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STUDY STRONG SHEFFER STROKE NON-ASSOCIATIVE MV-ALGEBRAS BY FUZZY FILTERS

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Abstract. In this paper, some types of fuzzy filters of a strong Sheffer stroke non-associative MV-algebra (for short, strong Sheffer stroke NMV-algebra) are introduced. By presenting new properties of filters, we define a prime filter in this algebraic structure. Then (prime) fuzzy filters of a strong Sheffer stroke NMV-algebra are determined and some features are proved. Finally, we built quotient strong Sheffer stroke NMV-algebra by a fuzzy filter.

1. Introduction

Sheffer operation was introduced by H. M. Sheffer as a single binary operation on a Boolean algebra restated all Boolean operations or formulas [16]. Since it has all diods on the chip forming processor in a computer, producing a single diod for this operation is simpler and cheaper than to produce different diods for other Boolean operations. Therefore, it is applied to algebraic structures such as Boolean algebras ([9], [16]), ortholattices [3], orthoimplication algebras [1], Hilbert algebras [11], UP-algebras [14] and BL-algebras [13]. In recent times, Chajda et al. introduced and studied non-associative MV-algebras (briefly, NMV-algebras) ([4], [5], [6]) because associativity of the binary relation of a MV-algebra causes serious problems in expert systems in artificial intelligence ([2], [6]). Also, Oner et al. analyzed filters and neutrosophic structures on strong Sheffer stroke NMV-algebras ([10], [15]). On the other side, the notion of fuzzy logic was originally introduced by Lotfi Zadeh [18] and has been developing expeditiously. Since these

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concepts have an important position in classic or nonclassic logical algebras, it leads to interesting results ([7], [8], [12], [17]).

In this study, basic concepts and new properties of a strong Sheffer stroke NMV-algebra are presented. Then a (prime) filter of strong Sheffer stroke NMV-algebras is defined and some features examined. It is shown that a filter of a strong sheffer stroke NMV-algebra is prime if and only if it is not contained by another filter of this algebraic structure. Indeed, it is proved that a filter of a strong Sheffer stroke NMV-algebra is prime if and only if the quotient structure defined by the filter is totally ordered strong Sheffer stroke NMV-algebra and its cardinality is less than or equals to 2. By describing a (prime) fuzzy filter of strong Sheffer stroke NMV-algebras, related notions are stated. It is proved that α is a (prime) fuzzy filter of a strong Sheffer stroke NMV-algebra if and only if $\alpha_a = \{x \in A : a \leq \alpha(x)\}$ is empty or a (prime) filter of A, for all $a \in [0, 1]$. Besides, it is shown that F is a (prime) filter of a strong Sheffer stroke NMV-algebra if and only if a fuzzy subset α_F defined by F is a (prime) fuzzy filter of this algebraic structure. It is demonstrated that a strong Sheffer stroke NMV-algebra is totally ordered if and only if every fuzzy filter is prime if and only if the filter $\{1\}$ is prime. Also, we prove that a fuzzy filter α of a strong Sheffer stroke NMV-algebra is prime if and only if α_h is a prime fuzzy filter of this algebra, for a surjective endomorphism h on this algebra, and that $\alpha_h = \alpha$ if and only if $h(\alpha_a) = \alpha_a$, for an automorphism h on this algebra and $a \in Im(\alpha)$. Finally, a congruence relation on a strong Sheffer stroke NMV-algebra is defined by a fuzzy filter, and so, a quotient strong Sheffer stroke NMV-algebra is constructed by means of the congruence relation. In fact, a fuzzy filter α of a strong Sheffer stroke NMV-algebra is prime if and only if the quotient structure is a totally ordered strong Sheffer stroke NMV-algebra and its cardinality is less than or equals to 2. In addition, it is shown that $\alpha \circ h$ is a fuzzy filter of A and the quotient structures defined by the fuzzy filters $\alpha \circ h$ and α are isomorphic, for strong Sheffer stroke NMV-algebras A and B, an epimorphism h between these algebras and a fuzzy filter α of B. Consequently, it is stated that the class of all fuzzy filters of a strong Sheffer stroke NMV-algebra forms a complete lattice since the interval $[0, 1]$ is a complete lattice and has important properties.

2. Preliminaries

In this section, basic definitions and notions about strong Sheffer stroke NMV-algebras are presented.

Definition 1. *[3] Let $\mathcal{A} = (A, |)$ be a groupoid. The operation $|$ on A is said to be a Sheffer stroke operation if it satisfies the following conditions:*

$(S1)$ $x|y = y|x$,
$(S2)$ $(x|x)|(x|y) = x$,
$(S3)$ $x|((y|z)|(y|z)) = ((x|y)|(x|y))|z$,
$(S4)$ $(x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x$.

Definition 2. *[4] A strong Sheffer stroke NMV-algebra is an algebra* $(A, |, 1)$ *of type* $(2, 0)$ *satisfying the identities for all* $x, y, z \in A$*:*

$(n1)$ $x|y \approx y|x$,
$(n2)$ $x|0 \approx 1$,
$(n3)$ $(x|1)|1 \approx x$,
$(n4)$ $((x|1)|y)|y \approx ((y|1)|x)|x$,
$(n5)$ $(x|1)|((x|y)|1) \approx 1$,
$(n6)$ $x|(((((x|y)|y)|z)|z)|1) \approx 1$

where 0 *denotes the algebraic constant* $1|1$.

Lemma 1. *[10] Let* $(A, |, 1)$ *be a strong Sheffer Stroke NMV-algebra. Then the binary relation* $\leq$ *defined by*

$$x \leq y \text{ if and only if } x|(y|1) \approx 1$$

is a partial order on A. *Hence,* $(A, \leq)$ *is a poset with the least element* 0 *and the greatest element* 1.

Lemma 2. *[10] In a strong Sheffer stroke NMV-algebra* A*, the following properties hold for all* $x, y, z \in A$*:*

(i) $x|(x|1) \approx 1$,
(ii) $x \leq y \Leftrightarrow y|1 \leq x|1$,
(iii) $y \leq x|(y|1)$,
(iv) $y|1 \leq x|y$,
(v) $x \leq (x|y)|y$,
(vi) $x \leq (((x|y)|y)|z)|z$,
(vii) $((x|y)|y)|y \approx x|y$,
$(viii)$ $x|1 \approx x|x$,
(ix) $x|(x|x) \approx 1$,
(x) $1|(x|x) \approx x$,
(xi) $x \leq y \Rightarrow y|z \leq x|z$,
(xii) $x|(y|1) \leq (y|(z|1))|((x|(z|1))|1)$,
$(xiii)$ $x|(y|1) \leq (z|(x|1))|((z|(y|1))|1)$.

Definition 3. *[10] A nonempty subset* $F \subseteq A$ *is called a filter of* A *if it satisfies the following properties:*
$(S_f - 1)$ $1 \in F$,
$(S_f - 2)$ *For all* $x, y \in A$, $x|(y|1) \in F$ *and* $x \in F$ *imply* $y \in F$.

Definition 4. *[10] Let* F *be a filter of a strong Sheffer stroke NMV-algebra* $(A, |, 1)$. *Define the binary relation* $\propto_F$ *on* A *as below: for all* $x, y \in A$

$$x \propto_F y \text{ if and only if } x|(y|1) \in F \text{ and } y|(x|1) \in F. \tag{1}$$

Definition 5. *[10] If* $x\xi y$ *implies* $x|k\xi y|k$, *for all* $x, y, k \in A$, *then the equivalence relation* ξ *is called a congruence relation on* A.

Lemma 3. *[10] An equivalence relation ξ is a congruence relation on A if and only if $x\xi y$ and $k_1\xi k_2$ imply $x|k_1\xi y|k_2$.*

Lemma 4. *[10] Let F be a filter of a strong Sheffer stroke NMV-algebra $(A,|,1)$ and the binary relation $\propto_F$ be defined as (1). Then $\propto_F$ is a congruence relation on A.*

Theorem 1. *[10] Let F be a filter of a strong Sheffer stroke NMV-algebra $(A,|,1)$ and $\propto$ be a congruence relation on A defined by F. Then $(A/\propto,|_\propto,[1]_\propto)$ is also a strong Sheffer stroke NMV-algebra where $A/F\equiv A/\propto=\{[x]_\propto : x\in A\}$, the strong Sheffer stroke $|_\propto$ on A/F is defined by $[x]_\propto|_\propto[y]_\propto\approx[x|y]_\propto$, for all $x,y\in A$ and $F\approx[1]_\propto$.*

Definition 6. *[10] Let $(A,|_A,1_A)$ and $(B,|_B,1_B)$ be strong Sheffer stroke NMV-algebras. A mapping $h:A\longrightarrow B$ is called a homomorphism if*

$$h(x|_Ay)=h(x)|_Bh(y),$$

for all $x,y\in A$.

3. Some Results in Strong Sheffer Stroke NMV-Algebras

In this section, new properties of strong Sheffer stroke NMV-algebras are given. Unless otherwise stated, A represents a strong Sheffer stroke NMV-algebra.

Lemma 5. *Let A be a strong Sheffer stroke NMV-algebra. Then $(A,\leq)$ is a bounded lattice with the least element 0 and the greatest element 1 of A, where $x\vee y\approx(x|(y|1))|(y|1)$ and $x\wedge y\approx(((x|1)|y)|y)|1$, for all $x,y\in A$.*

Proof. It is known from Lemma 1 that $(A,\leq)$ is a poset. Then $x\leq(x|(y|1))|(y|1)$ and $y\leq(x|(y|1))|(y|1)$ from Lemma 2 (v) and (iii), respectively. Thus, $(x|(y|1))|(y|1)$ is an upper bound of x and y. Let $x,y\leq z$. So, $x|(z|1)\approx 1$ and $y|(z|1)\approx 1$ from Lemma 1. Since

$$\begin{aligned}(x|(y|1))|(y|1)&\leq(z|(y|1))|(y|1)\\&\approx(((z|1)|1)|(y|1))|(y|1)\\&\approx(((y|1)|1)|(z|1))|(z|1)\\&\approx(y|(z|1))|(z|1)\\&\approx(z|1)|1\\&\approx z\end{aligned}$$

from Lemma 2 (i), (xi), (n1), (n3) and (n4), it follows that $(x|(y|1))|(y|1)$ is the least upper bound of x and y. Hence,$x\vee y\approx(x|(y|1))|(y|1)$, and similarly, $x\wedge y\approx(((x|1)|y)|y)|1$, for all $x,y\in A$.

Since $0|(x|1)\approx(x|1)|0\approx 1$ and $x|(1|1)\approx x|0\approx 1$ from (n1) and (n2), it is obtained from Lemma 1 that $0\leq x$ and $x\leq 1$, for all $x,y\in A$. Therefore, 0 is the least element and 1 is the greatest element of A □

Proposition 1. *Let A be a strong Sheffer stroke NMV-algebra. Then*

$$x|((y|(z|1))|1) \approx (x|(y|1))|((x|(z|1))|1),$$

for all $x, y, z \in A$.

Proof. Let A be a strong Sheffer stroke NMV-algebra.

$$\begin{aligned} x|((y|(z|1))|1) &\approx x|((y|(z|1))|(y|(z|1))) \\ &\approx y|((x|(z|1))|(x|(z|1))) \\ &\approx y|((x|(z|1))|1) \\ &\geq (x|(y|1))|((x|(z|1))|1) \end{aligned}$$

from Lemma 2 (viii), (iii), (xi), (S1) and (S3). Also,

$$\begin{aligned} x|((y|(z|1))|1) &\approx x|((y|(z|1))|(y|(z|1))) \\ &\approx y|((x|(z|1))|(x|(z|1))) \\ &\approx y|((x|(z|1))|1) \\ &\leq (x|(y|1))|((x|((x|(z|1))|1))|1) \\ &\approx (x|(y|1))|((x|((x|(z|1))|(x|(z|1))))|1) \\ &\approx (x|(y|1))|((((x|x)|(x|x))|(z|1))|1) \\ &\approx (x|(y|1))|((x|(z|1))|1) \end{aligned}$$

from Lemma 2 (viii), (xiii), (S1)-(S3).

Hence, $x|((y|(z|1))|1) \approx (x|(y|1))|((x|(z|1))|1)$, for all $x, y, z \in A$. □

Proposition 2. *Let A be a strong Sheffer stroke NMV-algebra. Then*

$$(x|y)|1 \leq x \text{ and } (x|y)|1 \leq y,$$

for all $x, y \in A$.

Proof. Let A be a strong Sheffer stroke NMV-algebra. Since $((x|y)|1)|(x|1) \approx (x|1)|((x|y)|1) \approx 1$ and $((x|y)|1)|(y|1) \approx (y|1)|((y|x)|1) \approx 1$ from (n1) and (n5), it is obtained from Lemma 1 that $(x|y)|1 \leq x$ and $(x|y)|1 \leq y$, for all $x, y \in A$. □

Lemma 6. *A nonempty subset F of A is a filter of A if and only if*
$(S_f - 3)$ $x, y \in F$ imply $(x|y)|1 \in F$,
$(S_f - 4)$ $x \in F$ and $x \leq y$ imply $y \in F$,
for all $x, y \in A$.

Proof. (⇒) Let F be a filter of A and $x, y \in A$. Since

$$\begin{aligned} x|(((x|y)|y)|1) &\approx x|(((x|y)|y)|((x|y)|y)) \\ &\approx (x|y)|((x|y)|(x|y)) \\ &\approx 1 \end{aligned}$$

from Lemma 2 (viii), (ix), (S1) and (S3), it follows from $(S_f - 2)$ that $(x|y)|y \in F$. Since $y|(((x|y)|1)|1) = (x|y)|y \in F$ from (n1) and (n3), respectively, it is obtained from (S_f-2) that $(x|y)|1 \in F$. Let $x \in F$ and $x \leq y$. Then $x|(y|1) \in F$ from Lemma 1 and (S_f-1). Thus, $y \in F$ from (S_f-2).

($\Leftarrow$) Let F be a nonempty subset of A satisfying (S_f-3) and (S_f-4). Assume that $x \in F$. Since $x \leq 1$ for all $x \in A$, it follows from (S_f-4) that $1 \in F$. Let $x|(y|1) \in F$ and $x \in F$. Then $(x|(x|(y|1)))|1 \in F$ from (S_f-3). Since

$$\begin{aligned}((x|(x|(y|1)))|1)|(y|1) &\approx ((((y|1)|x)|x)|1)|(y|1)\\ &\approx ((((x|1)|y)|y)|1)|(y|1)\\ &\approx (y|1)|((y|(y|(x|1)))|1)\\ &\approx 1\end{aligned}$$

from (n1), (n4) and (n5), it is obtained from Lemma 1 that $(x|(x|(y|1)))|1 \leq y$. Thus, $y \in F$ from (S_f-4). □

Lemma 7. *Let F be a filter of A. Then*

(*a*) $z|(((y|(x|1))|(x|1))|1) \in F$ *and* $z \in F$ *imply* $(x|(y|1))|(y|1) \in F$,
(*b*) $z|((y|(x|1))|1) \in F$ *and* $z \in F$ *imply* $((x|(y|1))|(y|1))|(x|1) \in F$ *and*
(*c*) $x|((y|(z|1))|1) \in F$ *and* $x|(y|1) \in F$ *imply* $x|(z|1) \in F$,

for all $x, y, z \in A$.

Proof. (*a*) Since

$$\begin{aligned}z|(((x|(y|1))|(y|1))|1) &\approx z|(((((x|1)|1)|(y|1))|(y|1))|1)\\ &\approx z|(((((y|1)|1)|(x|1))|(x|1))|1)\\ &\approx z|(((y|(x|1))|(x|1))|1) \in F\end{aligned}$$

from (n3) and (n4) and $z \in F$, it follows from (S_f-2) that $(x|(y|1))|(y|1) \in F$.

(*b*) Since

$$\begin{aligned}z|((((x|(y|1))|(y|1))|(x|1))|1) &\approx z|((((((x|1)|1)|(y|1))|(y|1))|(x|1))|1)\\ &\approx z|((((((y|1)|1)|(x|1))|(x|1))|(x|1))|1)\\ &\approx z|((((y|(x|1))|(x|1))|(x|1))|1)\\ &\approx z|((y|(x|1))|1) \in F\end{aligned}$$

from (n3), (n4) and Lemma 2 (vii) and $z \in F$, it is obtained from (S_f-2) that $((x|(y|1))|(y|1))|(x|1) \in F$.

(*c*) Since $(x|(y|1))|((x|(z|1))|1) \approx x|((y|(z|1))|1) \in F$ from Proposition 1 and $x|(y|1) \in F$, it follows from (S_f-2) that $x|(z|1) \in F$. □

Definition 7. *Let F be a filter of A. Then F is a prime filter of A if $x \vee y \in F$ implies $x \in F$ or $y \in F$, for all $x, y \in A$.*

Example 1. *Consider a strong Shefeer stroke NMV-algebra $(A, |, 1)$ where a set $A = \{0, a, b, c, d, e, f, 1\}$ and the operation $|$ on A has the following Cayley table ([10]):*

TABLE 1. Cayley table of $|$

$\vert$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	1	f	1	1	f	f	1	f
b	1	1	e	1	e	1	e	e
c	1	1	1	d	1	d	d	d
d	1	f	e	1	c	f	e	c
e	1	f	1	d	f	b	d	b
f	1	1	e	d	e	d	a	a
1	1	f	e	d	c	b	a	0

Then $\{a, d, e, 1\}$ *is a prime filter of* A *while* $\{e, 1\}$ *is not since* $a \notin \{e, 1\}$ *and* $c \notin \{e, 1\}$ *when* $a \vee c \approx (a|(c|1))|(c|1) \approx (a|d)|d \approx f|d \approx e \in \{e, 1\}$.

Lemma 8. *Let* F *be a filter of* A. *Then* F *is a prime filter of* A *if and only if* $x \in F$ *or* $x|1 \in F$, *for all* $x \in A$.

Proof. Let F be a prime filter of A. Since

$$\begin{aligned} x \vee (x|1) &\approx (x|((x|1)|1))|((x|1)|1) \\ &\approx x|(x|x) \\ &\approx 1 \in F \end{aligned}$$

from Lemma 5, (n1), (n3), Lemma 2 (ix) and $(S_f - 1)$, it is obtained that $x \in F$ or $x|1 \in F$, for all $x \in A$.

Conversely, let F be a filter of A such that $x \in F$ or $x|1 \in F$, for all $x \in A$. Assume that $x \vee y \in F$ such that $x \notin F$ and $y \notin F$, for some $x, y \in A$. Then $x|1 \in F$ and $y|1 \in F$. Since $x|1 \leq (y|1)|((x|1)|1) \approx x|(y|1)$ and $y|1 \leq (x|1)|((y|1)|1) \approx y|(x|1)$ from Lemma 2 (iii), (n1) and (n3), it follows from $(S_f - 4)$ that $x|(y|1) \in F$ and $y|(x|1) \in F$. Since $(x|(y|1))|(y|1) \approx x \vee y \in F$ and $(y|(x|1))|(x|1) \approx y \vee x \approx x \vee y \in F$ from Lemma 5, it is obtained from $(S_f - 2)$ that $x \in F$ and $y \in F$. This is a contradiction. Thus, $x \vee y \in F$ implies $x \in F$ or $y \in F$ which means that F is a prime filter of A. □

Lemma 9. *Let* F *be a filter of* A. *Then* F *is a prime filter of* A *if and only if* $x \notin F$ *and* $y \notin F$ *imply* $x|(y|1) \in F$ *and* $y|(x|1) \in F$, *for all* $x, y \in A$.

Proof. Let F be a prime filter of A, $x \notin F$ and $y \notin F$. Then $x|1 \in F$ and $y|1 \in F$. Since $x|1 \leq (y|1)|((x|1)|1) \approx x|(y|1)$ and $y|1 \leq (x|1)|((y|1)|1) \approx y|(x|1)$ from Lemma 2 (iii), (n1) and (n3), it follows from $(S_f - 4)$ that $x|(y|1) \in F$ and $y|(x|1) \in F$.

Conversely, let F be a filter of A such that $x \notin F$ and $y \notin F$ imply $x|(y|1) \in F$ and $y|(x|1) \in F$, for all $x, y \in A$. Assume that $x \notin F$ and $x|1 \notin F$, for some $x \in A$. Then $x|1 \approx x|x \approx x|((x|1)|1) \in F$ and $x \approx (x|x)|(x|x) \approx (1|((x|x)|(x|x)))|(1|((x|x)$

$|(x|x))) \approx (x|1)|(x|1) \in F$ from (n3), Lemma 2 (viii), (x) and (S1)-(S2). This is a contradiction. Thus, $x \in F$ or $x|1 \in F$, for all $x \in F$, i.e., F is a prime filter of A. □

Lemma 10. *Let F be a filter of A. Then*

(i) *$x \in F$ and $y \in F$ imply $x \wedge y \in F$,*
(ii) *F is a prime filter of A if and only if $x|(y|1) \in F$ or $y|(x|1) \in F$,*

for all $x, y \in A$.

Proof. (i) It is clear.

(ii) Let F be a prime filter of A. Since

$$\begin{aligned}(x|(y|1)) \vee (y|(x|1)) &\approx ((x|(y|1))|((y|(x|1))|1))|((y|(x|1))|1)\\ &\approx ((x|(y|1))|((y|(x|x))|(y|(x|x))))|((y|(x|x))|1)\\ &\approx ((((x|(y|1))|(x|x))|((x|(y|1))|(x|x)))|y)|((y|(x|x))|1)\\ &\approx (y|(x|x))|((y|(x|x))|1)\\ &\approx 1 \in F,\end{aligned}$$

from Lemma 5, Lemma 2 (i) and (viii), (S1)-(S3), it follows that $x|(y|1) \in F$ or $y|(x|1) \in F$

Conversely, let F be a filter of A such that $x|(y|1) \in F$ or $y|(x|1) \in F$, for all $x, y \in A$. Suppose that $x \vee y \in F$. If $x|(y|1) \in F$, then we have from $(S_f - 2)$ that $y \in F$ since $(x|(y|1))|(y|1) \approx x \vee y \in F$ from Lemma 5. Similarly, if $y|(x|1) \in F$, then we get from $(S_f - 2)$ that $x \in F$ since $(y|(x|1))|(x|1) \approx y \vee x \approx x \vee y \in F$ from Lemma 5. Hence, F is a prime filter of A.

□

Corollary 1. *Let F be a filter of A such that $F \neq A$. Then F is a prime filter of A if and only if $(x|(y|1)) \vee (y|(x|1)) \in F$, for all $x, y \in A$.*

Lemma 11. *Let F be a filter of A such that $F \neq A$. Then F is a prime filter of A if and only if there is no a filter G of A such that $F \subset G \subset A$.*

Proof. Let F be a prime filter of A. Assume that G is a filter of A such that $F \subset G \subset A$ and $y \in G$ such that $y \notin F$. Then $y|1 \in F$, and so, $y|1 \in G$. Since $y \in G$ and $y|1 \in G$, it follows from Lemma 2 (ix), (n1), Lemma 5 and Lemma 10 (i) that

$$\begin{aligned}0 &\approx 1|1\\ &\approx ((y|1)|((y|1)|(y|1)))|1\\ &\approx (((y|1)|(y|1))|(y|1))|1\\ &\approx y \wedge (y|1) \in G.\end{aligned}$$

Since $0 \in G$ and 0 is the least element of A, we have from $(S_f - 4)$ that $x \in G$, for all $x \in A$. Thus, $G = A$ which is a contradiction. Therefore, there is no a filter G of A such that $F \subset G \subset A$.

Conversely, let there be no a filter G of A such that $F \subset G \subset A$. Suppose that $x \vee y \in F$ such that $x, y \notin F$. Then there exists a filter G of A such that $x \in G$ or $y \in G$. Since $x, y \leq x \vee y$, we have from $(S_f - 4)$ that $x \vee y \in G$. Thus, $F \subset G$ which is a contradiction. Hence, $x \vee y \in F$ implies $x \in F$ or $y \in F$ which means that F is a prime filter of A. □

Lemma 12. *Let F be a filter of A and $\propto_F$ be a congruence relation on A defined by F. Define a relation $\subseteq$ on A/F by*

$$[x]_{\propto_F} \subseteq [y]_{\propto_F} \Leftrightarrow x|(y|1) \in F,$$

for all $x, y \in A$. Then the relation $\subseteq$ is a partial order on A/F.

Proof. Let F be a filter of A and $\propto_F$ be a congruence relation on A defined by F. Then $(A/F, |_{\propto_F}, F)$ is a strong Sheffer stroke NMV-algebra by Theorem 1.

• Since $x|(x|1) \approx 1 \in F$ from Lemma 2 (i) and $(S_f - 1)$, it follows that $[x]_{\propto_F} \subseteq [x]_{\propto_F}$, for all $x \in A$.

• Let $[x]_{\propto_F} \subseteq [y]_{\propto_F}$ and $[y]_{\propto_F} \subseteq [x]_{\propto_F}$. Then $x|(y|1) \in F$ and $y|(x|1) \in F$, and so, $x \propto_F y$. Thus, $[x]_{\propto_F} = [y]_{\propto_F}$.

• Let $[x]_{\propto_F} \subseteq [y]_{\propto_F}$ and $[y]_{\propto_F} \subseteq [z]_{\propto_F}$. Then $x|(y|1) \in F$ and $y|(z|1) \in F$. Since $x|(y|1) \leq (y|(z|1))|((x|(z|1))|1)$ from Lemma 2 (xii), it is obtained from $(S_f - 4)$ that $(y|(z|1))|((x|(z|1))|1) \in F$. Thus, it follows from $(S_f - 2)$ that $x|(z|1) \in F$ which implies that $[x]_{\propto_F} \subseteq [z]_{\propto_F}$.

Hence, the relation $\subseteq$ is a partial order on A/F. □

Theorem 2. *Let F be a filter of A and $\propto_F$ be a congruence relation on A defined by F. Then F is a prime filter of A if and only if $(A/F, |_{\propto_F}, F)$ is totally ordered and $|A/F| \leq 2$.*

Proof. Let F be a filter of A and $\propto_F$ be a congruence relation on A defined by F. Then $(A/F, |_{\propto_F}, F)$ is a strong Sheffer stroke NMV-algebra by Theorem 1. Let F be a prime filter of A. Then $x|(y|1) \in F$ or $y|(x|1) \in F$ by Lemma 10 (ii). Thus, $[x]_{\propto_F} \subseteq [y]_{\propto_F}$ or $[y]_{\propto_F} \subseteq [x]_{\propto_F}$ from Lemma 12. Hence, $(A/F, |_{\propto_F}, F)$ is totally ordered. Moreover, let $|A/F| > 2$. Then $[x]_{\propto_F} \in A/F$ such that $[0]_{\propto_F} \subset [x]_{\propto_F} \subset [1]_{\propto_F}$. Since F is a prime filter of A, it is known that $x \in F$ or $x|1 \in F$. Assume that $x|1 \in F$. Since $x|(0|1) \approx x|1 \in F$ and $0|(x|1) \approx 1 \in F$ from (n2), we get $[x]_{\propto_F} = [0]_{\propto_F}$ which is a contradiction. Therefore, $|A/F| \leq 2$.

Conversely, let $(A/F, |_{\propto_F}, F)$ be totally ordered. Then $[x]_{\propto_F} \subseteq [y]_{\propto_F}$ or $[y]_{\propto_F} \subseteq [x]_{\propto_F}$, for all $x, y \in A$. So, $x|(y|1) \in F$ or $y|(x|1) \in F$ by Lemma 12. Thus, F is a prime filter of A from Lemma 10 (ii). □

4. Fuzzy Filters of Strong Sheffer Stroke Nmv-Algebras

In this section, fuzzy filters strong Sheffer stroke NMV-algebras are introduced.

Definition 8. *A fuzzy subset α of A is called a fuzzy filter of A if*
$(FF1)$ $\alpha(x) \leq \alpha(1)$,
$(FF2)$ $\min\{\alpha(x), \alpha(x|(y|1))\} \leq \alpha(y)$,
for all $x, y \in A$.

Example 2. *Consider the strong Shefeer stroke NMV-algebra A in Example 1. Then a fuzzy subset α of A defined by*

$$\alpha(x) = \begin{cases} 0.19, & \text{if } x \approx 0, a, b, d \\ 0.81, & \text{otherwise} \end{cases}$$

is a fuzzy filter of A.

Lemma 13. *Let α be a fuzzy filter of A. Then*

(1) *if $x \leq y$, then $\alpha(x) \leq \alpha(y)$,*
(2) $\alpha(x|(y|1)) = \alpha(1)$ *implies* $\alpha(x) \leq \alpha(y)$,
(3) $\alpha((x|y)|1) = \alpha(x) \wedge \alpha(y)$,
(4) $\alpha(x \wedge y) = \alpha(x) \wedge \alpha(y)$,
(5) $\alpha(x) \wedge \alpha(x|1) = \alpha(0)$,
(6) $\alpha(x|(y|1)) \wedge \alpha(y|(z|1)) \leq \alpha(x|(z|1))$,
(7) $\alpha(x) \wedge \alpha(x|(y|1)) = \alpha(y) \wedge \alpha(y|(x|1)) = \alpha(x) \wedge \alpha(y)$ *and*
(8) $\alpha((((x|1)|y)|y)|1) = \alpha((((y|1)|x)|x)|1) = \alpha(x \wedge y)$,

for all $x, y, z \in A$.

Proof. (1) Let $x \leq y$. Then $x|(y|1) \approx 1$ from Lemma 1. Thus,

$$\begin{aligned} \alpha(x) &= \min\{\alpha(x), \alpha(1)\} \\ &= \min\{\alpha(x), \alpha(x|(y|1))\} \\ &\leq \alpha(y) \end{aligned}$$

from (FF1) and (FF2).

(2) Let $\alpha(x|(y|1)) = \alpha(1)$. Then

$$\begin{aligned} \alpha(x) &= \min\{\alpha(x), \alpha(1)\} \\ &= \min\{\alpha(x), \alpha(x|(y|1))\} \\ &\leq \alpha(y) \end{aligned}$$

from (FF1) and (FF2).

(3) Since $(x|y)|1 \leq x$ and $(x|y)|1 \leq y$ from Proposition 2, it follows from (1) that $\alpha((x|y)|1) \leq \alpha(x)$ and $\alpha((x|y)|1) \leq \alpha(y)$. Thus, $\alpha((x|y)|1) \leq \alpha(x) \wedge \alpha(y)$. Also,

$$\begin{aligned} \alpha(x) \wedge \alpha(y) &= \min\{\alpha(x), \alpha(y)\} \\ &\leq \min\{\alpha((x|y)|y), \alpha(y)\} \\ &= \min\{\alpha(y), \alpha(y|(((x|y)|1)|1))\} \\ &= \alpha((x|y)|1) \end{aligned}$$

from Lemma 2 (v), (1), (n1), (n3) and (FF2), respectively, Hence,

$$\alpha((x|y)|1) = \alpha(x) \wedge \alpha(y),$$

for all $x, y \in A$.

(4) Since $x \wedge y \leq x$ and $x \wedge y \leq y$, it is obtained from (1) that $\alpha(x \wedge y) \leq \alpha(x)$ and $\alpha(x \wedge y) \leq \alpha(y)$. So, $\alpha(x \wedge y) \leq \alpha(x) \wedge \alpha(y)$. Moreover, since $(x|y)|1 \leq x$ and $(x|y)|1 \leq y$ from Proposition 2, we have $(x|y)|1 \leq x \wedge y$. Thus, $\alpha(x) \wedge \alpha(y) = \alpha((x|y)|1) \leq \alpha(x \wedge y)$ from (3) and (1), respectively. Therefore, $\alpha(x \wedge y) = \alpha(x) \wedge \alpha(y)$, for all $x, y \in A$.

(5) $\alpha(x) \wedge \alpha(x|1) = \alpha((x|(x|1))|1) = \alpha(1|1) = \alpha(0)$ from (3) and Lemma 2 (i).

(6)

$$\begin{aligned}\alpha(x|(y|1)) \wedge \alpha(y|(z|1)) &= \min\{\alpha(x|(y|1)), \alpha(y|(z|1))\} \\ &= \min\{\alpha(x|(y|1)), \alpha(x|((y|(z|1))|1))\} \\ &= \min\{\alpha(x|(y|1)), \alpha((x|(y|1))|((x|(z|1))|1))\} \\ &\leq \alpha(x|(z|1))\end{aligned}$$

from Lemma 2 (iii), (1), Proposition 1 and (FF2).

(7)

$$\begin{aligned}\alpha(y) \wedge \alpha(y|(x|1)) &= \alpha((y|(y|(x|1)))|1) \\ &= \alpha((((x|1)|y)|y)|1) \\ &= \alpha(x \wedge y) \\ &= \alpha(x) \wedge \alpha(y),\end{aligned}$$

and similarly, $\alpha(x) \wedge \alpha(x|(y|1)) = \alpha(y) \wedge \alpha(x) = \alpha(x) \wedge \alpha(y)$ from (3), (n1), Lemma 5 and (4), respectively. Thus, $\alpha(x) \wedge \alpha(x|(y|1)) = \alpha(y) \wedge \alpha(y|(x|1)) = \alpha(x) \wedge \alpha(y)$, for all $x, y \in A$.

(8) It is proved Lemma 5.

□

Theorem 3. *Let α be a fuzzy subset of A. Then α is a fuzzy filter of A if and only if*

(*i*) *α is order-preserving,*

(*ii*) *$\alpha(x) \wedge \alpha(y) \leq \alpha((x|y)|1)$, for all $x, y \in A$.*

Proof. Let α be a fuzzy filter of A. Then it follows from Lemma 13 (1) and (3).

Conversely, let α be a fuzzy subset of A satisfying (i) and (ii). Since $x \leq 1$, it is obtained from (i) that $\alpha(x) \leq \alpha(1)$, for all $x \in A$.

$$\begin{aligned}\min\{\alpha(x), \alpha(x|(y|1))\} &= \alpha(x) \wedge \alpha(x|(y|1)) \\ &\leq \alpha((x|(x|(y|1)))|1) \\ &= \alpha(y \wedge x) \\ &\leq \alpha(y)\end{aligned}$$

from (ii), (n1), Lemma 5 and (i), respectively. Thus, α is a fuzzy filter of A. □

Theorem 4. *Let α be a fuzzy subset of A. Then α is a fuzzy filter of A if and only if $x \leq y|(z|1)$ implies $\alpha(x) \wedge \alpha(y) \leq \alpha(z)$, for all $x, y, z \in A$.*

Proof. Let α be a fuzzy filter of A and $x \leq y|(z|1)$. Then $x|((y|(z|1))|1) \approx 1$ from Lemma 1. Since

$$\begin{aligned}((x|y)|1)|(z|1) &\approx ((x|y)|(x|y))|(z|1) \\ &\approx x|((y|(z|1))|(y|(z|1))) \\ &\approx x|((y|(z|1))|1) \\ &\approx 1\end{aligned}$$

from Lemma 2 (viii) and (S3), it follows from Lemma 1 that $(x|y)|1 \leq z$. So, $\alpha(x) \wedge \alpha(y) = \alpha((x|y)|1) \leq \alpha(z)$ from Lemma 13 (3) and (1), respectively.

Conversely, let α be a fuzzy subset of A such that $x \leq y|(z|1)$ implies $\alpha(x) \wedge \alpha(y) \leq \alpha(z)$, for all $x, y, z \in A$. Since $x \leq 1 \approx x|0 \approx x|(1|1)$, from (n2), it is obtained that $\alpha(x) = \alpha(x) \wedge \alpha(x) \leq \alpha(1)$, for all $x \in A$. Since $x \leq x \vee y \approx (x|(y|1))|(y|1)$ from Lemma 5, it follows that $\min\{\alpha(x), \alpha(x|(y|1))\} = \alpha(x) \wedge \alpha(x|(y|1)) \leq \alpha(y)$, for all $x, y \in A$. Hence, α is a fuzzy filter of A. □

Theorem 5. *Let A be a strong Sheffer stroke NMV-algebra. Then α is a fuzzy filter of A if and only if $\alpha_a = \{x \in A : a \leq \alpha(x)\}$ is empty or a filter of A, for all $a \in [0, 1]$.*

Proof. Let α be a fuzzy filter of A and $\alpha_a = \{x \in A : a \leq \alpha(x)\} \neq \emptyset$. Suppose that $x \in \alpha_a$. Since $a \leq \alpha(x) \leq \alpha(1)$, we have $1 \in \alpha_a$. Let $x, x|(y|1) \in \alpha_a$. So, $a \leq \alpha(x)$ and $a \leq \alpha(x|(y|1))$. Since $a \leq \min\{\alpha(x), \alpha(x|(y|1))\} \leq \alpha(y)$, it is obtained that $y \in \alpha_a$. Hence, α_a is a filter of A.

Conversely, let $\alpha_a \neq \emptyset$ be a filter of A. Assume that $x \in \alpha_a$ such that $\alpha(1) < \alpha(x)$. If $a = 1/2(\alpha(1) + \alpha(x))$, then $\alpha(1) < a < \alpha(x)$. Thus, $1 \notin \alpha_a$ which is a contradiction with $(S_f - 1)$. Hence, $\alpha(x) \leq \alpha(1)$, for all $x \in A$. Suppose that $x, x|(y|1) \in \alpha_a$ such that $\alpha(y) < \min\{\alpha(x), \alpha(x|(y|1))\}$. If $a = 1/2(\alpha(y) + \min\{\alpha(x), \alpha(x|(y|1))\})$, then $\alpha(y) < a < \min\{\alpha(x), \alpha(x|(y|1))\} \leq \alpha(x)$ and $\alpha(y) < a < \min\{\alpha(x),\ \alpha(x|(y|1))\} \leq \alpha(x|(y|1))$. Thus, $y \notin \alpha_a$ which is a contradiction with $(S_f - 2)$. So, $\min\{\alpha(x), \alpha(x|(y|1))\} \leq \alpha(y)$, for all $x, y \in A$. Therefore, α is a fuzzy filter of A. □

Lemma 14. *Let α_a and α_b be two filter of A such that $a < b$. Then $\alpha_a = \alpha_b$ if and only if there exist no $x_0 \in A$ such that $a \leq \alpha(x_0) < b$.*

Proof. Let $\alpha_a = \alpha_b$ be such that $a < b$. Then $\alpha_a = \{x \in A : a \leq \alpha(x)\} = \{x \in A : b \leq \alpha(x)\} = \alpha_b$. If there exists $x_0 \in A$ such that $a \leq \alpha(x_0) < b$, then $x_0 \notin \alpha_b = \alpha_a$ which is a contradiction with $x_0 \in \alpha_a$. Thus, there exist no $x_0 \in A$ such that $a \leq \alpha(x_0) < b$.

Conversely, suppose that there exist no $x_0 \in A$ such that $a \leq \alpha(x_0) < b$. Let $\alpha_a \neq \alpha_b$ be such that $a < b$. Then there exist $x_0 \in A$ such that $a \leq c = \alpha(x_0) < b$ which is a contradiction. Hence, $\alpha_a = \alpha_b$. □

Corollary 2. *Let α be a fuzzy filter of A. Then $\alpha_a = \alpha_b$, for any $a, b \in Im(\alpha)$ if and only if $a = b$.*

Proof. It is obvious that $\alpha_a = \alpha_b$, for any $a, b \in Im(\alpha)$ if $a = b$.

Conversely, let $\alpha_a = \alpha_b$, for any $a, b \in Im(\alpha)$. Then there exist $x_0, x_1 \in A$ such that $\alpha(x_0) = a$ and $\alpha(x_1) = b$. So, $x_0 \in \alpha_a = \alpha_b$ and $x_1 \in \alpha_b = \alpha_a$. Thus, $b \leq \alpha(x_0) = a$ and $a \leq \alpha(x_1) = b$ which imply $a = b$. □

Lemma 15. *Let α be a fuzzy filter of A and $x_0 \in A$. Then $\alpha(x_0) = a$ if and only if $x_0 \in \alpha_a$ and $x_0 \notin \alpha_b$, for all $a < b$.*

Proof. Let $\alpha(x_0) = a$. Since $\alpha(x_0) = a < b$, we get $x_0 \in \alpha_a$ and $x_0 \notin \alpha_b$, for all $a < b$.

Conversely, let $x_0 \in \alpha_a$ and $x_0 \notin \alpha_b$, for all $a < b$. Then $a \leq \alpha(x_0) < b$. If $a \leq \alpha(x_0) = b_0$, then $x_0 \notin \alpha_{b_0}$ which is a contradiction. Hence, $\alpha(x_0) = a$. □

Let α be a fuzzy subset of A. Define a subset

$$A_\alpha = \{x \in A : \alpha(x) = \alpha(1)\}$$

of A.

Lemma 16. *Let F be a nonempty subset of A and α_F be a fuzzy subset of A by*

$$\alpha_F(x) = \begin{cases} a_1, & \text{if } x \in F \\ a_2, & \text{otherwise} \end{cases}$$

where $a_1, a_2 \in [0, 1]$ such that $a_1 > a_2$. Then α_F is a fuzzy filter of A if and only if F is a filter of A. Also, $A_{\alpha_F} = F$.

Proof. Let α_F be a fuzzy filter of A. Since $\alpha_F(1) = a_1$ by (FF1), we get $1 \in F$. Let $x, x|(y|1) \in F$. Then $\alpha_F(x) = a_1$ and $\alpha_F(x|(y|1)) = a_1$. Since $a_1 = \min\{\alpha_F(x), \alpha_F(x|(y|1))\} \leq \alpha(y)$, we have $\alpha_F(y) = a_1$, i.e., $y \in F$.

Conversely, let F be a filter of A. Since $1 \in F$, $\alpha_F(x) \leq \alpha_F(1) = a_1$, for all $x \in A$. Let $\min\{\alpha_F(x), \alpha_F(x|(y|1))\} = a_1$. Then $\alpha_F(x) = a_1 = \alpha_F(x|(y|1))$ which means that $x \in F$ and $x|(y|1) \in F$. So, $y \in F$ which implies $\alpha_F(y) = a_1$. Thus, $\min\{\alpha_F(x), \alpha_F(x|(y|1))\} \leq \alpha(y)$. Moreover, if $\min\{\alpha_F(x), \alpha_F(x|(y|1))\} = a_2$, then $\min\{\alpha_F(x), \alpha_F(x|(y|1))\} \leq \alpha(y)$, for all $x, y \in A$. Hence, α_F is a fuzzy filter of A.

Since F is a filter of A,

$$\begin{aligned} A_{\alpha_F} &= \{x \in A : \alpha_F(x) = \alpha_F(1)\} \\ &= \{x \in A : \alpha_F(x) = a_1\} \\ &= \{x \in A : x \in F\} \\ &= A \cap F = F. \end{aligned}$$

□

Definition 9. *Let α be a fuzzy filter of A. Then α is called a prime fuzzy filter of A if $\alpha(x \vee y) = \alpha(x) \vee \alpha(y)$, for all $x, y \in A$.*

Example 3. *Consider the strong Sheffer stroke NMV-algebra A in Example 1. Then a fuzzy subset α_1 of A defined by*

$$\alpha_1(x) = \begin{cases} 0.007, & \text{if } x \approx 0, a, c, e \\ 0.993, & \text{otherwise} \end{cases}$$

is a prime fuzzy filter of A.

However, a fuzzy subset α_2 of A defined by

$$\alpha_2(x) = \begin{cases} 0.92, & \text{if } x \approx 1 \\ 0.9, & \text{otherwise} \end{cases}$$

is not a prime fuzzy filter of A since $\alpha_2(b \vee e) = \alpha_2((b|(e|1))|(e|1)) = \alpha_2(b|(b|b)) = \alpha_2(b|e) = \alpha_2(1) \neq \alpha_2(b) = \alpha_2(b) \vee \alpha_2(e)$.

Theorem 6. *Let α be a fuzzy filter of A. Then α is a prime fuzzy filter of A if and only if $\alpha(x) = \alpha(1)$ or $\alpha(x|1) = \alpha(1)$, for all $x \in A$.*

Proof. Let α be a prime fuzzy filter of A. Since

$$\begin{aligned} \alpha(x) \vee \alpha(x|1) &= \alpha(x \vee (x|1)) \\ &= \alpha((x|((x|1)|1))|((x|1)|1)) \\ &= \alpha(x|(x|x)) \\ &= \alpha(1) \end{aligned}$$

from Lemma 5, (n1), (n3) and Lemma 2 (ix), it follows that $\alpha(x) = \alpha(1)$ or $\alpha(x|1) = \alpha(1)$, for all $x \in A$.

Conversely, let α be a fuzzy filter of A such that $\alpha(x) = \alpha(1)$ or $\alpha(x|1) = \alpha(1)$, for all $x \in A$. Since $x \leq x \vee y$ and $y \leq x \vee y$, it follows from Lemma 13 (1) that $\alpha(x) \leq \alpha(x \vee y)$ and $\alpha(y) \leq \alpha(x \vee y)$, and so, $\alpha(x) \vee \alpha(y) \leq \alpha(x \vee y)$, for all $x, y \in A$. If $\alpha(x) = \alpha(1)$ or $\alpha(y) = \alpha(1)$, then $\alpha(x \vee y) \leq \alpha(x) \vee \alpha(y)$ from (FF1). If $\alpha(x) \neq \alpha(1)$ and $\alpha(y) \neq \alpha(1)$, then $\alpha(x|1) = \alpha(1)$ and $\alpha(y|1) = \alpha(1)$. Since

$$\begin{aligned} \alpha(x \vee y) &= \alpha(y \vee x) \\ &= \alpha(1) \wedge \alpha(y \vee x) \\ &= \alpha(x|1) \wedge \alpha(y \vee x) \\ &= \alpha(((x|1)|(y \vee x))|1) \\ &= \alpha(((x|1)|((y|(x|1))|(x|1)))|1) \\ &= \alpha((y|(x|1))|1) \\ &\leq \alpha(y), \end{aligned}$$

and similarly, $\alpha(x \vee y) \leq \alpha(x)$ from Lemma 13 (1) and (3), Lemma 5, (n1), (n3), Lemma 2 (iv), (vii) and (ix), it is obtained that $\alpha(x \vee y) \leq \alpha(x) \vee \alpha(y)$. Hence, $\alpha(x \vee y) = \alpha(x) \vee \alpha(y)$, for all $x, y \in A$,, i.e., F is a prime fuzzy filter of A. □

Theorem 7. *Let α be a fuzzy filter of A. Then α is a prime fuzzy filter of A if and only if $\alpha(x) \neq \alpha(1)$ and $\alpha(y) \neq \alpha(1)$ imply $\alpha(x|(y|1)) = \alpha(1)$ and $\alpha(y|(x|1)) = \alpha(1)$, for all $x, y \in A$.*

Proof. Let α be a prime fuzzy filter of A and $\alpha(x) \neq \alpha(1)$ and $\alpha(y) \neq \alpha(1)$. Then $\alpha(x|1) = \alpha(1)$ and $\alpha(y|1) = \alpha(1)$ from Theorem 6. Since $(x|1)|((x|(y|1))|1) \approx 1$ and $(y|1)|((y|(x|1))|1) \approx 1$ from (n5), it follows from (FF2) that

$$\alpha(1) = \min\{\alpha(1), \alpha(1)\} = \min\{\alpha(x|1), \alpha((x|1)|((x|(y|1))|1))\} \leq \alpha(x|(y|1))$$

and

$$\alpha(1) = \min\{\alpha(1), \alpha(1)\} = \min\{\alpha(y|1), \alpha((y|1)|((y|(x|1))|1))\} \leq \alpha(y|(x|1)),$$

respectively. Thus, $\alpha(x|(y|1)) = \alpha(1)$ and $\alpha(y|(x|1)) = \alpha(1)$ from (FF1).

Conversely, let α be a fuzzy filter of A such that $\alpha(x) \neq \alpha(1)$ and $\alpha(y) \neq \alpha(1)$ imply $\alpha(x|(y|1)) = \alpha(1)$ and $\alpha(y|(x|1)) = \alpha(1)$, for all $x, y \in A$. If $\alpha(x) \neq \alpha(1)$ and $\alpha(1|1) = \alpha(0) \neq \alpha(1)$ for any $x \in A$, then $\alpha(x|1) = \alpha(x|(0|1)) = \alpha(1)$ and $\alpha(0|(x|1)) = \alpha(1)$ from (n1) and (n2). Also, if $\alpha(x|1) \neq \alpha(1)$ and $\alpha(1|1) = \alpha(0) \neq \alpha(1)$ for any $x \in A$, then $\alpha(x) = \alpha((x|1)|1) = \alpha((x|1)|(0|1)) = \alpha(1)$ and $\alpha(0|((x|1)|1)) = \alpha(1)$ from (n1)-(n3). Therefore, $\alpha(x) = \alpha(1)$ or $\alpha(x|1) = \alpha(1)$, for all $x \in A$. Hence, α is a prime fuzzy filter of A by Theorem 6. □

Corollary 3. *Let α be a fuzzy filter of A. Then α is a prime fuzzy filter of A if and only if $\alpha(x \vee (x|1)) = \alpha(1)$, for all $x, y \in A$.*

Theorem 8. *Let α be a fuzzy filter of A. Then α is a prime fuzzy filter of A if and only if $\alpha(x|(y|1)) = \alpha(1)$ or $\alpha(y|(x|1)) = \alpha(1)$, for all $x, y \in A$.*

Proof. Let α be a prime fuzzy filter of A. Since

$$\begin{aligned}
\alpha(x|(y|1)) \vee \alpha(y|(x|1)) &= \alpha((x|(y|1)) \vee (y|(x|1))) \\
&= \alpha(((x|(y|1))|((y|(x|1))|1))|((y|(x|1))|1)) \\
&= \alpha(((x|(y|y))|((y|(x|x))|(y|(x|x))))|((y|(x|x))|(y|(x|x)))) \\
&= \alpha(((((x|(y|y))|(x|x))|((x|(y|y))| \\
&\quad (x|x)))|y)|((y|(x|x))|(y|(x|x)))) \\
&= \alpha((y|(x|x))|((y|(x|x))|(y|(x|x)))) \\
&= \alpha(1)
\end{aligned}$$

from Lemma 5, Lemma 2 (viii), (ix) and (S1)-(S3), it follows that $\alpha(x|(y|1)) = \alpha(1)$ or $\alpha(y|(x|1)) = \alpha(1)$, for all $x, y \in A$.

Conversely, let α be a fuzzy filter of A such that $\alpha(x|(y|1)) = \alpha(1)$ or $\alpha(y|(x|1)) = \alpha(1)$, for all $x, y \in A$. By substituting $[y := x|1]$ in the hypothesis, we have $\alpha(1) = \alpha(x|((x|1)|1)) = \alpha(x|x) = \alpha(x|1)$ and $\alpha(1) = \alpha((x|1)|(x|1)) = \alpha((x|x)|(x|x)) = \alpha(x)$ from (n3), Lemma 2 (viii) and (S2). Thus, α is a prime fuzzy filter of A. □

Corollary 4. *Let α be a fuzzy filter of A. Then α is a prime fuzzy filter of A if and only if $\alpha(x|(y|1)) \vee \alpha(y|(x|1)) = \alpha(1)$, for all $x, y \in A$.*

Theorem 9. *Let A be a strong Sheffer stroke NMV-algebra. Then α is a prime fuzzy filter of A if and only if α_a is empty or a prime filter of A, for all $a \in [0, 1]$.*

Proof. Let α be a prime fuzzy filter of A and $\alpha_a \neq \emptyset$. Assume that $x \vee y \in \alpha_a$. Since $a \leq \alpha(x \vee y) = \alpha(x) \vee \alpha(y)$, it follows that $a \leq \alpha(x)$ or $a \leq \alpha(y)$. Thus, $x \in \alpha_a$ or $y \in \alpha_a$ which imply that α_a is a prime filter of A.

Conversely, $\alpha_a \neq \emptyset$ be a prime filter of A and $a = \alpha(x \vee y)$. Since $x \vee y \in \alpha_a$, it is obtained that $x \in \alpha_a$ or $y \in \alpha_a$. Hence, $a \leq \alpha(x)$ or $a \leq \alpha(y)$, and so, $\alpha(x \vee y) = a \leq \alpha(x) \vee \alpha(y)$. Since $x \leq x \vee y$ and $y \leq x \vee y$, we get from Lemma 13 (1) that $\alpha(x) \leq \alpha(x \vee y)$ and $\alpha(y) \leq \alpha(x \vee y)$. So, $\alpha(x) \vee \alpha(y) \leq \alpha(x \vee y)$. Therefore, $\alpha(x \vee y) = \alpha(x) \vee \alpha(y)$ which means that α is a prime fuzzy filter of A. □

Corollary 5. *Let A be a strong Sheffer stroke NMV-algebra. Then α is a (prime) fuzzy filter of A if and only if $\alpha_{\alpha(1)}$ is a (prime) filter of A.*

Corollary 6. *Let F be a nonempty subset of A. Then F is a (prime) filter of A if and only if the characteristic function χ_F of F is a (prime) fuzzy filter of A.*

Corollary 7. *Let F be a nonempty subset of A and α_F be a fuzzy subset of A by*

$$\alpha_F(x) = \begin{cases} a_1, & \text{if } x \in F \\ a_2, & \text{otherwise} \end{cases}$$

where $a_1, a_2 \in [0, 1]$ such that $a_1 > a_2$. Then α_F is a prime fuzzy filter of A if and only if F is a prime filter of A.

Proof. Let α_F be a prime fuzzy filter of A. It is obvious that F is a filter of A by Lemma 16. Since $\alpha_F(x) = \alpha_F(1) = a_1$ or $\alpha_F(x|1) = \alpha_F(1) = a_1$ from $(S_f - 1)$, it follows that $x \in F$ or $x|1 \in F$ which means that F is a prime filter of A by Lemma 8.

Let F be a prime filter of A. It is clear that α_F is a fuzzy filter of A by Lemma 16. Since $x \in F$ or $x|1 \in F$, for all $x \in A$, it is obtained from $(S_f - 1)$ that $\alpha_F(x) = a_1 = \alpha_F(1)$ or $\alpha_F(x|1) = a_1 = \alpha_F(1)$ which means that α_F is a prime fuzzy filter of A by Theorem 6. □

Theorem 10. *Let A be a strong Sheffer stroke NMV-algebra. Then the following conditions are equivalent:*
(1) A is totally ordered.
(2) Every fuzzy filter of A is prime.
(3) $\{1\}$ is a prime filter of A.

Proof. Let A be a strong Sheffer stroke NMV-algebra.

(1)⇒(2) Let A be totally ordered and α be a fuzzy filter of A. Then $x \leq y$ or $y \leq x$, for all $x, y \in A$. Since $x|(y|1) \approx 1$ or $y|(x|1) \approx 1$ from Lemma 1, it follows that $\alpha(x|(y|1)) = \alpha(1)$ or $\alpha(y|(x|1)) = \alpha(1)$ for all $x, y \in A$ which means that α is a prime fuzzy filter of A from Theorem 8.

(2)⇒(3) Let every fuzzy filter of A be prime. Then $\chi_{\{1\}}$ is a prime fuzzy filter of A. Thus, $\{1\}$ is a prime filter of A by Corollary 6.

(3)⇒(1) Let the filter $\{1\}$ of A be prime. Then $\chi_{\{1\}}$ is a prime fuzzy filter of A by Corollary 6. Since $\chi_{\{1\}}(x|(y|1)) \vee \chi_{\{1\}}(y|(x|1)) = \chi_{\{1\}}(1) = 1$ from Corollary 4, it follows that $\chi_{\{1\}}(x|(y|1)) = 1$ or $\chi_{\{1\}}(y|(x|1)) = 1$, for all $x, y \in A$. Thus, $x|(y|1) \approx 1$ or $y|(x|1) \approx 1$ which implies that $x \leq y$ or $y \leq x$ from Lemma 1. Hence, A is totally ordered. □

Let h be an endomorphism on A and α be a fuzzy subset of A. Define a new fuzzy subset of A by

$$\alpha_h(x) = \alpha(h(x)),$$

for all $x \in A$.

Theorem 11. *Let h be a surjective endomorphism on A. Then α is a (prime) fuzzy filter of A if and only if α_h is a (prime) fuzzy filter of A.*

Proof. (⇒) Let h be a surjective endomorphism on A and α be a fuzzy filter of A. Then $\alpha_h(x) = \alpha(h(x)) \leq \alpha(1) = \alpha(h(1)) = \alpha_h(1)$, for all $x \in A$. Also,

$$\begin{aligned}
\min\{\alpha_h(x), \alpha_h(x|(y|1))\} &= \min\{\alpha(h(x)), \alpha(h(x|(y|1)))\} \\
&= \min\{\alpha(h(x)), \alpha(h(x)|(h(y)|h(1)))\} \\
&\leq \alpha(h(y)) \\
&= \alpha_h(y),
\end{aligned}$$

for all $x, y \in A$. Thus, α_h is a fuzzy filter of A. If α is prime, then $\alpha_h(x) = \alpha(h(x)) = \alpha(1) = \alpha(h(1)) = \alpha_h(1)$ or $\alpha_h(x|1) = \alpha(h(x|1)) = \alpha(h(x)|h(1)) = \alpha(h(x)|1) = \alpha(1) = \alpha(h(1)) = \alpha_h(1)$, for all $x \in A$, for all $x \in A$ so that α_h is prime.

(⇐) Let h be a surjective endomorphism on A and α_h be a fuzzy filter of A. Then $\alpha(x) = \alpha(h(a)) = \alpha_h(a) \leq \alpha_h(1) = \alpha(h(1)) = \alpha(1)$ and

$$\begin{aligned}
\min\{\alpha(x), \alpha(x|(y|1))\} &= \min\{\alpha(h(a)), \alpha(h(a)|(h(b)|h(1)))\} \\
&= \min\{\alpha(h(a)), \alpha(h(a|(b|1)))\} \\
&= \min\{\alpha_h(a), \alpha_h(a|(b|1))\} \\
&\leq \alpha_h(b) \\
&= \alpha(h(b)) \\
&= \alpha(y)
\end{aligned}$$

where $x = h(a)$ and $y = h(b)$, for all $x, y, a, b \in A$. If α_h is prime, then $\alpha(x) = \alpha(h(a)) = \alpha_h(a) = \alpha_h(1) = \alpha(h(1)) = \alpha(1)$ or $\alpha(x|1) = \alpha(h(a)|h(1)) = \alpha(h(a|1)) = \alpha_h(a|1) = \alpha_h(1) = \alpha(h(1)) = \alpha(1)$, for all $x, a \in A$, for all $x \in A$. Hence, α is prime. □

Theorem 12. *Let h be an automorphism on A and α be a fuzzy filter of A. Then $\alpha_h = \alpha$ if and only if $h(\alpha_a) = \alpha_a$, for any $a \in Im(\alpha)$.*

Proof. Let $\alpha_h = \alpha$, $a \in Im(\alpha)$ and $x \in \alpha_a$. Then $h(x) \in h(\alpha_a)$. Since $a \leq \alpha(x) = \alpha_h(x) = \alpha(h(x))$, it follows that $h(x) \in \alpha_a$, i.e., $h(\alpha_a) \subseteq \alpha_a$. Let $x \in \alpha_a$ and $y \in A$ such that $h(y) = x$. Since $a \leq \alpha(x) = \alpha(h(y)) = \alpha_h(y) = \alpha(y)$, it is obtained that $y \in \alpha_a$. Then $x = h(y) \in h(\alpha_a)$ which implies that $\alpha_a \subseteq h(\alpha_a)$. Thus, $h(\alpha_a) = \alpha_a$, for any $a \in Im(\alpha)$.

Conversely, let $h(\alpha_a) = \alpha_a$, for any $a \in Im(\alpha)$ and $\alpha(x) = a$. By Lemma 15, $x \in \alpha_a$ and $x \notin \alpha_b$, for all $a \leq b$. Since $h(x) \in h(\alpha_a) = \alpha_a$, we have $a \leq \alpha(h(x)) = \alpha_h(x)$. Suppose that $\alpha_h(x) = b$. Then $\alpha(h(x)) = \alpha_h(x) = b$, and so, $h(x) \in \alpha_b = h(\alpha_b)$. Since h is an automorphism, we get $x \in \alpha_b$ which is a contradiction. Thus, $\alpha_h(x) = \alpha(h(x)) = a = \alpha(x)$, for all $x \in A$, i.e., $\alpha_h = \alpha$. □

Definition 10. *Let α be a fuzzy filter of A. Define the binary relation $\sim_\alpha$ on A by for all $x, y \in A$*

$$x \sim_\alpha y \ if \ and \ only \ if \ \alpha(x|(y|1)) = \alpha(1) = \alpha(y|(x|1)). \tag{2}$$

Example 4. *Consider the strong Sheffer stroke NMV-algebra A in Example 1. For a fuzzy filter α of A by*

$$\alpha(x) = \begin{cases} 0.87, & if\ x \approx d, 1 \\ 0.03, & otherwise, \end{cases}$$

$\sim_\alpha = \{(0,0), (a,a), (b,b), (c,c), (d,d), (e,e), (f,f), (1,1), (d,1), (1,d), (c,0), (0,c), (a,e), (e,a), (b,f), (f,b)\}$ *is a binary relation on A.*

Lemma 17. *Let α be a fuzzy filter of A and the binary relation $\sim_\alpha$ be defined as (2). Then $\sim_\alpha$ is a congruence relation on A.*

Proof. • Reflexive: Since $\alpha(x|(x|1)) = \alpha(1)$ from Lemma 2 (i), it follows that $x \sim_\alpha x$, for all $x \in A$.

• Let $x \sim_\alpha y$. Then $\alpha(x|(y|1)) = \alpha(1) = \alpha(y|(x|1))$. Since $\alpha(y|(x|1)) = \alpha(1) = \alpha(x|(y|1))$, we get $y \sim_\alpha x$.

• Let $x \sim_\alpha y$ and $y \sim_\alpha z$. Then $\alpha(x|(y|1)) = \alpha(1) = \alpha(y|(x|1))$ and $\alpha(y|(z|1)) = \alpha(1) = \alpha(z|(y|1))$. Since $\alpha(1) = \alpha(1) \wedge (1) = \alpha(x|(y|1)) \wedge \alpha(y|(z|1)) \leq \alpha(x|(z|1))$ and $\alpha(1) = \alpha(1) \wedge (1) = \alpha(z|(y|1)) \wedge \alpha(y|(x|1)) \leq \alpha(z|(x|1))$ from Lemma 13 (6), it is obtained that $\alpha(x|(z|1)) = \alpha(1) = \alpha(z|(x|1))$. Thus, $x \sim_\alpha z$.

Hence, $\sim_\alpha$ is an equivalence relation on A.

Let $x \sim_\alpha y$ and $z \sim_\alpha t$. Then $\alpha(x|(y|1)) = \alpha(1) = \alpha(y|(x|1))$ and $\alpha(z|(t|1)) = \alpha(1) = \alpha(t|(z|1))$.

(a) It follows from (n1), (n3) and Lemma 2 (xiii) that $x|(y|1) \approx (y|1)|((x|1)|1) \leq (z|((y|1)|1))|((z|((x|1)|1))|1) \approx (y|z)|((x|z)|1)$, and similarly, $y|(x|1) \leq (x|z)|((y|z)|1)$. Since $\alpha((x|z)|((y|z)|1)) = \alpha(1) = \alpha((y|z)|((x|z)|1))$ from Lemma 13 (1) and (FF1), it is obtained $x|z \sim_\alpha y|z$.

(b) By substituting $[x := z]$, $[y := t]$ and $[z := y]$ in (a), simultaneously, it follows from (n1) that $y|z \sim_\alpha y|t$.

Therefore, $x|z \sim_\alpha y|t$ from the transitivity of $\sim_\alpha$, and so, $\sim_\alpha$ is a congruence relation on A.

□

Theorem 13. *Let α be a fuzzy filter of A and $\sim$ be a congruence relation on A defined by α. Then $(A/\sim, |_\sim, [1]_\sim)$ is also a strong Sheffer stroke NMV-algebra where $A/\sim = \{[x]_\sim : x \in A\}$, the strong Sheffer stroke $|_\sim$ on $A/\sim$ is defined by $[x]_\sim|_\sim[y]_\sim = [x|y]_\sim$, for all $x, y \in A$. Also, a relation $\preceq$ defined by $[x]_\sim \preceq [y]_\sim \Leftrightarrow \alpha(x|(y|1)) = \alpha(1)$, for all $x, y \in A$, is a partial order on $A/\sim$ and $[1]_\sim$ is the greatest element and $[0]_\sim$ is the least element of $A/\sim$.*

Proof. Let α be a fuzzy filter of A, $\sim$ be a congruence relation on A defined by α and the binary operation $|_\sim$ be defined by $[x]_\sim|_\sim[y]_\sim = [x|y]_\sim$, for all $x, y \in A$. Since

(n1)(and (S1)): $[x]_\sim|_\sim[y]_\sim = [x|y]_\sim = [y|x]_\sim = [y]_\sim|_\sim[x]_\sim$,

(n2): $[x]_\sim|_\sim[0]_\sim = [x|0]_\sim = [1]_\sim$,

(n3): $([x]_\sim|_\sim[1]_\sim)|_\sim[1]_\sim = [(x|1)|1]_\sim = [x]_\sim$,

(n4):

$$\begin{aligned}(([x]_\sim|_\sim[1]_\sim)|_\sim[y]_\sim)|_\sim[y]_\sim &= [((x|1)|y)|y]_\sim \\ &= [((y|1)|x)|x]_\sim \\ &= (([y]_\sim|_\sim[1]_\sim)|_\sim[x]_\sim)|_\sim[x]_\sim,\end{aligned}$$

(n5): $([x]_\sim|_\sim[1]_\sim)|_\sim(([x]_\sim|_\sim[y]_\sim)|_\sim[1]_\sim) = [(x|1)|((x|y)|1)]_\sim = [1]_\sim$,

(n6):

$$\begin{aligned}&[x]_\sim|_\sim(((((([x]_\sim|_\sim[y]_\sim)|_\sim|_\sim[y]_\sim)|_\sim[z]_\sim)|_\sim[z]_\sim)|_\sim[1]_\sim) \\ &= [x|(((((x|y)|y)|z)|z)|1)]_\sim \\ &= [1]_\sim,\end{aligned}$$

(S2): $([x]_\sim|_\sim[x]_\sim)|_\sim([x]_\sim|_\sim[y]_\sim) = [(x|x)|(x|y)]_\sim = [x]_\sim$,

(S3):

$$\begin{aligned}[x]_\sim|_\sim(([y]_\sim|_\sim[z]_\sim)|_\sim([y]_\sim|_\sim[z]_\sim)) &= [x|((y|z)|(y|z))]_\sim \\ &= [((x|y)|(x|y))|z]_\sim \\ &= (([x]_\sim|_\sim[y]_\sim)|_\sim([x]_\sim|_\sim[y]_\sim))|_\sim[z]_\sim\end{aligned}$$

and

(S4):

$$\begin{aligned}&([x]_\sim|_\sim(([x]_\sim|_\sim[x]_\sim)|_\sim([y]_\sim|_\sim[y]_\sim)))|_\sim([x]_\sim|_\sim(([x]_\sim|_\sim[x]_\sim)|_\sim([y]_\sim|_\sim[y]_\sim))) \\ &= [(x|((x|x)|(y|y)))|(x|((x|x)|(y|y)))]_\sim \\ &= [x]_\sim,\end{aligned}$$

for all $x, y, z \in A$, the binary operation $|_\sim$ is a strong Sheffer stroke.

• Reflexive: $[x]_\sim \preceq [x]_\sim$ since $\alpha(x|(x|1)) = \alpha(1)$, from Lemma 2 (i).

• Antisymmetric: let $[x]_\sim \preceq [y]_\sim$ and $[y]_\sim \preceq [x]_\sim$. Since $\alpha(x|(y|1)) = \alpha(1) = \alpha(y|(x|1))$, we have $x \sim y$ which implies $[x]_\sim = [y]_\sim$.

• Transitive: let $[x]_\sim \preceq [y]_\sim$ and $[y]_\sim \preceq [z]_\sim$. Then $\alpha(x|(y|1)) = \alpha(1)$ and $\alpha(y|(z|1)) = \alpha(1)$. Since $\alpha(1) = \alpha(1) \wedge \alpha(1) = \alpha(x|(y|1)) \wedge \alpha(y|(z|1)) \leq \alpha(x|(z|1))$ from Lemma 13 (6), it follows from (FF1) that $\alpha(x|(z|1)) = \alpha(1)$, i.e., $[x]_\sim \preceq [z]_\sim$.

Thus, $\preceq$ is a partial order on $A/\sim$.

Since $\alpha(x|(1|1)) = \alpha(x|0) = \alpha(1)$ from (n2), it is ontained that $[x]_\sim \preceq [1]_\sim$, for all $x \in A$. Thus, $[1]_\sim$ is the greatest element, and so, $[0]_\sim = [1|1]_\sim = [1]_\sim|_\sim[1]_\sim$ is the least element of $A/\sim$. □

Example 5. *Consider the strong Shefeer stroke NMV-algebra A in Example 1. For a fuzzy filter α of A defined by*

$$\alpha(x) = \begin{cases} 1, & \text{if } x \approx f, 1 \\ 0.001, & \text{otherwise} \end{cases}$$

$\sim_\alpha = \{(0,0),(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(1,1),(f,1),(1,f),(a,0),(0,a),$ $(c,e),(e,c),(b,d),(d,b)\}$ *is a congruence relation on A. Then* $(A/\sim_\alpha, |_{\sim_\alpha}, [1]_\sim)$ *is also a strong Sheffer stroke NMV-algebra with the following Cayley table where* $A/\sim_\alpha = \{[0]_{\sim_\alpha}, [d]_{\sim_\alpha}, [e]_{\sim_\alpha}, [1]_{\sim_\alpha}\}$*:*

Table 2. Cayley table of $|_{\sim_\alpha}$

$\vert_{\sim_\alpha}$	$[0]_{\sim_\alpha}$	$[d]_{\sim_\alpha}$	$[e]_{\sim_\alpha}$	$[1]_{\sim_\alpha}$
$[0]_{\sim_\alpha}$	$[1]_{\sim_\alpha}$	$[1]_{\sim_\alpha}$	$[1]_{\sim_\alpha}$	$[1]_{\sim_\alpha}$
$[d]_{\sim_\alpha}$	$[1]_{\sim_\alpha}$	$[e]_{\sim_\alpha}$	$[1]_{\sim_\alpha}$	$[e]_{\sim_\alpha}$
$[e]_{\sim_\alpha}$	$[1]_{\sim_\alpha}$	$[1]_{\sim_\alpha}$	$[d]_{\sim_\alpha}$	$[d]_{\sim_\alpha}$
$[1]_{\sim_\alpha}$	$[1]_{\sim_\alpha}$	$[e]_{\sim_\alpha}$	$[d]_{\sim_\alpha}$	$[0]_{\sim_\alpha}$

Theorem 14. *Let α be a fuzzy filter of A. Then α is a prime fuzzy filter of A if and only if $A/\sim_\alpha$ is totally ordered and $|A/\sim_\alpha| \leq 2$.*

Proof. Let α be a prime fuzzy filter of A. By Theorem 8, $\alpha(x|(y|1)) = \alpha(1)$ or $\alpha(y|(x|1)) = \alpha(1)$. Then $[x]_\sim \preceq [y]_\sim$ or $[y]_\sim \preceq [x]_\sim$ which means that $A/\sim_\alpha$ is totally ordered. Also, let $|A/\sim_\alpha| > 2$. Then $[x]_{\sim_\alpha} \in A/\sim_\alpha$ such that $[0]_{\sim_\alpha} < [x]_{\sim_\alpha} < [1]_{\sim_\alpha}$. Since α is a prime fuzzy filter of A, we have $\alpha(x) = \alpha(1)$ or $\alpha(x|1) = \alpha(1)$. Assume that $\alpha(x|1) = \alpha(1)$. Since $\alpha(x|(0|1)) = \alpha(x|1) = \alpha(1)$ and $\alpha(0|(x|1)) = \alpha(1)$ from (n2), it follows that $[x]_{\sim_\alpha} = [0]_{\sim_\alpha}$ which is a contradiction. So, $|A/\sim_\alpha| \leq 2$.

Conversely, let $A/\sim_\alpha$ be totally ordered. Then $[x]_\sim \preceq [y]_\sim$ or $[y]_\sim \preceq [x]_\sim$, for all $x, y \in A$. Since $\alpha(x|(y|1)) = \alpha(1)$ or $\alpha(y|(x|1)) = \alpha(1)$, it is obtained from Theorem 8 that α is a prime fuzzy filter of A. □

Theorem 15. *Let $(A, |_A, 1_A)$ and $(B, |_B, 1_B)$ be strong Sheffer stroke NMV-algebras, $h : A \longrightarrow B$ be an epimorphism and α be a fuzzy filter of B. Then $\alpha \circ h$ is a fuzzy filter of A and $A/\sim_{\alpha\circ h} \cong B/\sim_\alpha$.*

Proof. Let $(A, |_A, 1_A)$ and $(B, |_B, 1_B)$ be strong Sheffer stroke NMV-algebras, $h : A \longrightarrow B$ be an epimorphism and α be a fuzzy filter of B. It is first shown that $\alpha \circ h$ is a fuzzy filter of A.

- $\alpha \circ h(x) = \alpha(h(x)) \leq \alpha(1_B) = \alpha(h(1_A)) = \alpha \circ h(1_A)$ and
-

$$\begin{aligned}\min\{\alpha \circ h(x), \alpha \circ h(x|_A(y|_A 1_A))\} &= \min\{\alpha(h(x)), \alpha(h(x|_A(y|_A 1_A)))\} \\ &= \min\{\alpha(h(x)), \alpha(h(x)|_B(h(y)|_B h(1_A)))\} \\ &= \min\{\alpha(h(x)), \alpha(h(x)|_B(h(y)|_B 1_B))\} \\ &\leq \alpha(h(y)) \\ &= \alpha \circ h(y),\end{aligned}$$

for all $x, y \in A$.

$A/\sim_{\alpha\circ h}$ and $B/\sim_\alpha$ are strong Sheffer stroke NMV-algebras by Theorem 13. Let $f : A/\sim_{\alpha\circ h} \longrightarrow B/\sim_\alpha$ be defined by $f([x]_{\sim_{\alpha\circ h}}) = [h(x)]_{\sim_\alpha}$, for all $x \in A$.

- f is well-defined and one-to-one: Let $[x]_{\sim_{\alpha\circ h}}, [y]_{\sim_{\alpha\circ h}} \in A/\sim_{\alpha\circ h}$. Then

$$\begin{aligned}[x]_{\sim_{\alpha\circ h}} = [y]_{\sim_{\alpha\circ h}} &\Leftrightarrow x \sim_{\alpha\circ h} y \\ &\Leftrightarrow \alpha \circ h(x|_A(y|_A 1_A)) = \alpha \circ h(1_A) = \alpha \circ h(y|_A(x|_A 1_A)) \\ &\Leftrightarrow \alpha(h(x)|_B(h(y)|_b h(1_A))) = \alpha(h(1_A)) \\ &\quad\; = \alpha(h(y)|_B(h(x)|_b h(1_A))) \\ &\Leftrightarrow \alpha(h(x)|_B(h(y)|_b 1_B)) = \alpha(1_B) = \alpha(h(y)|_B(h(x)|_b 1_B)) \\ &\Leftrightarrow h(x) \sim_\alpha h(y) \\ &\Leftrightarrow [h(x)]_{\sim_\alpha} = [h(y)]_{\sim_\alpha} \\ &\Leftrightarrow f([x]_{\sim_{\alpha\circ h}}) = f([y]_{\sim_{\alpha\circ h}}).\end{aligned}$$

- f is a homomorphism: Let $[x]_{\sim_{\alpha\circ h}}, [y]_{\sim_{\alpha\circ h}} \in A/\sim_{\alpha\circ h}$. Then

$$\begin{aligned}f([x]_{\sim_{\alpha\circ h}}|_{\sim_{\alpha\circ h}}[y]_{\sim_{\alpha\circ h}}) &= f([x|_A y]_{\sim_{\alpha\circ h}}) \\ &= [h(x|_A y)]_{\sim_\alpha} \\ &= [h(x)|_B h(y)]_{\sim_\alpha} \\ &= [h(x)]_{\sim_\alpha}|_{\sim_\alpha}[h(y)]_{\sim_\alpha} \\ &= f([x]_{\sim_{\alpha\circ h}})|_{\sim_\alpha} f([y]_{\sim_{\alpha\circ h}}).\end{aligned}$$

- f is onto: Let $[y]_{\sim_\alpha} \in B/\sim_\alpha$. Since h is an epimorphism, there exists $x \in A$ such that $h(x) = y$. Thus, there exists $[x]_{\sim_{\alpha\circ h}} \in A/\sim_{\alpha\circ h}$ such that $f([x]_{\sim_{\alpha\circ h}}) = [h(x)]_{\sim_\alpha} = [y]_{\sim_\alpha}$. □

Theorem 16. *The class $\mathcal{F}_A$ of all fuzzy filters of A forms a complete lattice.*

Proof. Since every fuzzy filter of A is a mapping from A to the interval $[0, 1]$ and $[0, 1]$ is a complete lattice where $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$, for all $a, b \in [0, 1]$, $\mathcal{F}_A$ forms a complete lattice. □

5. Conclusion

In present study, basic definitions and notions of a strong Sheffer stroke NMV-algebra are given. Then new properties, various filters, fuzzy filters of a strong Sheffer stroke NMV-algebra and the relationships between them are investigated. We prove that a filter of a strong Sheffer stroke NMV-algebra is prime if and only if it is not contained by another filter of this algebraic structure, and examine some features of a prime filter. Also, it is shown that the quotient structure of a strong Sheffer stroke NMV-algebra defined by a prime filter is totally ordered and it has at most 2 elements. Besides, we define a (prime) fuzzy filter of strong Sheffer stroke NMV-algebras and show that α is a (prime) fuzzy filter of a strong Sheffer stroke NMV-algebra if and only if $\alpha_a = \{x \in A : a \leq \alpha(x)\}$ is empty or a (prime) filter of A, for all $a \in [0, 1]$. It is demonstrated that a fuzzy subset α_F is a (prime) fuzzy filter of a strong Sheffer stroke NMV-algebra if and only if F is a (prime) filter of the algebra. Thus, the relationships between filters and fuzzy filters of a strong Sheffer stroke NMV-algebra are stated. We prove that a strong Sheffer stroke NMV-algebra is totally ordered if and only if every fuzzy filter is prime if and only if the filter $\{1\}$ is prime. It is shown that a fuzzy subset α_h of a strong Sheffer stroke NMV-algebra is a (prime) fuzzy filter defined by means of a (prime) fuzzy filter α and a surjective endomomorphism h on this algebra, and that $\alpha_h = \alpha$ if and only if $h(\alpha_a) = \alpha_a$ whenever h is an automorphism on this algebra and $a \in Im(\alpha)$. By describing a congruence relation on a strong Sheffer stroke NMV-algebra by a fuzzy filter, a quotient structure of a strong Sheffer stroke NMV-algebra is built via the congruence relation. Hence, it is shown that the structure forms a strong Sheffer stroke NMV-algebra. Indeed, we prove that the quotient structure defined by a prime fuzzy filter is totally ordered strong Sheffer stroke NMV-algebra and it has at most 2 elements. Moreover, we present that $\alpha \circ h$ is a fuzzy filter of A and the quotient structures defined by the fuzzy filters $\alpha \circ h$ and α are isomorphic when an epimorphism h between strong Sheffer stroke NMV-algebras A and B and a fuzzy filter α of B. Finally, it is easy to see that the class of all fuzzy filters of a strong Sheffer stroke NMV-algebra forms a complete lattice.

In the future works, we wish to investigate annihilators ans stabilizers on strong Sheffer stroke NMV-algebras.

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