameter of the null Cartan curve γ. Besides, the pseudo-orthonormal (n+2)-frame $\mathbf{L}^{sim}$, $\mathbf{N}^{sim}$, $\mathbf{W}_1^{sim}$, $\mathbf{W}_2^{sim}, \ldots, \mathbf{W}_n^{sim}$ is a shape Cartan frame of the null Cartan curve γ under the n-similarity transformation. $\qquad\square$

Remark 1. *In case of $\tilde{\kappa}_\gamma\left(\sigma\right) = 0$, the Cartan curvature $|\kappa_\gamma| = d$ is a positive real constant. Then, the parametrization of a null curve $\gamma : I \to \mathbb{M}^{n+2}$ with $\tilde{\kappa}_\gamma\left(\sigma\right) = 0$*

with respect to de Sitter parameter σ is given by

$$\gamma(\sigma) = x_0 + \frac{1}{2d} \int_{\sigma_0}^{\sigma} \mathbf{L}^{sim}(\sigma) \, d\sigma, \qquad \sigma \in I \tag{17}$$

from the equation (9) and (16).

Example 1. *Let's choose $\tilde{\kappa}_\gamma = -\tanh\left(\frac{\sigma}{2}\right)$ and $\tilde{\tau}_\gamma = 4$ for the shape Cartan curvatures of a null curve $\gamma : I \to \mathbb{M}^4$ with the initial conditions*

$$\mathbf{L}^{0sim} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right), \, \mathbf{N}^{0sim} = \left(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right), \tag{18}$$

$$\mathbf{W}_1^{0sim} = \left(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \, \mathbf{W}_2^{0sim} = \left(0, \frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right).$$

Then, we get $\tilde{\xi}_\gamma = 0$ and the system (15) determines a null vector $\mathbf{L}^{sim}$ given by

$$\mathbf{L}^{sim}(\sigma) = \frac{1}{\sqrt{2}}\left(\cosh(2\sigma), 0, \cosh(2\sigma), 0\right) \tag{19}$$

with $\mathbf{L}^{sim}(0) = \mathbf{L}^{0sim}$, in $\mathbb{M}^4$. The parametrization of null Cartan curve γ is found as

$$\gamma(\sigma) = \frac{2\sqrt{2}}{3}\left(3\sigma + \frac{12e^{2\sigma} + 21e^\sigma + 11}{(1+e^\sigma)^3}, 0, 3\sigma + \frac{12e^{2\sigma} + 21e^\sigma + 11}{(1+e^\sigma)^3}, 0\right)$$

for any $\sigma \in I$ by solving the equation (16).

5. Self-similar Null Cartan Curves

In this section, the concept of self-similarity is applied to null Cartan curves. Self-similar spacelike and timelike curves were studied by [27] and were defined as curves with constant p-shape curvatures. This idea can be extended to a null Cartan curve $\gamma : I \to \mathbb{M}^{n+2}$; that is, γ is called *self-similar* if shape Cartan curvatures of γ are constant.

A null curve is called a *null helix* if it has the constant Cartan curvatures which are not all zero in $\mathbb{M}^{n+2}$. A null helix is automotically a self-similar null Cartan curve in $\mathbb{M}^{n+2}$. Thus, null helices can be considered as a subclass of self-similar null Cartan curves.

5.1. **Self-similar null Cartan curves in $\mathbb{M}^4$.** Now, we determine the parametrizations of all self-similar null Cartan curves by means of the constant shape Cartan curvatures in the Minkowski space-time. They can be examined by separating into four different cases as follows. For each case, we choose the initial conditions (18) in the example 1.

- **Case 1:** Let's take $\tilde{\kappa}_{\gamma_1} = 0$ and $\tilde{\tau}_{\gamma_1} = 0$, which means $\tilde{\xi}_{\gamma_1} = \pm\frac{1}{2}$.
- **Case 2:** Let's take $\tilde{\kappa}_{\gamma_1} = 0$ and $\tilde{\tau}_{\gamma_1} = a \neq 0$, which means $\tilde{\xi}_{\gamma_1} = \pm\frac{1}{2}$.

- **Case 3:** Let's take $\tilde{\kappa}_{\gamma_1} = b \neq 0$ and $\tilde{\tau}_{\gamma_1} = 0$, which means $\tilde{\xi}_{\gamma_1}$ is also a constant different from 0 and $\pm\frac{1}{2}$.
- **Case 4:** Let's take $\tilde{\kappa}_{\gamma_1} = b \neq 0$ and $\tilde{\tau}_{\gamma_1} = a \neq 0$, which means $\tilde{\xi}_{\gamma_1}$ is also a constant different from 0 and $\pm\frac{1}{2}$.

All the cases above correspond to the following two general cases:

- **GCase 1:** Let's take $\tilde{\xi}_{\gamma_1} = c \neq 0$ and $\tilde{\tau}_{\gamma_1} = 0$, such that it corresponds to the Case 1 and Case 3.
- **GCase 2:** Let's take $\tilde{\xi}_{\gamma_1} = c \neq 0$ and $\tilde{\tau}_{\gamma_1} = a \neq 0$, such that it corresponds to the Case 2 and Case 4.

GCase 1: Using the equation (15), we obtain the following differential equation

$$\left(\mathbf{L}^{sim}\right)''' + 2c\left(\mathbf{L}^{sim}\right)' = 0$$

and by solving this equation, we conclude that if $c > 0$

$$\mathbf{L}^{sim}\left(\sigma\right) = \frac{1}{\sqrt{2}}\left(1, 0, \cos\left(\sqrt{2c}\sigma\right), \sin\left(\sqrt{2c}\sigma\right)\right)$$

and if $c < 0$

$$\mathbf{L}^{sim}\left(\sigma\right) = \frac{1}{\sqrt{2}}\left(\cosh\left(\sqrt{-2c}\sigma\right), \sinh\left(\sqrt{-2c}\sigma\right), 1, 0\right).$$

If the **Case 1** is valid, we use the equation (17) so that we get the following parametrization of the self-similar null Cartan curve

$$\gamma_1\left(\sigma\right) = \frac{1}{2\sqrt{2}d}\left(\sigma, 0, \frac{\sin\left(\sqrt{2c}\sigma\right)}{\sqrt{2c}}, -\frac{\cos\left(\sqrt{2c}\sigma\right)}{\sqrt{2c}}\right)$$

when $c > 0$ and

$$\gamma_2\left(\sigma\right) = \frac{1}{2\sqrt{2}d}\left(\frac{\sinh\left(\sqrt{-2c}\sigma\right)}{\sqrt{-2c}}, \frac{\cosh\left(\sqrt{-2c}\sigma\right)}{\sqrt{-2c}}, \sigma, 0\right)$$

when $c < 0$.

Since $\tilde{\xi}_{\gamma_1} = c = \pm\frac{1}{2}$ for the Case 1, we obtain

$$\gamma_1\left(\sigma\right) = \frac{1}{2\sqrt{2}d}\left(\sigma, 0, \sin\left(\sigma\right), -\cos\left(\sigma\right)\right)$$

for $c = 1/2$ and

$$\gamma_2\left(\sigma\right) = \frac{1}{2\sqrt{2}d}\left(\sinh\left(\sigma\right), \cosh\left(\sigma\right), \sigma, 0\right)$$

for $c = -1/2$.

If the **Case 3** is valid, we use the equation (16) so that we get the following parametrization of the self-similar null Cartan curve

$$\gamma_3\left(\sigma\right) = \frac{1}{2\sqrt{2}}(\frac{e^{2b\sigma}}{2b}, 0, \frac{e^{2b\sigma}}{4b^2+2c}\left(2b\cos\left(\sqrt{2c}\sigma\right) + \sqrt{2c}\sin\left(\sqrt{2c}\sigma\right)\right),$$
$$\frac{e^{2b\sigma}}{4b^2+2c}\left(2b\sin\left(\sqrt{2c}\sigma\right) - \sqrt{2c}\cos\left(\sqrt{2c}\sigma\right)\right))$$

when $c > 0$ and

$$\gamma_4\left(\sigma\right) = \frac{1}{4\sqrt{2}}(\frac{\cosh\left(m_1\sigma\right) + \sinh\left(m_1\sigma\right)}{m_1} - \frac{\cosh\left(m_2\sigma\right) + \sinh\left(m_2\sigma\right)}{m_2},$$
$$\frac{\cosh\left(m_1\sigma\right) + \sinh\left(m_1\sigma\right)}{m_1} + \frac{\cosh\left(m_2\sigma\right) + \sinh\left(m_2\sigma\right)}{m_2}, \frac{e^{2b\sigma}}{b}, 0)$$

when $c < 0$, where $m_1 = 2b + \sqrt{-2c}$, $m_2 = 2b - \sqrt{-2c} \neq 0$ and $c = \pm\frac{1}{2} - \frac{b^2}{2}$.

GCase 2: Using the equation (15), we obtain the following differential equation

$$\left(\mathbf{L}^{sim}\right)^{iv} + 2c\left(\mathbf{L}^{sim}\right)'' - a^2\mathbf{L}^{sim} = 0$$

and by solving this equation, we conclude that

$$\mathbf{L}^{sim}\left(\sigma\right) = \frac{1}{\sqrt{2}}\left(\cosh\left(q_1\sigma\right), \sinh\left(q_1\sigma\right), \cos\left(q_2\sigma\right), \sin\left(q_2\sigma\right)\right)$$

where $q_1 = \sqrt{-c + \sqrt{c^2 + a^2}}$, $q_2 = \sqrt{c + \sqrt{c^2 + a^2}}$. and $c = \pm\frac{1}{2} - \frac{b^2}{2}$.

If the **Case 2** is valid, we use the equation (17) so that we get the following parametrization of the self-similar null Cartan curve

$$\gamma_5\left(\sigma\right) = \frac{1}{2d\sqrt{2}}\left(\frac{\sinh\left(q_1\sigma\right)}{q_1}, \frac{\cosh\left(q_1\sigma\right)}{q_1}, \frac{\sin\left(q_2\sigma\right)}{q_2}, -\frac{\cos\left(q_2\sigma\right)}{q_2}\right).$$

If the **Case 4** is valid, we use the equation (16) so that we get the following parametrization of the self-similar null Cartan curve

$$\gamma_6\left(\sigma\right) = \frac{1}{4\sqrt{2}}(\frac{\cosh\left(n_1\sigma\right) + \sinh\left(n_1\sigma\right)}{n_1} + \frac{\cosh\left(n_2\sigma\right) + \sinh\left(n_2\sigma\right)}{n_2},$$
$$\frac{\cosh\left(n_1\sigma\right) + \sinh\left(n_1\sigma\right)}{n_1} - \frac{\cosh\left(n_2\sigma\right) + \sinh\left(n_2\sigma\right)}{n_2},$$
$$\frac{4be^{2b\sigma}\cos\left(q_2\sigma\right) + 2q_2e^{2b\sigma}\sin\left(q_2\sigma\right)}{4b^2 + q_2^2}, \frac{-2q_2e^{2b\sigma}\cos\left(q_2\sigma\right) + 4be^{2b\sigma}\sin\left(q_2\sigma\right)}{4b^2 + q_2^2})$$

where $n_1 = 2b + q_1 \neq 0$, $n_2 = 2b - q_1 \neq 0$.

From the above calculations, we obtain the following result.

Theorem 4. *Let γ be a null Cartan curve in $\mathbb{M}^4$. Then γ is a self-similar null Cartan curve if and only if it is congruent to one of the curves γ_1, γ_2 γ_3, γ_4, γ_5 and γ_6.*

In [4], null helices were defined and found their parametrizations in $\mathbb{M}^4$. When we compare the parametrizations of null helices with self-similar null curves, we conclude that null helices are a special class of self-similar null Cartan curves in $\mathbb{M}^4$. For example, a null helix satisfying $\tau \neq 0$ are expressed by

$$\alpha\left(s\right) = \sqrt{\frac{1}{v^2 + r^2}}\left(\frac{1}{v}\sinh vs, \frac{1}{v}\cosh vs, \frac{1}{r}\sin rs, -\frac{1}{r}\cos rs\right) \qquad (20)$$

where $v = \sqrt{\sqrt{\kappa^2 + \tau^2} - \kappa}$ and $r = \sqrt{\sqrt{\kappa^2 + \tau^2} + \kappa}$ and this curve is a kind of self-similar null Cartan curve γ_5 (see also [10] for null helices).

In [10], Theorem 3.2 says that a null Cartan curve γ lies on $\mathbb{S}_1^3\left(r\right)$ iff $\tau_\gamma \neq 0$ is a constant in $\mathbb{M}^4$. Then, we conclude that the self-similar null Cartan curve lying on $\mathbb{S}_1^3\left(r\right)$ is similar to γ_5 because of the definitions $\tilde{\kappa}_\gamma$ and $\tilde{\tau}_\gamma$. On the other hand, in [7], Theorem 3.10 states that there are no null curves lying on $\mathbb{H}_0^3\left(r\right)$ in $\mathbb{M}^4$, which means that there is no a self-similar (similar) null Cartan curve lying on $\mathbb{H}_0^3\left(r\right)$.

6. Concluding Remarks

In the current paper, the similarity geometry of a null Cartan curve in Minkowski-Lorentzian spaces was investigated and self-similar null Cartan curves were studied in Minkowski space-time. Next study will be about self-similar null Cartan curves in Lorentzian space forms like null helices studied in [15].

The motions of curves in $\mathbb{E}^2$, $\mathbb{E}^3$ and $\mathbb{E}^n$ ($n > 3$) yield the mKdV hierarchy, Schrödinger hierarchy and a multi- component generalization of mKdV-Schrödinger hierarchies, respectively. KS. Chou and C. Qu [9] showed that the motions of curves in two-, three- and n-dimensional ($n > 3$) similarity geometries correspond to the Burgers hierarchy, Burgers-mKdV hierarchy and a multi-component generalization of these hierarchies by using the similarity invariants of curves in comparison with its invariants under the Euclidean motion. Also, they [8] found that many 1+1-dimensional integrable equations like KdV, Burgers, Sawada-Kotera, Harry-Dym hierarchies and Camassa-Holm equations arise from motions of plane curves in centro-affine, similarity, affine and fully affine geometries. The motion of curves on two-dimensional surfaces in $\mathbb{E}_1^3$ was considered by Gürses [18]. Therefore, the motion of Lorentzian similar (null and nonnull) curves in Lorentzian-Minkowski similarity geometries will be investigated as well.

Declaration of Competing Interests The author declares that he has no competing interest.

References

[1] Ateş, F., Kaya, S., Yaylı, Y., Ekmekci, F. N., Generalized similar Frenet curves, *Mathematical Sciences and Applications E-notes*, 5(2) (2017), 26-35. https://doi.org/10.36753/mathenot.421731

[2] Bejancu, A., Lightlike curves in Lorentz manifolds, *Publ. Math. Debrecen*, 44(1–2) (1994), 145–155.

[3] Berger, M., Geometry I, Springer, New York, 1998.

[4] Bonnor, W. B., Null curves in a Minkowski space-time, *Tensor N.S.,* 20 (1969), 229–242.

[5] Brook, A., Bruckstein, A. M., Kimmel, R., On Similarity-Invariant Fairness Measures. In: Kimmel, R., Sochen, N. A., Weickert, J., (eds) Scale Space and PDE Methods in Computer Vision. Scale-Space, Lecture Notes in Computer Science, 3459, Springer, Berlin, Heidelberg, 2005. https://doi.org/10.1007/11408031_39

[6] Budinich, P., Null vectors, spinors, and strings, *Comm. Math. Phys.*, 107(3) (1986), 455-465.

[7] Camci, Ç., İlarslan, K., Sucurovic, E., On pseudohyperbolical curves in Minkowski Space-time, *Turk. J. Math.*, 27(2) (2003), 315–328.

[8] Chou, K. S., Qu, C., Integrable equations arising from motions of plane curves, *Proceedings of Institute of Mathematics of NAS of Ukraine*, 43(1) (2002), 281–290.

[9] Chou, K. S., Qu, C., Motions of curves in similarity geometries and Burgers-mKdV hierarchies, *Chaos, Solitons & Fractals,* 19 (2004), 47-53. https://doi.org/10.1016/S0960-0779(03)00060-2

[10] Cöken, A. C., Ciftci, Ü., On the Cartan curvatures of a null curve in Minkowski space-time, *Geom. Dedicata*, 114 (2005), 71-78. https://doi.org/10.1007/s10711-005-4804-1

[11] Duggal, K.L., Bejancu, A., Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications, volume 364 of Mathematics and its Aplications. Kluwer Academic Publishers Group, Dordrecht, The Netherlands, 1996.

[12] Encheva, R., Georgiev, G., Curves on the shape sphere, *Results in Mathematics*, 44 (2003), 279-288.

[13] Encheva, R., Georgiev, G., Similar Frenet curves, *Results in Mathematics*, 55(3-4) (2009), 359–372. https://doi.org/10.1007/s00025-009-0407-8

[14] Falconer, K., Fractal Geometry: Mathematical Foundations and Applications, Second Edition, John Wiley & Sons, Ltd., 2003.

[15] Ferrandez, A., Gimenez, A., Lucas, P., Null helices in Lorentzian space forms, *Int. J. Mod. Phys. A,* 16 (2001), 4845-4863. Doi: 10.1142/S0217751X01005821

[16] Ferrandez, A., Gimenez, A., Lucas, P., Geometrical particle models on 3D null curves, *Phys. Lett. B,* 543(3-4) (2002), 311–317. Doi: 10.1016/S0370-2693(02)02450-4

[17] Grigorchuk, R., Šunic, Z., Self Similarity an Branching Group Theory, Groups St Andrews, London Mathematical Society Lecture Note Series: 339(1), 2005.

[18] Gürses, M., Motion of curves on two-dimensional surfaces and soliton equations, *Physics Letters A*, 241(6) (1998), 329-334. https://doi.org/10.1016/S0375-9601(98)00151-0

[19] Hughston, L.P., Shaw, W.T., Real classical strings, *Proc. Roy. Soc. London Ser. A*, 414 (1987), 415-422.

[20] Hughston, L.P., Shaw, W.T., Classical strings in ten dimensions, *Proc. Roy. Soc. London Ser. A*, 414 (1987), 423-431.

[21] Hutchinson, J.E., Fractals and self-similarity, *Indiana University Mathematics Journal*, 30(5) (1981).

[22] Kamishima, Y., Lorentzian similarity manifolds, *Cent. Eur. J. Math.*, 10(5) (2012), 1771-1788. Doi: 10.2478/s11533-012-0076-9

[23] Li, S.Z., Invariant Representation, Matching and Pose Estimation of 3D Space Curves Under Similarity Transformation, *Pattern Recognition*, 30(3) (1997), 447-458. https://doi.org/10.1016/S0031-3203(96)00089-1

[24] Nersessian, A., Ramos, E., Massive spinning particles and the geometry of null curves, *Phys. Lett. B*, 445 (1998), 123–128. https://doi.org/10.1016/S0370-2693(98)01408-7

[25] O'Neill, B., Semi-Riemannian Geometry with Applications to Relativity. Academic Press Inc., London, 1983.

[26] Sahbi, H., Kernel PCA for similarity invariant shape recognition, *Neurocomputing*, 70 (2007), 3034-3045. https://doi.org/10.1016/j.neucom.2006.06.007

[27] Şimşek, H., Özdemir, M., Similar and self-similar curves in Minkowski n-space, *Bull. Korean Math. Soc.* 52(6) (2015), 2071-2093. https://doi.org/10.4134/BKMS.2015.52.6.2071