



APPLICATION OF THE RATIONAL (G'/G)-EXPANSION METHOD FOR SOLVING SOME COUPLED AND COMBINED WAVE EQUATIONS

Mustafa EKİCİ¹ and Metin ÜNAL²

¹Computer Technology Department, Çanakkale Onsekiz Mart University, Çanakkale, TURKEY

²Department of Mathematics and Science Education, Uşak University, Uşak, TURKEY

ABSTRACT. In this paper, we explore the travelling wave solutions for some nonlinear partial differential equations by using the recently established rational (G'/G)-expansion method. We apply this method to the combined KdV-mKdV equation, the reaction-diffusion equation and the coupled Hirota-Satsuma KdV equations. The travelling wave solutions are expressed by hyperbolic functions, trigonometric functions and rational functions. When the parameters are taken as special values, the solitary waves are also derived from the travelling waves. We have also given some figures for the solutions.

1. INTRODUCTION

In the past decades, the travelling wave solutions of nonlinear partial differential equations (NLPDEs) play an effective role in physics, engineering and applied mathematics. The mathematical models of these subjects give important information about the behaviour of the physical event. Therefore, it is very important to obtain the traveling wave solutions of NLPDEs [32]. The NLPDEs have interesting structures that deals with many phenomena in physics, chemistry and engineering, for example; in fluid flow, plasma waves, mechanics, solid state physics, oceanic phenomena, atmospheric phenomena and so on. Many researchers have been proposed various different methods to find solutions for nonlinear partial differential equations and nonlinear fractional differential equations [36–40]. Such as the inverse scattering transform method [1], the Hirota's bilinear method [2], truncated

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✉ mustafa.ekici@comu.edu.tr-Corresponding author; metin.unal@usak.edu.tr

ORCID 0000-0003-2494-8229; 0000-0002-4462-0872.

Painlevé expansion method [3], the tanh-function expansion method [4], the Jacobi elliptic function expansion method [5], the homogeneous balance method [6–8], the trial function method [9], the exp-function method [10, 34], differential transform method [33], the Bäcklund transform method [11], the generalized Riccati equation method [12–15], the sub-ODE method [17–20], the original (G'/G) -expansion method [16, 29], the double $(G'/G, 1/G)$ -expansion method [35] etc.. Since there is not a common method that can be used to solve all types of nonlinear evolution equations.

Some researchers established several powerful and direct methods. Wang et al. [16] first introduced the (G'/G) -expansion method to find travelling wave solutions of nonlinear evolution equations. Later Islam et al. [21] proposed the rational (G'/G) -expansion method which aims to derive closed form travelling wave solutions. In this paper we use the rational (G'/G) -expansion method and apply for the combined KdV-mKdV equation, the reaction-diffusion equation, and the coupled Hirota-Satsuma KdV equations. We derived abundant solutions for each equation that is different from the solutions in the literature.

2. DESCRIPTION OF THE METHOD

Suppose that $u = u(x, t)$ is an unknown function depends on the x and t variables and we define the polynomial P in $u(x, t)$ and its various order partial derivatives and nonlinear terms as

$$P(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, \dots) = 0. \quad (1)$$

We use the following steps, to solve Eq.(1) by means of the rational (G'/G) -expansion method.

Step 1: We assign a new variable $U(\xi)$ in terms of x and t variables and a new transformation:

$$u(x, t) = U(\xi) \quad , \quad \xi = x - st + \xi_0 \quad (2)$$

where is ξ_0 a constant and s is the velocity of the wave. The transformation in Eq.(2) transforms Eq.(1) into an ordinary differential equation (ODE) for $u = U(\xi)$.

$$Q(U, U', -sU', U'', s^2U'', -sU'', \dots) = 0 \quad (3)$$

where U and its derivatives with respect to ξ are the elements of the Q polynomial of $U(\xi)$.

Step 2: Next we integrate Eq.(3) one or twice as possible. Suppose that the solution of Eq.(3) can be written in the following form

$$u(\xi) = \frac{\sum_{j=0}^n a_j (G'/G)^j}{\sum_{j=0}^n b_j (G'/G)^j} \quad (4)$$

where a_j and b_j ($j = 0, 1, 2, \dots, n$), ($a_n \neq 0$, $b_n \neq 0$) are arbitrary coefficient to be found later. Next we write, the $G = G(\xi)$ function, which satisfies the following second order ODE;

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0 \quad (5)$$

where λ and μ are real constants. We convert Eq.(5) into (G'/G) form,

$$\frac{d}{d\xi}(G'/G) = -(G'/G)^2 - \lambda(G'/G) - \mu. \quad (6)$$

From Eq.(5) or Eq.(6) the solution for (G'/G) as follows

$$(G'/G) = \begin{cases} -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left(\frac{c_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) + c_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right)}{c_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) + c_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right)} \right) & ; \lambda^2 - 4\mu > 0, \\ -\frac{\lambda}{2} + \frac{\sqrt{4\mu - \lambda^2}}{2} \left(\frac{-c_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right) + c_2 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right)}{c_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right) + c_2 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right)} \right) & ; \lambda^2 - 4\mu < 0, \\ -\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2\xi} & ; \lambda^2 - 4\mu = 0, \end{cases} \quad (7)$$

where c_1 and c_2 are constants.

Step 3: To determine the value of n , which is the degree of $U(\xi)$, in Step 2, we apply the homogeneous balance method, that is balancing between the highest order nonlinear terms and the highest order derivatives in Eq.(3). The degree of other terms in Eq.(3) can be written as in the following form [21]

$$\deg \left[\frac{d^m u(\xi)}{d\xi^m} \right] = n + m, \quad \deg \left[u^m \left(\frac{d^l u(\xi)}{d\xi^l} \right)^p \right] = mn + p(n + l)$$

where $\deg[U(\xi)]$ is the degree of $U(\xi)$.

Step 4: After determining the value of n , we substitute Eq.(4) along with Eq.(5) into Eq.(3). Equating the coefficients of (G'/G) to zero, gives a system of algebraic equations. In order to solve these equations we use the computer software programme such as Maple or Mathematica. If there is a possible solution, we obtain values for a_i , b_i , λ , μ and s ($i = 0, 1, 2, \dots, n$).

Step 5: Finally we substitute the values of a_i , b_i ($i = 0, 1, 2, \dots, n$), λ , μ , s and the solutions given in Eq.(7), into Eq.(4), hence the solutions of the nonlinear Eq.(1) are derived.

3. APPLICATION OF THE METHOD

Example 1. *The combined KdV-mKdV equation*

The KdV and mKdV equations are widely studied popular soliton equations. The nonlinear terms appearing in the KdV and mKdV equations often exist in applied science and engineering, such as in plasma physics, ocean dynamics and quantum field theory [22–24]. If we combine the quadratic nonlinear term of the KdV equation and the cubic nonlinear term of the mKdV equation, then we get the combined KdV-mKdV equation or the Gardner equation [25]

$$u_t + \alpha u u_x + \beta u^2 u_x + u_{xxx} = 0 \quad (8)$$

where α and β are nonzero parameters. This equation describes the wave propagation of bounded particle, sound wave and thermal pulse [26–28].

The travelling wave transformation $u(x, t) = U(\xi)$, $\xi = x - st + \xi_0$, transforms Eq.(8) into to the following ODE

$$-sU' + \alpha U U' + \beta U^2 U' + U''' = 0 \quad (9)$$

where s is the velocity of the wave and the superscript of U shows the derivative of U with respect to ξ . Next, we integrate Eq.(9) and deduce the following equation

$$C - sU + \frac{1}{2}\alpha U^2 + \frac{1}{3}\beta U^3 + U'' = 0 \quad (10)$$

where C is an integration constant to be found later. We use homogeneous balance method, such as balancing the terms U'' and U^3 in Eq.(10) we get $n = 1$, so we can write Eq.(4) as

$$U(\xi) = \frac{a_0 + a_1(G'/G)}{b_0 + b_1(G'/G)} \quad (11)$$

Next we substitute Eq.(11) into Eq.(10) and organize the equation in terms of the powers of (G'/G) . Hence equating the coefficients of (G'/G) and its powers to zero in the resulting equation, gives a system of algebraic equations for a_0, b_0, a_1, b_1, s and C . Solving the set of equations by using the computer programme Maple, we get the following set of solutions.

Set 1

$$\begin{aligned} a_0 &= \mp \frac{1}{2} \frac{b_0(\pm\sqrt{-\frac{6}{\beta}}\alpha + 6\lambda)}{\beta\sqrt{-\frac{6}{\beta}}}, \quad a_1 = \pm\sqrt{-\frac{6}{\beta}}b_0, \quad b_1 = 0 \\ s &= -\frac{2\beta\lambda^2 + \alpha^2 - 8\beta\mu}{4\beta}, \quad C = \frac{\alpha(6\beta\lambda^2 + \alpha^2 - 24\beta\mu)}{24\beta^2} \end{aligned} \quad (12)$$

where $b_0, \lambda, \alpha, \beta$ and μ are all arbitrary constants. Substituting Eq.(12) into Eq.(11) we get the following solution

$$U(\xi) = \pm\sqrt{-\frac{6}{\beta}}(G'/G) - \frac{\alpha}{2\beta} \mp \frac{3\lambda}{\beta}\sqrt{-\frac{\beta}{6}} \quad (13)$$

where

$$\xi = x + \left(\frac{\alpha^2}{4\beta} + \frac{\lambda^2 - 4\mu}{2} \right) t + \xi_0 \quad (14)$$

and (G'/G) is given in Eq.(7). Substituting Eq.(7) into Eq.(13), we deduce the following travelling wave solutions.

Case 1: If $\lambda^2 - 4\mu > 0$, then we have

$$\begin{aligned} U(\xi) = & \pm \frac{1}{2} \sqrt{-\frac{6(\lambda^2 - 4\mu)}{\beta}} \left(\frac{c_1 \cosh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi) + c_2 \sinh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi)}{c_1 \sinh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi) + c_2 \cosh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi)} \right) \\ & \mp \frac{\lambda}{2} \sqrt{-\frac{6}{\beta}} - \frac{\alpha}{2\beta} \mp \frac{3\lambda}{\beta} \sqrt{-\frac{\beta}{6}}. \end{aligned} \quad (15)$$

If we choose $c_1 = \sinh(\xi_0)$ and $c_2 = \cosh(\xi_0)$, we get the following hyperbolic solution for the Eq.(10)

$$U(\xi) = \pm \frac{1}{2} \sqrt{-\frac{6(\lambda^2 - 4\mu)}{\beta}} \tanh \left(\frac{\xi}{2} \sqrt{\lambda^2 - 4\mu} + \xi_0 \right) \mp \frac{\lambda}{2} \sqrt{-\frac{6}{\beta}} - \frac{\alpha}{2\beta} \mp \frac{3\lambda}{\beta} \sqrt{-\frac{\beta}{6}}$$

The plot of the solution for the values $(\lambda = 5, \mu = 4, \alpha = 3, \beta = -4, \xi_0 = 2)$ is given in Fig 1.

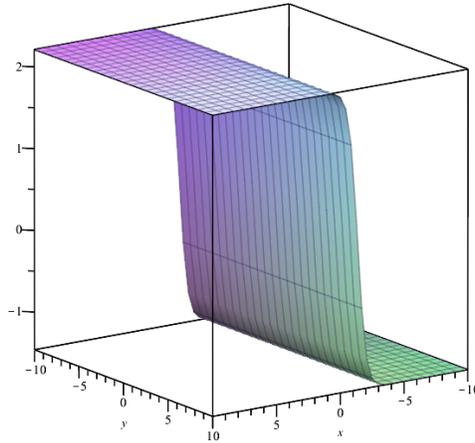


FIGURE 1. Hyperbolic solution for Eq.(8)

Case 2: If $\lambda^2 - 4\mu < 0$, then we have

$$U(\xi) = \pm \frac{1}{2} \sqrt{-\frac{6(4\mu - \lambda^2)}{\beta}} \left(\frac{-c_1 \cos(\frac{\xi}{2} \sqrt{4\mu - \lambda^2}) + c_2 \sin(\frac{\xi}{2} \sqrt{4\mu - \lambda^2})}{c_1 \sin(\frac{\xi}{2} \sqrt{4\mu - \lambda^2}) + c_2 \cos(\frac{\xi}{2} \sqrt{4\mu - \lambda^2})} \right) \\ \mp \frac{\lambda}{2} \sqrt{-\frac{6}{\beta}} - \frac{\alpha}{2\beta} \mp \frac{3\lambda}{\beta} \sqrt{-\frac{\beta}{6}}$$

If we choose $c_1 = \sin(\xi_0)$ and $c_2 = \cos(\xi_0)$, we get the following trigonometric solution for the Eq.(10)

$$U(\xi) = \pm \frac{1}{2} \sqrt{-\frac{6(4\mu - \lambda^2)}{\beta}} \tan \left(\frac{\xi}{2} \sqrt{4\mu - \lambda^2} + \xi_0 \right) \mp \frac{\lambda}{2} \sqrt{-\frac{6}{\beta}} - \frac{\alpha}{2\beta} \mp \frac{3\lambda}{\beta} \sqrt{-\frac{\beta}{6}}$$

The plot of the solution for the values ($\lambda = 4, \mu = 5, \alpha = 3, \beta = -6, \xi_0 = 2$) is given in Fig 2.

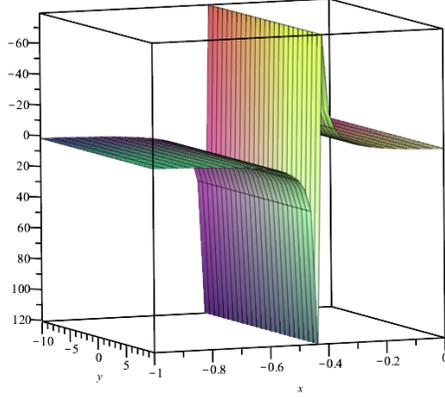


FIGURE 2. Trigonometric solution for Eq.(8)

Case 3: If $\lambda^2 - 4\mu = 0$, then we have

$$U(\xi) = \pm \sqrt{-\frac{6}{\beta}} \left(\frac{c_2}{c_1 + c_2 \xi} \right) \mp \frac{\lambda}{2} \sqrt{-\frac{6}{\beta}} - \frac{\alpha}{2\beta} \mp \frac{3\lambda}{\beta} \sqrt{-\frac{\beta}{6}}$$

The plot of the solution for the values ($\lambda = 4, \mu = 4, \alpha = 3, \beta = -6, \xi_0 = 2$) is given in Fig 3. In particular, if $c_1 = 0$ and $c_2 \neq 0$ and $\lambda > 0$ and $\mu = 0$, then

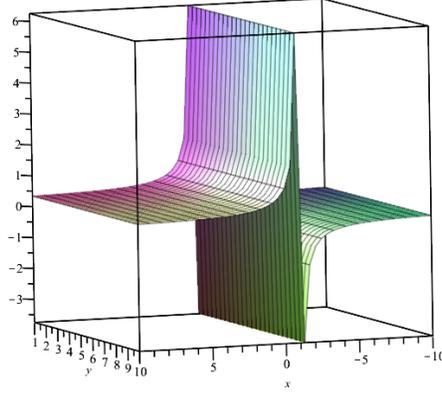


FIGURE 3. Rational solution for Eq.(8)

Eq.(15) becomes

$$U(\xi) = \pm \frac{\lambda}{2} \sqrt{-\frac{6}{\beta}} \tanh\left(\frac{\lambda}{2}\xi\right) \mp \frac{\lambda}{2} \sqrt{-\frac{6}{\beta}} - \frac{\alpha}{2\beta} \mp \frac{3\lambda}{\beta} \sqrt{-\frac{\beta}{6}} \quad (16)$$

or if $c_1 \neq 0$ and $c_2 = 0$ and $\lambda > 0$ and $\mu = 0$, then Eq.(15) becomes

$$U(\xi) = \pm \frac{\lambda}{2} \sqrt{-\frac{6}{\beta}} \coth\left(\frac{\lambda}{2}\xi\right) \mp \frac{\lambda}{2} \sqrt{-\frac{6}{\beta}} - \frac{\alpha}{2\beta} \mp \frac{3\lambda}{\beta} \sqrt{-\frac{\beta}{6}} \quad (17)$$

where

$$\xi = x + \left(\frac{\alpha^2}{4\beta} + \frac{\lambda^2}{2}\right)t + \xi_0.$$

Note that Eq.(16) and Eq.(17) represents the solitary wave solutions of the combined KdV–mKdV equation Eq.(8)

Set 2

$$\begin{aligned} a_0 &= \frac{(-\lambda\alpha \mp \sqrt{-96\beta\mu^2 - 6\lambda^4\beta + 48\mu\lambda^2\beta})b_1}{4\beta}, \quad a_1 = \frac{-\alpha b_1}{2\beta}, \quad b_0 = \frac{b_1\lambda}{2} \\ s &= \frac{-2\lambda^2\beta + 8\mu\beta - \alpha^2}{4\beta}, \quad C = \frac{\alpha(6\lambda^2\beta - 24\mu\beta + \alpha^2)}{24\beta^2} \end{aligned} \quad (18)$$

where b_1, λ and μ are arbitrary constants. Substituting Eq.(18) into Eq.(11) we get the following solution

$$U(\xi) = \frac{-2\alpha(G'/G) + (-\lambda\alpha \mp \sqrt{-96\beta\mu^2 - 6\lambda^4\beta + 48\mu\lambda^2\beta})}{4\beta(G'/G) + 2\lambda\beta} \quad (19)$$

where

$$\xi = x + \left(2\mu - \frac{2\lambda^2\beta + \alpha^2}{4\beta}\right)t + \xi_0 \quad (20)$$

and (G'/G) is given in Eq.(7).

Set 3

$$\begin{aligned} a_0 &= \frac{-6b_0^2\lambda + 12b_1\mu b_0 + 3b_0b_1\lambda^2 - 6\lambda\mu b_1^2}{\mp\sqrt{-6b_1^2\lambda^2\beta + 24b_1\beta b_0\lambda - 24b_0^2\beta}} + \frac{\alpha b_0}{2\beta} \\ a_1 &= \frac{-\alpha b_1 \pm \sqrt{-6b_1^2\lambda^2\beta + 24b_1\beta b_0\lambda - 24b_0^2\beta}}{2\beta} \\ s &= \frac{-2\lambda^2\beta + 8\mu\beta - \alpha^2}{4\beta}, \quad C = \frac{\alpha(6\lambda^2\beta - 24\mu\beta + \alpha^2)}{24\beta^2} \end{aligned} \quad (21)$$

where $b_0, b_1, \lambda, \alpha, \beta$ and μ are arbitrary constants. Substituting the values of constants from Eq.(21) into Eq.(11) gives

$$U(\xi) = \frac{\left(\frac{-\alpha b_1 \pm \sqrt{-6b_1^2\lambda^2\beta + 24b_1\beta b_0\lambda - 24b_0^2\beta}}{2\beta}\right)(G'/G) + \left(\frac{-6b_0^2\lambda + 12b_1\mu b_0 + 3b_0b_1\lambda^2 - 6\lambda\mu b_1^2}{\mp\sqrt{-6b_1^2\lambda^2\beta + 24b_1\beta b_0\lambda - 24b_0^2\beta}} + \frac{\alpha b_0}{2\beta}\right)}{b_1(G'/G) + b_0}$$

where $\xi = x + \left(\frac{2\beta\lambda^2 + \alpha^2 - 8\beta\mu}{4\beta}\right)t + \xi_0$.

Example 2. The reaction-diffusion equation

We have the reaction-diffusion equation [30]

$$u_{tt} + \alpha u_{xx} + \beta u + \gamma u^3 = 0 \quad (22)$$

where α, β and γ are nonzero constants. The traveling wave variable Eq.(2) reduces the Eq.(22) into an ODE

$$(\alpha + s^2)U'' + \beta U + U^3 = 0, \quad (23)$$

where s is the velocity of the wave. Next we express the solution of the Eq.(23) in terms of (G'/G) as it is written in Eq.(4), where $G = G(\xi)$ satisfies the second order linear ODE in Eq.(23). We use homogeneous balance method, such as balancing the terms U'' and U^3 in Eq.(23) we get $n = 1$, hence from Eq.(4), we have

$$U(\xi) = \frac{a_0 + a_1(G'/G)}{b_0 + b_1(G'/G)}. \quad (24)$$

Substituting Eq.(23) into Eq.(22) and write the left hand side in terms of (G'/G) . Hence equating the coefficients of the resulting equation to zero, gives a system of algebraic equations for a_0, b_0, a_1, b_1 and s . Solving the set of equations by using the computer programme, we get the following set of solutions:

Set 1

$$\begin{aligned} a_0 &= \pm \frac{1}{2} \sqrt{\frac{-\beta(\lambda^2-4\mu)}{\gamma}} b_1, \quad b_0 = \frac{1}{2} \lambda b_1, \\ a_1 &= 0, \quad s = \pm \sqrt{\frac{2\beta}{\lambda^2-4\mu} - \alpha} \end{aligned} \quad (25)$$

where b_1, λ and μ are all arbitrary constants. Substituting Eq.(25) into Eq.(24) we get the following solution

$$U(\xi) = \frac{\pm \frac{1}{2} \sqrt{\frac{-\beta(\lambda^2-4\mu)}{\gamma}}}{(G'/G) + \lambda/2} \quad (26)$$

where

$$\xi = x \pm \left(\sqrt{\frac{2\beta}{\lambda^2-4\mu} - \alpha} \right) t + \xi_0 \quad (27)$$

and (G'/G) is given in Eq. (7). Substituting Eq.(7) into Eq.(26), we deduce the following travelling wave solutions.

Case 1: If $\lambda^2 - 4\mu > 0$, then we have

$$U(\xi) = \pm \sqrt{-\frac{\beta}{\gamma}} \left(\frac{c_1 \sinh(\frac{\xi}{2} \sqrt{\lambda^2 - 4\mu}) + c_2 \cosh(\frac{\xi}{2} \sqrt{\lambda^2 - 4\mu})}{c_1 \cosh(\frac{\xi}{2} \sqrt{\lambda^2 - 4\mu}) + c_2 \sinh(\frac{\xi}{2} \sqrt{\lambda^2 - 4\mu})} \right).$$

If we choose $c_1 = \cosh(\xi_0)$ and $c_2 = \sinh(\xi_0)$, we get the following hyperbolic solution for the Eq.(22)

$$U(\xi) = \pm \sqrt{-\frac{\beta}{\gamma}} \tanh \left(\frac{\xi}{2} \sqrt{\lambda^2 - 4\mu} + \xi_0 \right).$$

Case 2: If $\lambda^2 - 4\mu < 0$, then we have

$$U(\xi) = \pm \sqrt{-\frac{\beta}{\gamma}} \left(\frac{c_1 \sin(\frac{\xi}{2} \sqrt{4\mu - \lambda^2}) + c_2 \cos(\frac{\xi}{2} \sqrt{4\mu - \lambda^2})}{-c_1 \cos(\frac{\xi}{2} \sqrt{4\mu - \lambda^2}) + c_2 \sin(\frac{\xi}{2} \sqrt{4\mu - \lambda^2})} \right).$$

If we choose $c_1 = \cos(\xi_0)$ and $c_2 = \sin(\xi_0)$, we get the following trigonometric solution for the Eq.(22)

$$U(\xi) = \mp \sqrt{-\frac{\beta}{\gamma}} \tan \left(\frac{\xi}{2} \sqrt{4\mu - \lambda^2} + \xi_0 \right).$$

Case 3: If $\lambda^2 - 4\mu = 0$, then we have trivial solution for the Eq.(22)

$$U(\xi) = 0.$$

Set 2

$$\begin{aligned} a_0 &= \pm \frac{\lambda b_0 \sqrt{\beta}}{\sqrt{\gamma(4\mu - \lambda^2)}} , \quad a_1 = \pm 2 \sqrt{\frac{\beta}{\gamma(4\mu - \lambda^2)}} b_0 , \\ b_1 &= 0 , \quad s = \pm \sqrt{\frac{2\beta}{\lambda^2 - 4\mu} - \alpha} \end{aligned} \quad (28)$$

where b_0, λ, β and μ are arbitrary constants. Substituting Eq.(28) into Eq.(24) we get the following solution

$$U(\xi) = \pm \frac{2\sqrt{\beta}}{\sqrt{\gamma(4\mu - \lambda^2)}} \left((G'/G) + \frac{\lambda}{2} \right), \quad (29)$$

where

$$\xi = x \pm \left(\sqrt{\frac{2\beta}{\lambda^2 - 4\mu} - \alpha} \right) t + \xi_0 \quad (30)$$

and (G'/G) is given in Eq.(7).

Set 3

$$\begin{aligned} a_0 &= \pm \sqrt{\frac{\beta}{\gamma(4\mu - \lambda^2)}} (\lambda b_0 - 2\mu b_1) , \\ a_1 &= \pm \sqrt{\frac{\beta}{\gamma(4\mu - \lambda^2)}} (\lambda b_1 - 2b_0) , \quad s = \pm \sqrt{\frac{2\beta}{\lambda^2 - 4\mu} - \alpha} \end{aligned} \quad (31)$$

where b_0, b_1, λ and μ are arbitrary constants. Substituting Eq.(31) into Eq.(24) we get the following solution

$$U(\xi) = \pm \frac{\sqrt{\frac{\beta}{\gamma(4\mu - \lambda^2)}} (\lambda b_1 - 2b_0) (G'/G) + \sqrt{\frac{\beta}{\gamma(4\mu - \lambda^2)}} (\lambda b_0 - 2\mu b_1)}{b_1 (G'/G) + b_0}$$

where

$$\xi = x \pm \left(\sqrt{\frac{2\beta}{\lambda^2 - 4\mu} - \alpha} \right) t + \xi_0$$

and (G'/G) is given in Eq.(7).

Set 4

$$\begin{aligned} a_0 &= \pm \sqrt{\frac{\beta}{\gamma(4\mu - \lambda^2)}} \left(\left(\frac{\lambda}{2} \pm \frac{1}{6} \sqrt{3\lambda^2 - 12\mu} \right) - 2\mu \right) b_1 , \quad a_1 = \pm \sqrt{\frac{-\beta}{3\gamma}} b_1 , \\ b_0 &= \left(\frac{\lambda}{2} \pm \frac{1}{6} \sqrt{3\lambda^2 - 12\mu} \right) b_1 , \quad s = \pm \sqrt{\frac{2\beta}{\lambda^2 - 4\mu} - \alpha} \end{aligned} \quad (32)$$

where b_1, λ and μ are arbitrary constants. Substituting Eq.(32) into Eq.(24) we get the following solution

$$U(\xi) = \pm \frac{\sqrt{\frac{-\beta}{3\gamma}}(G'/G) + \sqrt{\frac{\beta}{\gamma(4\mu-\lambda^2)}} \left(\left(\frac{\lambda}{2} \pm \frac{1}{6} \sqrt{3\lambda^2 - 12\mu} \right) - 2\mu \right)}{(G'/G) + \frac{\lambda}{2} \pm \frac{1}{6} \sqrt{3\lambda^2 - 12\mu}}$$

where

$$\xi = x \pm \left(\sqrt{\frac{2\beta}{\lambda^2 - 4\mu}} - \alpha \right) t + \xi_0$$

and (G'/G) is given in Eq.(7).

Example 3. *The coupled Hirota-Satsuma KdV equations*

The coupled Hirota-Satsuma KdV equations (CHSK) describes an interaction of two long waves with different dispersion relations [31]. We will consider the CHSK equations in the following form

$$\begin{aligned} u_t &= \frac{1}{4}u_{xxx} + 3uu_x - 6vv_x, \\ v_t &= -\frac{1}{2}v_{xxx} - 3uv_x. \end{aligned} \quad (33)$$

Making the transformations $u(x, t) = U(\xi)$, $v(x, t) = V(\xi)$, $\xi = x - st + \xi_0$, where s is the velocity of the wave to be determined later. We get the CHSK equations in the following form

$$\begin{aligned} -sU' &= \frac{1}{4}U''' + 3UU' - 6VV', \\ -sV' &= -\frac{1}{2}V''' - 3UV'. \end{aligned} \quad (34)$$

By balancing the highest order derivatives and nonlinear terms in Eq.(34), we get $n = 2$ and from Eq.(4) we write the solutions of Eq.(33) as

$$\begin{aligned} U(\xi) &= \frac{a_0 + a_1(G'/G) + a_2(G'/G)^2}{b_0 + b_1(G'/G) + b_2(G'/G)^2}, \\ V(\xi) &= \frac{e_0 + e_1(G'/G) + e_2(G'/G)^2}{d_0 + d_1(G'/G) + d_2(G'/G)^2} \end{aligned} \quad (35)$$

Substituting Eq.(35) into Eq.(34), and we convert Eq.(34) into a polynomial in (G'/G) . Equating the coefficients of the same power of (G'/G) to zero, yields a set of simultaneous algebraic equations. Solving the set of equations for $a_i, b_i, e_i, d_i (i = 0, 1, 2)$ and s by using the computer programme, we get the following set of solutions

Set 1

$$\begin{aligned} a_2 &= -2b_0, \quad a_1 = -2\lambda b_0, \quad b_1 = b_2 = 0, \quad d_1 = d_2 = 0 \\ e_0 &= -\frac{e_2(\lambda^2 b_0 + 8\mu b_0 + 8a_0)}{4b_0}, \quad e_1 = \lambda e_2, \quad d_0 = e_2 \\ s &= \frac{\lambda^2 b_0 + 8\mu b_0 + 6a_0}{2b_0} \end{aligned} \quad (36)$$

where a_0, b_0, e_2, λ and μ are constants. Substituting Eq.(36) into Eq.(35), hence we reach the following solutions

$$U(\xi) = -2 \left[(G'/G)^2 + \lambda(G'/G) \right] + \frac{a_0}{b_0} \quad (37)$$

$$V(\xi) = (G'/G)^2 + \lambda(G'/G) - \frac{\lambda^2 + 8\mu}{4b_0} - 2\frac{a_0}{b_0}$$

where

$$\xi = x - \left(\frac{\lambda^2 + 8\mu}{2b_0} + 3\frac{a_0}{b_0} \right) t + \xi_0$$

and (G'/G) is given in Eq.(7). Substituting Eq.(7) into Eq.(37), we deduce the following travelling wave solutions.

Case 1: If $\lambda^2 - 4\mu > 0$ and if we choose $c_1 = \cosh(\xi_0)$, $c_2 = \sinh(\xi_0)$, then we have the hyperbolic solutions for the Eq.(34)

$$U(\xi) = \frac{a_0}{b_0} + \frac{\lambda^2}{2} - \frac{\lambda^2 - 4\mu}{2} \coth^2 \left(\frac{\xi}{2} \sqrt{\lambda^2 - 4\mu} + \xi_0 \right),$$

$$V(\xi) = \frac{-2a_0}{b_0} - \frac{\lambda^2}{4} - \frac{\lambda^2 + 8\mu}{4b_0} + \frac{\lambda^2 - 4\mu}{4} \coth^2 \left(\frac{\xi}{2} \sqrt{\lambda^2 - 4\mu} + \xi_0 \right).$$

Case 2: If $\lambda^2 - 4\mu < 0$ and if we choose $c_1 = \cos(\xi_0)$, $c_2 = \sin(\xi_0)$, then we have the trigonometric solutions for the Eq.(34)

$$U(\xi) = \frac{a_0}{b_0} + \frac{\lambda^2}{2} - \frac{\lambda^2 - 4\mu}{2} \cot^2 \left(\frac{\xi}{2} \sqrt{4\mu - \lambda^2} + \xi_0 \right),$$

$$V(\xi) = \frac{-2a_0}{b_0} - \frac{\lambda^2}{4} - \frac{\lambda^2 + 8\mu}{4b_0} + \frac{\lambda^2 - 4\mu}{4} \cot^2 \left(\frac{\xi}{2} \sqrt{4\mu - \lambda^2} + \xi_0 \right).$$

Case 3: If $\lambda^2 - 4\mu = 0$, , then we have rational solutions for the Eq.(34)

$$U(\xi) = \frac{a_0}{b_0} + \frac{\lambda^2}{2} - \frac{2c_2^2}{(c_1 + c_2\xi)^2},$$

$$V(\xi) = \frac{-2a_0}{b_0} - \frac{\lambda^2}{4} - \frac{\lambda^2 + 8\mu}{4b_0} + \left(\frac{c_2}{c_1 + c_2\xi} \right)^2.$$

Set 2

$$a_2 = -b_0, \quad a_1 = -\lambda b_0, \quad b_1 = b_2 = 0, \quad d_2 = 0, \quad e_0 = \frac{\lambda d_0 e_2}{2d_1}$$

$$a_0 = -\frac{b_0(\lambda^2 d_1^2 + 8\mu d_1^2 + 4e_2^2)}{8d_1^2}, \quad e_1 = \frac{e_2(\lambda d_1 + 2d_0)}{2d_1} \quad (38)$$

$$s = \frac{\lambda^2 d_1^2 - 4\mu d_1^2 - 12e_2^2}{8d_1^2}$$

where $b_0, d_0, d_1, e_2, \lambda$ and μ are constants. Substituting Eq.(38) into Eq.(35), hence we reach the following solutions

$$U(\xi) = - \left[(G'/G)^2 + \lambda(G'/G) \right] - \left(\frac{\lambda^2 + 8\mu}{8} \right) - \left(\frac{e_2^2}{2d_1^2} \right)$$

$$V(\xi) = e_2 \frac{(G'/G)^2 + \left(\frac{\lambda d_1 + 2d_0}{2d_1} \right) (G'/G) + \frac{\lambda d_0}{2d_1}}{d_1(G'/G) + d_0}$$

where

$$\xi = x - \left(\frac{\lambda^2 - 4\mu}{8} + \frac{4}{3} \frac{e_2^2}{d_1^2} \right) t + \xi_0$$

and (G'/G) is given in Eq.(7).

Set 3

$$\begin{aligned} a_2 &= -b_0, \quad a_1 = -\lambda b_0, \quad b_1 = b_2 = 0 \\ e_0 &= \frac{e_2 d_0}{d_2}, \quad e_1 = \frac{e_2 d_1}{d_2} \\ s &= -\frac{\lambda^2 b_0 + 8\mu b_0 + 12a_0}{4b_0} \end{aligned} \quad (39)$$

where $a_0, b_0, d_0, d_1, d_2, e_2, \lambda$ and μ are constants. Substituting Eq.(39) into Eq.(35), hence we reach the following solutions

$$\begin{aligned} U(\xi) &= - \left[(G'/G)^2 + \lambda(G'/G) \right] + \frac{a_0}{b_0} \\ V(\xi) &= e_2 \frac{(G'/G)^2 + \frac{d_1}{d_2}(G'/G) + \frac{d_0}{d_2}}{d_2(G'/G)^2 + d_1(G'/G) + d_0} \end{aligned}$$

where

$$\xi = x + \left(\frac{\lambda^2 + 8\mu}{4} + 3 \frac{a_0}{b_0} \right) t + \xi_0$$

and (G'/G) is given in Eq.(7).

Set 4

$$\begin{aligned} a_0 &= \frac{\lambda^4 b_0 - 8\lambda^2 \mu b_0 + 16\mu^2 a_2 + 16\mu^2 b_0}{4\lambda^2} \\ a_1 &= \frac{\lambda^4 b_0 - 8\lambda^2 \mu b_0 + 16\mu^2 a_2 + 16\mu^2 b_0}{4\mu\lambda}, \quad d_2 = 0 \\ b_1 &= \frac{\lambda b_0}{\mu}, \quad b_2 = \frac{\lambda^2 b_0}{4\mu^2}, \quad e_0 = e_1 = e_2 = 0 \\ s &= -\frac{\lambda^4 b_0 - 16\lambda^2 b_0 \mu + 48\mu^2 a_2 + 48\mu^2 b_0}{4b_0 \lambda^2} \end{aligned} \quad (40)$$

where a_2, b_0, λ and μ are constants. Substituting Eq.(40) into Eq.(35), hence we reach the following solutions

$$\begin{aligned} U(\xi) &= \frac{a_2(G'/G)^2 + \left(\frac{\lambda^4 b_0 - 8\lambda^2 \mu b_0 + 16\mu^2 a_2 + 16\mu^2 b_0}{4\mu\lambda} \right) (G'/G) + \left(\frac{\lambda^4 b_0 - 8\lambda^2 \mu b_0 + 16\mu^2 a_2 + 16\mu^2 b_0}{4\lambda^2} \right)}{\frac{\lambda^2 b_0}{4\mu^2} (G'/G)^2 + \frac{\lambda b_0}{\mu} (G'/G) + b_0} \\ V(\xi) &= 0 \end{aligned}$$

where

$$\xi = x + \left(\frac{\lambda^2 - 16\mu}{4} + \frac{12\mu^2 a_2 + 12\mu^2 b_0}{b_0 \lambda^2} \right) t + \xi_0$$

and (G'/G) is given in Eq.(7).

Set 5

$$\begin{aligned}
a_0 &= \frac{\lambda^4 b_0 - 8\lambda^2 b_0 \mu + 16\mu^2 a_2 + 16\mu^2 b_0}{4\lambda^2}, & e_1 &= e_2 = 0 \\
a_1 &= \frac{\lambda^4 b_0 - 8\lambda^2 b_0 \mu + 16\mu^2 a_2 + 16\mu^2 b_0}{4\lambda\mu}, & d_1 &= d_2 = 0 \\
b_1 &= \frac{\lambda b_0}{\mu}, & b_2 &= \frac{\lambda^2 b_0}{4\mu^2}, & s &= -\frac{\lambda^4 b_0 - 16\lambda^2 b_0 \mu + 48\mu^2 a_2 + 48\mu^2 b_0}{4b_0 \lambda^2}
\end{aligned} \tag{41}$$

where a_2, b_0, λ and μ are constants. Substituting Eq.(41) into Eq.(35), hence we reach the following solutions

$$U(\xi) = \frac{a_2(G'/G)^2 + \left(\frac{\lambda^4 b_0 - 8\lambda^2 b_0 \mu + 16\mu^2 a_2 + 16\mu^2 b_0}{4\lambda\mu}\right)(G'/G) + \left(\frac{\lambda^4 b_0 - 8\lambda^2 b_0 \mu + 16\mu^2 a_2 + 16\mu^2 b_0}{4\lambda}\right)}{\frac{\lambda^2 b_0}{4\mu^2}(G'/G)^2 + \frac{\lambda b_0}{\mu}(G'/G) + b_0}$$

$$V(\xi) = \frac{e_0}{d_0}$$

where e_0, d_0 are constants and

$$\xi = x + \left(\frac{\lambda^2 - 16\mu}{4} + \frac{12\mu^2 a_2 + 12\mu^2 b_0}{b_0 \lambda^2}\right)t + \xi_0$$

and (G'/G) is given in Eq.(7).

4. CONCLUSION

In this paper, we have obtained various types of travelling wave solutions for the combined KdV-mKdV equation, the reaction-diffusion equation, and the coupled Hirota-Satsuma KdV equations that are solved by using the rational (G'/G) -expansion method. The main idea of this method is to reduce the partial differential equation to an ODE by using the travelling wave transformation (Eq.(2)), after integrating the ODE in Eq.(3), once or twice, then express the ODE in a compact form. This ODE can be written by a n-th degree polynomial in terms of (G'/G) , where $G = G(\xi)$ is the general solution of the second order LODE in Eq.(5). In order to find the positive integer, we use the homogeneous balance method, that is balancing between the highest order derivative term and nonlinear term. The coefficients of the polynomials can be obtained by solving a set of algebraic equations. Generally, the resulted algebraic equations can be solved by using Maple software program. It is mostly possible to find a solution of the algebraic equations, but it is generally unable to guarantee the existence of a solution. Despite of this, the rational (G'/G) -expansion method is still powerful method for finding travelling wave solutions of nonlinear evolution equations. The rational (G'/G) -expansion method is also direct, concise, elementary that the general solution of the second order ODE Eq.(5) is well known and effective that it can be used for many other nonlinear evolution equations, such as the generalized shallow water wave equation, the compound KdV-Burgers equations, the Klein-Gordon equation, the generalized KPP equation, the approximate long water wave equations, the coupled nonlinear

Klein-Gordon-Zakharov equations, and so on. Therefore, various explicit solutions of these nonlinear evolution equations can be obtained by this method.

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