Eastern Anatolian Journal of Science
Volume VII, Issue II, 2021, 42-48
Eastern Anatolian Journal of Science

A Note on Spherical Product Surface with L_1-Pointwise 1-Type Gauss Map

İlim Kişi[1], Günay Öztürk[2]
[1]*Department of Mathematics, Kocaeli University, Kocaeli, Turkey,*
ilim.ayvaz@kocaeli.edu.tr
[1]*Department of Mathematics, İzmir Democracy University, İzmir, Turkey,*
gunay.ozturk@idu.edu.tr

Abstract

In this study, we deal with the spherical product surface whose Gauss map G satisfies the equality $L_1 G = f(G+C)$ where L_1 is the Cheng-Yau operator in Galilean 3-space G_3. We obtain the necessary and sufficient conditions for spherical product surface to have L_1-pointwise 1-type Gauss map in G_3.

Keywords: Spherical product surface, Cheng-Yau operator, Galilean 3-space.

1. *Introduction*

Finite type immersions are first given by Chen (Chen 1983), (Chen 1984), (Chen 1996). Let M be a submanifold in m-dimensional Euclidean space IE^m. Isometric immersion $x: M \to IE^m$ is called as a finite type if it can be written as a finite sum of eigenvectors of the Laplacian Δ of M for a constant map x_0, and non-constant maps $x_1, x_2, ..., x_k$, i.e.,

$$x = x_0 + \sum_{i=1}^{k} x_i.$$

Here, $\Delta x = \lambda_i x_i$, $\lambda_i \in IR$, $1 \le i \le k$. The submanifold is said to be k-type if the numbers λ_i s are different (Chen 1984).

Then, by Chen and Piccinni, these immersions are generalized to the Gauss map G of M as

$$\Delta G = a(G+C)$$

for a constant vector C and a real number a in (Chen and Piccini 1987). A submanifold that satisfies the last equality has $1-$type Gauss map. After that, in the last equality, a non-constant differentiable function f is taken instead of a. Namely, the last equality turns into

$$\Delta G = f(G+C). \tag{1}$$

A submanifold that satisfies the equation (1) is said to have pointwise $1-$type Gauss map. Also, if the vector C is zero, the pointwise $1-$type Gauss map is called as the first kind. Otherwise, it is called as the second kind (Chen et al. 2005). Surfaces satisfying the equation (1) are the subject of many studies such as (Arslan et al. 2011), (Arslan et al. 2014), (Arslan and Milousheva 2016), (Choi and Kim 2001), (Kişi and Öztürk 2018), (Kişi and Öztürk 2019a), (Kişi and Öztürk 2019b).

In (Alias and Gürbüz 2006), (Kashani 2009), the notion of finite type submanifolds is generalised by replacing the Laplacian operator with operators L_k $(k = 1,2,...,n-1)$ that represent the linear operators of the first variation of the $(k+1)$-th mean curvature of a submanifold. Here, $L_0 = -\Delta$ and L_1 is the Cheng-Yau operator. Recently, some papers published about the surfaces having L_1-pointwise 1-type Gauss map in some spaces, such as (Güler and Turgay 2019), (Kim et al 2016), (Kim and Turgay 2013), (Kim and Turgay 2017), (Mohammadpouri 2018), (Qian and Kim 2015), (Yoon et al. 2015).

Embeddings of product spaces defined by Kuiper with a new type in the IE^{m+n+d} as follows:

Received: 22.10.2021
Accepted: 08.12.2021
Published: 10.12.2021
**Corresponding author:* İlim KİŞİ, *PhD*
Kocaeli University, Faculty of Arts and Sciencess, Department of Mathematics, Kocaeli Turkey
E-mail: ilim.ayvaz@kocaeli.edu.tr
Cite this article as: İ. Kişi and G. Öztürk, A Note on Spherical Product Surface with L_1-Pointwise 1-Type Gauss Map,, *Eastern Anatolian Journal of Science, Vol. 7, Issue 2, 42-48, 2021.*

Let $g:S^n \to IE^{n+1}$ and $f:M \to IE^{m+d}$ be embeddings from n-sphere into IE^{n+1} and from an m-dimensional manifold M into IE^{m+d}, respectively. Then, the new type embedding is

$$X = f \otimes g : M \times S^n \to IE^{m+n+d}, \qquad (2)$$

$$X(u,v) = (f_1(u), f_2(u), ..., f_{m+d-1}(u), f_{m+d}(u)g(v))$$

(Kuiper 1970).

In (Arslan et al. 2008), (Bulca et al. 2012), authors entitled this type embedding as rotational embedding. They deal with a special case of the rotational embedding which is called as spherical product surfaces in Euclidean spaces. They obtained this surface by writing $m = n = 1$, and $d = 1,2$ in (2), respectively. Moreover, in Galilean 3-space, spherical product surface is studied in (Aydın and Öğrenmiş 2016).

2. Preliminaries

Here, some preliminaries about Galilean geometry are given. For more detailed information, the studies (Röschel 1986), (Yaglom 1979) can be examined.

The scalar product and the cross product of the two vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ in G_3 are defined as

$$\langle a,b\rangle = \begin{cases} a_1 b_1, & if \;\; a_1 \neq 0 \;\; or \;\; b_1 \neq 0 \\ a_2 b_2 + a_3 b_3 & if \;\; a_1 = 0 \;\; and \;\; b_1 = 0, \end{cases} \qquad (3)$$

and

$$a \times b = \begin{vmatrix} 0 & \mathrm{e}_2 & \mathrm{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \qquad (4)$$

respectively. Here, e_2 =(0,1,0) and e_3 =(0,0,1) are orthonormal unit vectors. The length of the vector $a = (a_1, a_2, a_3)$ is given as follows:

$$\|a\| = \begin{cases} |a_1|, & if \;\; a_1 \neq 0 \\ \sqrt{a_2^2 + a_3^2}, & if \;\; a_1 = 0 \end{cases} \qquad (5)$$

(Pavkovic and Kameranovic 1987).

An admissible unit speed curve $\alpha : I \subset IR \to G_3$ is given with the parametrization

$$\alpha(u) = (u, y(u), z(u)).$$

For an isometric immersion $\mathrm{X}:\mathrm{M} \to \tilde{\mathrm{M}}$ from a hypersurface M from an $(n+1)$-dimensional Riemannian manifold $\tilde{\mathrm{M}}$, and for the Levi-Civita connections $\tilde{\nabla}$ of $\tilde{\mathrm{M}}$ and ∇ of M, the Gauss formula is given by

$$\tilde{\nabla}_{\mathrm{X}}\mathrm{Y} = \nabla_{\mathrm{X}}\mathrm{Y} + \langle \mathrm{S}(\mathrm{X}), \mathrm{Y}\rangle,$$

where $X, Y \in \chi(M)$ and S is the shape operator of M. It is known that the eigenvalues $k_1, k_2, ..., k_n$ of S are called as the principal curvatures of M. For a smooth function f on *M*, linear operators $L_k s$ are defined as

$$L_k(f) = div(P_k(\nabla f))$$

where ∇ is the gradient, div is the divergence operator and

$$P_k = \sum_{i=0}^{k}(-1)^i s_{k-i} S^i$$

is the Newton k-th transformation, $s_k = \begin{pmatrix} n \\ k \end{pmatrix} H_k$ is the k-th mean curvature (Cheng and Yau 1977). Thus, for $k = 0$, $P_0 = I_n$ (I_n is the identity matrix), and for $k = 1$, $P_1 = tr(S) I_n - S$.

Now, let M be a surface, e_1, e_2 be the principal directions correspond to the curvatures k_1, k_2 of M. From (2), for a smooth function f, the Cheng-Yau operator $L_1 f$ can be given as

$$L_1 f = div(P_1(\nabla f))$$
$$= e_1[k_2]e_1 f + e_2[k_1]e_2 f + k_2\left(e_1 e_1 - \nabla_{e_2} e_2\right) f + k_1\left(e_2 e_2 - \nabla_{e_1} e_1\right) f.$$

Hence, the Cheng-Yau operator L_1 can be given by

$$L_1 = e_1[k_2]\tilde{\nabla} e_1 + e_2[k_1]\tilde{\nabla} e_2 + k_2\left(\tilde{\nabla} e_1 \tilde{\nabla} e_1 - \tilde{\nabla}_{\nabla_{e_2} e_2}\right) + k_1\left(\tilde{\nabla} e_2 \tilde{\nabla} e_2 - \tilde{\nabla}_{\nabla_{e_1} e_1}\right)$$

(Kim and Turgay 2013).

Let the surface M parametrized with

$$X(u_1, u_2) = (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2))$$

in G_3. To represent the partial derivatives, we use

$$x,_i=\frac{\partial x}{\partial u_i} \quad and \quad x,_{ij}=\frac{\partial^2 x}{\partial u_i \partial u_j}, \quad 1\le i,j\le 2.$$

If $x,_i \neq 0$ for some $i=1,2$, then the surface is admissible (i.e. having not any Euclidean tangent planes). The first fundamental form I of the surface M is defined as

$$I=(g_1 d_{u_1}+g_2 d_{u_2})^2+\varepsilon(h_{11}d_{u_1}^2+2h_{12}d_{u_1}d_{u_2}+h_{22}d_{u_2}^2),$$

where $g_i=x,_i$, $h_{ij}=y,_i\, y,_j+z,_i\, z,_j$; $i,j=1,2$

and

$$\varepsilon=\begin{cases}0, & if \quad d_{u_1}:d_{u_2} \quad is \quad non-isotropic,\\ 1, & if \quad d_{u_1}:d_{u_2} \quad is \quad isotropic.\end{cases}$$

Let a function W is given by

$$W=\sqrt{\left(x,_1 z,_2-x,_2 z,_1\right)^2+\left(x,_2 y,_1-x,_1 y,_2\right)^2} \quad (6)$$

Then, the unit normal vector field is given as

$$G=\frac{1}{W}(0,-x,_1 z,_2+x,_2 z,_1, x,_1 y,_2-x,_2 y,_1). \quad (7)$$

Similarly, the second fundamental form II of the surface M is defined as

$$II=L_{11}d_{u_1}^2+2L_{12}d_{u_1}d_{u_2}+L_{22}d_{u_2}^2,$$

where

$$L_{ij}=\frac{1}{g_1}\left\langle g_1(0,y,_{ij},z,_{ij})-g_{i,j}(0,y,_1,z,_1),N\right\rangle, \qquad g_1\neq 0$$

or

$$L_{ij}=\frac{1}{g_2}\left\langle g_2(0,y,_{ij},z,_{ij})-g_{i,j}(0,y,_2,z,_2),N\right\rangle, \qquad g_2\neq 0.$$

The Gaussian and the mean curvatures of M are defined as

$$K=\frac{L_{11}L_{22}-L_{12}^2}{W^2} \text{ and } H=\frac{g_2^2 L_{11}-2g_1 g_2 L_{12}+g_1^2 L_{22}}{2W^2}.$$

(8)

A surface is called as flat (resp. minimal) if its Gaussian (resp. mean) curvatures vanish (Aydın et al. 2019), (Röschel 1986).

Lemma 2.1 (Kim and Turgay 2013) Let M be an oriented surface in IE^3 and K and H be the Gaussian and the mean curvatures of M, respectively. Then the Gauss map G of M satisfies

$$L_1 G=-\nabla K-2HKG. \quad (9)$$

Definition 2.2 (Kim and Turgay 2013) Let M be an oriented surface in IE_3. Then, M is said to have L_1-harmonic Gauss map if its Gauss map satisfies $L_1 G=0$.

Definition 2.3 (Kim and Turgay 2013) Let M be an oriented surface in IE^3. Then, M is said to have L_1-pointwise 1-type Gauss map if its Gauss map satisfies

$$L_1 G=f\left(G+C\right) \quad (10)$$

for a smooth function f and a constant vector C. Moreover, if the vector C is zero, the pointwise L_1-type Gauss map is called as the first kind. Otherwise, it is called as the second kind.

3. *Spherical Product Surface with L_1 Pointwise 1-Type Gauss Map in G_3*

For the smooth functions p and q, let $\alpha(u)=(u,p(u))$ and $\beta(u)=(v,q(v))$ be the unit speed plane curves in Galilean 2-space G_2. Then the spherical product patch is given as

$$\begin{aligned}X&=\alpha\otimes\beta: G_2\to G_3,\\ X(u,v)&=(u,p(u)v,p(u)q(v)).\end{aligned} \quad (11)$$

The surface M given with (11) is called as spherical product surface in G_3.

From the parametrization of the surface,

$$g_1=u,_1=1, \quad g_2=u,_2=0. \quad (12)$$

An orthonormal frame $\{e_1,e_2,G\}$ of M is given by

$$e_1=\frac{X_u}{\|X_u\|}=\left(1,p'(u)v,p'(u)q(v)\right), \|X_u\|=1 \quad (13)$$

$$e_2=\frac{X_v}{\|X_v\|}=\frac{\left(0,1,q'(v)\right)}{\sqrt{1+\left(q'(v)\right)^2}}, \ \|X_v\|=p\sqrt{1+\left(q'(v)\right)^2}$$

and

$$G=\frac{\left(0,-q',1\right)}{\sqrt{1+\left(q'\right)^2}}, \quad (14)$$

where $W = p\sqrt{1+\left(q^{'}\right)^2}$. The coefficients of the second fundamental form are obtained as

$$L_{11} = \frac{p^{''}A}{\sqrt{1+\left(q^{'}\right)^2}}, \quad L_{12} = 0, \quad L_{22} = \frac{pq^{''}}{\sqrt{1+\left(q^{'}\right)^2}}, \qquad (15)$$

where $A = A(v) = -vq^{'}(v) + q(v)$. From, (12) and (15), curvature functions of M are obtained as

$$K = \frac{p^{''}q^{''}A}{p\left(1+\left(q^{'}\right)^2\right)^2}, \qquad H = \frac{q^{''}}{2p\left(1+\left(q^{'}\right)^2\right)^{\frac{3}{2}}} \qquad (16)$$

(Aydın and Öğrenmiş 2016).

Theorem 3.4 (Aydın and Öğrenmiş 2016) Let M be a spherical product surface given with the parametrization (11) in G_3. Then, M is minimal if and only if the curve β is a line and the surface is an open part of an isotropic plane.

Theorem 3.5 Aydın and Öğrenmiş 2016) Let M be a spherical product surface given with the parametrization (11) in G_3. Then, M is flat if and only if it is either an isotropic plane or the generating curve α is a line.

By (13) and (16), we write the gradient of the Gaussian curvature

$$\nabla K = \frac{q^{''}AB}{p^2\left(1+\left(q^{'}\right)^2\right)^2}e_1 + \frac{p^{''}\left(1+\left(q^{'}\right)^2\right)(q^{'''}A+q^{''}A^{'}) - 4p^{''}\left(q^{''}\right)^2 q^{'}A}{p\left(1+\left(q^{'}\right)^2\right)^3}e_2,$$

(17)

where $B = B(u) = p^{'''}(u)p(u) - p^{''}(u)p^{'}(u)$.

Thus, from (9), (16) and (17), we obtain the Cheng-Yau operator of the Gauss map as

$$L_1G = \frac{-q^{''}AB}{p^2\left(1+\left(q^{'}\right)^2\right)^2}e_1 \qquad (18)$$

$$+\frac{-p^{''}(1+\left(q^{'}\right)^2)(q^{'''}A+q^{''}A^{'})+4p^{''}\left(q^{''}\right)^2 q^{'}A}{p\left(1+\left(q^{'}\right)^2\right)^3}e_2$$

$$+\frac{-p^{''}\left(q^{''}\right)^2 A}{p^2\left(1+\left(q^{'}\right)^2\right)^{\frac{7}{2}}}G.$$

We assume that the spherical product surface M has L_1 -pointwise 1-type Gauss map of the first kind, i.e. $L_1G = fG$ for a smooth function f . Then, from (13), (14) and (18),

$$\frac{-1}{p^2\left(1+\left(q^{'}\right)^2\right)^4}\left\{\begin{array}{c} q^{''}\left(1+\left(q^{'}\right)^2\right)^2 AB\left(1, p^{'}v, p^{'}q\right) \\ +\left(pp^{''}\left(1+\left(q^{'}\right)^2\right)^{\frac{3}{2}}(q^{'''}A+q^{''}A^{'}) - 4pp^{''}\left(q^{''}\right)^2 q^{'}A\right)\left(0,1,q^{'}\right) \\ +p^{''}\left(q^{''}\right)^2 A\left(0,-q^{'},1\right). \end{array}\right.$$

$$= \frac{f}{\sqrt{1+\left(q^{'}\right)^2}}\left(0,-q^{'},1\right) \qquad (19)$$

From (19), $q^{''}AB = 0$. Here, $q^{'} \neq 0$, $q^{''} \neq 0$, $A \neq 0$, and $p^{'} \neq 0$, $p^{''} \neq 0$, otherwise f would be zero. Then, $p^{''} = pc^2$ and so $p(u) = c_1e^{cu} + c_2e^{-cu}$. By the second and the third components of (21), we have

$$pp^{''}\left(1+\left(q^{'}\right)^2\right)^{\frac{3}{2}}(q^{'''}A+q^{''}A^{'}) - 4pp^{''}\left(q^{''}\right)^2 q^{'}A - p^{''}q^{'}\left(q^{''}\right)^2 A = fp^2q^{'}\left(1+\left(q^{'}\right)^2\right)^{\frac{7}{2}}$$

(20)

and

$$pp^{''}q^{'}\left(1+\left(q^{'}\right)^2\right)^{\frac{3}{2}}(q^{'''}A+q^{''}A^{'}) - 4pp^{''}\left(q^{''}\right)^2 (q^{'})^2 A + p^{''}\left(q^{''}\right)^2 A = -fp^2\left(1+\left(q^{'}\right)^2\right)^{\frac{7}{2}}.$$

(21)

Multiplying (20) with $-q^{'}$, and then adding the obtained equation by (21), we get

$$p^{''}\left(q^{''}\right)^2 A = -fp^2\left(1+\left(q^{'}\right)^2\right)^{\frac{7}{2}},$$

where $p^{''} = pc^2$. Hence, we have

$$f = \frac{-p^{''}\left(q^{''}\right)^2 A}{p^2\left(1+\left(q^{'}\right)^2\right)^{\frac{7}{2}}}. \qquad (22)$$

Then, multiplying (21) with $q^{'}$, and then adding the obtained equation by (20), we get

$$\left(1+\left(q^{'}\right)^2\right)^{\frac{3}{2}}\left(q^{'''}(-q^{'}v+q) - \left(q^{''}\right)^2 v\right) - 4\left(q^{''}\right)^2 q^{'}(-q^{'}v+q) = 0 \qquad (23)$$

Then, we give the following theorems:

Theorem 3.6 Let M be a spherical product surface given with the parametrization (11) in G_3. M has L_1 -pointwise 1-type Gauss map of the first (20) kind if and only if $p(u) = c_1e^{cu} + c_2e^{-cu}$ and $q(v)$ satisfies the differential equation (23).

Theorem 3.7 Let M be a spherical product surface given with the parametrization (11) in G_3. M has L_1-pointwise 1-type Gauss map of the first kind if and only if $K = \dfrac{c^2 q'' A}{\left(1+(q')^2\right)^2}$ which is a function of v and $f = \dfrac{-Kq''}{p\left(1+(q')^2\right)^{\frac{3}{2}}}$.

Now, we consider that the spherical product surface M has L_1-pointwise 1-type Gauss map of the second kind, i.e. $L_1 G = f(G+C)$ for a smooth function f and a nonzero constant vector C. Differentiating (13) and (14) with respect to e_1 and e_2, we obtain the derivatives

$$
\begin{aligned}
\tilde{\nabla}_{e_1} e_1 &= \frac{p''(qq'+v)}{\left(1+(q')^2\right)^{\frac{1}{2}}} e_2 + \frac{p'' A}{\left(1+(q')^2\right)^{\frac{1}{2}}} G, \\
\tilde{\nabla}_{e_1} e_2 &= 0, \\
\tilde{\nabla}_{e_1} G &= 0, \qquad\qquad (24) \\
\tilde{\nabla}_{e_2} e_1 &= \frac{p'}{p} e_2, \\
\tilde{\nabla}_{e_2} e_2 &= \frac{q''}{p\left(1+(q')^2\right)^{\frac{3}{2}}} G, \\
\tilde{\nabla}_{e_2} G &= \frac{-q''}{p\left(1+(q')^2\right)^{\frac{3}{2}}} e_2.
\end{aligned}
$$

From the equation (10) and (18), we can write the vector C as

$$
\begin{aligned}
C = {} & \frac{-q'' AB}{fp^2\left(1+(q')^2\right)^2} e_1 \\
& + \frac{-p''\left(1+(q')^2\right)(q''' A + q'' A') + 4p''(q'')^2 q' A}{fp\left(1+(q')^2\right)^3} e_2 \\
& + \left(\frac{-p''(q'')^2 A}{fp^2\left(1+(q')^2\right)^{\frac{7}{2}}} - 1 \right) G. \qquad (25)
\end{aligned}
$$

Since C is a nonzero constant vector, $\tilde{\nabla}_{e_1} C = 0$ and $\tilde{\nabla}_{e_2} C = 0$. From (24) and (25), we get

$$
\begin{aligned}
0 = \tilde{\nabla}_{e_1} C = {} & e_1\left[\frac{-q'' AB}{fp^2\left(1+(q')^2\right)^2} \right] e_1 \\
& + \left(e_1\left[\frac{D}{fp\left(1+(q')^2\right)^3} \right] - \frac{p'' q''(qq'+v)AB}{fp^2\left(1+(q')^2\right)^{\frac{5}{2}}} \right) e_2 \\
& + \left(e_1\left[\frac{-p''(q'')^2 A}{fp^2\left(1+(q')^2\right)^{\frac{7}{2}}} - 1 \right] - \frac{p'' q'' A^2 B}{fp^2\left(1+(q')^2\right)^{\frac{5}{2}}} \right) G, \qquad (26)
\end{aligned}
$$

and

$$
\begin{aligned}
0 = \tilde{\nabla}_{e_2} C = {} & e_2\left[\frac{-q'' AB}{fp^2\left(1+(q')^2\right)^2} \right] e_1 \\
& + \left(e_2\left[\frac{D}{fp\left(1+(q')^2\right)^3} \right] + \frac{q''}{p\left(1+(q')^2\right)^{\frac{3}{2}}} \left(\frac{p''(q'')^2 A}{fp^2\left(1+(q')^2\right)^{\frac{7}{2}}} + 1 \right) - \frac{-q'' p' AB}{fp^3\left(1+(q')^2\right)^2} \right) e_2 \\
& + \left(e_2\left[\frac{-p''(q'')^2 A}{fp^2\left(1+(q')^2\right)^{\frac{7}{2}}} - 1 \right] + \frac{q'' D}{p\left(1+(q')^2\right)^{\frac{3}{2}}} \right) G, \qquad (27)
\end{aligned}
$$

where $D = D(u,v) = -p''\left(1+(q')^2\right)(q''' A + q'' A') + 4p''(q'')^2 q' A.$.

Since

$$
e_1\left[\frac{-q'' AB}{fp^2\left(1+(q')^2\right)^2} \right] = e_2\left[\frac{-q'' AB}{fp^2\left(1+(q')^2\right)^2} \right] = 0,
$$

we get

$$\frac{-q'' AB}{fp^2\left(1+\left(q'\right)^2\right)^2}=c, \quad c \quad const.$$

Thus, we have

$$f=\frac{-q'' AB}{cp^2\left(1+\left(q'\right)^2\right)^2}. \tag{28}$$

Theorem 3.8 Let M be a spherical product surface given with the parametrization (11) in G_3. M has L_1-pointwise 1-type Gauss map of the second kind if and only if f is a constant function given with (28).

References

ALIAS LJ, GÜRBÜZ N (2006). An extension of Takahashi theorem for the linearized operators of the higher order mean curvatures. *Geom. Dedicata*, 121, 113–127.

ARSLAN K, BAYRAM BK, BULCA B, KIM YH, MURATHAN C, ÖZTÜRK G (2011). Rotational embeddings in IE^4 with pointwise 1-type Gauss map. *Turk. J. Math*. 35, 493–499.

ARSLAN K, BULCA B, BAYRAM BK, ÖZTÜRK G, UGAIL H (2009). On spherical product surfaces in IE^3. *International Conference on CyberWorlds. 1,* 132–137.

ARSLAN K, BULCA B, MILOUSHEVA V (2014). Meridian surfaces in IE^4 with pointwise 1-type Gauss map. *Bull. Korean Math. Soc*. 51, 911–922.

AYDIN ME, KÜLAHÇI M, ÖĞRENMIŞ AO (2019). Constant curvature translation surfaces in Galilean 3-space. *International Electronic Journal of Geometry*. 12, 9–19.

ARSLAN K, MILOUSHEVA V (2016). Meridian surfaces of elliptic or hyperbolic type with pointwise 1-type Gauss map in Minkowski 4-space. *Taiwanese Journal of Mathematics*. 20, 311–332.

AYDIN ME, ÖĞRENMIŞ AO (2016). Spherical product surfaces in the Galilean space. *Konuralp Journal of Mathematics*. 4, 290–298.

BULCA B, ARSLAN K, BAYRAM BK, ÖZTÜRK G (2012). Spherical product surfaces in IE^4. *An. St. Univ. Ovidius Constanta*. 20, 41–54.

CHEN BY (1983). On submanifolds of finite type. Soochow J. Math. 9, 65–81.

CHEN BY (1984). *Total Mean Curvature and Submanifolds of Finite Type, Series in Pure Mathematics.* 1. World Scientific Publishing Co. Singapore.

CHEN BY (1996). A report on submanifolds of finite type. *Soochow J. Math.* 22, 117–337.

CHEN BY, PICCINNI P (1987). Submanifolds with finite type Gauss map. *Bull. Austral. Math. Soc*. 35, 161–186.

CHEN BY, CHOI M, KIM YH (2005). Surfaces of revolution with pointwise 1-type Gauss map. *J. Korean Math.* 42, 447–455.

CHENG SY, YAU ST (1977). Hypersurfaces with constant scalar curvature. *Math. Ann.* 225, 195–204.

CHOI M, KIM YH (2001). Characterization of the helicoid as ruled surfaces with pointwise 1-type Gauss map. *Bull. Korean Math. Soc.* 38, 753–761.

GÜLER E, TURGAY NC (2019). Cheng-Yau operator and Gauss map of rotational hypersurfaces in 4-space. *Mediterr. J. Math.* 16, 1–16.

KASHANI SMB (2009). On some L_1-finite type (hyper)surfaces in IR^{n+1}. *Bull. Korean Math. Soc.* 46, 35–43.

KIM DS, KIM JR, KIM YH (2016). Cheng-Yau operator and Gauss map of surfaces of revolution. *Bull. Malays. Math. Sci. Soc.* 39, 1319–1327.

KIM YH, TURGAY NC (2013). Surfaces in IE^3 with L_1-pointwise 1-type Gauss map. *Bulletin of the Korean Mathematical Society*. 50, 935–949.

KIM YH, TURGAY NC (2017). On the ruled surfaces with L_1-pointwise 1-type Gauss map. *Kyungpook Math. J*. 57, 133–144.

KIŞI İ, ÖZTÜRK G (2018). A new type of tubular surface having pointwise 1-type Gauss map in Euclidean 4-space IE^4. *J. Korean Math. Soc.* 55, 923–938.

KIŞI İ, ÖZTÜRK G (2019a). Spherical product surface having pointwise 1-type Gauss map in Galilean 3-space G_3. International Journal of Geometric Methods in Modern Physics. 16, 1–10.

KIŞI İ, ÖZTÜRK G (2019b). Tubular surface having pointwise 1-type Gauss map in Euclidean 4-space. *International Electronic Journal of Geometry*. 12, 202–209.

KUIPER NH (1970). Minimal total absolute curvature for immersions. *Invent. Math.* 10, 209–238.

MOHAMMADPOURI A (2018). Hypersurfaces with L_r-pointwise 1-type Gauss map. *Journal of Mathematical Physics, Analysis, Geometry*. 14, 67–77.

PAVKOVIC BJ, KAMENAROVIC I (1987). The equiform differential geometry of curves in the Galilean space G_3. *Glasnik Matematicki*. 22, 449–457.

ROSCHEL O. *Die Geometrie Des Galileischen Raumes*. Forschungszentrum Graz Research Centre, Austria, 1986.

QIAN J, KIM YH (2015). Classifications of canal surfaces with L_1-pointwise 1-type Gauss map. *Milan J. Math.* 83, 145–155.

YAGLOM IM (1979). *A Simple Non-Euclidean Geometry and Its Physical Basis*. Springer-Verlag Inc., New York.

YOON DW, KIM YH, JUNG JS (2015). Rotation surfaces with L_1-pointwise 1-type Gauss map in pseudo-Galilean space. *Annales Polonici Mathematici*. 113, 255–267.