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Surfaces Using a Common Geodesic Curve With an Alternative Moving Frame in The 3-Dimensional Lie Group

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ABSTRACT

Our purpose in this research is to use an alternative moving frame in the 3-dimensional Lie group G to construct the problem of how to characterize a surface family and derive the conditions from a given common geodesic curve as an isoparametric curve. We also derive the relation about developability along the common geodesic of a ruled surface as a member of the surface family. Finally, we will give some examples to show some applications of the method.

Keywords: Alternative Moving frame, Lie group, Geodesic, Ruled surface.

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Introduction

The curve and surface theory are the important areas of the different ambient spaces. In particular, these theories and their geometric properties have been examined by many researchers in the Lie group, [1-8].

On the other hand, the geodesic principle has played an important role in lots of areas, such as the geometric design of the hull, multi-scale analysis of images, the relativistic description of gravity. Nowadays, a good deal of research on surface theory is focused on a surface family having a given isogeodesic curve which is both a geodesic and an isoparametric curve. From this aspect, many researchers have derived a parametric representation of a surface family whose members share the same isogeodesic curve, [9-17].

In this paper, we define the surface family as a linear combination of the components of the alternative moving frame in the Lie group by utilizing this $\{N, C, W\}$ frame along the given geodesic and deriving the conditions for the coefficients to satisfy both the geodesic and the isoparametric requirements. We also present some examples to give the surface family and a ruled surface possessing a given common isogeodesic curve.

Materials and Methods

In this section, we will review some concepts related to the Lie group. For general information about the Lie group, we refer to [2, 6, 7].

The Frenet formulas in the Lie Group for a unit speed curve $\alpha(s)$

$$T' = \kappa N,$$
$$N' = -\kappa T + (\tau - \tau_G)B,$$
$$B' = -(\tau - \tau_G)N,$$

where κ and τ are the curvature functions of $\alpha(s)$ and τ_G, which is introduced [2, 7], is the Lie group torsion of $\alpha(s)$.

On the other hand, the alternative moving frame along the $a(s)$ is defined by the three vectors N,

$$C = \frac{N'}{\|N'\|} = -\frac{1}{\sqrt{1+h^2}}T + \frac{h}{\sqrt{1+h^2}}B$$

and $W = \frac{h}{\sqrt{1+h^2}}T + \frac{1}{\sqrt{1+h^2}}B$, which called the unit principal normal vector, the derivative of the principal normal vector and the unit Darboux vector, respectively. Then the derivatives of this frame is given by

$$N' = f(s)C,$$
$$C' = -f(s)N + g(s)W, \tag{1}$$
$$W' = -g(s)C,$$

where $f(s) = \kappa\sqrt{1+h^2}$, $g(s) = f(s)\frac{1}{\sigma_N} = \frac{h'}{1+h^2}$ and $\sigma_N(s) = \frac{\kappa(1+h^2)^{\frac{3}{2}}}{h'}$.

So $f(s)$ and $g(s)$ are curvatures of $\alpha(s)$ in terms of alternative moving frame in Lie Group G, [3, 5].

Here, $h=\frac{\tau-\tau_G}{\kappa}$ is denoted the harmonic curvature function of $\alpha(s)$ which is introduced in [6].

Theorem 1.

The curve is a general helix in the Lie Group G if and only if its harmonic function is a constant, [2,6].

Results

In this section, we will give an algorithm for constructing surfaces from an isogeodesic curve in the Lie Group.

Surfaces

Let $\alpha(s)$ be an arc length parametrized curve on a surface $P(s,t)$ in G. Then the curve α is called an isoparametric curve if it is a parameter curve, that is, there exists a parameter t_0 such that $\alpha(s)=P(s,t_0)$. Also the curve $\alpha(s)$ on the surface $P(s,t)$ is geodesic if and only if $N(s)\|\eta(s,t_0)$ where $\eta(s,t)$ is the normal vector of the surface $P(s,t)$. Then, a given curve $\alpha(s)$ is called an isogeodesic of $P(s,t)$ if it is both a geodesic and an isoparametric curve on $P(s,t)$.

Let $P=P(s,t)$ be a parametric surface through a curve $\alpha(s)$. This surface is given based on the curve $\alpha(s)$ and the alternative moving frame $\{N,C,W\}$ as follows

$$P(s,t)=\alpha(s)+\lambda_1(s,t)N(s)+\lambda_2(s,t)C(s)+\lambda_3(s,t)W(s), \quad (2)$$
$$0\le s\le L \text{ and } 0\le t\le T,$$

where $\lambda_1(s,t)$, $\lambda_2(s,t)$ and $\lambda_3(s,t)$ are all C^1 functions. These functions are called the marching-scale functions.

Since $\alpha(s)$ is an isoparametric curve on this surface, there exists a parameter $t=t_0\in[0,T]$ such that $\alpha(s)=P(s,t_0)$, $0\le s\le L$, that is,

$$\lambda_1(s,t_0)=\lambda_2(s,t_0)=\lambda_3(s,t_0)=0, \quad (3)$$
$$0\le s\le L \text{ and } t_0\in[0,T].$$

On the other hand, we know that the curve $\alpha(s)$ on the surface $P(s,t)$ is a geodesic if and only if $N(s)\|\eta(s,t_0)$.

To compute $\eta(s,t_0)$, using the alternative Frenet formula (1), one can easily get that

$$\frac{\partial P(s,t)}{\partial s}=\left(\frac{\partial\lambda_1(s,t)}{\partial s}-f(s)\lambda_2(s,t)\right)N(s)$$
$$+\left(-\frac{\kappa(s)}{f(s)}+f(s)\lambda_1(s,t)+\frac{\partial\lambda_2(s,t)}{\partial s}-g(s)\lambda_3(s,t)\right)C(s)$$
$$+\left(\frac{\kappa(s)h(s)}{f(s)}+g(s)\lambda_2(s,t)+\frac{\partial\lambda_3(s,t)}{\partial s}\right)W(s)$$

And

$$\frac{\partial P(s,t)}{\partial t}=\frac{\partial\lambda_1(s,t)}{\partial t}N(s)+\frac{\partial\lambda_2(s,t)}{\partial t}C(s)+\frac{\partial\lambda_3(s,t)}{\partial t}W(s),$$

using the value of T in terms of the alternative moving frame as $T=-\frac{\kappa(s)}{f(s)}C+\frac{\kappa(s)h(s)}{f(s)}W$.

So, the normal $\eta(s,t)$ can be calculated as

$$\eta(s,t)=\frac{\partial P(s,t)}{\partial s}\times\frac{\partial P(s,t)}{\partial t}$$

$$=\left[\left(-\frac{\kappa(s)}{f(s)}+f(s)\lambda_1(s,t)+\frac{\partial\lambda_2(s,t)}{\partial s}-g(s)\lambda_3(s,t)\right)\frac{\partial\lambda_3(s,t)}{\partial t}\right.$$
$$\left.-\left(\frac{\kappa(s)h(s)}{f(s)}+g(s)\lambda_2(s,t)+\frac{\partial\lambda_3(s,t)}{\partial s}\right)\frac{\partial\lambda_2(s,t)}{\partial t}\right]N(s)$$
$$+\left[-\left(\frac{\partial\lambda_1(s,t)}{\partial s}-f(s)\lambda_2(s,t)\right)\frac{\partial\lambda_3(s,t)}{\partial t}\right.$$
$$\left.-\left(\frac{\kappa(s)h(s)}{f(s)}+g(s)\lambda_2(s,t)+\frac{\partial\lambda_3(s,t)}{\partial s}\right)\frac{\partial\lambda_1(s,t)}{\partial t}\right]C(s)$$
$$+\left[\left(\frac{\partial\lambda_1(s,t)}{\partial s}-f(s)\lambda_2(s,t)\right)\frac{\partial\lambda_2(s,t)}{\partial t}\right.$$
$$\left.-\left(-\frac{\kappa(s)}{f(s)}+f(s)\lambda_1(s,t)+\frac{\partial\lambda_2(s,t)}{\partial s}-g(s)\lambda_3(s,t)\right)\frac{\partial\lambda_1(s,t)}{\partial t}\right]W(s).$$

Then, from (3), we get

$$\eta(s,t_0)=P_1(s,t_0)N(s)+P_2(s,t_0)C(s)+P_3(s,t_0)W(s),$$

where

$$P_1(s,t_0)=-\frac{\kappa(s)}{f(s)}\frac{\partial\lambda_3(s,t_0)}{\partial t}-\frac{\kappa(s)h(s)}{f(s)}\frac{\partial\lambda_2(s,t_0)}{\partial t}, \quad (4)$$

$$P_2(s,t_0)=\frac{\kappa(s)h(s)}{f(s)}\frac{\partial\lambda_1(s,t_0)}{\partial t},$$

$$P_3(s,t_0)=\frac{\kappa(s)}{f(s)}\frac{\partial\lambda_1(s,t_0)}{\partial t}.$$

On the other hand, we know that $N(s)\|\eta(s,t_0)$ if and only if

$$P_1(s,t_0)\neq 0, P_2(s,t_0)=0 \text{ and } P_3(s,t_0)=0$$

As a result of combining (3) and (4), we have obtained the conditions for $P(s,t)$ to have the curve α as an isogeodesic in the Lie group as shown by the following theorem.

Theorem 2.

The given curve $\alpha(s)$ is an isogeodesic curve on the surface $P(s,t)$ which given by (2) in the Lie group if the following conditions are satisfied

$$\lambda_1(s,t_0)=\lambda_2(s,t_0)=\lambda_3(s,t_0)=0,$$

$$\frac{\partial\lambda_1(s,t_0)}{\partial t}=0 \text{ and } \frac{\kappa(s)}{f(s)}\frac{\partial\lambda_3(s,t_0)}{\partial t}+\frac{\kappa(s)h(s)}{f(s)}\frac{\partial\lambda_2(s,t_0)}{\partial t}=const.\neq 0, \tag{5}$$

where $0\leq s\leq L$ and $t_0\in[0,T]$.

For the aim of simplification and analysis, $\lambda_1(s,t)$, $\lambda_2(s,t)$ and $\lambda_3(s,t)$ can be decomposed into two factors

$$\lambda_1(s,t)=\vartheta(s)X(t),$$
$$\lambda_2(s,t)=\rho(s)Y(t),$$
$$\lambda_3(s,t)=\sigma(s)Z(t),$$

where $\theta(s),\rho(s),\sigma(s)$, $X(t),Y(t)$, and $Z(t)$ are all C^1 functions and $\theta(s),\rho(s)$ and $\sigma(s)$ are not identically zero.

If $\alpha(s)$ is an isogeodesic curve on the surface $P(s,t)$, then the sufficient conditions can be easily given as follows

$$X(t_0)=Y(t_0)=Z(t_0)=0,$$
$$\theta(s)=0 \text{ or } \left.\frac{dX}{dt}\right|_{t_0}=0, \tag{6}$$
$$\frac{\kappa(s)\sigma(s)}{f(s)}\frac{dZ(t_0)}{dt}+\frac{\kappa(s)h(s)\rho(s)}{f(s)}\frac{dY(t_0)}{dt}=c=const.\neq 0,$$

where $0\leq s\leq L$ and $t_0\in[0,T]$.

Ruled Surface with Common Geodesic Curve in Lie Group

Let $P(s,t)$ be a ruled surface with the directrix $\alpha(s)$ which is also an isoparametric curve on P. Then there exists a parameter $t=t_0$ such that $P(s,t_0)=\alpha(s)$. Thus, the surface $P(s,t)$ can be written as follows

$$P(s,t)-P(s,t_0)=(t-t_0)\zeta(s),$$

where $0\leq s\leq L$, $0\leq t\leq T$ and $t_0\in[0,T]$ and $\zeta(s)$ denotes the direction of the rulings.

By using (2), the surface is equivalent to

$$(t-t_0)\zeta(s)=\lambda_1(s,t)N(s)+\lambda_2(s,t)C(s)+\lambda_3(s,t)W(s). \tag{7}$$

Since the curve $\alpha(s)$ is the isogeodesic curve, by using the conditions given Theorem 2, one can easily write that

$$\lambda_1(s,t)=0,\ \lambda_2(s,t)=\mu(s)(t-t_0),\ \lambda_3(s,t)=\nu(s)(t-t_0),$$

where $\mu(s)$ and $\nu(s)$ are some real functions.

So, substituting these values of $\lambda_1(s,t)$, $\lambda_2(s,t)$ and $\lambda_3(s,t)$ into (7), we easily obtain

$$\zeta(s)=\mu(s)C(s)+\nu(s)W(s), \tag{8}$$

for all $s\in[0,L]$.

The ruled surface with the given isogeodesic directrix $\alpha(s)$ is written by

$$P(s,t)=\alpha(s)+(t-t_0)(\mu(s)C(s)+\nu(s)W(s)), \tag{9}$$

where $\mu(s)$ and $\nu(s)$ are two controlling functions of the surface.

Corollary 3.

If the given ruled surface (9) is developable in the Lie group, then μ equals zero.

Proof. If the surface $P(s,t)$ is developable, then we know that $\det[\alpha',\zeta,\zeta']=0$. Using (1), we get

$$\det[\alpha',\zeta,\zeta']=\mu(s)f(s)\left(\frac{\nu(s)\kappa(s)}{f(s)}+\frac{\mu(s)\kappa(s)h(s)}{f(s)}\right).$$

Since $f(s)\neq 0$ and by using the conditions given Theorem 2, if the above equation equals zero, then we get $\mu(s)=0$.

So, the developable surface with the curve α as a isogeodesic can be written

$$P(s,t)=\alpha(s)+(t-t_0)\frac{cf(s)}{\kappa(s)}W(s), \tag{10}$$

where $c=const.\neq 0$. Then $\zeta(s)$ can be given as

$$\zeta(s)=\frac{cf(s)}{\kappa(s)}W(s). \tag{11}$$

Corollary 4.

The developable surface $P(s,t)$ is a cylinder surface if and only if $\alpha(s)$ is a general helix.

Proof. If the developable surface $P(s,t)$ is cylindrical, then we know that $\zeta \times \zeta' = 0$. The derivatives of (11) and from (1), we get

$$\zeta \times \zeta' = \frac{c^2 f^2(s)}{\kappa^2(s)} g(s) N(s) = 0.$$

If the above equation equals zero vector, since $f(s) \neq 0$ and $\kappa(s) \neq 0$, then we get $g(s) = 0$. This means that $h = const.$ From Theorem 1, we get that the curve is a general helix.

Conclusion 5. General helices on the cylinder surface in the Lie Group G are isogeodesic.

Example 6. Let

$$\alpha(s) = (0, -\sin s, \cos s) \tag{12}$$

be a unit speed curve. Then the curve α is framed in terms of the alternative frame as follows

$$N(s) = (0, \sin s, -\cos s),$$
$$C(s) = \frac{2}{\sqrt{5}}\left(-\frac{1}{2}, \cos s, \sin s\right),$$
$$W(s) = \frac{2}{\sqrt{5}}\left(1, \frac{1}{2}\cos s, \frac{1}{2}\sin s\right),$$

where $f(s) = \frac{\sqrt{5}}{2}, g(s) = 0$ and $\kappa = 1$, $h = -\frac{1}{2}$.

i) Taking $c = 1$, $t_0 = 0$ and from (10), then we get a ruled surface $P_1(s,t)$ that is a developable cylindrical with an isogeodesic curve $\alpha(s)$ as shown in Figure 1:

$$P_1(s,t) = \left(t, -\sin s + \frac{t}{2}\cos s, \cos s + \frac{t}{2}\sin s\right),$$

where $0 \le s \le 2\pi$ and $0 \le t \le 4$.

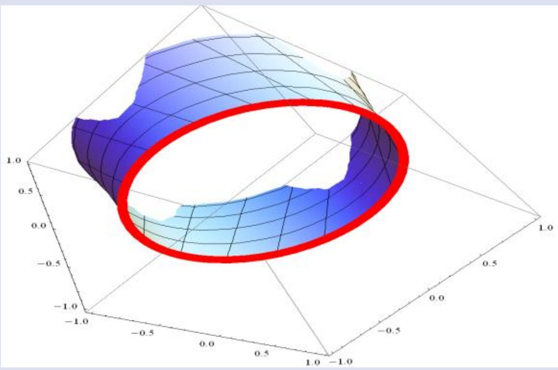

Figure 1. The cylinder surface P_1 (s, t) with isogeodesic curve α (s).

ii) Considering $\lambda_1(s,t) = 0$, $\lambda_2(s,t) = \sqrt{5}t$, $\lambda_3(s,t) = \sqrt{5}t$ and $t_0 = 0$ such as the Theorem 2 is satisfied. Then we get another surface $P_2(s,t)$, that is also the cylinder surface, with an isogeodesic curve $\alpha(s)$ as shown in Figure 2.

$$P_2(s,t) = (t, -\sin s + 3t\cos s, \cos s + 3t\sin s),$$

where $0 \le s \le 2\pi$ and $0 \le t \le 4$.

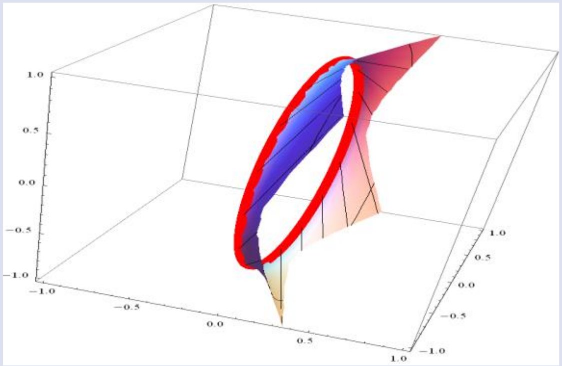

Figure 2. The surface P2 (s, t) with isogeodesic curve α (s)

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Conflicts of interest

The authors state that they did not have conflict of interests.

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