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One Parameter Elliptic Motions in Three-Dimensional Space

Ayşe Zeynep Azak*

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ABSTRACT

Elliptic motions have been defined by three different right-handed coordinate systems. The motion of these coordinate systems depends on the time parameter which has great importance in robotics. In particular, it is used in a model of a robot arm manipulator to achieve high performance. Hence, we have expressed some theorems and results concerning this elliptic motion. Besides, the special cases of this motion have been discussed.

Keywords: *Kinematics, elliptic motion, one parameter motion.*

AMS Subject Classification (2020): *53A17; 53A25; 53A40.*

1. Introduction

Designing and analyzing motions of objects in three dimensional space appears in the fields of robotics, computer graphics, 3D computer games and CAD applications [3]. One of the most important problem in three dimensional spatial kinematics is the representation of spherical displacements and motions. In the study of these motions, time can be thought of a motion parameter. Thus, Müller has presented one parameter spherical motions in the Euclidean space. He has obtained the relations for absolute, sliding, relative velocities and pole curves of these motions [12]. Then, Tosun et al. have introduced one-parameter Lorentzian spherical motions and have obtained the relations between fixed and moving pole curves for these motions [15]. In 2008, one-parameter dual Lorentzian spherical motions in three dimensional Lorentz space have been introduced. After, the relations and theorems with respect to velocities, instantaneous rotation axis, acceleration, acceleration center and acceleration axis have been found by Güngör and Tosun [5]. Yaylı et al. have discussed E-Study maps of circles which lie on the dual hyperbolic and Lorentzian unit spheres and have given some geometrical results [18]. In addition to this, Abdel-Baky and El-Ghefari have developed explicit expressions based on the E. Study's dual line coordinates for the one-parameter dual spherical motions. Applying this new technique, the Disteli formulae of the axodes have been derived [1]. Furthermore, some authors have examined studies with respect to one-parameter motions in different spaces e.g. [6, 7, 9, 16]

Especially an ellipsoid has a great number of applications to various domains of mathematics, physics, geodesy, crystallography. Thus, ellipsoids play an essential role in such areas as probability and statistics [4], fluid dynamics and mechanics [2, 8, 17], reference ellipsoid [14], thermal ellipsoid [11]. Except for the applications of ellipsoids as we mentioned before, ellipsoids are well studied in various areas of geometry, too. For example, Özdemir has given the generation of elliptical rotations with the help of the elliptical scalar product and elliptical vector product for a given ellipsoid. For this purpose, an elliptical orthogonal matrix and an elliptical skew-symmetric matrix have been defined for this elliptical scalar product. Thereby, he has examined the motion of a point on the ellipsoid using elliptical rotation matrices [13].

This present paper emphasizes expansion of one-parameter Euclidean spherical motions to one parameter elliptic motions. We have considered two moving and one fixed tri-axial ellipsoids associated with elliptically orthonormal frames in order to examine the motion of a point X. Absolute, relative and sliding velocities are directly derived by using the differentiation of X with respect to moving and fixed ellipsoids. Then, the

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**** Corresponding author***

fundamental theorem is proven that states the relationship between the absolute, relative and sliding velocities. As a result, it is shown that this theorem corresponds to one-parameter spherical motion in three dimensional Euclidean space E^3 in the case of a_1, a_2, a_3 equal to one. Finally, some theorems with respect to pole points of this motion are presented.

2. Preliminaries

Let we consider a tri-axial ellipsoid which is centered at the origin and is given in the standard form. Thus, its equation is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

where $a, b, c \in R$.

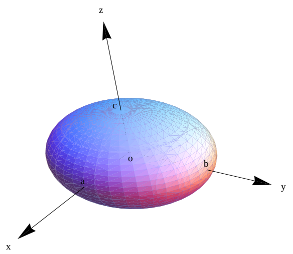

Figure 1. Tri-axial Ellipsoid

The parametric equations of the tri-axial ellipsoid can be written as

$$\gamma(\theta,\beta)=(a\cos\theta\cos\beta, b\cos\theta\sin\beta, c\sin\theta)$$

where the angles are compatible with the parameters $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\beta \in [-\pi, \pi]$.
The motion is described by considering the displacements of a point. Therefore, the vectors have particularly an important role while describing these displacements. For this, with the help of this given ellipsoid we will choose a proper scalar product which doesn't change the distance between any point on the ellipsoid and origin. Because this ellipsoid is considered to equivalent to a sphere for the scalar product space.
The elliptical inner product or B-inner product for the vectors $\vec{u}=(u_1,u_2,u_3)$ and $\vec{w}=(w_1,w_2,w_3) \in R^3$

$$B(\vec{u},\vec{w})=a_1 u_1 w_1 + a_2 u_2 w_2 + a_3 u_3 w_3,$$

where $a_1, a_2, a_3 \in R^+$. This scalar product is positive definite and non-degenerate. Moreover, it can be written as $B(\vec{u},\vec{w})=u^t\Omega w$ where the associated matrix Ω is defined as follows

$$\Omega=\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

where $a_1=\frac{1}{a^2},\ a_2=\frac{1}{b^2},\ a_3=\frac{1}{c^2}$ [13]. Thus, the real vector space R^3 equipped with the elliptical inner product will be denoted by $R^3_{a_1,a_2,a_3}$ and the number $\Delta=\sqrt{\det\Omega}$ will be called constant of the scalar products [13].
The elliptical norm of a vector $\vec{u} \in R^3$ is defined to be $\|\vec{u}\|_B=\sqrt{B(\vec{u},\vec{u})}$. Moreover, two vectors $\vec{u}$ and $\vec{w}$ are called B-orthogonal or elliptically orthogonal vectors if $B(\vec{u},\vec{w})=0$. In addition to that if their norms become 1, then these vectors are called elliptically orthonormal or B-orthonormal. If $\{\overrightarrow{u},\overrightarrow{v},\overrightarrow{w}\}$ is a B-orthonormal base of $R^3_{a_1,a_2,a_3}$, then $\det(\overrightarrow{u},\overrightarrow{v},\overrightarrow{w})=\Delta^{-1}$. The cosine of the angle between two vectors $\vec{u}$ and $\vec{w}$ is defined as,

$$\cos\theta=\frac{B(\vec{u},\vec{w})}{\|\vec{u}\|_B\,\|\vec{w}\|_B}$$

where θ is compatible with the parameters of the angular parametric equations of ellipse or ellipsoid [13]. Let $\vec{u} = (u_1, u_2, u_3), \vec{v} = (v_1, v_2, v_3) \in R^3$ and $\{\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}\}$ be standard unit vectors for B. Then, the elliptical vector product is defined as

$$\overrightarrow{u} \wedge \overrightarrow{v} = \frac{1}{\Delta^2} \begin{vmatrix} a_1\overrightarrow{e_1} & a_2\overrightarrow{e_2} & a_3\overrightarrow{e_3} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

The elliptical norm of the vector $\overrightarrow{u} \wedge \overrightarrow{v}$ is given by

$$\|\overrightarrow{u} \wedge \overrightarrow{v}\|_B = \|\overrightarrow{u}\|_B \|\overrightarrow{v}\|_B \sin\theta.$$

Let B be a non-degenerate scalar product. Then, any matrix $C \in R^{3\times 3}$ is called a B-orthogonal matrix, if it satisfies the matrix equality $C^t \Omega C = \Omega$. Here, Ω is the associated matrix of B. Moreover, all rows (or columns) are B-orthonormal to each other. The set of B-orthogonal matrices are denoted by $O_B(3)$. That is,

$$O_B(3) = \left\{ C \in R^{3\times 3} : C^t \Omega C = \Omega \quad \text{and} \quad \det C = \pm 1 \right\}.$$

If $\det C = 1$, then we call it a B-rotation matrix or an elliptical rotation matrix. If $\det C = -1$, we call it an elliptical reflection matrix. The set of the B-rotation matrices of R^3 can be expressed as follows:

$$SO_B(3) = \left\{ C \in R^{3\times 3} : C^t \Omega C = \Omega \quad \text{and} \quad \det C = 1 \right\}.$$

$SO_B(3)$ is a subgroup of $O_B(3)$, [10].

The matrix $T \in R^{3\times 3}$ is called a B-skew-symmetric matrix, if it satisfies the matrix equality $T^t \Omega = -\Omega T$. Thus, the set of B-skew symmetric matrices are defined by

$$L = \left\{ T \in R^{3\times 3} : B(T\vec{u}, \vec{w}) = -B(\vec{u}, T\vec{w}) \quad \text{for all} \quad \vec{u}, \vec{w} \in R^3 \right\}$$

In the scalar product space $R^3_{a_1,a_2,a_3}$, the skew-symmetric matrix is represented as follows [13]:

$$T = \Delta \begin{bmatrix} 0 & \frac{x}{a_1} & -\frac{y}{a_1} \\ -\frac{x}{a_2} & 0 & \frac{z}{a_2} \\ \frac{y}{a_3} & -\frac{z}{a_3} & 0 \end{bmatrix}$$

Here, we can omit the scalar product constant Δ. Because this condition doesn't affect to become skew-symmetric matrix.

3. Velocities on the One Parameter Elliptic Motions

Let H and H' be the common centered, tri-axial moving and fixed ellipsoids and $\{O; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and $\{O; \vec{e}'_1, \vec{e}'_2, \vec{e}'_3\}$ represent the elliptically orthogonal frames, respectively. These frames are rigidly linked to these ellipsoids and move with respect to each other.

$$B(\vec{e}_i, \vec{e}_j) = \begin{cases} a_i, & if\ i = j \\ 0, & if\ i \neq j \end{cases} \tag{3.1}$$

and

$$B\left(\vec{e}'_i, \vec{e}'_j\right) = \begin{cases} a_i, & if\ i = j \\ 0, & if\ i \neq j \end{cases} \tag{3.2}$$

Without thinking none of these systems are privileged, let us introduce another elliptically orthogonal relative frame $\{O; \vec{r}_1, \vec{r}_2, \vec{r}_3\}$. So, we have

$$B(\vec{r}_i, \vec{r}_j) = \begin{cases} a_i, & if\ i = j \\ 0, & if\ i \neq j \end{cases} \tag{3.3}$$

Let each of these orthogonal frames has the same orientation. Namely, one frame is obtained by using another. Thus, the following relation can be written shortly between the relative frame and moving frame as follows

$$\vec{r}_j = \sum_{k=1}^{3} a_{jk} \vec{e}_k \tag{3.4}$$

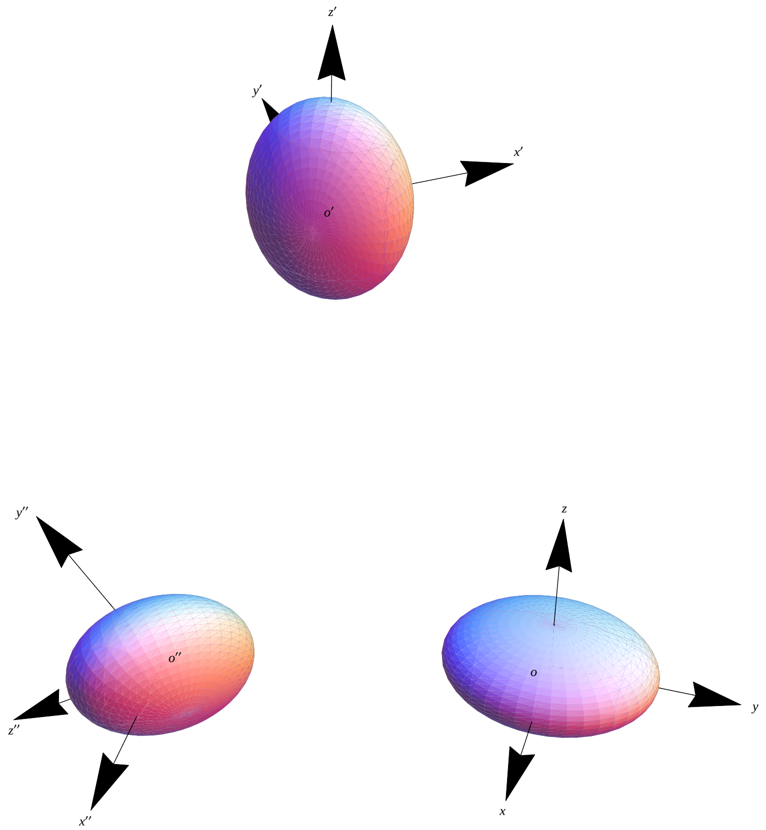

Figure 2. One-parameter motion of tri-axial ellipsoids

In a similar way, the relation between the relative frame and the fixed frame can be expressed as follows

$$\vec{r}_j = \sum_{k=1}^{3} {a'}_{jk} \vec{e}'_k \tag{3.5}$$

where $A = [a_{jk}]$ and $A' = [{a'}_{jk}]$ matrices are elliptically orthogonal. Furthermore, these matrices correspond to transitive between the relative, moving and fixed frames for H and H', respectively. By using the following abbreviations

$$E = \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix}, \qquad R = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{bmatrix}, \qquad E' = \begin{bmatrix} \vec{e}'_1 \\ \vec{e}'_2 \\ \vec{e}'_3 \end{bmatrix}$$

we get

$$R = AE, \quad R = A'E'. \tag{3.6}$$

Here, the elements of the matrices A and A' are differentiable functions of the parameter t. Thus, the motion which is determined by the matrix $A = A(t)$ (or $A' = A'(t)$) is called as one-parameter elliptic motion.

If we consider the equation (3.6) and make necessary calculations, the following equations can be found between the moving frame $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and the relative frame $\{\vec{r}_1, \vec{r}_2, \vec{r}_3\}$

$$\vec{e}_k = \sum_{\ell=1}^{3} \frac{a_\ell}{a_k} a_{\ell k} \vec{r}_\ell \tag{3.7}$$

Similarly, the equations between the fixed frame $\{\vec{e}_1', \vec{e}_2', \vec{e}_3'\}$ and the relative frame $\{\vec{r}_1, \vec{r}_2, \vec{r}_3\}$ are

$$\vec{e}_k' = \sum_{\ell=1}^{3} \frac{a_\ell}{a_k} a'_{\ell k} \vec{r}_\ell \tag{3.8}$$

Now, let us calculate the differentials of vectors $\overrightarrow{r_j}$ with respect to H and H', respectively. If we consider equation (3.3), then the differential of the relative orthogonal coordinate frame R with respect to H and H' is found as

$$d\vec{r}_j = \sum_{k,\ell=1}^{3} da_{jk}\, a_{\ell k} \frac{a_\ell}{a_k} \vec{r}_\ell \tag{3.9}$$

and

$$d'\vec{r}_j = \sum_{k,\ell=1}^{3} da'_{jk}\, a'_{\ell k} \frac{a_\ell}{a_k} \vec{r}_\ell \tag{3.10}$$

If we want to denote the above equalities by matrix form, we can easily get that

$$dR = dAA^{-1}R, \quad d'R = dA'(A')^{-1}R \tag{3.11}$$

where if we take into account $\omega_{j\ell} = da_{jk}\, a_{\ell k}\, a_\ell$ and similarly $\omega'_{j\ell} = da'_{jk}\, a'_{\ell k}\, a_\ell$ for $(1 \le j, \ell, k \le 3)$, then the permutations of the indices $j, \ell, k = 1,2,3; 2,3,1; 3,2,1$ can be denoted by $\omega_{j\ell} = \omega_k$. Therefore, equations (3.9) and (3.10) can be expressed clearly as

$$\begin{aligned} d\overrightarrow{r_1} &= \frac{\sqrt{a_2a_3}}{\sqrt{a_1}}\omega_3\overrightarrow{r_2} - \frac{\sqrt{a_2a_3}}{\sqrt{a_1}}\omega_2\overrightarrow{r_3} \\ d\overrightarrow{r_2} &= -\frac{\sqrt{a_1a_3}}{\sqrt{a_2}}\omega_3\overrightarrow{r_1} + \frac{\sqrt{a_1a_3}}{\sqrt{a_2}}\omega_1\overrightarrow{r_3} \\ d\overrightarrow{r_3} &= \frac{\sqrt{a_1a_2}}{\sqrt{a_3}}\omega_2\overrightarrow{r_1} - \frac{\sqrt{a_1a_2}}{\sqrt{a_3}}\omega_1\overrightarrow{r_2} \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} d'\overrightarrow{r_1} &= \frac{\sqrt{a_2a_3}}{\sqrt{a_1}}\omega'_3\overrightarrow{r_2} - \frac{\sqrt{a_2a_3}}{\sqrt{a_1}}\omega'_2\overrightarrow{r_3} \\ d'\overrightarrow{r_2} &= -\frac{\sqrt{a_1a_3}}{\sqrt{a_2}}\omega'_3\overrightarrow{r_1} + \frac{\sqrt{a_1a_3}}{\sqrt{a_2}}\omega'_1\overrightarrow{r_3} \\ d'\overrightarrow{r_3} &= \frac{\sqrt{a_1a_2}}{\sqrt{a_3}}\omega'_2\overrightarrow{r_1} - \frac{\sqrt{a_1a_2}}{\sqrt{a_3}}\omega'_1\overrightarrow{r_2} \end{aligned} \tag{3.13}$$

By choosing $dAA^{-1} = S$ and $dA'(A')^{-1} = S'$, equation (3.12) can be rewritten as follows:

$$dR = SR, \quad d'R = S'R \tag{3.14}$$

It can be easily seen that both S and S' matrices are skew-symmetric in the sense of elliptic, i.e. $S^t\Omega = -\Omega S$ where Ω is the associated matrix with related to the inner product. Thus, from the equations (3.12) and (3.14), this elliptical skew-symmetric matrix is given by

$$S = \Delta \begin{bmatrix} 0 & \frac{\omega_3}{a_1} & -\frac{\omega_2}{a_1} \\ -\frac{\omega_3}{a_2} & 0 & \frac{\omega_1}{a_2} \\ \frac{\omega_2}{a_3} & -\frac{\omega_1}{a_3} & 0 \end{bmatrix}. \tag{3.15}$$

In a similar way, from the equations (3.13) and (3.14) skew-symmetric matrix S' in the sense of elliptic is given by

$$S' = \Delta \begin{bmatrix} 0 & \frac{\omega'_3}{a_1} & -\frac{\omega'_2}{a_1} \\ -\frac{\omega'_3}{a_2} & 0 & \frac{\omega'_1}{a_2} \\ \frac{\omega'_2}{a_3} & -\frac{\omega'_1}{a_3} & 0 \end{bmatrix}. \tag{3.16}$$

Let we take a point $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ according to the relative system to analyze the elliptic motions on the ellipsoid. Then, we have the following equality

$$\overrightarrow{OX} = \overrightarrow{X} = x_1\,\overrightarrow{r_1} + x_2\,\overrightarrow{r_2} + x_3\,\overrightarrow{r_3} \tag{3.17}$$

If we consider the equation (3.3), it is easily be seen that the point is located on the ellipsoid. Namely, we have

$$\|X\|_B^2 = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 1.$$

Now, we will compute the differentials of X with respect to moving ellipsoid H and fixed ellipsoid H'.
If we consider the equation (3.17), we obtain the differentiation of X with respect to moving ellipsoid as follows

$$d\overrightarrow{X} = dx_1\,\overrightarrow{r_1} + x_1\,d\overrightarrow{r_1} + dx_2\,\overrightarrow{r_2} + x_2\,d\overrightarrow{r_2} + dx_3\,\overrightarrow{r_3} + x_3\,d\overrightarrow{r_3}. \tag{3.18}$$

Substituting the equation (3.12) into the equation (3.18) and rearranging it, we find

$$\begin{aligned} d\overrightarrow{X} = &\left(dx_1 - \tfrac{\sqrt{a_1a_3}}{\sqrt{a_2}}\omega_3x_2 + \tfrac{\sqrt{a_1a_2}}{\sqrt{a_3}}\omega_2x_3\right)\overrightarrow{r_1} + \left(dx_2 + \tfrac{\sqrt{a_2a_3}}{\sqrt{a_1}}\omega_3x_1 - \tfrac{\sqrt{a_1a_2}}{\sqrt{a_3}}\omega_1x_3\right)\overrightarrow{r_2} \\ &+ \left(dx_3 - \tfrac{\sqrt{a_2a_3}}{\sqrt{a_1}}\omega_2x_1 + \tfrac{\sqrt{a_1a_3}}{\sqrt{a_2}}\omega_1x_2\right)\overrightarrow{r_3}. \end{aligned} \tag{3.19}$$

The velocity vector of X with respect to the moving ellipsoid H is called the relative velocity of this point and is denoted by $\overrightarrow{V_r} = \frac{d\overrightarrow{x}}{dt}$. If $\overrightarrow{V_r} = 0$, namely $d\overrightarrow{X} = 0$, then the point X becomes fixed on the moving ellipsoid H. Then, from equation (3.19), the condition that the point X is fixed on H is given by

$$\begin{aligned} dx_1 &= \tfrac{\sqrt{a_1a_3}}{\sqrt{a_2}}\omega_3x_2 - \tfrac{\sqrt{a_1a_2}}{\sqrt{a_3}}\omega_2x_3, \quad dx_2 = \tfrac{\sqrt{a_1a_2}}{\sqrt{a_3}}\omega_1x_3 - \tfrac{\sqrt{a_2a_3}}{\sqrt{a_1}}\omega_3x_1, \\ dx_3 &= \tfrac{\sqrt{a_2a_3}}{\sqrt{a_1}}\omega_2x_1 - \tfrac{\sqrt{a_1a_3}}{\sqrt{a_2}}\omega_1x_2\,. \end{aligned} \tag{3.20}$$

Differentiating the point X with respect to fixed ellipsoid H' and using the equation (3.13), we have

$$\begin{aligned} d'\overrightarrow{X} = &\left(dx_1 - \tfrac{\sqrt{a_1a_3}}{\sqrt{a_2}}\omega'_3x_2 + \tfrac{\sqrt{a_1a_2}}{\sqrt{a_3}}\omega'_2x_3\right)\overrightarrow{r_1} + \left(dx_2 + \tfrac{\sqrt{a_2a_3}}{\sqrt{a_1}}\omega'_3x_1 - \tfrac{\sqrt{a_1a_2}}{\sqrt{a_3}}\omega'_1x_3\right)\overrightarrow{r_2} \\ &+ \left(dx_3 - \tfrac{\sqrt{a_2a_3}}{\sqrt{a_1}}\omega'_2x_1 + \tfrac{\sqrt{a_1a_3}}{\sqrt{a_2}}\omega'_1x_2\right)\overrightarrow{r_3}. \end{aligned} \tag{3.21}$$

So, the velocity vector of X with respect to the fixed ellipsoid H' is called the absolute velocity of this point and is denoted by $\overrightarrow{V_a} = \frac{d'\overrightarrow{x}}{dt}$. If $\overrightarrow{V_a} = 0$, namely $d'\overrightarrow{X} = 0$, then the point X becomes fixed on the fixed ellipsoid H'. Hence, the condition that the point X is fixed in H' is given by

$$\begin{aligned} dx_1 &= \tfrac{\sqrt{a_1a_3}}{\sqrt{a_2}}\omega'_3x_2 - \tfrac{\sqrt{a_1a_2}}{\sqrt{a_3}}\omega'_2x_3, \quad dx_2 = \tfrac{\sqrt{a_1a_2}}{\sqrt{a_3}}\omega'_1x_3 - \tfrac{\sqrt{a_2a_3}}{\sqrt{a_1}}\omega'_3x_1, \\ dx_3 &= \tfrac{\sqrt{a_2a_3}}{\sqrt{a_1}}\omega'_2x_1 - \tfrac{\sqrt{a_1a_3}}{\sqrt{a_2}}\omega'_1x_2. \end{aligned} \tag{3.22}$$

If the point X is fixed on moving ellipsoid H then the velocity of X with respect to H' is called sliding velocity of X and denoted by $\overrightarrow{V_f} = \frac{d_f\overrightarrow{X}}{dt}$. If the equation (3.20) is substituted into the equation (3.21), we find

$$\begin{aligned} d_f\overrightarrow{X} = &\left(\tfrac{\sqrt{a_1a_2}}{\sqrt{a_3}}\left(\omega'_2 - \omega_2\right)x_3 - \tfrac{\sqrt{a_1a_3}}{\sqrt{a_2}}\left(\omega'_3 - \omega_3\right)x_2\right)\overrightarrow{r_1} \\ &+ \left(\tfrac{\sqrt{a_2a_3}}{\sqrt{a_1}}\left(\omega'_3 - \omega_3\right)x_1 - \tfrac{\sqrt{a_1a_2}}{\sqrt{a_3}}\left(\omega'_1 - \omega_1\right)x_3\right)\overrightarrow{r_2} \\ &+ \left(\tfrac{\sqrt{a_1a_3}}{\sqrt{a_2}}\left(\omega'_1 - \omega_1\right)x_2 - \tfrac{\sqrt{a_2a_3}}{\sqrt{a_1}}\left(\omega'_2 - \omega_2\right)x_1\right)\overrightarrow{r_3}. \end{aligned} \tag{3.23}$$

Hence, the sliding velocity vector is

$$\begin{aligned} \overrightarrow{V_f} = &\left(\tfrac{\sqrt{a_1a_2}}{\sqrt{a_3}}\Psi_2\,x_3 - \tfrac{\sqrt{a_1a_3}}{\sqrt{a_2}}\Psi_3\,x_2\right)\overrightarrow{r_1} \\ &+ \left(\tfrac{\sqrt{a_2a_3}}{\sqrt{a_1}}\Psi_3\,x_1 - \tfrac{\sqrt{a_1a_2}}{\sqrt{a_3}}\Psi_1\,x_3\right)\overrightarrow{r_2} \\ &+ \left(\tfrac{\sqrt{a_1a_3}}{\sqrt{a_2}}\Psi_1\,x_2 - \tfrac{\sqrt{a_2a_3}}{\sqrt{a_1}}\Psi_2\,x_1\right)\overrightarrow{r_3}. \end{aligned}$$

The matrix representation of the above equation is

$$\overrightarrow{V_f} = X^t\Psi R \tag{3.24}$$

where $\Psi = S' - S$ and $R = \begin{bmatrix} \overrightarrow{r_1} \\ \overrightarrow{r_2} \\ \overrightarrow{r_3} \end{bmatrix}$.
If the Pfaffian vector $\overrightarrow{\Psi}$ is taken to be

$$\overrightarrow{\Psi} = a_1\Psi_1\,\vec{r}_1 + a_2\Psi_2\,\vec{r}_2 + a_3\Psi_3\,\vec{r}_3, \quad \Psi_i = \omega_i' - \omega_i, \quad 1 \leq i \leq 3 \tag{3.25}$$

then, the following equality can be given

$$\overrightarrow{V_f} = \Delta\left(\overrightarrow{\Psi} \wedge \overrightarrow{X}.\right) \tag{3.26}$$

Theorem 3.1. *In the one-parameter elliptic motion, the absolute velocity vector of the point X is the sum of the relative and sliding velocity vectors of the point X.*

Proof. Taking into account the equations (3.19), (3.21) and (3.23), the following identity can be easily seen

$$d_f\overrightarrow{X} = d'\overrightarrow{X} - d\overrightarrow{X}. \tag{3.27}$$

Moreover, we know that the identities $\overrightarrow{V_f} = \frac{d_f\overrightarrow{X}}{dt}$, $\overrightarrow{V_a} = \frac{d'\overrightarrow{x}}{dt}$, $\overrightarrow{V_r} = \frac{d\overrightarrow{x}}{dt}$ denotes the sliding velocity vector, absolute velocity vector and the relative velocity vector of the motion, respectively. Thus equation (3.27) yields to

$$\overrightarrow{V_a} = \overrightarrow{V_r} + \overrightarrow{V_f}. \tag{3.28}$$

□

Therefore, the following special case can be given.
Special Case: In the case of $a_1 = 1,\ a_2 = 1,\ a_3 = 1$, one-parameter elliptic motion corresponds to one-parameter spherical motion in three-dimensional Euclidean space. This motion was studied by Müller, [12].
In physics, the angular velocity of an object is the rate at which it rotates around a chosen center point. Privately, this center of rotation is chosen by origin. Angular velocity is the time rate of change of its angular displacement relative to the origin. In three dimensional space, the angular velocity is a pseudovector, with its magnitude measuring the rate of rotation, and its direction pointing along the axis of rotation (perpendicular to the radius and velocity vectors). Furthermore, the Darboux vector is known as the angular velocity vector of the Frenet frame of a space curve. Because if we consider the rigid object moving smoothly along the regular curve, the object will be seen to rotate the same way as its Frenet frame. Consequently, the following result can be given.

Result 3.1. *In the one-parameter elliptic motion, the infinitesimal rotational motion occurs at every point X belong to moving ellipsoid at any time t. In this rotation motion, the Pfaffian vector $\overrightarrow{\Psi}$ plays the role of Darboux rotation vector.*

The following theorems can be given considering the geometrical relationship between the Darboux rotation vector and the Pfaffian vector.

Theorem 3.2. *In the one-parameter elliptic motion, there exists pole points P and $\hat{P}$ (P is rotation pole and $\hat{P}$ is its opposite point) on the ellipsoid at any time t. Namely, these points remain stable on both of the ellipsoidal surfaces.*

Proof. Now, let us take an elliptical unit vector $\overrightarrow{p}$ in the direction of the rotation vector $\overrightarrow{\Psi}$. So,

$$\overrightarrow{\Psi} = \overrightarrow{p} \cdot \sqrt{a_1^2\Psi_1^2 + a_2^2\Psi_2^2 + a_3^2\Psi_3^2} \quad \text{and} \quad \|\overrightarrow{p}\|_B^2 = 1.$$

where $\Psi = \pm\left\|\overrightarrow{\Psi}\right\|_B = \sqrt{a_1^2\Psi_1^2 + a_2^2\Psi_2^2 + a_3^2\Psi_3^2}$ represents the infinitesimal rotation angle at the time period t. Moreover, $\overrightarrow{OP} = \overrightarrow{p}$ is the instantaneous rotation pole of the point P. This point is characterized by the sliding velocity vector becomes zero. If we consider the equation (3.26), the following identity can be written

$$\Delta\left(\overrightarrow{\Psi} \times \overrightarrow{X}\right) = 0.$$

Since $\Delta \neq 0$ and $\left\|\overrightarrow{X}\right\|_B = 1$, the above equality is satisfied when $\overrightarrow{X} = \pm\overrightarrow{p}$.
The rotation pole point P and its opposite point $\hat{P}$ remain stable. □

Therefore, the following theorems with respect to pole points of the one-parameter elliptic motion can be given.

Theorem 3.3. *Every point of the moving ellipsoid rotates around the pole point P (and its opposite point $\hat{P}$) with the angular velocity $\Psi : dt$ at every time t. Namely, one-parameter motion occurs such a rotation that the whole moving ellipsoidal surface H rotates with respect to the fixed ellipsoidal surface H' at a time t.*

Proof. Let us consider a rotation motion around an axis. Assume that this axis passes perpendicularly through the origin of the coordinate system. Let us denote the direction vector of this axis by $\overrightarrow{d}$ and take any point X on the ellipse which is affected by this rotation motion. Thus, (3.26) implies that the sliding vector $d_f\overrightarrow{x}$ is orthogonal to both $\overrightarrow{X}$ and $\overrightarrow{\Psi}$. Taking the norm of equation (3.26), we have

$$\begin{aligned} \|d_f\overrightarrow{x}\|_B &= \Delta\left\|\overrightarrow{\Psi}\right\|_B \left\|\overrightarrow{X}\right\|_B \sin\alpha \\ &= \pm\omega\, r. \end{aligned} \tag{3.29}$$

where $\left\|\overrightarrow{\Psi}\right\|_B = \pm\omega$ is the angular velocity, r is the distance of the point X from the rotation axis and α is an elliptical rotation angle between the direction vector $\overrightarrow{\Psi}$ and the position vector $\overrightarrow{X}$. Thus, we deduce that $d_f\overrightarrow{x}$ is the velocity vector of X and rotates around the axis $\overrightarrow{\Psi}$ with the angular velocity $\pm\omega$. If the angular velocity is positive, a right helix movement is occurred by the rotation in the direction of given a point X. Here, the sign of Ψ depends on the direction of the vector $\overrightarrow{p}$. $\square$

Thus, the following theorem can be given.

Theorem 3.4. *In the one-parameter elliptic motion, a point X of the moving ellipsoidal surface H draws a trajectory on the fixed ellipsoidal surface H' which passes every time from the rotation pole P (and its opposite point $\hat{P}$).*

Proof. The vectors $\overrightarrow{\Psi}$ and $\overrightarrow{X} = \overrightarrow{OX}$ form a plane which passes through the points P, $\hat{P}$ and X of the ellipsoid. This plane cuts ellipsoid through a great ellipse. If the point X is on the moving ellipsoidal surface H, from equation (3.26) the direction of progress $d_f\overrightarrow{x}$ of X becomes orthogonal to this great ellipse. Thus, the point X which is on the moving ellipsoidal surface H draws a trajectory on the fixed ellipsoidal surface H' that passes every time from the rotation pole point P (and its opposite point $\hat{P}$). $\square$

4. Conclusions

One-parameter elliptic motion is occurred by a point on an ellipse through an angle around a vector. To define this motion, first we have found the absolute, relative and sliding velocity vectors by means of three tri-axial ellipsoids. Then, the relationship has been presented between the Pfaffian vector $\overrightarrow{\Psi}$ and the sliding vector $d_f\overrightarrow{x}$ by means of theorems. Moreover, the pole points of the elliptical motion are obtained. Furthermore, it is emphasized that one parameter elliptical motion includes the one-parameter Euclidean spherical motion. So, we hope that this study will benefit the sciences of mathematics, physics and engineering.

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Affiliations

AYŞE ZEYNEP AZAK
ADDRESS: Sakarya University, Department of Mathematics and Science Education, 54300, Sakarya, Turkey.
E-MAIL: apirdal@sakarya.edu.tr
ORCID ID: 0000-0002-2686-6043