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On Third Order Hyperbolic Jacobsthal Numbers

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Abstract

In this paper, we introduce the hyperbolic third order Jacobsthal and Jacobsthal-Lucas numbers and we present recurrence relations, Binet's formulas, generating functions and the summation formulas for these numbers.

Keywords: *Third order Jacobsthal numbers, third order Jacobsthal-Lucas numbers, third order hyperbolic Jacobsthal numbers, third order hyperbolic Jacobsthal-Lucas numbers.*
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1. Introduction and Preliminaries

Hyperbolic numbers have been studied by various authors recently. First, Torunbalcı defined it for Fibonacci numbers and Dikmen defined it for Jacobsthal numbers. Soykan defined it for generalized Fibonacci numbers and Pell numbers. Then Taşyurdu defined it for Tribonacci and Tribonacci-Lucas Sequences (see [5-9]). In this paper, we define hyperbolic third order Jacobsthal and Jacobsthal-Lucas numbers in the next section and give some properties of them. First, in this section, we present some background about hyperbolic numbers (see [2-3]) and third order Jacobsthal and third order Jacobsthal-Lucas numbers (see [10-12]).

Third order Jacobsthal sequence $\{J_n^{(3)}\}_{n\geq 0}$ and and third order Jacobsthal-Lucas numbers $\{j_n^{(3)}\}_{n\geq 0}$ are defined by the third-order recurrence relations

$$J_n^{(3)} = J_{n-1}^{(3)} + J_{n-2}^{(3)} + 2J_{n-3}^{(3)} \tag{1.1}$$

with the initial values $J_0^{(3)} = 0$, $J_1^{(3)} = 1$, $J_2^{(3)} = 1$ and

$$j_n^{(3)} = j_{n-1}^{(3)} + j_{n-2}^{(3)} + 2j_{n-3}^{(3)}$$

with the initial values $j_0^{(3)} = 2$, $j_1^{(3)} = 1$, $j_2^{(3)} = 5$. Some of the third order Jacobsthal and third order Jacobsthal-Lucas numbers are

$$0, 1, 1, 2, 5, 9, 18, 37, 73, 146, 293, \ldots$$

and

$$2, 1, 5, 10, 17, 37, 74, 145, 293, 586, 1169, \ldots$$

respectively (see [1]).

Third order Jacobsthal and third order Jacobsthal-Lucas numbers for negative subscripts can be given as follows

$$J_{-n}^{(3)} = \frac{1}{2}\left(J_{3-n}^{(3)} - J_{2-n}^{(3)} - J_{1-n}^{(3)}\right), \tag{1.2}$$

$$j_{-n}^{(3)} = \frac{1}{2}\left(j_{3-n}^{(3)} - j_{2-n}^{(3)} - j_{1-n}^{(3)}\right) \tag{1.3}$$

for $n \geq 1$. Therefore, some of the third order Jacobsthal and third order Jacobsthal-Lucas numbers for negative subscripts are

$$\ldots, \frac{-9}{64}, \frac{-9}{32}, \frac{7}{16}, \frac{-1}{8}, \frac{-1}{4}, \frac{1}{2}, 0$$

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and

$$..., \frac{-41}{32}, \frac{7}{16}, \frac{7}{8}, \frac{-5}{4}, \frac{1}{2}, 1, -1, 1$$

respectively.
Using standard techniques for solving recurrence relations, the auxiliary equation, and its roots are given by

$$x^3 - x^2 - x - 2 = 0;\ x = 2 \text{ and } x = \frac{-1 \mp i\sqrt{3}}{2}.$$

Since the last two complex numbers are the cube root of one we can show them by $\omega_1 = \frac{-1+i\sqrt{3}}{2}$ and $\omega_2 = \frac{-1-i\sqrt{3}}{2}$. Then, Binet's formula for the third order Jacobsthal and third order Jacobsthal-Lucas numbers can be given by

$$J_n^{(3)} = \frac{2}{7}2^n + \left(\frac{-3-2i\sqrt{3}}{21}\right)\omega_1^n + \left(\frac{-3+2i\sqrt{3}}{21}\right)\omega_2^n$$

and

$$j_n^{(3)} = \frac{8}{7}2^n + \left(\frac{3+2i\sqrt{3}}{7}\right)\omega_1^n + \left(\frac{3-2i\sqrt{3}}{7}\right)\omega_2^n$$

respectively (see [12]).
Generating functions of third order Jacobsthal and Jacobsthal-Lucas numbers are

$$\sum_{n=0}^{\infty} J_n^{(3)} x^n = \frac{x}{1-x-x^2-2x^3},$$
$$\sum_{n=0}^{\infty} j_n^{(3)} x^n = \frac{2-x+2x^2}{1-x-x^2-2x^3}$$

respectively (see [12]).
Some additional identities for third order Jacobsthal and Jacobsthal-Lucas numbers was given by Cook and Bacon in [12] .These are:
Summation formula:

$$\sum_{k=0}^{n} J_k^{(3)} = \begin{cases} J_{n+1}^{(3)}, & \text{if } n \neq 0\,(mod 3) \\ J_{n+1}^{(3)} - 1, & \text{if } n = 0\,(mod 3) \end{cases}, \tag{1.4}$$

$$\sum_{k=0}^{n} j_k^{(3)} = \begin{cases} j_{n+1}^{(3)} - 2, & \text{if } n \neq 0\,(mod 3) \\ j_{n+1}^{(3)} + 1, & \text{if } n = 0\,(mod 3) \end{cases}, \tag{1.5}$$

and

$$3J_n^{(3)} + j_n^{(3)} = 2^{n+1}, \tag{1.6}$$

$$j_n^{(3)} - 3J_n^{(3)} = 2j_{n-3}^{(3)}, \tag{1.7}$$

$$J_{n+2}^{(3)} - 4J_n^{(3)} = \begin{cases} -2, & \text{if } n = 1\,(mod 3) \\ 1, & \text{if } n \neq 1\,(mod 3) \end{cases}, \tag{1.8}$$

$$j_n^{(3)} - 4J_n^{(3)} = \begin{cases} 2, & \text{if } n = 0\,(mod 3) \\ -3, & \text{if } n = 1\,(mod 3) \\ 1, & \text{if } n = 2\,(mod 3) \end{cases}, \tag{1.9}$$

$$j_{n+1}^{(3)} + j_n^{(3)} = 3J_{n+2}^{(3)}, \tag{1.10}$$

$$j_n^{(3)} - J_{n+2}^{(3)} = \begin{cases} 1, & \text{if } n = 0\,(mod 3) \\ -1, & \text{if } n = 1\,(mod 3) \\ 0, & \text{if } n = 2\,(mod 3) \end{cases}, \tag{1.11}$$

$$\left(j_n^{(3)}\right)^2 - 9\left(J_n^{(3)}\right)^2 = 2^{n+2} j_{n-3}^{(3)}. \tag{1.12}$$

2. Hyperbolic Numbers

The set of hyperbolic numbers $\mathbb{H}$ can be defined by

$$\mathbb{H} = \{z = x + hy \mid h \notin \mathbb{R},\ h^2 = 1, x, y \in \mathbb{R}\}.$$

Addition, substraction and multiplication of any two hyperbolic numbers z_1 and z_2 are defined by

$$z_1 \pm z_2 = (x_1 + hy_1) \pm (x_2 + hy_2) = (x_1 \pm x_2) + h(y_1 \pm y_2),$$
$$z_1 \times z_2 = (x_1 + hy_1) \times (x_2 + hy_2) = x_1x_2 + y_1y_2 + h(x_1y_2 + y_1x_2).$$

and the division of two hyperbolic numbers are given by

$$\frac{z_1}{z_2}=\frac{x_1+hy_1}{x_2+hy_2}=\frac{(x_1+hy_1)(x_2-hy_2)}{(x_2+hy_2)(x_2-hy_2)}=\frac{x_1x_2+y_1y_2}{x_2^2-y_2^2}+h\frac{x_1y_2+y_1x_2}{x_2^2-y_2^2}.$$

The hyperbolic conjugation of $z=x+hy$ is defined by

$$\bar{z}=z^{\dagger}=x-hy.$$

Note that $\bar{\bar{z}}=z$. Note also that for any hyperbolic numbers z_1, z_2, z we have

$$\begin{aligned}\overline{z_1+z_2} &= \overline{z_1}+\overline{z_2},\\ \overline{z_1\times z_2} &= \overline{z_1}\times\overline{z_2},\\ \|z\|^2 &= z\times\bar{z}=x^2-y^2.\end{aligned}$$

For more information on hyperbolic numbers, see for example, [2] and [3].

3. Hyperbolic Third Order Jacobsthal and Jacobsthal-Lucas Sequence

In [5], author defined hyperbolic Fibonacci sequence and investigated its properties. In [8], author defined hyperbolic Jacobsthal sequence and investigated its properties.Then Taşyurdu, Soykan and Göcen defined hyperbolic numbers for Tribonacci, Tribonacci-Lucas, third order Pell, Pell-Lucas sequences (see [7,9]. Gerda-Morales defined dual third order Jacobsthal and Jacobsthal-Lucas sequence and investigated its properties (see [11]). In this work we define hyperbolic third order Jacobsthal and Jacobsthal-Lucas sequence and investigated its properties. The nth hyperbolic third order Jacobsthal number $\widetilde{J}_n^{(3)}$ and the nth hyperbolic third order Jacobsthal-Lucas numbers $\widetilde{j}_n^{(3)}$ are defined by

$$\widetilde{J}_n^{(3)} = J_n^{(3)}+hJ_{n+1}^{(3)} \tag{3.1}$$

$$\widetilde{j}_n^{(3)} = j_n^{(3)}+hj_{n+1}^{(3)} \tag{3.2}$$

with initial conditions $\widetilde{J}_0^{(3)}=h$, $\widetilde{J}_1^{(3)}=1+h$ and $\widetilde{j}_0^{(3)}=2+h$, $\widetilde{j}_1^{(3)}=1+5h$ where $h^2=1$ $(h\notin\mathbb{R})$. Then some of the hyperbolic third order Jacobsthal numbers and the hyperbolic third order Jacobsthal-Lucas numbers are

$$h,1+h,1+2h,2+5h,5+9h,9+18h,18+37h,37+73h,...$$
$$2+h,1+5h,5+10h,10+17h,17+37h,37+74h,74+145h,...$$

respectively. It can be easily shown that

$$\widetilde{J}_n^{(3)}=\widetilde{J}_{n-1}^{(3)}+\widetilde{J}_{n-2}^{(3)}+2\widetilde{J}_{n-3}^{(3)}. \tag{3.3}$$

To see this, we have

$$\begin{aligned}\widetilde{J}_n^{(3)} &= J_n^{(3)}+hJ_{n+1}^{(3)}=J_{n-1}^{(3)}+J_{n-2}^{(3)}+2J_{n-3}^{(3)}+h(J_n^{(3)}+J_{n-1}^{(3)}+2J_{n-2}^{(3)})\\ &= (J_{n-1}^{(3)}+hJ_n^{(3)})+(J_{n-2}^{(3)}+hJ_{n-1}^{(3)})+2(J_{n-3}^{(3)}+hJ_{n-2}^{(3)})=\widetilde{J}_{n-1}^{(3)}+\widetilde{J}_{n-2}^{(3)}+2\widetilde{J}_{n-3}^{(3)}.\end{aligned}$$

Similarly it can be shown that

$$\widetilde{j}_n^{(3)}=\widetilde{j}_{n-1}^{(3)}+\widetilde{j}_{n-2}^{(3)}+2\widetilde{j}_{n-3}^{(3)}. \tag{3.4}$$

Now, we start to give some results starting with the following theorem.

Theorem 3.1. *If $\widetilde{J}_n^{(3)}$ is a hyperbolic third order Jacobsthal number, then*

$$\lim_{n\to\infty}\frac{\widetilde{J}_{n+1}^{(3)}}{\widetilde{J}_n^{(3)}}=2.$$

Proof. For the third order Jacobsthal sequence $J_n^{(3)}$ we have

$$\lim_{n\to\infty}\frac{J_{n+1}^{(3)}}{J_n^{(3)}}=2.$$

Then using this limit value for the hyperbolic third order Jacobsthal numbers $\widetilde{J}_n^{(3)}$, we obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{\widetilde{J}_{n+1}^{(3)}}{\widetilde{J}_n^{(3)}} &= \lim_{n\to\infty}\frac{J_{n+1}^{(3)}+hJ_{n+2}^{(3)}}{J_n^{(3)}+hJ_{n+1}^{(3)}}\\ &= \lim_{n\to\infty}\frac{J_{n+1}^{(3)}+h(J_{n+1}^{(3)}+J_n^{(3)}+2J_{n-1}^{(3)})}{J_n^{(3)}+hJ_{n+1}^{(3)}}\\ &= \lim_{n\to\infty}\frac{\frac{J_{n+1}^{(3)}}{J_n^{(3)}}+h\left(\frac{J_{n+1}^{(3)}}{J_n^{(3)}}+1+2\frac{J_{n-1}^{(3)}}{J_n^{(3)}}\right)}{1+h\frac{J_{n+1}^{(3)}}{J_n^{(3)}}}\\ &= \frac{2+h(2+1+1)}{1+2h}=\frac{2+4h}{1+2h}=2.\end{aligned}$$

□

Next, we present Binet formula.

Theorem 3.2. *The Binet formula for the hyperbolic third order Jacobsthal sequence and third order Jacobsthal-Lucas sequence are given as*

$$\widehat{J}_n^{(3)} = \frac{2}{7}2^n(1+2h) + \left(\frac{-3-2i\sqrt{3}}{21}\right)\omega_1^n(1+\omega_1 h) + \left(\frac{-3+2i\sqrt{3}}{21}\right)\omega_2^n(1+\omega_2 h) \tag{3.5}$$

and

$$\widehat{j}_n^{(3)} = \frac{8}{7}2^n(1+2h) + \left(\frac{3+2i\sqrt{3}}{7}\right)\omega_1^n(1+\omega_1 h) + \left(\frac{3-2i\sqrt{3}}{7}\right)\omega_2^n(1+\omega_2 h) \tag{3.6}$$

where ω_1 and ω_2 are cube roots of one as before.

Proof. Using Binet formula

$$J_n^{(3)} = \frac{2}{7}2^n + \left(\frac{-3-2i\sqrt{3}}{21}\right)\omega_1^n + \left(\frac{-3+2i\sqrt{3}}{21}\right)\omega_2^n$$

and

$$j_n^{(3)} = \frac{8}{7}2^n + \left(\frac{3+2i\sqrt{3}}{7}\right)\omega_1^n + \left(\frac{3-2i\sqrt{3}}{7}\right)\omega_2^n$$

we see that

$$\begin{aligned}
\widehat{J}_n^{(3)} &= J_n^{(3)} + hJ_{n+1}^{(3)} \\
&= \frac{2}{7}2^n + \left(\frac{-3-2i\sqrt{3}}{21}\right)\omega_1^n + \left(\frac{-3+2i\sqrt{3}}{21}\right)\omega_2^n + \\
&\quad + h\left(\frac{2}{7}2^{n+1} + \left(\frac{-3-2i\sqrt{3}}{21}\right)\omega_1^{n+1} + \left(\frac{-3+2i\sqrt{3}}{21}\right)\omega_2^{n+1}\right) \\
&= \frac{2}{7}2^n(1+2h) + \left(\frac{-3-2i\sqrt{3}}{21}\right)\omega_1^n(1+\omega_1 h) + \left(\frac{-3+2i\sqrt{3}}{21}\right)\omega_2^n(1+\omega_2 h).
\end{aligned}$$

Similarly it can be shown that

$$\begin{aligned}
\widehat{j}_n^{(3)} &= j_n^{(3)} + hj_{n+1}^{(3)}. \\
&= \frac{8}{7}2^n + \left(\frac{3+2i\sqrt{3}}{7}\right)\omega_1^n + \left(\frac{3-2i\sqrt{3}}{7}\right)\omega_2^n + \\
&\quad + h\left(\frac{8}{7}2^{n+1} + \left(\frac{3+2i\sqrt{3}}{7}\right)\omega_1^{n+1} + \left(\frac{3-2i\sqrt{3}}{7}\right)\omega_2^{n+1}\right) \\
&= \frac{8}{7}2^n(1+2h) + \left(\frac{3+2i\sqrt{3}}{7}\right)\omega_1^n(1+\omega_1 h) + \left(\frac{3-2i\sqrt{3}}{7}\right)\omega_2^n(1+\omega_2 h).
\end{aligned}$$

□

It is useful to let Binet formula for the hyperbolic third order Jacobsthal sequence as follows.

Corollary 3.3. *Binet formula for the hyperbolic third order Jacobsthal sequence and for the hyperbolic third order Jacobsthal-Lucas sequence can be given as*

$$\widehat{J}_n^{(3)} = \begin{cases} \frac{1}{7}\left(2^{3k+1} - 2 + h\left(2^{3k+2}+3\right)\right) & , \text{ if } n = 3k \\ \frac{1}{7}\left(2^{3k+2} + 3 + h\left(2^{3k+3}-1\right)\right) & , \text{ if } n = 3k+1 \\ \frac{1}{7}\left(2^{3k+3} - 1 + h\left(2^{3k+4}-2\right)\right) & , \text{ if } n = 3k+2 \end{cases}, \tag{3.7}$$

$$\widehat{j}_n^{(3)} = \begin{cases} \frac{1}{7}\left(2^{3k+3} + 6 + h\left(2^{3k+4}-9\right)\right) & , \text{ if } n = 3k \\ \frac{1}{7}\left(2^{3k+4} - 9 + h\left(2^{3k+5}+3\right)\right) & , \text{ if } n = 3k+1 \\ \frac{1}{7}\left(2^{3k+5} + 3 + h\left(2^{3k+6}+6\right)\right) & , \text{ if } n = 3k+2 \end{cases} \tag{3.8}$$

respectively.

Proof. Using Binet formula for the hyperbolic third order Jacobsthal sequence (3.5), we see that, if $n=3k$, then $\omega_1^n=1$ and $\omega_2^n=1$

$$\begin{aligned}
\widetilde{J}_n^{(3)} &= \frac{2}{7}2^n(1+2h)+\left(\frac{-3-2i\sqrt{3}}{21}\right)(1+\omega_1 h)+\left(\frac{-3+2i\sqrt{3}}{21}\right)(1+\omega_2 h) \\
&= \left(\frac{2}{7}2^{3k}+\frac{-3-2i\sqrt{3}}{21}+\frac{-3+2i\sqrt{3}}{21}\right)+h\left(\frac{2^{3k+2}}{7}+\frac{-3-2i\sqrt{3}}{21}\omega_1+\frac{-3+2i\sqrt{3}}{21}\omega_2\right) \\
&= \frac{1}{7}\left(2^{3k+1}-2+h\left(2^{3k+2}+3\right)\right),
\end{aligned}$$

if $n=3k+1$, then $\omega_1^n=\omega_1=\frac{-1+i\sqrt{3}}{2}$ and $\omega_2^n=\omega_2=\frac{-1-i\sqrt{3}}{2}$

$$\begin{aligned}
\widetilde{J}_n^{(3)} &= \frac{2}{7}2^n(1+2h)+\left(\frac{-3-2i\sqrt{3}}{21}\right)\omega_1(1+\omega_1 h)+\left(\frac{-3+2i\sqrt{3}}{21}\right)\omega_2(1+\omega_2 h) \\
&= \left(\frac{2}{7}2^{3k+1}+\frac{-3-2i\sqrt{3}}{21}\omega_1+\frac{-3+2i\sqrt{3}}{21}\omega_2\right)+h\left(\frac{2^{3k+3}}{7}+\frac{-3-2i\sqrt{3}}{21}\omega_1^2+\frac{-3+2i\sqrt{3}}{21}\omega_2^2\right) \\
&= \frac{1}{7}\left(2^{3k+2}+3+h\left(2^{3k+3}-1\right)\right),
\end{aligned}$$

if $n=3k+2$, then $\omega_1^n=\omega_1^2=\frac{-1-i\sqrt{3}}{2}$ and $\omega_2^n=\omega_2^2=\frac{-1+i\sqrt{3}}{2}$

$$\begin{aligned}
\widetilde{J}_n^{(3)} &= \frac{2}{7}2^n(1+2h)+\left(\frac{-3-2i\sqrt{3}}{21}\right)\omega_1^2(1+\omega_1 h)+\left(\frac{-3+2i\sqrt{3}}{21}\right)\omega_2^2(1+\omega_2 h) \\
&= \left(\frac{2}{7}2^{3k+2}+\frac{-3-2i\sqrt{3}}{21}\omega_1^2+\frac{-3+2i\sqrt{3}}{21}\omega_2^2\right)+h\left(\frac{2^{3k+4}}{7}+\frac{-3-2i\sqrt{3}}{21}+\frac{-3+2i\sqrt{3}}{21}\right) \\
&= \frac{1}{7}\left(2^{3k+3}-1+h\left(2^{3k+4}-2\right)\right).
\end{aligned}$$

Binet formula for the hyperbolic third order Jacobsthal-Lucas sequence can be proven similarly. □

Next, we present the generating function for the hyperbolic third order Jacobsthal and Jacobsthal-Lucas numbers. For the generating function for third order Jacobsthal and Jacobsthal-Lucas numbers see [12].

Theorem 3.4. *The generating function for the hyperbolic third order Jacobsthal numbers and for the hyperbolic third order Jacobsthal-Lucas numbers are*

$$\sum_{n=0}^{\infty}\widetilde{J}_n^{(3)}x^n=\frac{h+x}{1-x-x^2-2x^3} \tag{3.9}$$

and

$$\sum_{n=0}^{\infty}\widetilde{j}_n^{(3)}x^n=\frac{2+h+(-1+4h)x+(2+4h)x^2}{1-x-x^2-2x^3}$$

respectively.

Proof. Let

$$g(x)=\sum_{n=0}^{\infty}\widetilde{J}_n^{(3)}x^n$$

be generating function of hyperbolic third order Jacobsthal numbers. Then

$$\begin{aligned}
(1-x-x^2-2x^3)g(x) &= \sum_{n=0}^{\infty}\widetilde{J}_n^{(3)}x^n-x\sum_{n=0}^{\infty}\widetilde{J}_n^{(3)}x^n-x^2\sum_{n=0}^{\infty}\widetilde{J}_n^{(3)}x^n-2x^3\sum_{n=0}^{\infty}\widetilde{J}_n^{(3)}x^n \\
&= \sum_{n=0}^{\infty}\widetilde{J}_n^{(3)}x^n-\sum_{n=0}^{\infty}\widetilde{J}_n^{(3)}x^{n+1}-\sum_{n=0}^{\infty}\widetilde{J}_n^{(3)}x^{n+2}-2\sum_{n=0}^{\infty}\widetilde{J}_n^{(3)}x^{n+3} \\
&= \sum_{n=0}^{\infty}\widetilde{J}_n^{(3)}x^n-\sum_{n=1}^{\infty}\widetilde{J}_{n-1}^{(3)}x^n-\sum_{n=2}^{\infty}\widetilde{J}_{n-2}^{(3)}x^n-2\sum_{n=3}^{\infty}\widetilde{J}_{n-3}^{(3)}x^n \\
&= \left(\widetilde{J}_0^{(3)}+\widetilde{J}_1^{(3)}x+\widetilde{J}_2^{(3)}x^2\right)-\widetilde{J}_0^{(3)}x-\widetilde{J}_1^{(3)}x^2-\widetilde{J}_0^{(3)}x^2 \\
&\quad+\sum_{n=3}^{\infty}\left(\widetilde{J}_n^{(3)}-\widetilde{J}_{n-1}^{(3)}-\widetilde{J}_{n-2}^{(3)}-2\widetilde{J}_{n-3}^{(3)}\right)x^n \\
&= \widetilde{J}_0^{(3)}+\left(\widetilde{J}_1^{(3)}-\widetilde{J}_0^{(3)}\right)x-\left(\widetilde{J}_0^{(3)}+\widetilde{J}_1^{(3)}-\widetilde{J}_2^{(3)}\right)x^2.
\end{aligned}$$

Rearranging the above equation, we obtain

$$g(x)=\frac{\widetilde{J}_0^{(3)}+\left(\widetilde{J}_1^{(3)}-\widetilde{J}_0^{(3)}\right)x-\left(\widetilde{J}_0^{(3)}+\widetilde{J}_1^{(3)}-\widetilde{J}_2^{(3)}\right)x^2}{1-x-x^2-2x^3}=\frac{h+x}{1-x-x^2-2x^3}$$

since $\widetilde{J}_0^{(3)}=h$, $\widetilde{J}_1^{(3)}=1+h$ and $\widetilde{J}_2^{(3)}=1+2h$.
Similarly, let

$$f(x)=\sum_{n=0}^{\infty}\widetilde{j}_n^{(3)}x^n$$

be generating function of hyperbolic third order Jacobsthal-Lucas numbers. Then as above

$$f(x)=\frac{\widetilde{j}_0^{(3)}+\left(\widetilde{j}_1^{(3)}-\widetilde{j}_0^{(3)}\right)x-\left(\widetilde{j}_0^{(3)}+\widetilde{j}_1^{(3)}-\widetilde{j}_2^{(3)}\right)x^2}{1-x-x^2-2x^3}=\frac{2+h+(-1+4h)x+(2+4h)x^2}{1-x-x^2-2x^3}$$

since $\widetilde{j}_0^{(3)}=2+h$, $\widetilde{j}_1^{(3)}=1+5h$ and $\widetilde{j}_2^{(3)}=5+10h$. □

Next, we give linear sum identitity of hyperbolic third order Jacobsthal and Jacobsthal-Lucas numbers.

Theorem 3.5. *For $n\geq 0$, we have the following summation formulas:*

$$\sum_{k=0}^{n}\widetilde{J}_k^{(3)} = \frac{1}{3}\left(\widetilde{J}_{n+2}^{(3)}+2\widetilde{J}_n^{(3)}-\widetilde{J}_2^{(3)}+\widetilde{J}_0^{(3)}\right), \tag{3.10}$$

$$\sum_{k=0}^{n}\widetilde{j}_k^{(3)} = \frac{1}{3}\left(\widetilde{j}_{n+2}^{(3)}+2\widetilde{j}_n^{(3)}-\widetilde{j}_2^{(3)}+\widetilde{j}_0^{(3)}\right). \tag{3.11}$$

Proof. First proof follows from the summing formula

$$\sum_{k=0}^{n}J_k^{(3)}=\frac{1}{3}\left(J_{n+2}^{(3)}+2J_n^{(3)}-1\right).$$

$$\begin{aligned}
\sum_{k=0}^{n}\widetilde{J}_k^{(3)} &= \sum_{k=0}^{n}\left(J_k^{(3)}+hJ_{k+1}^{(3)}\right)\\
&= \sum_{k=0}^{n}J_k^{(3)}+h\sum_{k=0}^{n}J_{k+1}^{(3)}\\
&= \frac{1}{3}\left(J_{n+2}^{(3)}+2J_n^{(3)}-1\right)+\frac{h}{3}\left(J_{n+2}^{(3)}+2J_n^{(3)}+3J_{n+1}^{(3)}-1\right)\\
&= \frac{1}{3}\left(J_{n+2}^{(3)}+2\left(J_n^{(3)}+hJ_{n+1}^{(3)}\right)-1+h\left(J_{n+2}^{(3)}+J_{n+1}^{(3)}+2J_n^{(3)}-1\right)\right)\\
&= \frac{1}{3}\left(J_{n+2}^{(3)}+2\widetilde{J}_n^{(3)}-1+h\left(J_{n+3}^{(3)}-1\right)\right)\\
&= \frac{1}{3}\left(J_{n+2}^{(3)}+hJ_{n+3}^{(3)}+2\widetilde{J}_n^{(3)}-(1+h)\right)\\
&= \frac{1}{3}\left(\widetilde{J}_{n+2}^{(3)}+2\widetilde{J}_n^{(3)}-(1+h)\right).
\end{aligned}$$

Therefore,

$$\sum_{k=0}^{n}\widetilde{J}_k^{(3)}=\frac{1}{3}(\widetilde{J}_{n+2}^{(3)}+2\widetilde{J}_n^{(3)}-\widetilde{J}_2^{(3)}+\widetilde{J}_0^{(3)}).$$

Similarly, second proof follows from the following formula:

$$\sum_{k=0}^{n}j_k^{(3)}=\frac{1}{3}\left(j_{n+2}^{(3)}+2j_n^{(3)}-3\right).$$

$$\begin{aligned}
\sum_{k=0}^{n}\widetilde{j}_k^{(3)} &= \sum_{k=0}^{n}\left(j_k^{(3)}+hj_{k+1}^{(3)}\right)\\
&= \sum_{k=0}^{n}j_k^{(3)}+h\sum_{k=0}^{n}j_{k+1}^{(3)}\\
&= \frac{1}{3}\left(j_{n+2}^{(3)}+2j_n^{(3)}-3\right)+\frac{h}{3}\left(j_{n+2}^{(3)}+2j_n^{(3)}+3j_{n+1}^{(3)}-3j_0^{(3)}-3\right)\\
&= \frac{1}{3}\left(j_{n+2}^{(3)}+2\left(j_n^{(3)}+hj_{n+1}^{(3)}\right)-3+h\left(j_{n+2}^{(3)}+j_{n+1}^{(3)}+2j_n^{(3)}-9\right)\right)\\
&= \frac{1}{3}\left(j_{n+2}^{(3)}+2\widetilde{j}_n^{(3)}-3+h\left(j_{n+3}^{(3)}-9\right)\right)\\
&= \frac{1}{3}\left(j_{n+2}^{(3)}+hj_{n+3}^{(3)}+2\widetilde{j}_n^{(3)}-3(1+3h)\right)\\
&= \frac{1}{3}\left(\widetilde{j}_{n+2}^{(3)}+2\widetilde{j}_n^{(3)}-3(1+3h)\right).
\end{aligned}$$

Therefore,

$$\sum_{k=0}^{n}\widetilde{j}_k^{(3)}=\frac{1}{3}\left(\widetilde{j}_{n+2}^{(3)}+2\widetilde{j}_n^{(3)}-\widetilde{j}_2^{(3)}+\widetilde{j}_0^{(3)}\right).$$

□

Corollary 3.6. *For $n\geq 0$, the summation formulas of hyperbolic third order Jacobsthal and Jacobsthal-Lucas numbers can be given respectively as follows:*

$$\begin{aligned}\sum_{k=0}^{n}\widetilde{J}_k^{(3)} &= \begin{cases} J_{n+1}^{(3)}(1+2h), & \text{if } n\neq 0\,(mod3)\\ J_{n+1}^{(3)}(1+2h)-(1+h), & \text{if } n=0\,(mod3)\end{cases},\\ \sum_{k=0}^{n}\widetilde{j}_k^{(3)} &= \begin{cases} \left(j_{n+1}^{(3)}-2\right)(1+2h), & \text{if } n\neq 0\,(mod3)\\ j_{n+1}^{(3)}(1+2h)+1-h, & \text{if } n=0\,(mod3)\end{cases}.\end{aligned}$$

Proof. Using the formulas given in (1.4) and (1.5), we have

$$\begin{aligned}\sum_{k=0}^{n}\widetilde{J}_k^{(3)} &= \sum_{k=0}^{n}\left(J_k^{(3)}+hJ_{k+1}^{(3)}\right)\\ &= \sum_{k=0}^{n}J_k^{(3)}+h\sum_{k=0}^{n}J_{k+1}^{(3)}\\ &= \begin{cases} J_{n+1}^{(3)}+h, & \text{if } n\neq 0\,(mod3)\\ J_{n+1}^{(3)}-1, & \text{if } n=0\,(mod3)\end{cases}+h\begin{cases} J_{n+1}^{(3)}, & \text{if } n\neq 0\,(mod3)\\ J_{n+1}^{(3)}-1, & \text{if } n=0\,(mod3)\end{cases}+hJ_{n+1}^{(3)}-hJ_0^{(3)}\\ &= \begin{cases} J_{n+1}^{(3)}(1+h)+hJ_{n+1}^{(3)}-hJ_0^{(3)}, & \text{if } n\neq 0\,(mod3)\\ J_{n+1}^{(3)}(1+h)-1-h+hJ_{n+1}^{(3)}-hJ_0^{(3)}, & \text{if } n=0\,(mod3)\end{cases}\\ &= \begin{cases} J_{n+1}^{(3)}(1+2h), & \text{if } n\neq 0\,(mod3)\\ J_{n+1}^{(3)}(1+2h)-(1+h), & \text{if } n=0\,(mod3)\end{cases}.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{k=0}^{n}\widetilde{j}_k^{(3)} &= \sum_{k=0}^{n}\left(j_k^{(3)}+hj_{k+1}^{(3)}\right)\\ &= \sum_{k=0}^{n}j_k^{(3)}+h\sum_{k=0}^{n}j_{k+1}^{(3)}\\ &= \begin{cases} j_{n+1}^{(3)}-2, & \text{if } n\neq 0\,(mod3)\\ j_{n+1}^{(3)}+1, & \text{if } n=0\,(mod3)\end{cases}+h\begin{cases} j_{n+1}^{(3)}-2, & \text{if } n\neq 0\,(mod3)\\ j_{n+1}^{(3)}+1, & \text{if } n=0\,(mod3)\end{cases}+hj_{n+1}^{(3)}-hj_0^{(3)}\\ &= \begin{cases} \left(j_{n+1}^{(3)}-2\right)(1+h)+hj_{n+1}^{(3)}-2h, & \text{if } n\neq 0\,(mod3)\\ \left(j_{n+1}^{(3)}+1\right)(1+h)+hj_{n+1}^{(3)}-2h, & \text{if } n=0\,(mod3)\end{cases}\\ &= \begin{cases} \left(j_{n+1}^{(3)}-2\right)(1+2h), & \text{if } n\neq 0\,(mod3)\\ j_{n+1}^{(3)}(1+2h)+1-h, & \text{if } n=0\,(mod3)\end{cases}.\end{aligned}$$

We now present a few special identities for the hyperbolic third order Jacobsthal sequence $\{\widetilde{J}_n^{(3)}\}$. □

Theorem 3.7. *For the hyperbolic third order Jacobsthal sequence $\{\widetilde{J}_n^{(3)}\}$ and for the hyperbolic third order Jacobsthal-Lucas sequence $\{\widetilde{j}_n^{(3)}\}$ we have the following inequalities:*

$$\begin{aligned}3\widetilde{J}_n^{(3)}+\widetilde{j}_n^{(3)} &= 2^n(1+2h),\\ \widetilde{j}_n^{(3)}-3\widetilde{J}_n^{(3)} &= 2\widetilde{j}_{n-3}^{(3)},\\ \widetilde{J}_{n+2}^{(3)}-4\widetilde{J}_n^{(3)} &= \begin{cases}1-2h, & \text{if } n=0\,(mod3)\\ -2+h, & \text{if } n=1\,(mod3)\\ 1+h, & \text{if } n=2\,(mod3)\end{cases},\\ \widetilde{j}_n^{(3)}-4\widetilde{J}_n^{(3)} &= \begin{cases}2-3h, & \text{if } n=0\,(mod3)\\ -3+h, & \text{if } n=1\,(mod3)\\ 1+2h, & \text{if } n=2\,(mod3)\end{cases},\\ \widetilde{j}_{n+1}^{(3)}+\widetilde{j}_n^{(3)} &= 3\widetilde{J}_{n+2}^{(3)},\\ \widetilde{j}_n^{(3)}-\widetilde{J}_{n+2}^{(3)} &= \begin{cases}1-h, & \text{if } n=0\,(mod3)\\ -1, & \text{if } n=1\,(mod3)\\ h, & \text{if } n=2\,(mod3)\end{cases},\\ \left(\widetilde{j}_n^{(3)}\right)^2-9\left(\widetilde{J}_n^{(3)}\right)^2 &= 2^{n+2}\left(j_{n-3}^{(3)}+2j_{n-2}^{(3)}\right)+2h\left(j_n^{(3)}j_{n+1}^{(3)}-9J_n^{(3)}J_{n+1}^{(3)}\right).\end{aligned}$$

Proof. We use the formulas (1.6) given in [2]. Using (1.6), we have

$$\begin{aligned} 3\widetilde{J}_n^{(3)}+\widetilde{j}_n^{(3)} &= 3\left(J_n^{(3)}+hJ_{n+1}^{(3)}\right)+j_n^{(3)}+hj_{n+1}^{(3)} \\ &= 3J_n^{(3)}+j_n^{(3)}+h\left(3J_{n+1}^{(3)}+j_{n+1}^{(3)}\right) \\ &= 2^{n+1}+h2^{n+2}=2^{n+1}\left(1+2h\right). \end{aligned}$$

Using (1.7), we have

$$\begin{aligned} \widetilde{j}_n^{(3)}-3\widetilde{J}_n^{(3)} &= j_n^{(3)}+hj_{n+1}^{(3)}-3\left(J_n^{(3)}+hJ_{n+1}^{(3)}\right) \\ &= j_n^{(3)}-3J_n^{(3)}+h\left(j_{n+1}^{(3)}-3J_{n+1}^{(3)}\right) \\ &= 2j_{n-3}^{(3)}+2hj_{n-2}^{(3)}. \end{aligned}$$

Using (1.8), we have

$$\begin{aligned} \widetilde{J}_{n+2}^{(3)}-4\widetilde{J}_n^{(3)} &= \left(J_{n+2}^{(3)}+hJ_{n+3}^{(3)}\right)-4\left(J_n^{(3)}+hJ_{n+1}^{(3)}\right) \\ &= \left(J_{n+2}^{(3)}-4J_n^{(3)}\right)+h\left(J_{n+3}^{(3)}-4J_{n+1}^{(3)}\right) \\ &= \begin{cases} -2, & \text{if } n=1\,(mod3) \\ 1, & \text{if } n\neq 1\,(mod3) \end{cases} +h\begin{cases} -2, & \text{if } n=0\,(mod3) \\ 1, & \text{if } n\neq 0\,(mod3) \end{cases} \\ &= \begin{cases} 1-2h, & \text{if } n=0\,(mod3) \\ -2+h, & \text{if } n=1\,(mod3) \\ 1+h, & \text{if } n=2\,(mod3) \end{cases} \end{aligned}$$

Using (1.9), we have

$$\begin{aligned} \widetilde{j}_n^{(3)}-4\widetilde{J}_n^{(3)} &= \left(j_n^{(3)}+hj_{n+1}^{(3)}\right)-4\left(J_n^{(3)}+hJ_{n+1}^{(3)}\right) \\ &= \left(j_n^{(3)}-4J_n^{(3)}\right)+h\left(j_{n+1}^{(3)}-4J_{n+1}^{(3)}\right) \\ &= \begin{cases} 2, & \text{if } n=0\,(mod3) \\ -3, & \text{if } n=1\,(mod3) \\ 1, & \text{if } n=2\,(mod3) \end{cases} +h\begin{cases} 2, & \text{if } n=2\,(mod3) \\ -3, & \text{if } n=0\,(mod3) \\ 1, & \text{if } n=1\,(mod3) \end{cases} \\ &= \begin{cases} 2-3h, & \text{if } n=0\,(mod3) \\ -3+h, & \text{if } n=1\,(mod3) \\ 1+2h, & \text{if } n=2\,(mod3) \end{cases} \end{aligned}$$

Using (1.10), we have

$$\begin{aligned} \widetilde{j}_{n+1}^{(3)}+\widetilde{j}_n^{(3)} &= j_{n+1}^{(3)}+hj_{n+2}^{(3)}+j_n^{(3)}+hj_{n+1}^{(3)} \\ &= j_{n+1}^{(3)}+j_n^{(3)}+h\left(j_{n+2}^{(3)}+j_{n+1}^{(3)}\right) \\ &= 3J_{n+2}^{(3)}+3hJ_{n+3}^{(3)}, \end{aligned}$$

Using (1.11) , we have

$$\begin{aligned} \widetilde{j}_n^{(3)}-\widetilde{J}_{n+2}^{(3)} &= \left(j_n^{(3)}+hj_{n+1}^{(3)}\right)-\left(J_{n+2}^{(3)}+hJ_{n+3}^{(3)}\right) \\ &= \left(j_n^{(3)}-J_{n+2}^{(3)}\right)+h\left(j_{n+1}^{(3)}-J_{n+3}^{(3)}\right) \\ &= \begin{cases} 1, & \text{if } n=0\,(mod3) \\ -1, & \text{if } n=1\,(mod3) \\ 0, & \text{if } n=2\,(mod3) \end{cases} +h\begin{cases} 1, & \text{if } n=2\,(mod3) \\ -1, & \text{if } n=0\,(mod3) \\ 0, & \text{if } n=1\,(mod3) \end{cases} \\ &= \begin{cases} 1-h, & \text{if } n=0\,(mod3) \\ -1, & \text{if } n=1\,(mod3) \\ h, & \text{if } n=2\,(mod3) \end{cases} \end{aligned}$$

Using (1.12), we have

$$\begin{aligned} \left(\widetilde{j}_n^{(3)}\right)^2-9\left(\widetilde{J}_n^{(3)}\right)^2 &= \left(j_n^{(3)}+hj_{n+1}^{(3)}\right)^2-9\left(J_n^{(3)}+hJ_{n+1}^{(3)}\right)^2 \\ &= \left(j_n^{(3)}\right)^2+\left(j_{n+1}^{(3)}\right)^2+2hj_n^{(3)}j_{n+1}^{(3)}-9\left[\left(J_n^{(3)}\right)^2+\left(J_{n+1}^{(3)}\right)^2+2hJ_n^{(3)}J_{n+1}^{(3)}\right] \\ &= \left[\left(j_n^{(3)}\right)^2-9\left(J_n^{(3)}\right)^2\right]+\left[\left(j_{n+1}^{(3)}\right)^2-9\left(J_{n+1}^{(3)}\right)^2\right]+2h\left(j_n^{(3)}j_{n+1}^{(3)}-9J_n^{(3)}J_{n+1}^{(3)}\right) \\ &= 2^{n+2}\left(j_{n-3}^{(3)}+2j_{n-2}^{(3)}\right)+2h\left(j_n^{(3)}j_{n+1}^{(3)}-9J_n^{(3)}J_{n+1}^{(3)}\right). \end{aligned}$$

□

4. Conclusion

The hyperbolic third order Jacobsthal and Jacobsthal-Lucas numbers are defined by

$$\begin{aligned}\widetilde{J}_n^{(3)} &= J_n^{(3)} + hJ_{n+1}^{(3)},\\ \widetilde{j}_n^{(3)} &= j_n^{(3)} + hj_{n+1}^{(3)},\end{aligned}$$

with initial conditions $\widetilde{J}_0^{(3)} = h, \widetilde{J}_1^{(3)} = 1 + h$ and $\widetilde{j}_0^{(3)} = 2 + h, \widetilde{j}_1^{(3)} = 1 + 5h$ where $h^2 = 1\ (h \notin \mathbb{R})$.
We introduced the hyperbolic third order Jacobsthal numbers and we presented recurrence relations, Binet's formulas, generating functions and the summation formulas for these numbers.
There are new studies on dual hyperbolic, Tribonacci and Tribonacci-Lucas numbers (see $[5,9]$), and on hyperbolic generalized Pell numbers (see $[6,7]$) by other authors. Dual form of these numbers given by Cerda-Morales in $[11]$. This work can be continued to define the hyperbolic third order Jacobsthal vectors and investigate dot vector and mix product of these vectors.

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Competing Interests

Authors have declared that no competing interests exist.

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