A NUMERICAL METHOD ON BAKHVALOV SHISHKIN MESH FOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITH A BOUNDARY LAYER

Hayriye GUCKIR CAKIR,[1] Firat CAKIR,[2] and Musa CAKIR[3]

[1]Department of Mathematics, Adıyaman University, Adıyaman, TURKEY
[2]Department of Mathematics, Batman University, Batman, TURKEY
[3]Department of Mathematics, Van Yüzüncü Yıl University, Van, TURKEY

ABSTRACT. We construct a finite difference scheme for a first-order linear singularly perturbed Volterra integro-differential equation (SPVIDE) on Bakhvalov-Shishkin mesh. For the discretization of the problem, we use the integral identities and deal with the emerging integrals terms with interpolating quadrature rules which also yields remaining terms. The stability bound and the error estimates of the approximate solution are established. Further, we demonstrate that the scheme on Bakhvalov-Shishkin mesh is $O(N^{-1})$ uniformly convergent, where N is the mesh parameter. The numerical results are also provided for a couple of examples.

1. INTRODUCTION

In this present work, we are specifically consider the following class of the singularly perturbed linear Volterra integro-differential equations (SPVIDEs)

$$Lu := \varepsilon u' + a(x)u + \lambda \int_0^x K(x,t)u(t)dt = f(x), \quad x \in I = [0,\ell], \qquad (1)$$

subject to

$$u(0) = A, \qquad (2)$$

where $0 < \varepsilon \ll 1$ is a small perturbation parameter. We assume $a(x) \geq \alpha > 0$, $f(x)(x \in I)$ and $K(x,t)((x,t) \in I \times I)$ are sufficiently smooth functions such

that the initial layer for the solution $u(x)$ occurs at $x = 0$ for small values of ε. Volterra integro-differential equations (VIDEs) are an important class of equations which are extensively used to model many sciencitific problems such as population dynamics [13], filament streching [5] and epidemics [37]. Many techniques have been introduced to solve VIDEs analytically. The variational iteration method, the Adomian decomposition method and the homotopy perturba-tion method are some well-known analytical methods to solve VIDEs([40], [9], [17]). Recently, a new approach on the variational analytical method has been introduced to solve Volterra-Fredholm Integral equations which does not require construction of the variational principle [18]. Further, a finite difference scheme is utilized to examine the numerical solutions of a non-linear VIDE in [11].

Singularly perturbed differential equations, which have the highest order derivative term multi- plied with a small positive number ε, possess solutions with interior or boundary layers. Boundary layers are regions where rapid changes occur which makes solving such problems more challenging. Since standard schemes fail to give the accurate results for problems with boundary layer for small ε values, numerical solutions of such problems have been of interest to many researchers([12], [15], [16], [22], [28], [29], [31], [34], [38], [35]). Singularly perturbed Volterra integro-differential equations (SPVIDEs) have been widely used to model problems in many science fields such as epidemic dynamics, synchronous control systems, filament stretching and heat transfer ([6], [7], [14], [20], [21], [32], [33]). A review on the literature of the SPVIDEs was given in [25]. Further, asymptotic expansions derivation of the solutions to SPVIDEs are studied in [6], [7], [25]. In [32] a problem of nonlinear SPVIDE modelling the elongation ratio of filament is studied and the qualitative properties of the solution is discussed under some physically interesting assumptions. In [5], a specific integro-differential equation with a boundary layer which describes filament stretching process is considered and the leading order behavior of the problem is examined by an asymptotic method. Singularly perturbed integro differential equations have been also an interest to many researchers. In [23] and [24], the numerical solutions of singularly perturbed integro-differential and integro-differential-algebraic equations are analyzed by the implicit Runge-Kutta methods. An exponential finite difference method is applied for the inner and outer layers and a type of implicit Runge-Kutta method is performed to obtain the outer layer solutions of SPVIDEs in [36]. A finite Legendre expansion is constructed to solve different kinds of integral equations and integro-differential equations [26]. In [19], tension spline collocation methods are utilized to numerically discretize singularly perturbed Volterra integral and integro-differential equations. In [39], the authors present different types of exponential schemes to solve SPVIDEs and the stability analysis of the schemes is examined. Fitted difference schemes are also proven to provide accurate results in the solution process of different types of singularly perturbed problems. In [2], an exponentially fitted difference method is designed on a uniform mesh to solve linear SPVIDEs. First-order convergent

finite difference schemes are developed to solve linear first order SPVIDEs with delay in [4], [27]. In [3], using a fitted difference operator a second-order difference scheme is constructed on a piecewise uniform mesh to solve linear SPVIDEs.

In this present work, we mainly construct a uniform convergent difference scheme on a Bakhvalov-Shishkin mesh for the problem (1)-(2). Bakhvalov-Shishkin mesh is a mixed version of the Shishkin mesh and Bakhvalov mesh which are known to yield accurate results for singularly perturbed problems with boundary layers. In [30], the author demonstrated that the results from an upwind difference scheme on Bakhvalov-Shishkin mesh applied to a linear convection-diffusion equation are more accurate than the results from the upwind scheme on a Shishkin mesh. Further, a finite difference scheme on Bakhvalov-Shishkin mesh is utilized to deal with a singularly perturbed boundary value problem in [10].

The rest of the paper is organized in the following order. In Section 2, the asymptotic estimates on the exact solution to (1)-(2) are established. In Section 3, we define the Bakhvalov-Shishkin mesh points according to the boundary layer conditions of the problem (1)-(2) and derive a finite difference scheme utilizing the integral identities with exponential basis functions and then applying interpolating quadrature rules provided in [1] to the integral terms. In Section 4, we establish the stability bounds and the error estimates of the numerical solution and as a result we show that the scheme demonstrates $O(N^{-1})$ uniform convergence with respect to the perturbation parameter. We also provide the numerical results in Section 5.

2. Asymptotic Behavior of the Solution

In the following lemma, we establish a priori estimates for the asymptotic behavior of the solution to the problem (1)-(2).

Lemma 1. *Let $a, f \in C(I)$ and $K \in C(I \times I)$. The solution u to the problem (1)-(2) holds*

$$\|u\|_\infty \leq C, \tag{3}$$

where

$$C = (A + \alpha^{-1}\|f\|_\infty)e^{\lambda \bar{K} \alpha^{-1} \ell},$$

and $\bar{K} = \max\limits_{I \times I} |K(x,t)|$. In addition, if $a, f \in C^1(I)$ and $K \in C^1(I \times I)$ with

$$\left|\frac{\partial}{\partial x} K(x,t)\right| \leq \bar{K}_1 < \infty, \tag{4}$$

then the solution $u(x)$ satisfies

$$|u'(x)| \leq C\left(1 + \frac{1}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}}\right), x \in I. \tag{5}$$

Proof. To establish the first estimate given in (3) we start by rewriting (1) as

$$\varepsilon u' + a(x)u = F(x), \tag{6}$$

where

$$F(x) = f(x) - \lambda \int_0^x K(x,t)u(t)dt. \tag{7}$$

Solving the equation (6) with $u(0) = A$ yields

$$u(x) = Ae^{-\frac{1}{\varepsilon}\int_0^x a(s)ds} + \frac{1}{\varepsilon}\int_0^x F(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^x a(s)ds} d\xi,$$

and further we calculate

$$|u(x)| \le |A|e^{-\frac{1}{\varepsilon}\int_0^x a(s)ds} + \frac{1}{\varepsilon}\int_0^x |F(\xi)|e^{-\frac{1}{\varepsilon}\int_\xi^x a(s)ds} d\xi.$$

Since we have $a(x) \ge \alpha > 0$, it follows

$$|u(x)| \le |A|e^{-\frac{1}{\varepsilon}\int_0^x \alpha ds} + \frac{1}{\varepsilon}\int_0^x |F(\xi)|e^{-\frac{1}{\varepsilon}\int_\xi^x \alpha ds} d\xi$$

$$= |A|e^{-\frac{\alpha x}{\varepsilon}} + \frac{1}{\varepsilon}\int_0^x |F(\xi)|e^{-\frac{\alpha(x-\xi)}{\varepsilon}} d\xi. \tag{8}$$

Here, by the definition of $F(x)$ in (7), we get

$$|F(x)| \le \|f\|_\infty + \lambda\bar{K}\int_0^x |u(t)|dt. \tag{9}$$

Substituting (9) into (8) yields

$$|u(x)| \le |A|e^{-\frac{\alpha x}{\varepsilon}} + \frac{1}{\varepsilon}\int_0^x \left(\|f\|_\infty + \lambda\bar{K}\int_0^\xi |u(t)|dt\right)e^{-\frac{\alpha(x-\xi)}{\varepsilon}} d\xi$$

$$= |A|e^{-\frac{\alpha x}{\varepsilon}} + \frac{1}{\varepsilon}\|f\|_\infty \int_0^x e^{-\frac{\alpha(x-\xi)}{\varepsilon}} d\xi + \frac{\lambda\bar{K}}{\varepsilon}\int_0^x \int_0^\xi |u(t)|dt\, e^{-\frac{\alpha(x-\xi)}{\varepsilon}} d\xi$$

We integrate by parts the last term with double integral here

$$|u(x)| \le |A|e^{-\frac{\alpha x}{\varepsilon}} + \alpha^{-1}\|f\|_\infty\left(1 - e^{-\frac{\alpha x}{\varepsilon}}\right) + \alpha^{-1}\lambda\bar{K}\left(1 - e^{-\frac{\alpha x}{\varepsilon}}\right)\int_0^x |u(t)|dt$$

$$\le |A| + \alpha^{-1}\|f\|_\infty + \alpha^{-1}\lambda\bar{K}\int_0^x |u(t)|dt. \tag{10}$$

An application of the Gronwall's inequality to (10) provides

$$|u(x)| \le \left(|A| + \alpha^{-1}\|f\|_\infty\right)e^{\alpha^{-1}\lambda\bar{K}x}$$
$$\le \left(|A| + \alpha^{-1}\|f\|_\infty\right)e^{\alpha^{-1}\lambda\bar{K}\ell},$$

which leads to the desired result in (3).

For the next estimate provided in (5), we first differentiate the equation (1) and have

$$\varepsilon u'' + a'(x)u + a(x)u' + \lambda K(x,x)u + \lambda\int_0^x \frac{\partial}{\partial x}K(x,t)u(t)dt = f'(x).$$

Then, letting

$$v(x) = u'(x),$$

and

$$g(x) = f'(x) - a'(x)u - \lambda K(x,x)u - \lambda\int_0^x \frac{\partial}{\partial x}K(x,t)u(t)dt, \tag{11}$$

we have

$$\varepsilon v' + a(x)v = g(x). \tag{12}$$

In a similar manner to the previous work above, we solve (12)

$$v(x) = v(0)e^{-\frac{1}{\varepsilon}\int_0^x a(s)ds} + \frac{1}{\varepsilon}\int_0^x g(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^x a(s)ds}d\xi.$$

Then, we have

$$|v(x)| \le |v(0)|e^{-\frac{1}{\varepsilon}\int_0^x a(s)ds} + \frac{1}{\varepsilon}\int_0^x |g(\xi)|e^{-\frac{1}{\varepsilon}\int_\xi^x a(s)ds}d\xi$$
$$\le |v(0)|e^{-\frac{1}{\varepsilon}\int_0^x \alpha ds} + \frac{1}{\varepsilon}\int_0^x |g(\xi)|e^{-\frac{1}{\varepsilon}\int_\xi^x \alpha ds}d\xi \tag{13}$$
$$\le |v(0)|e^{-\frac{\alpha x}{\varepsilon}} + \frac{1}{\varepsilon}\int_0^x |g(\xi)|e^{-\frac{\alpha(x-\xi)}{\varepsilon}}d\xi.$$

Here, by the formula of $g(x)$ given in (11), from (3) and knowing that $a, f \in C^1(I)$, $K \in C^1(I \times I)$ and from (4) we obtain

$$
\begin{aligned}
|g(x)| &\leq ||f'||_\infty + ||a'||_\infty |u| + \lambda \bar{K} |u| + \lambda \bar{K}_1 \int_0^x |u(t)| dt \\
&\leq ||f'||_\infty + C\Big(||a'||_\infty + \lambda \bar{K}_1 + \ell \Big),
\end{aligned}
\tag{14}
$$

which implies $||g||_\infty \leq C_*$ for a $C_* \in \mathbb{R}$. Hence, utilizing this estimate on $g(x)$ in (13) provides

$$
\begin{aligned}
|v(x)| &\leq |v(0)| e^{-\frac{\alpha x}{\varepsilon}} + \frac{1}{\varepsilon} ||g||_\infty \int_0^x e^{-\frac{\alpha(x-\xi)}{\varepsilon}} d\xi \\
&\leq |v(0)| e^{-\frac{\alpha x}{\varepsilon}} + \alpha^{-1} C_* (1 - e^{-\frac{\alpha x}{\varepsilon}}).
\end{aligned}
\tag{15}
$$

On the other hand, inserting $x = 0$ in (1) and since $a, f \in C^1(I)$ it follows that

$$
|v(0)| = |u'(0)| = \frac{1}{\varepsilon} |f(0) - Aa(0)| \leq \frac{c}{\varepsilon}.
$$

Substituting this into (15) yields

$$
|v(x)| \leq \frac{c}{\varepsilon} e^{-\frac{\alpha x}{\varepsilon}} + \alpha^{-1} C_* (1 - e^{-\frac{\alpha x}{\varepsilon}}),
$$

which provides the desired result. $\qquad\square$

3. DIFFERENCE SCHEME

3.1. **Notation.** Before we proceed to the definition of the mesh points and discretization of the problem we provide the notation we use throughout the paper. Let $\bar{\omega}_h = \{0 = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = \ell\}$ denote a non-uniform mesh on $[0, \ell]$. For each $i = 0, \cdots, N$, let $h_i = x_i - x_{i-1}$ denote the step size. For any continuous mesh function v_i defined on ω_h we use the notation

$$
v_{\bar{x}, i} = \frac{v_i - v_{i-1}}{h_i}
$$

for backward difference.

3.2. **Discretization.** In this section, we construct our difference scheme based on Bakhvalov-Shishkin mesh. According to this mesh construction, we divide the domain into two subintervals $[0, \sigma]$ and $[\sigma, \ell]$, where σ is the transition parameter. For a positive even discretization parameter N, we determine the transition parameter σ as

$$
\sigma = \min \Big\{ \frac{\ell}{2}, \varepsilon \alpha^{-1} \ln N \Big\}.
\tag{16}
$$

We assume $\varepsilon \ll N^{-1}$ as it is used in practice. We define a set of mesh points as the following

$$x_i = \begin{cases} -\alpha^{-1}\varepsilon \ln[1 - 2(1 - N^{-1})\frac{i}{N}], & x_i \in [0, \sigma], i = 0, 1, \cdots, \frac{N}{2}, \\ \sigma + \left(i - \frac{N}{2}\right)h, \quad h = \frac{2(\ell - \sigma)}{N}, & x_i \in [\sigma, \ell], i = \frac{N}{2} + 1, \cdots, N. \end{cases} \tag{17}$$

To derive the difference approximation, we use the following integral identity

$$\chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} Lu(x)\varphi_i(x)dx = \chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} f(x)\varphi_i(x)dx, \tag{18}$$

with the exponential basis function

$$\varphi_i(x) = e^{-\frac{a_i}{\varepsilon}(x_i - x)}, \quad i = 1, \cdots, N,$$

where

$$\chi_i = h_i^{-1} \int_{x_{i-1}}^{x_i} \varphi_i(x)dx = \frac{1 - e^{-a_i\rho_i}}{a_i\rho_i}, \quad \rho_i = \frac{h_i}{\varepsilon}.$$

We remark that φ_i solves the equation

$$-\varepsilon\varphi_i'(x) + a_i\varphi_i(x) = 0, \quad x_{i-1} \le x \le x_i \tag{19}$$
$$\varphi_i(x_i) = 1.$$

To obtain the difference scheme from (18), we proceed by evaluating the integrals term by term applying the interpolating quadrature rules with weight functions and obtain the remainder terms as provided in [1]. In the following, we handle the differential term on the left-hand side of (18),

$$\chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} \left[\varepsilon u'(x) + a(x)u(x)\right]\varphi_i(x)dx = \chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} [\varepsilon u'(x) + a_i u(x)]\varphi_i(x)dx$$
$$+ \chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} [a(x) - a_i]u(x)\varphi_i(x)dx$$
$$= \varepsilon\theta_i u_{\bar{x},i} + a_i u_i + R_i^{(1)}, \tag{20}$$

where

$$\theta_i = \frac{a_i\rho_i e^{-a_i\rho_i}}{1 - e^{-a_i\rho_i}}, \tag{21}$$

and

$$R_i^{(1)} = \chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} [a(x) - a_i]u(x)\varphi_i(x)dx. \tag{22}$$

Further, applying the first quadrature rules provided in [1] to the integral term in (18) twice we obtain

$$\chi_i^{-1} h_i^{-1} \lambda \int_{x_{i-1}}^{x_i} \varphi_i(x) \int_0^x K(x,t)u(t)dtdx = \lambda \int_0^{x_i} K(x_i,t)u(t)dt + R_i^{(2)}, \tag{23}$$

where

$$R_i^{(2)} = \lambda \int_{x_{i-1}}^{x_i} \frac{\partial}{\partial \xi} \Big(\int_0^\xi K(\xi, t)u(t)dt \Big) \Big[T_0(x - \xi) - h_i^{-1}(x - x_{i-1}) \Big] d\xi, \qquad (24)$$

and $T_0(\lambda) = 1$ for $\lambda \geq 0$ and $T_0(\lambda) = 0$ for $\lambda < 0$. Here, we apply the composite right-side rectangle rule to the integral term in the right-hand side of (27) and get

$$\lambda \int_0^{x_i} K(x_i, t)u(t)dt = \lambda \sum_{j=1}^i h_j K(x_i, x_j)u_j + R_i^{(3)}, \qquad (25)$$

where

$$R_i^{(3)} = -\lambda \sum_{j=1}^i \int_{x_{j-1}}^{x_j} (\xi - x_{j-1})\frac{\partial}{\partial \xi} \Big(K(x_i, \xi)u(\xi) \Big) d\xi. \qquad (26)$$

Then, inserting (25) in (23) provides

$$\chi_i^{-1} h_i^{-1} \lambda \int_{x_{i-1}}^{x_i} \varphi_i(x) \int_0^x K(x, t)u(t)dtdx = \lambda \sum_{j=1}^i h_j K(x_i, x_j)u_j + R_i^{(2)} + R_i^{(3)}. \qquad (27)$$

On the other hand, the right-hand side of (18) gets the in the form

$$\chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} f(x)\varphi_i(x)dx = f_i + R_i^{(4)}, \qquad (28)$$

where

$$R_i^{(4)} = \chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} [f(x) - f(x_i)]\varphi_i(x)dx. \qquad (29)$$

Inserting the relations (20), (27) and (28) in (18), we obtain the difference problem for the problem (1)-(2) as

$$\varepsilon \theta_i u_{\bar{x},i} + a_i u_i + \lambda \sum_{j=1}^i h_j K(x_i, x_j)u_j = f_i - R_i, \quad i = 1, 2, \cdots, N, \qquad (30)$$

$$u_0 = A,$$

where

$$R_i = R_i^{(1)} + R_i^{(2)} + R_i^{(3)} - R_i^{(4)}. \qquad (31)$$

As a result, neglecting the error term R_i in (30) provides the following difference scheme

$$L_N y_i := \varepsilon \theta_i y_{\bar{x},i} + a_i y_i + \lambda \sum_{j=1}^i h_j K(x_i, x_j)y_j = f_i, \quad i = 1, 2, \cdots, N, \qquad (32)$$

$$y_0 = A, \qquad (33)$$

where θ_i defined by (21).

4. Stability, Error Estimates and Convergence Results

Here, we establish the stability bound and the error estimates of the approximate solution y. Further, the convergence of the difference scheme provided in (32)-(33) is analyzed.

Lemma 2. *Assume that $|F_i| \leq \mathcal{F}_i$ and $\mathcal{F}_i$ be a non-decreasing function. The solution to the problem*

$$\ell_N v_i := \varepsilon \theta_i v_{\bar{x},i} + a_i v_i = F_i, \quad 1 \leq i \leq N,$$
$$v_0 = A.$$

$$|v_i| \leq |A| + \alpha^{-1} \mathcal{F}_i, \quad 1 \leq i \leq N.$$

Proof. The proof follows from the maximum principle for difference operators. Details can be found in [27]. □

Lemma 3. *Let y_i be the solution of the problem (32)-(33). Then, y_i satisfies*

$$\|y\|_\infty \leq C_0 \big(|A| + \|f\|_\infty \big). \tag{34}$$

Proof. The difference scheme equation given in (32) can be rewritten in the form

$$\theta_i \varepsilon y_{\bar{x},i} + a_i y_i = F_i, \tag{35}$$

where

$$F_i = f_i - \lambda \sum_{j=1}^{i} h_j K(x_i, x_j) y_j.$$

For F_i, we have the estimate

$$|F_i| \leq |f_i| + \lambda \Big| \sum_{j=1}^{i} h_j K(x_i, x_j) y_j \Big|$$

$$\leq |f_i| + \lambda \bar{K} \sum_{j=1}^{i} h_j |y_j|$$

$$\leq \|f\|_\infty + \lambda \bar{K} \sum_{j=1}^{i} h_j |y_j|.$$

Then, applying Lemma 2 to (35) and utilizing this estimate provide

$$|y_i| \leq |A| + \alpha^{-1} \|f\|_\infty + \alpha^{-1} \lambda \bar{K} \sum_{j=1}^{i} h_j |y_j|. \tag{36}$$

Further, applying the difference analogue of the Gronwall's inequality to (36) we have

$$|y_i| \leq \Big(|A| + \alpha^{-1} \|f\|_\infty \Big) e^{\alpha^{-1} \lambda \bar{K} \ell},$$

which yields the result in (34). □

The error of the difference problem is given by the solution to the problem

$$L_N z_i = R_i, \quad 1 \le i \le N, \tag{37}$$

$$z_0 = 0. \tag{38}$$

Lemma 4. *Suppose that z_i be the solution of (37)-(38). Then, z_i holds the estimate*

$$\|z\|_\infty \le C\|R\|_\infty. \tag{39}$$

Proof. The result follows from Lemma 3 taking $A = 0$ and $f = R$. $\square$

Lemma 5. *Let $a, f \in C^1(I)$ and $K \in C^1(I \times I)$ with*

$$\bar{K} = \max_{I \times I} |K(x,t)|, \tag{40}$$

$$\left| \frac{\partial}{\partial x} K(x,t) \right| \le \bar{K}_1 < \infty, \tag{41}$$

and

$$\left| \frac{\partial}{\partial t} K(x,t) \right| \le \bar{K}_2 < \infty. \tag{42}$$

Then, the truncation error R_i satisfies the estimate

$$\|R\|_\infty \le CN^{-1}. \tag{43}$$

Proof. To establish the estimate given in (43), we proceed by bounding each term in R_i provided in (31). For $R_i^{(1)}$, we have

$$|R_i^{(1)}| \le \chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} |(a'(s)(x - x_i))u(x)|\varphi_i(x)dx,$$

where $s \in [x, x_i]$ comes from the Mean Value Theorem. Then, since $a \in C^1(I)$ and from (3) we get

$$|R_i^{(1)}| \le C_1 h_i. \tag{44}$$

Further, for $R_i^{(2)}$ we take into account of (40), (41) and $|T_0(\lambda)| \le 1$, so

$$|R_i^{(2)}| \le \lambda \int_{x_{i-1}}^{x_i} \left| (1 + h_i^{-1}(x - x_i)) \frac{\partial}{\partial \xi} \left(\int_0^\xi K(\xi,t)u(t)dt \right) \right| d\xi$$

$$\le 2\lambda \int_{x_{i-1}}^{x_i} \left| \frac{\partial}{\partial \xi} \left(\int_0^\xi K(\xi,t)u(t)dt \right) \right| d\xi. \tag{45}$$

Then, applying the Leibnitz formula to (45) yields

$$|R_i^{(2)}| \le 2\lambda \left(\int_{x_{i-1}}^{x_i} \left| K(\xi,\xi) \right| \left| u(\xi) \right| \right) + \int_{x_{i-1}}^{x_i} \int_0^\xi \left| \frac{\partial}{\partial \xi} K(\xi,t)u(t) \right| dt d\xi \right)$$

$$\le 2\lambda(C\bar{K} + C\bar{K}_1)h_i$$

$$\le C_2 h_i. \tag{46}$$

On the other hand, by the Leibnitz formula and from (40), (42) and (5) we have

$$
\begin{aligned}
|R_i^{(3)}| &\leq \lambda \sum_{j=1}^{i} \int_{x_{j-1}}^{x_j} \left(\left| \frac{\partial}{\partial \xi} K(x_i, \xi) u(\xi) \right| + \left| K(x_i, \xi) u'(\xi) \right| \right) d\xi \\
&\leq \lambda \sum_{j=1}^{i} \left(C \bar{K}_2 + \bar{K} \int_{x_{j-1}}^{x_j} \left(1 + \frac{1}{\varepsilon} e^{-\frac{\alpha \xi}{\varepsilon}} \right) \right) d\xi \\
&= \lambda \sum_{j=1}^{i} \left(C \bar{K}_2 h_j + \bar{K} h_j + \alpha^{-1} \bar{K} \left(e^{-\frac{\alpha x_{j-1}}{\varepsilon}} - e^{-\frac{\alpha x_j}{\varepsilon}} \right) \right).
\end{aligned}
\tag{47}
$$

Then, by the Mean Value Theorem applied to the exponential term in (47) with $s \in [x_{j-1}, x_j]$ it follows that

$$
\begin{aligned}
|R_i^{(3)}| &\leq \lambda \sum_{j=1}^{i} \left(C \bar{K}_2 h_j + \bar{K} h_j + \alpha^{-1} \bar{K} h_j e^{-\frac{\alpha s}{\varepsilon}} \right) \\
&\leq C_3^* i |h^*|,
\end{aligned}
\tag{48}
$$

where $h^* = \max\limits_{1 \leq j \leq i} h_j$. Lastly, for $R_i^{(4)}$, similarly to the work above and since $f \in C^1(I)$ we have

$$
\begin{aligned}
|R_i^{(4)}| &\leq \chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} |f'(s)(x - x_i)| \varphi_i(x) dx \\
&\leq C_4 h_i,
\end{aligned}
\tag{49}
$$

where $s \in [x_{i-1}, x_i]$ by the Mean Value Theorem.

Further in the proof, we need to evaluate each estimate above on the sub-intervals $[0, \sigma]$ and $[\sigma, \ell]$. For this, we first establish the bounds on the step-size h_i on each interval. In the first sub-interval $[0, \sigma]$ with $\sigma \leq \frac{\ell}{2}$,

$$
x_i = -\alpha^{-1} \varepsilon \ln[1 - 2(1 - N^{-1}) \frac{i}{N}], \quad i = 1, \cdots, N/2
$$

and hence,

$$
h_i = -\alpha^{-1} \varepsilon \ln[1 - 2(1 - N^{-1}) \frac{i}{N}] + \alpha^{-1} \varepsilon \ln[1 - 2(1 - N^{-1}) \frac{i-1}{N}].
$$

Then, we apply the Mean Value Theorem to h_i with $i_* \in [i-1, i]$ and get

$$
h_i \leq \alpha^{-1} \varepsilon \frac{2(1 - N^{-1})N^{-1}}{1 - 2i_*(1 - N^{-1})N^{-1}} \leq C N^{-1}.
\tag{50}
$$

In the second sub-interval $[\sigma, \ell]$, we have

$$
x_i = \sigma + \left(i - \frac{N}{2} \right) h, \quad i = N/2 + 1, \cdots, N,
$$

where $\sigma \leq \frac{\ell}{2}$ and

$$h_i = \frac{2(\ell - \sigma)}{N} \leq CN^{-1}. \tag{51}$$

Inserting the bounds (50) and (51) in (44), (46), (48) and (49), we have

$$|R_i^{(k)}| \leq CN^{-1}, \quad k = 1, 2, 3, 4.$$

which implies the desired result (43). $\qquad\square$

Theorem 1. *Let u be the exact solution of (1)-(2) and y be the solution of (32)-(33). If the assumptions on the functions a, f and K from Lemma 5 hold, then*

$$\|y - u\|_\infty \leq CN^{-1}.$$

Proof. The proof follows from Lemma 4 and Lemma 5. $\qquad\square$

5. Algorithm and Numerical Results

In this section, we present the numerical results on an example with an exact solution and an example with an unknown solution. The results include graphs of the approximate solutions, error estimates and the convergence values of the approximate solution to the exact solution. In our algorithm, we consider the following elimination method

$$y_i^{(n)} = \frac{1}{\varepsilon\theta_i + h_i a_i}\left[\varepsilon\theta_i y_{i-1}^{(n} + h_i\big(f_i - \lambda\sum_{j=1}^{i} h_j K(x_i, x_j)y_j^{(n-1)}\big)\right], \tag{52}$$

$$y_0^{(n)} = A, \tag{53}$$

$$y_i^{(0)} = A. \tag{54}$$

where $y_i^{(0)}$ is the initial process.

Example 1. *We study the following initial value problem*

$$\varepsilon u'(x) + u(x) + \int_0^x xu(t)dt = 2\varepsilon(x - 1) + (x - 1)^2 - \varepsilon x e^{-\frac{x}{\varepsilon}} + \frac{x(x - 1)^3}{3}$$

$$+ (\varepsilon - 1 + x)e^{-x} + (\varepsilon - \frac{2}{3})x, \quad 0 \leq x \leq 2,$$

$$u(0) = 1.$$

The exact solution of this problem is

$$u(x) = e^{-\frac{x}{\varepsilon}} + (x - 1)^2 - e^{-x}.$$

The exact error is calculated by the formula

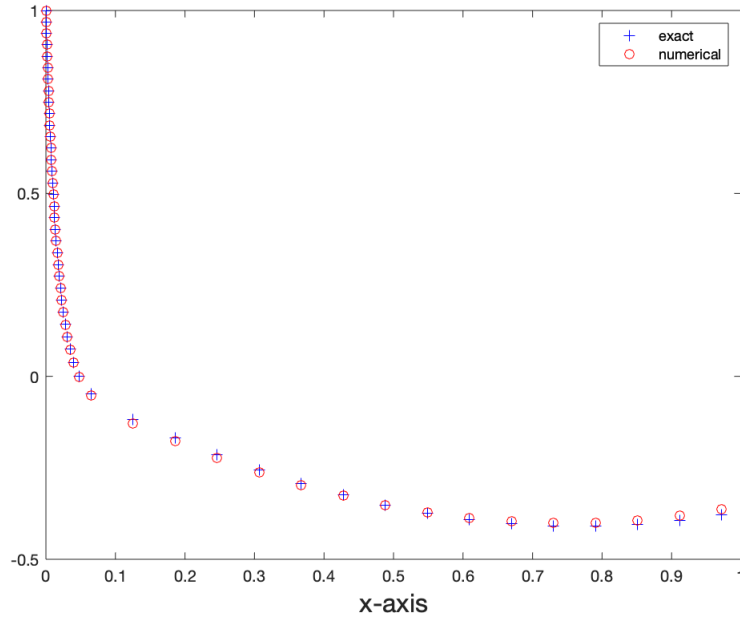$$e_\varepsilon^N = \|y^N - u\|_\infty,$$

FIGURE 1. The figure depicts the graphs for the exact solution and the approximate solution for $N = 32$.

where y^N is the numerical approximation of u for different N and ε values. We compute the convergence rate by

$$r^N = \frac{\ln\left(e^N/e^{2N}\right)}{\ln 2}.$$

In Table 1, we provide the errors e^N, e^{2N} and the convergence rates of the approximate solution for various N and $\varepsilon = 2^{-i}$ values.

Example 2. *Consider the following test problem*

$$\varepsilon u' + (x+1)u + \int_0^x xt(x-t)^2 u(t)dt = x - e^{2x}, \quad 0 \le x \le 2,$$

$$u(0) = 1.$$

The exact solution to this problem is not known. To compute the approximate solution and estimate the errors, we utilize the double mesh principle, that is calculating the error of the approximate solution on mesh size N with the approximate solution

TABLE 1. Errors e^N, e^{2N}, and rate of convergence r for Example 1.

ε		$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
	e^N	0.065 181 2	0.036 774 9	0.018 718 3	0.009 187 0	0.004 326 4
2^{-12}	e^{2N}	0.029 862 0	0.016 962 6	0.008 791 5	0.004 253 9	0.001 871 4
	r	1.126 145 3	1.116 368 7	1.090 268 5	1.110 475 2	
	e^N	0.065 374 3	0.036 984 5	0.018 926 7	0.009 393 2	0.004 531 1
2^{-18}	e^{2N}	0.030 078 8	0.017 176 5	0.009 001 5	0.004 462 0	0.002 076 7
	r	1.119 980	1.106 485 5	1.072 189 3	1.073 941 7	
	e^N	0.065 377 7	0.036 987 8	0.018 929 9	0.009 396 4	0.004 534 3
2^{-24}	e^{2N}	0.030 082 1	0.017 179 8	0.009 004 8	0.004 465 2	0.002 080 0
	r	1.119 883 7	1.106 332 4	1.071 911 6	1.073 390 5	

computed on double mesh $2N$, namely

$$e_\varepsilon^N = \|y^N - y^{2N}\|_\infty,$$

where y^N is the approximate solution on mesh N and y^{2N} is the approximate solution on mesh $2N$. The convergence rate is calculated as it is in Example 1.

In Table 2, the errors and the convergence rates of the approximate solution for various N and $\varepsilon = 2^{-i}$ values are presented.

TABLE 2. Errors e^N, e^{2N}, and rate of convergence r for Example 2.

ε		$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
	e^N	0.031 218 4	0.015 601 2	0.007 796 0	0.003 895 5	0.001 946 6
2^{-12}	e^{2N}	0.015 601 2	0.007 796 0	0.003 895 5	0.001 946 6	0.000 972 9
	r	1.000 741 7	1.000 858 3	1.000 922 3	1.000 865 3	
	e^N	0.031 249 5	0.015 624 6	0.007 812 2	0.003 906 1	0.001 953 0
2^{-18}	e^{2N}	0.015 624 6	0.007 812 2	0.003 906 1	0.001 953 0	0.000 488 2
	r	1.000 012 1	1.000 014 6	1.000 017 2	1.000 019 8	
	e^N	0.031 250 0	0.015 625 0	0.007 812 5	0.003 906 3	0.001 953 1
2^{-24}	e^{2N}	0.015 625 0	0.007 812 5	0.003 906 3	0.001 953 1	0.000 976 6
	r	1.000 000 2	1.000 000 2	1.000 000 3	1.000 000 3	

6. CONCLUSION

To sum up, we constructed a finite difference scheme on a Bakhvalov-Shishkin mesh to obtain the numerical solution of an initial value problem for a linear first-order singularly perturbed Volterra integro-differential equation with a boundary layer. We proved that the method is first-order uniformly convergent with respect to the perturbation parameter. As we can see in Table 1, Table 2 and Figure 1, the numerical results of the test problems are also consistent with the analysis on the error estimates and convergence order and hence, it is confirmed that the convergence order of the scheme $O(N^{-1})$. For future work, we suggest that this difference scheme method on Bakhvalov-Shishkin mesh can be applied to the singularly perturbed linear or non-linear problems with delay to obtain accurate numerical solutions. Further, our proposed scheme can be modified to handle integro-differential

equations with fractal derivatives which are studied in [8].

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