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Nonholonomic Frame for a Deformed (α, β)-metric

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ABSTRACT

Recently, in paper [14], we have introduced the following deformed (α, β)-metric:

$$F_\epsilon(\alpha, \beta) = \frac{\beta^2 + \alpha^2(a+1)}{\alpha} + \epsilon\beta$$

where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric; $\beta = b_iy^i$ is a 1-form, $|\epsilon| < 2\sqrt{a+1}$ is a real parameter and $a \in \left(\frac{1}{4}, +\infty\right)$ is a real positive scalar. The aim of this paper is to find the nonholonomic frame for this important kind of (α, β)-metric and also to investigate the Frobenius norm for the Hessian generated by this kind of metric.

Keywords: *Finsler (α, β)-metric, nonholonomic frame, projectively flat.*
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1. Introduction

The main purpose of this paper is to study the new perturbed (α, β)-metric

$$F_\epsilon(\alpha, \beta) = \frac{\beta^2 + \alpha^2(a+1)}{\alpha} + \epsilon\beta \tag{1.1}$$

where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric; $\beta = b_iy^i$ is a 1-form, $|\epsilon| < 2\sqrt{a+1}$ is a real parameter and $a \in \left(\frac{1}{4}, +\infty\right)$ is a real positive scalar. This metric was introduced in [14]. The perturbed (α, β)-metrics was first introduced by Matsumoto in [7] and since then, the theory of this kind of metrics in Finsler geometry was developed in a lot of papers (for example please see [17], [18]).

The theory of nonholonomic Finsler frame is important not just in Finsler geometry but also in physics, especially in quantum physics. The nonholonomic frame was studied by P.R.Holand when he analyzed the movement of a charged particle in an external electromagnetic field. First, let's recall some important results regarding the (α, β)-metrics:

As we know, (see [1]), the (α, β)-metric is defined in the following form: $F = \alpha\phi(s)$, where $s = \frac{\beta}{\alpha}$.
The function $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$ and it satisfies the following condition (see [12])

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \leq b < b_0$$

Also its a well known fact that F is a Finsler metric if and only if $||\beta_x||_\alpha < b_0$ for any $x \in M$.
The relationship between the geodesic coefficients of an (α, β)-metric F and α, namely G^i and G^i_α is presented in [9] in the following form:

$$G^i = G^i_\alpha + \frac{F_{|k}y^k}{2F}y^i + \frac{F}{2}g^{ij}\left(\frac{\partial F_{|k}}{\partial y^j}y^k - F_{|j}\right) \tag{1.2}$$

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Lemma 1.1. *([1]) The spray coefficients G^i are related to $\overline{G}^i$ by:*

$$G^i = G^i_\alpha + \alpha Q s^i_0 + J\{-2Q\alpha s_0 + r_{00}\}\frac{y^i}{\alpha} + H\{-2Q\alpha s_0 + r_{00}\}\left\{b_i - s\frac{y^i}{\alpha}\right\}$$

where $Q = \frac{\phi'}{\phi - s\phi'}$; $J = \frac{\phi'(\phi - s\phi')}{2\phi(\phi - s\phi' + (b^2 - s^2)\phi'')}$; $H = \frac{\phi''}{2(\phi - s\phi' + (b^2-s^2)\phi'')}$;
$s = \frac{\beta}{\alpha}$, $b = ||\beta_x||_\alpha$; $s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i})$, $s_{l0} = s_{li}y^i$, $s_0 = s_{l0}b^l$;
$G^i = \frac{g^{il}}{4}\left\{\left[F^2\right]_{x^k y^l} y^k - \left[F^2\right]_{x^k}\right\}$; $\overline{G}^i = \frac{a^{il}}{4}\left\{\left[\alpha^2\right]_{x^k y^l} y^k - \left[\alpha^2\right]_{x^k}\right\}$; $(g_{ij}) = \frac{1}{2}\left[F^2\right]_{y^i y^j}$ and $(a_{ij}) = (a_{ij})^{-1}$. Also, $r_{ij} = \frac{1}{2}\left(b_{i|j} + b_{j|i}\right)$, $r_{00} = r_{ij}y^i y^j$.

From paper [11], we know that a Finsler metric $F = F(x, y)$ on an open set $U \subset \mathbb{R}^n$ is projectively flat if and only if

$$F_{x^k y^l} y^k - F_{x^l} = 0.$$

In this respect, the following result holds

Lemma 1.2. *([11]) An (α, β)-metric $F = \alpha\phi(s)$, where $s = \frac{\beta}{\alpha}$ is projectively flat on an open subset $U \subset \mathbb{R}^n$, if and only if*

$$(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \alpha^3 Q s_{l0} + H\alpha(-2\alpha Q s_0 + r_{00})(b_l\alpha - s y_l) = 0$$

The homogenous polynomials in y^i of degree r, are denoted by $hp(r)$.
A well known result is the following one: A Finsler space F^n with an (α, β)- metric is a Douglas space if and only if $B^{ij} = B^i y^j - B^j y^i$ is hp(3). Also from [2], we know that:

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^*(b^i y^j - b^j y^i).$$

where

$$2G^i = \gamma^i_{00} + 2B^i$$

$$B^i = \frac{\alpha L_\beta}{L_\alpha} s^i_0 + C^*\left\{\frac{\beta L_\beta}{\alpha L} y^i - \frac{\alpha L_{\alpha\alpha}}{L_\alpha}\left(\frac{1}{\alpha}y^i - \frac{\alpha}{\beta}b^i\right)\right\}$$

$$\beta^2 + L_\alpha + \alpha\gamma^2 L_{\alpha\alpha} \neq 0; \quad \gamma^2 = b^2\alpha^2 - \beta^2; \quad L_\alpha = \frac{\partial L}{\partial \alpha}$$

$$L_\beta = \frac{\partial L}{\partial \beta}; L_{\alpha\alpha} = \frac{\partial L_\alpha}{\partial \alpha}$$

the subscript 0 means contraction by y^i and

$$C^* = \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}$$

1.1. Finsler spaces with (α, β)-metric

Definition 1.1. A Finsler space $F^n = (M, F(x, y))$ is endowed with (α, β) metric if there exist a 2-homogeneous function L of two variables such that the Finsler metric $F : TM \to \mathbb{R}$ is given by:

$$F^2(x, y) = L(\alpha(x, y), \beta(x, y)), \tag{1.3}$$

where $\alpha^2(x, y) = a_{ij}y^i y^j$, α is a Riemannian metric and $\beta(x, y) = b_i(x)y^i$ is a 1-form on M.

For the fundamental tensor of Finsler space $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$, the following results are well known from literature:

$$p^i = \frac{1}{\alpha}y^i = a_{ij}\frac{\partial \alpha}{\partial y^j}; \quad p_i = a_{ij}p^j = \frac{\partial \alpha}{\partial y^i};$$

$$l^i = \frac{1}{L}y^i = g_{ij}\frac{\partial L}{\partial y^j}; \quad l_i = g^{ij}\frac{\partial L}{\partial y^j} = p_i + b_i;$$

$$l_i = \frac{1}{L}p^i; \quad l^i l_j = p^i p_i = 1; \quad l^i p_i = \frac{\alpha}{L};$$
$$p^i l_i = \frac{L}{\alpha}; \quad b_i p^i = \frac{\beta}{\alpha}; \quad b_i l^i = \frac{\beta}{L}.$$

The relations between the metric tensors (as we know from [9], are: a_{ij} and g_{ij}) and are given by:

$$g_{ij} = \frac{L}{\alpha}a_{ij} + b_i p_j + p_i b_j + b_i b_j - \frac{\beta}{\alpha}p_i p_j = \frac{L}{\alpha}(a_{ij} - p_i p_j) + l_i l_j.$$

Let U be an open set of the tangent bundle of a Finsler manifold M endowed with the Finsler metric F and $V_i : u \in U \to V_i(u) \in V_u TM$, $i \in \{1, ..., n\}$ be a vertical frame over U. If $V_i(u)\frac{\partial}{\partial y^j}|_u$, then $V_i^j(u)$ are the entries of a invertible matrix for all $u \in U$. Denote by $\tilde{V}_k^j(u)$ the inverse of this matrix. This means that $V_j^i \tilde{V}_k^j = \delta_k^i$, $\tilde{V}_i^j(u) V_k^j = \delta_k^i$. We call V_j^i a nonholonomic Finsler frame.
In [9] is presented the following Theorem

Theorem 1.1. *([9]) For a Finsler space (M, F), with the (α, β)-metric $F = \alpha\phi(s)$, consider the matrix with the entries*

$$Y_i^j = \sqrt{\frac{\alpha}{L}}\left(\delta_j^i - l_i l_j + \sqrt{\frac{\alpha}{L}}p^i p_j\right) \tag{1.4}$$

defined on TM. Then, $Y_j = Y_j^i\left(\frac{\partial}{\partial y^i}\right)$, $j \in \{1, 2, ..., n\}$ is a nonholonomic frame.

Theorem 1.2. *([9]) With respect to this frame, the holonomic components of the Finsler metric tensor (g_{ij}) are given by*

$$g_{ij} = Y_i^\alpha Y_j^\beta a_{\alpha\beta}.$$

From [11], we know that for a Finsler space with (α, β)-metric $F^2(x, y) = L(\alpha(x, y), \beta(x, y))$, we have the following Finsler invariants

$$\rho_1 = \frac{1}{2\alpha}\frac{\partial L}{\partial \alpha}; \quad \rho_0 = \frac{1}{2}\frac{\partial^2 L}{\partial \beta^2}; \quad \rho_{-1} = \frac{1}{2\alpha}\frac{\partial^2 L}{\partial \alpha \partial \beta}; \quad \rho_{-2} = \frac{1}{2\alpha^2}\left(\frac{\partial^2 L}{\partial \alpha^2} - \frac{1}{\alpha}\frac{\partial L}{\partial \alpha}\right), \tag{1.5}$$

where subscripts $-2, -1, 0, 1$, gives us the degree of homogenity of these invariants. For a Finsler space with (α, β)-metric, we know from [11], that

$$\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0 \tag{1.6}$$

and the metric tensor g_{ij} of a Finsler space with (α, β)-metric, is given by:

$$g_{ij}(x, y) = \rho_1 a_{ij}(x) + \rho_0 b_i(x) + \rho_{-1}\left(b_i(x) y_j + b_j(x) y_i\right) + \rho_{-2} y_i y_j. \tag{1.7}$$

From (1.5), we see that g_{ij} can be obtained using two Finsler deformations

$$\begin{cases} a_{ij} \to h_{ij} = \rho_1 a_{ij} + \frac{1}{\rho_{-2}}\left(\rho_{-1} b_i + \rho_{-2} y_i\right)\left(\rho_{-1} b_j + \rho_{-2} y_j\right) \\ h_{ij} \to g_{ij} = h_{ij} + \frac{1}{\rho_{-2}}\left(\rho_0 \rho_{-2} - \rho_{-1}^2\right) b_i b_j. \end{cases} \tag{1.8}$$

The Finslerian nonholonomic frame that corresponds to the first and second deformation respectively, according to [3], is given by

$$X_j^i = \sqrt{\rho_1}\delta_j^i - \frac{1}{B^2}\left(\sqrt{\rho_1} \pm \sqrt{\rho_1 + \frac{B^2}{\rho_{-2}}}\right)(\rho_{-1} b^i + \rho_{-2} y^i)(\rho_{-1} b^j + \rho_{-2} y^j); \tag{1.9}$$

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\left(1 \pm \sqrt{1 + \frac{\rho_{-2} C^2}{\rho_0 \rho_{-2}} - \rho_{-1}^2}\right) b_i b_j, \tag{1.10}$$

where B and C, are given by

$$B^2 = a_{ij}(\rho_{-1} b^i + \rho_{-2} y^i)(\rho_{-1} b^j + \rho_{-2} y^j) = \rho_{-1}^2 b^2 + \beta \rho_1 \rho_{-2};$$

$$C^2 = h_{ij} b^i b^j = \rho_1 b^2 + \frac{1}{\rho_{-2}}(\rho_{-1} b^2 + \rho_{-2}\beta)^2.$$

Remark 1.1. The metric tensors a_{ij} and h_{ij} are related by $h_{ij} = X_i^k X_j^l a_{kl}$. Also, the metric tensors h_{ij} and g_{ij} are related by $g_{mn} = Y_m^i Y_n^j h_{ij}$.

Some other important results regarding the above mentioned results also in Finsler geometry, are contained in the following papers: [4], [5], [6], [7], [8], [10], [13], [16].

2. Main Results

First, in this section we will compute the odd part of the metric (1.1). We do that to verify the fact that our assumption regarding the parameter ϵ is true. For all $|s| < 1$ and the Riemannian metric α, we consider the 1-form β with $\|\beta\| < 1$. The odd part F_a of the metric (1.1) could be computed as follows

$$F_a(x, y) = \frac{F(x, y) - F(x, -y)}{2} = \alpha(x, y)\phi_a\left(\frac{\beta(x, y)}{\alpha(x, y)}\right).$$

The function $\phi(s)$, associated with the metric (1.1), is $\phi(s) = s^2 + \epsilon s + a + 1$ with $|\epsilon| < 2\sqrt{a+1}$.
So, one obtains:

$$\phi_a(s) = \frac{\phi(s) - \phi(-s)}{2} = \frac{(s^2 + \epsilon s + a + 1) - (s^2 - \epsilon s + a + 1)}{2} = s\epsilon.$$

Next, we choose a local bundle coordinate (x^i, y^i) on the tangent space of the manifold. Then, one obtains

$$F_a\left(F_a\right)_{y^i y^j} b^i b^j = \left(\phi_a^2(t) - t\phi_a(t) \cdot \phi_a'(t)\right) t^2 + t^4 \phi_a(t)\phi_a''(t).$$
$$= \left((t^2 + \epsilon t + a + 1)^2 - t(t^2 + \epsilon t + a + 1)\epsilon\right) t^2 > 0$$

This inequality hold for the above assumption $|\epsilon| < 2\sqrt{a+1}$, because the above inequality is equivalent with

$$t^2\left(\left(t + \frac{\epsilon}{2}\right)^2 + 2\left(a + 1 - \frac{\epsilon^2}{4}\right)\right) + \left(a + 1 + \frac{\epsilon t}{2}\right)^2 > 0,$$

and from this inequality, the conclusion follows easily. In the view of Lemma 1.1, the link between the spray coefficients G^i of the metric (1.1) and the G^i_α of the metric α, is presented in [14].

Next, we will compute the following Finsler invariants, attached for the metric (1.1)

$$\rho_1 = \frac{1}{2\alpha}\frac{\partial L}{\partial \alpha}; \quad \rho_0 = \frac{1}{2}\frac{\partial^2 L}{\partial \beta^2}; \quad \rho_{-1} = \frac{1}{2\alpha}\frac{\partial^2 L}{\partial \alpha \partial \beta}; \quad \rho_{-2} = \frac{1}{2\alpha^2}\left(\frac{\partial^2 L}{\partial \alpha^2} - \frac{1}{\alpha}\frac{\partial L}{\partial \alpha}\right).$$

First, let us remark that

$$L\left(\alpha(x, y), \beta(x, y)\right) = \left(\frac{\beta^2 + \alpha^2(a+1)}{\alpha} + \epsilon\beta\right)^2.$$

We will use the results from subsection 1.1. presented in Introduction of this paper.

Now, we will compute the Finsler invariants for the (α, β)-metric (1.1), using (1.4). After tedious computations, one obtains:

$$\rho_1 = \frac{\left((a+1)\,\alpha^2 + \varepsilon\,\beta\,\alpha + \beta^2\right)\left((a+1)\,\alpha^2 - \beta^2\right)}{\alpha^4} \tag{2.1}$$

$$\rho_{-1} = \frac{\varepsilon\,(a+1)\,\alpha^3 - 3\,\alpha\,\beta^2\varepsilon - 4\,\beta^3}{\alpha^4} \tag{2.2}$$

$$\rho_{-2} = -\frac{\beta\,\left(\varepsilon\,(a+1)\,\alpha^3 - 3\,\alpha\,\beta^2\varepsilon - 4\,\beta^3\right)}{\alpha^6} \tag{2.3}$$

$$\rho_0 = \frac{\left(\varepsilon^2 + 2\,a + 2\right)\alpha^2 + 6\,\varepsilon\,\beta\,\alpha + 6\,\beta^2}{\alpha^2} \tag{2.4}$$

Using (1.8) and (1.9), we can formulate now: the nonholonomic Finsler frame that corresponds to the first deformation and second deformation respectively for metric (1.1) is as follows

$$X^i_j = \sqrt{\rho_1}\delta^i_j - \frac{1}{B^2}\left(\sqrt{\rho_1} \pm \sqrt{\rho_1 + \frac{B^2}{\rho_{-2}}}\right)(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b^j + \rho_{-2}y^j); \tag{2.5}$$

$$Y^i_j = \delta^i_j - \frac{1}{C^2}\left(1 \pm \sqrt{1 + \frac{\rho_{-2}C^2}{\rho_0\rho_{-2}} - \rho_{-1}^2}\right) b_i b_j, \tag{2.6}$$

Now, using this relations, we can formulate the following

Theorem 2.1. *For a Finsler space endowed with the metric* $L = F^2 = \left(\beta + \frac{a\alpha^2+\beta^2}{\alpha}\right)^2$*, the following frame*

$$V_j^i = X_k^i Y_j^k$$

represents a Finslerian nonholonomic frame with the components X_k^i*, respectively* Y_j^k*, given by (2.5) and (2.6) and respectively (2.1)-(2.g4).*

Next, we will investigate the Hessian associated with the above metric (1.1)

$$HL(\alpha,\beta) = \begin{pmatrix} \frac{\partial^2 L}{\partial\alpha^2} & \frac{\partial^2 L}{\partial\alpha\partial\beta} \\ \frac{\partial^2 L}{\partial\alpha\partial\beta} & \frac{\partial^2 L}{\partial\beta^2} \end{pmatrix} \tag{2.7}$$

where

$$\frac{\partial^2 L}{\partial\alpha^2} = \frac{2\,(a+1)^2\,\alpha^4 + 4\,\beta^3\alpha\,\epsilon + 6\,\beta^4}{\alpha^4}$$

$$\frac{\partial^2 L}{\partial\alpha\partial\beta} = \frac{2\,\varepsilon\,\,(a+1)\,\,\alpha^3 - 6\,\alpha\,\beta^2\,\varepsilon - 8\,\beta^3}{\alpha^3}$$

$$\frac{\partial^2 L}{\partial\beta^2} = \frac{\left(2\,\epsilon^2 + 4\,a + 4\right)\alpha^2 + 12\,\epsilon\,\beta\,\alpha + 12\,\beta^2}{\alpha^2}.$$

First of all, let us find the determinant of the Hessian matrix associated with the Finsler metric: After computations using also Maple software, one obtains:

$$detHL(\alpha,\beta) = 8\,\frac{\left((a+1)\,\alpha^2 + \varepsilon\,\beta\,\alpha + \beta^2\right)^3}{\alpha^6} \tag{2.8}$$

Now we can prove the following theorem

Theorem 2.2. *For the metric (1.1), the determinant of the associated Hessian matrix is positive, i.e.*

$$detHL(\alpha,\beta) > 0$$

for $|\epsilon| < 2\sqrt{a+1}$*.*

Proof. Rewriting the above determinant, one obtains

$$detHL(\alpha,\beta) = \frac{8}{\alpha^6}\left(\alpha^2\left(a+1-\frac{\epsilon^2}{4}\right) + \left(\beta + \frac{\epsilon\alpha}{2}\right)^2\right)^3,$$

and we can observe immediately that the equation hold for $|\epsilon| < 2\sqrt{a+1}$ and this conclude the proof of the theorem. □

Let us recall first, some properties of Frobenius (Hilbert-Schmidt) norms from [15].
The Frobenius norm of a matrix $A = (A_{ij})$, is defined as follows

$$\|A\|_{HS} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}^2}.$$

Some of its properties are:

- $\|A \cdot B\|_{HS} \le \|A\|_{HS}\,\|B\|_{HS}$;
- $\|A \cdot B\|_{HS}^2 = \sum_{j=1}^{n} \|Ab_j\| \le \|A\|_{HS}^2 \cdot \|B\|_{HS}^2$;
- $\|FG\|_{HS}^2 = Tr\left(FGG^T F^T\right) = Tr\left(F^T FGG^T\right)$.

Here we denote by $\|\cdot\|_{HS}$ the Hilbert-Schmidt norm, which is also called Frobenius norm.
Let us recall the above mentioned Hessian matrix and let us put this metric as follows:

$$HL(\alpha,\beta) = \begin{pmatrix} \frac{2\,(a+1)^2\alpha^4 + 4\,\beta^3\alpha\,\epsilon + 6\,\beta^4}{\alpha^4} & \frac{2\,\varepsilon\,\,(a+1)\,\,\alpha^3 - 6\,\alpha\,\beta^2\,\varepsilon - 8\,\beta^3}{\alpha^3} \\ \frac{2\,\varepsilon\,\,(a+1)\,\,\alpha^3 - 6\,\alpha\,\beta^2\,\varepsilon - 8\,\beta^3}{\alpha^3} & \frac{\left(2\,\epsilon^2 + 4\,a + 4\right)\alpha^2 + 12\,\epsilon\,\beta\,\alpha + 12\,\beta^2}{\alpha^2} \end{pmatrix} \tag{2.9}$$

Next, we will give a proof to the following theorem, in which we will established a link between the Frobenius norm of the Hessian of the metric and the determinant of the Hessian of the same metric $L(\alpha,\beta)$.

Theorem 2.3. *For the metric*

$$L(\alpha(x,y), \beta(x,y)) = \left(\frac{\beta^2 + \alpha^2(a+1)}{\alpha} + \epsilon\beta\right)^2$$

associated with the metric (1.1) F_ϵ, the following inequality holds

$$\|HL(\alpha,\beta)\|_{HS} \leq$$

$$\sqrt{4\,\frac{\left(\left(a+1\right)^2\alpha^4 + 2\,\beta^3\alpha\,\epsilon + 3\,\beta^4\right)^2}{\alpha^8} + \frac{\left(2\,\epsilon\,\left(a+1\right)\alpha^3 - 6\,\alpha\,\beta^2\epsilon - 8\,\beta^3\right)^2}{\alpha^6} + \frac{\left(detHL\left(\alpha,\beta\right)\right)^2\alpha^4}{2\,\left(a+1\right)^2\alpha^4 + 4\,\beta^3\alpha\,\epsilon + 6\,\beta^4}}. \quad (2.10)$$

where

$$\frac{2\,\left(a+1\right)^2\alpha^4 + 4\,\beta^3\alpha\,\epsilon + 6\,\beta^4}{\alpha^4} \in [-1,0) \cup (0,1].$$

Here $\|\cdot\|_{HS}$ represent the Frobenius (Hilbert-Schmidt) norm for the matrix $HL(\alpha,\beta)$.

Proof. First of all, we will recall that the Gauss decomposition for a matrix M of second order can be done as follows

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & \frac{ad-bc}{a} \end{pmatrix}$$

where $ad - bc \neq 0$ and $a \neq 0$. Taking into account the above mentioned properties of the Frobenius norms, let us remark the following

$$\|M\|_{HS} \leq \sqrt{2 + \frac{c^2}{a^2}} \cdot \sqrt{a^2 + b^2 + \frac{(ad-bc)^2}{a^2}}.$$

From this, we can deduce the following

$$\sqrt{2 + \frac{c^2}{a^2}} \cdot \sqrt{a^2 + b^2 + \frac{(ad-bc)^2}{a^2}} \geq \sqrt{a^2 + b^2 + \frac{(ad-bc)^2}{a^2}} \geq ad - bc.$$

The last inequality is equivalent with

$$a^2 + b^2 + \frac{(ad-bc)^2(1-a^2)}{a^2} \geq 0,$$

and is easy to see that this inequality holds just for $1 - a^2 \geq 0$, or $a \in [-1, 1]$.
For our metric, this inequality and condition is equivalent with

$$\frac{2\,\left(a+1\right)^2\alpha^4 + 4\,\beta^3\alpha\,\epsilon + 6\,\beta^4}{\alpha^4} \in [-1,0) \cup (0,1].$$

Here we exclude the zero value because this represent another important assumed condition. Having in mind the above results, let us apply those results for the Hessian matrix (2.9). We can remark that this Hessian matrix can be rewritten as product of two matrices using the Gauss decomposition, as follows

$$HL(\alpha,\beta) = \begin{pmatrix} 1 & 0 \\ \frac{\left(\epsilon\,(a+1)\alpha^3 - 3\,\alpha\,\beta^2\epsilon - 4\,\beta^3\right)\alpha}{(a+1)^2\alpha^4 + 2\,\beta^3\alpha\,\epsilon + 3\,\beta^4} & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{2\,(a+1)^2\alpha^4 + 4\,\beta^3\alpha\,\epsilon + 6\,\beta^4}{\alpha^4} & \frac{2\,\epsilon\,(a+1)\alpha^3 - 6\,\alpha\,\beta^2\epsilon - 8\,\beta^3}{\alpha^3} \\ 0 & 8\,\frac{\left((a+1)\alpha^2 + \epsilon\,\beta\,\alpha + \beta^2\right)^3}{\alpha^2\left(2\,(a+1)^2\alpha^4 + 4\,\beta^3\alpha\,\epsilon + 6\,\beta^4\right)} \end{pmatrix}.$$

We will denote by A respectively with B the above two metrics and we will apply the properties of the Frobenius norms. In this respect, we have

$$A = \begin{pmatrix} 1 & 0 \\ \frac{\left(\epsilon\,(a+1)\alpha^3 - 3\,\alpha\,\beta^2\epsilon - 4\,\beta^3\right)\alpha}{(a+1)^2\alpha^4 + 2\,\beta^3\alpha\,\epsilon + 3\,\beta^4} & 1 \end{pmatrix}.$$

and

$$B = \begin{pmatrix} \frac{2\,(a+1)^2\alpha^4 + 4\,\beta^3\alpha\,\epsilon + 6\,\beta^4}{\alpha^4} & \frac{2\,\epsilon\,(a+1)\alpha^3 - 6\,\alpha\,\beta^2\epsilon - 8\,\beta^3}{\alpha^3} \\ 0 & 8\,\frac{\left((a+1)\alpha^2 + \epsilon\,\beta\,\alpha + \beta^2\right)^3}{\alpha^2\left(2\,(a+1)^2\alpha^4 + 4\,\beta^3\alpha\,\epsilon + 6\,\beta^4\right)} \end{pmatrix}.$$

Now taking into account that the following inequality holds for every Frobenius norms:

$$\|A \cdot B\|_{HS} \le \|A\|_{HS} \cdot \|B\|_{HS};$$

and also taking into account the above proved inequalities, we conclude that the following inequality take place

$$\|HL(\alpha,\beta)\|_{HS} \le$$

$$\sqrt{4\,\frac{\left((a+1)^2\alpha^4+2\,\beta^3\alpha\,\epsilon+3\,\beta^4\right)^2}{\alpha^8}+\frac{(2\,\epsilon\,(a+1)\,\alpha^3-6\,\alpha\,\beta^2\epsilon-8\,\beta^3)^2}{\alpha^6}+\frac{(detHL\,(\alpha,\beta))^2\,\alpha^4}{2\,(a+1)^2\,\alpha^4+4\,\beta^3\alpha\,\epsilon+6\,\beta^4}}$$

with

$$\frac{2\,(a+1)^2\,\alpha^4+4\,\beta^3\alpha\,\epsilon+6\,\beta^4}{\alpha^4} \in [-1,0)\cup(0,1].$$

So, the proof of the theorem is done. □

Example 2.1. In the above theorem if we choose $a=\epsilon=1$, one obtains the following inequality for the Frobenius norm of the Hessian of the function $L(\alpha,\beta)$:

$$\|HL(\alpha,\beta)\|_{HS} \le$$

$$\sqrt{\frac{(8\,\alpha^4+4\,\beta^3\alpha+6\,\beta^4)^2}{\alpha^8}+2\,\frac{(4\,\alpha^3-6\,\alpha\,\beta^2-8\,\beta^3)^2}{\alpha^6}+64\,\frac{(2\,\alpha^2+\beta\,\alpha+\beta^2)^6}{\alpha^8\,(8\,\alpha^4+4\,\beta^3\alpha+6\,\beta^4)^2}}$$

and from this we get

$$\|HL(\alpha,\beta)\|_{HS} \le \sqrt{\frac{(8\,\alpha^4+4\,\beta^3\alpha+6\,\beta^4)^4+2\,(4\,\alpha^3-6\,\alpha\,\beta^2-8\,\beta^3)^2\,\alpha^2+64\,(2\,\alpha^2+\beta\,\alpha+\beta^2)^6}{\alpha^8\,(8\,\alpha^4+4\,\beta^3\alpha+6\,\beta^4)^2}}$$

Finally, let us remark that the right term in this inequality could be easy minimized and this means that the Frobenius norm of the Hessian matrix $L(\alpha,\beta)$ of this metric could be bounded. This aspect is very interesting especially in Finsler geometry and could be a starting point to study the bound of Frobenius norms of the Hessian of Finsler metrics.

3. Conclusion

In this paper we have continued our investigation on the deformed (α,β)-metric (1.1) and we succeed to obtain a nonholonomic frame for this kind of metric. Also, we have obtined an important and strong result concerning an inequality between the Frobenius norm and the determinant of the Hessian matrix for this kind of deformed metric. As we have seen this could lead us to establish some results regarding the bound of Frobenius norms of the Hessian of a Finsler metric. In our future papers we will try to investigate other types of Finsler (α,β)-metrics from the above points of view.

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