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On a Type of Semi-Symmetric Non-Metric Connection in HSU-Unified Structure Manifold

Shivani Sundriyal* and Jaya Upreti

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ABSTRACT

In the present paper, we have studied some properties of a semi-symmetric non-metric connection in HSU-unified structure manifold and HSU-Kahler manifold. Some new results on such manifolds have been obtained.

Keywords: *Semi-symmetric non-metric connection; Levi-Civita connection; HSU-unified structure manifold; HSU-Kahler manifold; Nijenhuis tensor.*
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1. Introduction

The idea of a metric connection on a Riemannian manifold was given by Hyden in 1932[6]. A linear connection ∇ is said to be metric on a manifold M^n if $\nabla g = 0$; otherwise it is non-metric. In 1970, Yano[13] introduced semi-symmetric metric connection on Riemannian manifold. Smaranda[2], Agashe and Chafle[1], Sengupta[12], Chaubey[3][4] and many others [7][8][9][10][11] studied various and important properties of semi-symmetric metric and non-metric connections on several differentiable manifolds and also defined some new type of connections on Riemannian manifold.
Chaubey[5] studied a new type of semi-symmetric non-metric connection in 2019. He established that such connection on a Riemannian manifold is projectively invariant under certain conditions.
In the present paper, we have studied some properties of semi-symmetric non-metric connection defined in [5] on a HSU-unified structure manifold. Further, we also studied some properties of HSU-Kahler manifold with the same connection.

2. Preliminaries

Let M^n be an even dimensional differentiable manifold of class C^∞. Let there is a vector valued real linear function ϕ of differentiablity class C^∞ satisfying

$$\phi^2 X = a^r X \tag{2.1}$$

for some arbitrary vector field X. Also, a Riemannian metric g, such that

$$g(\overline{X}, \overline{Y}) = a^r g(X, Y) \tag{2.2}$$

where $\overline{X} = \phi X$; $0 \le r \le n$ and a is a real or complex number. Then M^n is said to be HSU-unified structure manifold [11].

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*** Corresponding author***

Now, let us define a 2-form F in M^n such that

$$F(X,Y) = F(Y,X) = g(\overline{X},Y) = g(X,\overline{Y}) \tag{2.3}$$

Then it is clear that,

$$F(\overline{X},\overline{Y}) = a^r F(X,Y) \tag{2.4}$$

from equation (2.3) it is clear that

$$F(\overline{X},Y) = a^r g(X,Y) \tag{2.5}$$

The 2-form is symmetric in M^n. If HSU-unified structure manifold M^n satisfies the condition

$$(\nabla_X \phi)Y = 0 \tag{2.6}$$

Then M^n will said to be HSU-Kahler manifold.

From equation (2.6) it is clear that,

$$\nabla_X \overline{Y} - \overline{\nabla_X Y} \Leftrightarrow \overline{\nabla_X \overline{Y}} = a^r(\nabla_X Y) \tag{2.7}$$

where ∇ is a linear Riemannian connection.

3. A semi-symmetric non-metric connection

Let (M^n, g) be a Riemannian manifold of dimension n endowed with a Levi-Civita connection ∇ corresponding to the Riemannian metric g. A linear connection $\tilde{\nabla}$ on (M^n, g) defined by [5]

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\{\eta(Y)X - \eta(X)Y\} \tag{3.1}$$

for arbitrary vector fields X and Y on M^n is a semi-symmetric non-metric connection. The torsion tensor $\tilde{T}$ on M^n with respect to $\tilde{\nabla}$ satisfies the equation

$$\tilde{T}(X,Y) = \eta(Y)X - \eta(X)Y \tag{3.2}$$

where η is 1-form associated with the vector field ξ and satisfies,

$$\eta(X) = g(X,\xi) \tag{3.3}$$

and the metric g holds the relation

$$(\tilde{\nabla}_X g)(Y,Z) = \frac{1}{2}\{2\eta(X)g(Y,Z) - \eta(Y)g(X,Z) - \eta(Z)g(X,Y)\} \tag{3.4}$$

4. HSU-unified structure manifold equipped with a semi-symmetric non-metric connection

Theorem 4.1. *Let (M^n, g) be a HSU-unified structure manifold. Then there exist a unique linear semi-symmetric non-metric connection $\tilde{\nabla}$ on M^n, given by equation (3.1) and satisfy equations (3.2) and (3.4).*

Proof. Suppose (M^n, g) is a HSU-unified structure manifold of dimension n equipped with connection $\tilde{\nabla}$. Let $\tilde{\nabla}$ and Levi-Civita connection ∇ are connected by the relation

$$\tilde{\nabla}_X Y = \nabla_X Y + U(X,Y) \tag{4.1}$$

for arbitrary vector fields $X, Y \in M^n$, where $U(X,Y)$ is a tensor field of type $(1,2)$. By definition of the torsion tensor $\tilde{T}$ of $\tilde{\nabla}$ and from equation (4.1) we have

$$\tilde{T}(X,Y) = U(X,Y) - U(Y,X) \tag{4.2}$$

so we have,

$$g(\tilde{T}(X,Y),Z) = g(U(X,Y),Z) - g(U(Y,X),Z) \tag{4.3}$$

from equations (3.2) and (4.3)

$$g(U(X,Y),Z) - g(U(Y,X),Z) = \eta(Y)g(X,Z) - \eta(X)g(Y,Z) \tag{4.4}$$

from eqution (3.4), we conclude that

$$(\tilde{\nabla}_X g)(Y,Z) = -U'(X,Y,Z) \tag{4.5}$$

where $U'(X,Y,Z) = g(U(X,Y),Z) + g(U(X,Z),Y)$.

Hence, by using equations (4.2), (4.3) and (4.5), we have

$$g(\tilde{T}(X,Y),Z) + g(\tilde{T}(Z,X),Y) + g(\tilde{T}(Z,Y),X) = 2g(U(X,Y),Z) - U'(X,Y,Z) + U'(Z,X,Y) - U'(Y,X,Z) \tag{4.6}$$

Using equations (3.4) and (4.5) in equation (4.6), we have

$$g(\tilde{T}(X,Y),Z) + g(\tilde{T}'(X,Y),Z) + g(\tilde{T}'(Y,X),Z) = 2g(U(X,Y),Z) - 2\eta(Z)g(X,Y) + \eta(X)g(Y,Z) + \eta(Y)g(X,Z) \tag{4.7}$$

where

$$g(\tilde{T}'(X,Y),Z) = g(\tilde{T}(Z,X),Y) = \eta(X)g(Z,Y) - \eta(Z)g(X,Y) \tag{4.8}$$

From equations (4.7) and (4.8) we get,

$$U(X,Y) = \frac{1}{2}(\eta(Y)X - \eta(X)Y) \tag{4.9}$$

and from equations (4.9) and (4.1) we have (3.1).

Conversely, we can show that if $\tilde{\nabla}$ satisfies equation (3.1), then it will also satisfy equations (3.2) and (3.4).

Hence, the theorem.

Theorem 4.2. *On an n-dimensional HSU-unified structure manifold (M^n, g) endowed with a semi-symmetric non-metric connection $\tilde{\nabla}$, the following relations hold;*

(i) $\tilde{T}(\overline{X},\overline{Y},Z) + \tilde{T}(\overline{Y},\overline{X},Z) = 0$

(ii) $\tilde{T}(\overline{X},\overline{Y},\overline{Z}) + \tilde{T}(\overline{Y},\overline{Z},\overline{X}) + \tilde{T}(\overline{Z},\overline{X},\overline{Y}) = 0$

(iii) $\tilde{T}(\overline{\overline{X}},Y,Z) = \tilde{T}(X,\overline{\overline{Y}},Z) = \tilde{T}(X,Y,\overline{\overline{Z}})$

(iv) $\tilde{T}(\overline{\overline{X}},Y,Z) + \tilde{T}(\overline{\overline{Y}},X,Z) = 0$

(v) $\tilde{T}(\overline{\overline{X}},Y,\overline{\overline{Z}}) = a^{2r}\tilde{T}(X,Y,Z) = \tilde{T}(X,\overline{\overline{Y}},\overline{\overline{Z}})$

Proof. From equation (3.2) we have, $\tilde{T}(X,Y) = \eta(Y)X - \eta(X)Y$.

Also,

$$\tilde{T}(X,Y,Z) = g(\tilde{T}(X,Y),Z) \tag{4.10}$$

So that,

$$\tilde{T}(X,Y,Z) = \eta(Y)g(X,Z) - \eta(X)g(Y,Z) \tag{4.11}$$

Replacing X by $\overline{X}$ and Y by $\overline{Y}$ in above equation

$$\tilde{T}(\overline{X},\overline{Y},Z) = \eta(\overline{Y})g(\overline{X},Z) - \eta(\overline{X})g(\overline{Y},Z) \tag{4.12}$$

$$\tilde{T}(\overline{Y},\overline{X},Z) = \eta(\overline{X})g(\overline{Y},Z) - \eta(\overline{Y})g(\overline{X},Z) \tag{4.13}$$

From equations (4.12) and (4.13), we get

$$\tilde{T}(\overline{X},\overline{Y},Z) + \tilde{T}(\overline{Y},\overline{X},Z) = 0$$

Hence the result (i).

Now, $\tilde{T}(\overline{X},\overline{Y},\overline{Z}) = \eta(\overline{Y})g(\overline{X},\overline{Z}) - \eta(\overline{X})g(\overline{Y},\overline{Z})$

Using equation (2.2),

$$\tilde{T}(\overline{X},\overline{Y},\overline{Z}) = a^r\eta(\overline{Y})g(X,Z) - a^r\eta(\overline{X})g(Y,Z) \tag{4.14}$$

Similarly, we get,

$$\tilde{T}(\overline{Y},\overline{Z},\overline{X}) = a^r\eta(\overline{Z})g(Y,X) - a^r\eta(\overline{Y})g(Z,X) \tag{4.15}$$

$$\tilde{T}(\overline{Z},\overline{X},\overline{Y}) = a^r\eta(\overline{X})g(Z,Y) - a^r\eta(\overline{Z})g(X,Y) \tag{4.16}$$

From equations (4.14),(4.15) and (4.16) we have the required result (ii).

Now,

$$\tilde{T}(\overline{\overline{X}},Y,Z) = \eta(Y)g(\overline{\overline{X}},Z) - \eta(\overline{\overline{X}})g(Y,Z) = a^r\eta(Y)g(X,Z) - a^r\eta(X)g(Y,Z)$$

Hence, we have

$$\tilde{T}(\overline{\overline{X}},Y,Z) = a^r\tilde{T}(X,Y,Z) \tag{4.17}$$

similarly,

$$\tilde{T}(X,\overline{\overline{Y}},Z) = a^r\tilde{T}(X,Y,Z) \tag{4.18}$$

$$\tilde{T}(X,Y,\overline{\overline{Z}}) = a^r\tilde{T}(X,Y,Z) \tag{4.19}$$

From equations (4.17),(4.18) and (4.19) we have the required result (iii).

From equation (4.17), we can get (iv).

Now,

$$\tilde{T}(\overline{\overline{X}},\overline{\overline{Y}},Z) = a^{2r}\eta(Y)g(X,Z) - a^{2r}\eta(X)g(Y,Z)$$

Hence, we have

$$\tilde{T}(\overline{\overline{X}},\overline{\overline{Y}},Z) = a^{2r}\tilde{T}(X,Y,Z) \tag{4.20}$$

similarly,

$$\tilde{T}(X,\overline{\overline{Y}},\overline{\overline{Z}}) = a^{2r}\tilde{T}(X,Y,Z) \tag{4.21}$$

From euqtions (4.20) and (4.21), it is clear that result (v) is also verified.

Hence, the theorem 4.2.

Theorem 4.3. *A HSU-unified structure manifold* (M^n, g) *endowed with a semi-symmetric non-metric connection* $\tilde{\nabla}$, *satisfies the following relations;*

(i) $(\tilde{\nabla}_X\phi)Y = (\nabla_X\phi)Y + \frac{1}{2}\{\eta(\overline{Y})X - \eta(Y)\overline{X}\}$

(ii) $(\tilde{\nabla}_{\overline{X}}\phi)\overline{Y} = (\nabla_{\overline{X}}\phi)\overline{Y} + \frac{1}{2}[a^r\{\eta(Y)\overline{X} - \eta(\overline{Y})X\}]$

Proof. We have,

$$(\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X(\phi Y) - \phi(\tilde{\nabla}_X Y) \tag{4.22}$$

Using the equation (3.1) in equation (4.22), we get

$$(\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X(\phi Y) - \phi(\nabla_X Y + \frac{1}{2}\{\eta(Y)X - \eta(X)Y\})$$

which implies,

$$(\tilde{\nabla}_X \phi)Y = (\nabla_X \phi)Y + \frac{1}{2}\{\eta(\overline{Y})X - \eta(Y)\overline{X}\}$$

Hence, the result (i).

Replacing X by $\overline{X}$ and Y by $\overline{Y}$ in result (i), we get,

$$(\tilde{\nabla}_{\overline{X}} \phi)\overline{Y} = (\nabla_{\overline{X}} \phi)\overline{Y} + \frac{1}{2}[a^r\{\eta(Y)\overline{X} - \eta(\overline{Y})X\}]$$

Hence the theorem.

Theorem 4.4. *If a HSU-unified structure manifold (M^n, g) admits a semi-symmetric non-metric connection $\tilde{\nabla}$, then the Nijenhuis tensor of Levi-Civita connection ∇ and $\tilde{\nabla}$ coincide.*

Proof. The Nijenhuis tensor with respect to ϕ is a vector valued bilinear function defined as, [7][10].

$$\tilde{N}(X,Y) = [\overline{X},\overline{Y}] - \overline{[\overline{X},Y]} - \overline{[X,\overline{Y}]} + \overline{\overline{[X,Y]}}$$

Since, for $X \in M^n$, $\overline{\overline{X}} = a^r X$. Hence,

$$\tilde{N}(X,Y) = [\overline{X},\overline{Y}] - \overline{[\overline{X},Y]} - \overline{[X,\overline{Y}]} + a^r[X,Y] \tag{4.23}$$

The Nijenhuis tensor with respect to Levi-Civita connection ∇ is given by,

$$N(X,Y) = (\nabla_{\overline{X}}\phi)Y - (\nabla_{\overline{Y}}\phi)X - \overline{((\nabla_X\phi)Y)} + \overline{(\nabla_Y\phi)X} \tag{4.24}$$

Using the result from theorem 4.3, we have

$$(\nabla_X \phi)Y = (\tilde{\nabla}_X \phi)Y - \frac{1}{2}\{\eta(\overline{Y})X - \eta(Y)\overline{X}\} \tag{4.25}$$

Replacing X by $\overline{X}$ in equation (4.25)

$$(\nabla_{\overline{X}} \phi)Y = (\tilde{\nabla}_{\overline{X}} \phi)Y - \frac{1}{2}\{\eta(\overline{Y})\overline{X} - a^r\eta(Y)X\} \tag{4.26}$$

Interchanging X and Y in equation (4.26)

$$(\nabla_{\overline{Y}} \phi)X = (\tilde{\nabla}_{\overline{Y}} \phi)X - \frac{1}{2}\{\eta(\overline{X})\overline{Y} - a^r\eta(X)Y\} \tag{4.27}$$

Operating ϕ on both side of equation (4.25)

$$\overline{(\nabla_X \phi)Y} = \overline{(\tilde{\nabla}_X \phi)Y} - \frac{1}{2}\{\eta(\overline{Y})\overline{X} - a^r\eta(Y)X\} \tag{4.28}$$

Interchanging X and Y in equation (4.28)

$$\overline{(\nabla_Y \phi)X} = \overline{(\tilde{\nabla}_Y \phi)X} - \frac{1}{2}\{\eta(\overline{X})\overline{Y} - a^r\eta(X)Y\} \tag{4.29}$$

Put the value of equation (4.26),(4.27),(4.28) and (4.29) in equation (4.24) we get

$$N(X,Y) = \tilde{N}(X,Y)$$

Hence, the theorem is proved.

5. HSU-Kahler manifold with a semi-symmetric non-metric connection $\tilde{\nabla}$

As we discussed in section 2, that a HSU-unified structure manifold M^n is said to be HSU-Kahler manifold if it satisfies the condition (2.6). That is;

$$(\nabla_X \phi)Y = 0$$

In this section we will discuss some properties of HSU-Kahler manifold with a semi-symmetric non-metric connection $\tilde{\nabla}$.

Theorem 5.1. *If M^n be a HSU-Kahler manifold equipped with a semi-symmetric non-metric connection $\tilde{\nabla}$, then*

(i) $(\tilde{\nabla}_{\overline{X}}\phi)\overline{Y} = \frac{a^r}{2}\{\eta(Y)\overline{X} - \eta(\overline{Y})X\}$

(ii) $(\tilde{\nabla}_X\phi)Y = 0$ iff $\eta(\overline{Y})X = \eta(Y)\overline{X}$

Proof. From theorem 4.3 and equation (2.6), we have

$$(\tilde{\nabla}_X\phi)Y = \frac{1}{2}\{\eta(\overline{Y})X - \eta(Y)\overline{X}\} \tag{5.1}$$

Replacing X by $\overline{X}$ and Y by $\overline{Y}$ in above equation, we have

$$(\tilde{\nabla}_{\overline{X}}\phi)\overline{Y} = \frac{a^r}{2}\{\eta(Y)\overline{X} - \eta(\overline{Y})X\}$$

Hence, the result (i). From equation (5.1) it is obvious that result (ii) will hold good in both sides.

Theorem 5.2. *A HSU-Kahler manifold M^n with a semi-symmetric non-metric connection $\tilde{\nabla}$ satisfies the following relation*

$$dF(X, Y, Z) = 0$$

Proof. We know that

$$dF(X, Y, Z) = (\tilde{\nabla}_X F)(Y, Z) + (\tilde{\nabla}_Y F)(Z, X) + (\tilde{\nabla}_Z F)(X, Y) \tag{5.2}$$

From equation (2.3) we have

$$F(Y, Z) = g(\overline{Y}, Z) \tag{5.3}$$

Differentiating (5.3) covariantly with respect to X we get

$$\tilde{\nabla}_X F(Y, Z) = \tilde{\nabla}_X g(\overline{Y}, Z)$$

This implies,

$$(\tilde{\nabla}_X F)(Y, Z) + F(\tilde{\nabla}_X Y, Z) + F(Y, \tilde{\nabla}_X Z) = (\tilde{\nabla}_X g)(\overline{Y}, Z) + g(\tilde{\nabla}_X \overline{Y}, Z) + g(\overline{Y}, \tilde{\nabla}_X Z)$$

Using the equation (3.4), (5.1) and (5.3), we get

$$(\tilde{\nabla}_X F)(Y, Z) = \eta(X)g(\overline{Y}, Z) - \frac{\eta(Z)}{2}g(X, \overline{Y}) - \frac{\eta(Y)}{2}g(\overline{X}, Z) \tag{5.4}$$

Similarly,

$$(\tilde{\nabla}_Y F)(Z, X) = \eta(Y)g(\overline{Z}, X) - \frac{\eta(X)}{2}g(Y, \overline{Z}) - \frac{\eta(Z)}{2}g(\overline{Y}, X) \tag{5.5}$$

$$(\tilde{\nabla}_Z F)(X, Y) = \eta(Z)g(\overline{X}, Y) - \frac{\eta(Y)}{2}g(Z, \overline{X}) - \frac{\eta(X)}{2}g(\overline{Z}, Y) \tag{5.6}$$

Put the values from (5.4),(5.5) and (5.6) in equation (5.2) we have the required result.

Theorem 5.3. *The Nijenhuis tensor with respect to a semi-symmetric non-metrc connection $\tilde{\nabla}$ in a HSU-Kahler manifold M^n vanishes, $i.e$; the manifold is integrable over $\tilde{\nabla}$.*

Proof. The Nijenhuis tensor with respect to the connection $\tilde{\nabla}$ is defined as,

$$\tilde{N}(X,Y) = (\tilde{\nabla}_{\overline{X}}\phi)Y - (\tilde{\nabla}_{\overline{Y}}\phi)X - \overline{((\tilde{\nabla}_X\phi)Y)} + \overline{(\tilde{\nabla}_Y\phi)X} \tag{5.7}$$

Replacing X by $\overline{X}$ in equation (5.1), we have

$$(\tilde{\nabla}_{\overline{X}}\phi)Y = \frac{1}{2}\{\eta(\overline{Y})\overline{X} - a^r\eta(Y)X\} \tag{5.8}$$

Interchanging X and Y in equation (5.8)

$$(\tilde{\nabla}_{\overline{Y}}\phi)X = \frac{1}{2}\{\eta(\overline{X})\overline{Y} - a^r\eta(X)Y\} \tag{5.9}$$

Operating ϕ on both sides of equation (5.1)

$$\overline{(\tilde{\nabla}_X\phi)Y} = \frac{1}{2}\{\eta(\overline{Y})\overline{X} - a^r\eta(Y)X\} \tag{5.10}$$

Interchanging X and Y in above equation

$$\overline{(\tilde{\nabla}_Y\phi)X} = \frac{1}{2}\{\eta(\overline{X})\overline{Y} - a^r\eta(X)Y\} \tag{5.11}$$

Putting values from equations (5.8), (5.9), (5.10), and (5.11) in equation (5.7), we get

$$\tilde{N}(X,Y) = 0$$

Hence, the theorem is proved.

References

[1] Agashe, N.S.: *A semi-symmetric non-metric connection on a Riemannian manifold*. Indian J. pure appl. Math. **23**, 399-409 (1992).
[2] Andonie, OC., Smaranda, D.: *Certaines connexions semisymetriques.* Tensor. **31**, 8-12 (1977).
[3] Chaubey, S. K., Ojha, R.H.: *On a semi-symmetric non-metric and quarter symmetric metric connexions.* Tensor N.S. **70** (2), 202-203 (2008).
[4] Chaubey, S. K., Ojha, R.H.: *On a semi-symmetric non-metric connection.* Filomat. **26** (2), 269-275 (2012).
[5] Chaubey, S. K., Yildiz, A.: *Riemannian manifolds admitting a new type of semisymmetric nonmetric connection.* Turkish Journal of Mathematics. **43** (4), 1887-1904 (2019).
[6] Hyden, A.: *Sub-Spaces of a Space with Torsion.* Proceedings of London Mathematical Society. **2** (1), 27-50 (1932).
[7] Kobayashi, S., Nomizu, K.: Foundation of differential geometry, Vol. I and II. Interscience Publisher, London (1969).
[8] Kumar, S., Kandpal, D., Upreti, J.: *On a HSU-unified Structure Manifold with a Recurrent Metric Connection.* Journal of Computer and Mathematical Sciences. **8** (8), 366-372 (2017).
[9] Kumar, S., Upreti, J.: *A new connection in an almost para-contact manifold.* Journal of National Academy of Mathematics, Gorakhpur. **28**, 42-52 (2014).
[10] MIshra, R.S.: Structures on a differentiable manifold and their applications. Chandrama Prakashan, 50-A Balrampur Hause, Allahabad, India (1984).
[11] Nivas, R., Agnihotri, A.: *On HSU-unified Structure Manifold with a Quarter-symmetric Non-metric Connection.* Bulletin of Mathematical Sciences and Applications. **3**, 63-70 (2013).
[12] Sengupta, J., De, U.C, Binh, T.: *On a type of semi-symmetric non-metric connection on a Riemannian manifold*. Indian Journal of Pure and Applied Mathematics. **31** (12) 1659-1670 (2000).
[13] Yano, K.: *On semi-symmetric metric connections.* Revue Roumaine de Mathematiques Pures et Appliquees. **15** 1579-1586 (1970).

Affiliations

SHIVANI SUNDRIYAL
ADDRESS: S.S.J Campus, Kumaun University, Dept. of Mathematics, 263601, Almora-India.
E-MAIL: shivani.sundriyal5@gmail.com
ORCID ID: 0000-0001-6195-2572

JAYA UPRETI
ADDRESS: S.S.J Campus, Kumaun University, Dept. of Mathematics, 263601, Almora-India.
E-MAIL: prof.upreti@gmail.com
ORCID ID:0000-0001-8615-1819