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Solution of Parabolic Problem with Inverse Coefficient s(t) with Periodic and Integral Conditions

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Abstract

In this publication, We examine the inverse parabolic parabolik with nonlocal and integral conditional. Firstly, finding the existence, uniqueness and problem of stability, numerical analysis will be done by using the finite difference method for the numerical approximation of this problem.The solution is found examining the Fourier and the iteration method and also numerical solution are given using the finite difference method and results will be mentioned in the discussion section.

1. Introduction

The inverse problems are an area of great interest to many researchers [1, 3, 5, 4]. Especially, periodic conditions are very used conditions especially in physics and engineering [2, 1, 3, 6].

$$s(t)\omega_t = \omega_{xx} + h(x,t), \tag{1}$$

$$\omega(x,0) = \vartheta(x), \ \ x \in [0,\pi], \tag{2}$$

$$\omega(0,t) = \omega(\pi,t), \omega_x(0,t) = \omega_x(\pi,t), \ 0 \le t \le T, \tag{3}$$

$$L(t) = \int_0^{\pi} x\omega(x,t)dx, 0 \le t \le T, \tag{4}$$

where (1) is the inverse coefficient problem, (2) is the initial condition, (3) are the periodic conditions and (4) is the integral data,the domain $W := \{0 < x < \pi, 0 < t < T\}$, $\vartheta(x)$ and $f(x,t)$ are known data on $[0,\pi]$ and $\overline{W} \times (-\infty,\infty)$.

Nomenclature
$\vartheta(x)$ initial condition of x
$s(t)$ inverse coefficient
$L(t)$ energy of material
$w(x,t)$ dissipation of heat
$\omega_0(t), \omega_{ck}(t), \omega_{sk}(t)$ Fourier coefficients
$W := \{0 < x < \pi, 0 < t < T\}$ domain for x,t

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2. Analysis of the Problem

If the conditions are met, the problem will be solved.

(S1) $L(t) \in C^1[0,T]$.

(S2) $\vartheta(x) \in C^2[0,\pi]$,

$\vartheta(0) = \vartheta(\pi),\ \vartheta'(0) = \vartheta'(\pi)$,

$\vartheta''(0) = \vartheta''(\pi),\ L(0) = \int_0^\pi x\vartheta(x)dx$,

(S3)

(1) $h(x,t) \in C^3[0,\pi],\ t \in 0,T]$,

(2) $h(x,t)\big|_{x=0} = h(x,t)\big|_{x=\pi}$,

$h_{x=0} = h_x(\pi,t)\big|_{x=\pi}, h_{xx}(0,t)\big|_{x=0} = h_{xx}(\pi,t)\big|_{x=\pi}$,

(3) $\pi\int_0^\pi h(\xi,t)d\xi - \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k}\int_0^\pi h(\xi,t)\sin 2k\xi d\xi \neq 0$,

$\forall t \in 0,T]$. According to the Fourier method, we obtain

$$\omega(x,t) = \frac{\omega_0(t)}{2} + \sum_{k=1}^{\infty}\left[\omega_{ck}(t)\cos 2kx + \omega_{sk}(t)\sin 2kx\right],$$

$$\omega_0(t) = \vartheta_0 + \frac{2}{\pi}\int_0^t\int_0^\pi \frac{1}{s(\tau)} fg(\xi,\tau)d\xi d\tau,$$

$$\omega_{ck}(t) = \vartheta_{ck}e^{-(2k)^2 t} + \frac{2}{\pi}\int_0^t\int_0^\pi \frac{1}{s(\tau)} h(\xi,\tau)\cos 2k\xi e^{-(2k)^2(t-\tau)}d\xi d\tau,$$

$$\omega_{sk}(t) = \vartheta_{sk}e^{-(2k)^2 t} + \frac{2}{\pi}\int_0^t\int_0^\pi \frac{1}{s(\tau)} h(\xi,\tau)\sin 2k\xi e^{-(2k)^2(t-\tau)}d\xi d\tau.$$

$$\omega(x,t) = \frac{\vartheta_0}{2} + \frac{1}{2}\int_0^t \frac{1}{s(\tau)}h_0(\tau,u)d\tau + \sum_{k=1}^{\infty}\left[\vartheta_{ck}e^{-(2k)^2 t} + \int_0^t \frac{1}{s(\tau)}h_{ck}(\tau)e^{-(2k)^2(t-\tau)}d\tau\right]\cos 2kx \quad (5)$$

$$+ \sum_{k=1}^{\infty}\left[\vartheta_{sk}e^{-(2k)^2 t} + \int_0^t \frac{1}{s(\tau)}h_{sk}(\tau)e^{-(2k)^2(t-\tau)}d\tau\right]\sin 2kx,$$

where

$$\vartheta_0 = \frac{2}{\pi}\int_0^\pi \vartheta(x)dx,$$

$$\vartheta_{ck} = \frac{2}{\pi}\int_0^\pi \vartheta(x)\cos 2kxdx,$$

$$\vartheta_{sk} = \frac{2}{\pi}\int_0^\pi \vartheta(x)\sin 2kxdx,$$

$$h_0(t) = \frac{2}{\pi}\int_0^\pi h(x,t)dx,$$

$$h_{ck}(t) = \frac{2}{\pi}\int_0^\pi h(x,t)\cos 2kxdx,$$

$$h_{sk}(t) = \frac{2}{\pi}\int_0^\pi h(x,t)\sin 2kxdx,\ k = 1,2,3,\ldots\ .$$

According to the condition (S1)-(S3), differentiating (4), we obtain

$$\int_0^\pi x\omega_t(x,t)dx = L'(t), 0 \le t \le T. \quad (6)$$

(5) and (6)

$$\frac{1}{s(t)} = \frac{L'(t) - \pi\sum_{k=1}^{\infty}2k\left(\vartheta_{sk}e^{-(2k)^2 t} + \int_0^t \frac{1}{s(\tau)}h_{sk}(\tau)e^{-(2k)^2(t-\tau)}d\tau\right)}{\pi\int_0^\pi h(\xi,t)d\xi - \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k}\int_0^\pi h(\xi,t)\sin 2k\xi d\xi} \quad (7)$$

Theorem 1. *The problem (1)-(4) has a unique solution if the conditions (S1)-(S3) are hold.*

Proof. Let's apply the absolute value to (5),we get

$$|\omega(x,t)| \le \frac{|\vartheta_0|}{2} + \left|\int_0^t \frac{1}{s(\tau)} f_0(\tau)d\tau\right|$$

$$+\sum_{k=1}^{\infty}\left(|\vartheta_{ck}(t)|+|\vartheta_{sk}(t)|\right)$$

$$+\sum_{k=1}^{\infty}\left|\left(\int_0^t \frac{1}{s(\tau)} h_{ck}(\tau)e^{-(2k)^2(t-\tau)}d\tau\right)\right|$$

$$+\sum_{k=1}^{\infty}\left|\left(\int_0^t \frac{1}{s(\tau)} h_{sk}(\tau)e^{-(2k)^2(t-\tau)}d\tau\right)\right|$$

Taking Cauchy inequality of the equation,we have

$$|\omega(x,t)| \le \frac{|\vartheta_0|}{2} + \left|\left(\int_0^t d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left(\int_0^{\pi}\frac{h^2(\xi,\tau)}{s^2(\tau)}d\xi\right)d\tau\right)\right|$$

$$+\sum_{k=1}^{\infty}\left(|\varphi_{ck}(t)|+|\varphi_{sk}(t)|\right)$$

$$+\sum_{k=1}^{\infty}\left|\left(\int_0^t e^{-2(2k)^2(t-\tau)}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left(\int_0^{\pi}\frac{h^2(\xi,\tau)}{s^2(\tau)}\cos 2k\xi d\xi\right)^{\frac{1}{2}}d\tau\right)\right|$$

$$+\sum_{k=1}^{\infty}\left|\left(\int_0^t e^{-2(2k)^2(t-\tau)}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left(\int_0^{\pi}\frac{h^2(\xi,\tau)}{s^2(\tau)}\sin 2k\xi d\xi\right)^{\frac{1}{2}}d\tau\right)\right|$$

Applying Hölder inequality of the equation,we have

$$|\omega(x,t)| \le \frac{|\vartheta_0|}{2}$$

$$+\sum_{k=1}^{\infty}\left(|\vartheta_{ck}(t)|+|\vartheta_{sk}(t)|\right)$$

$$+\frac{\sqrt{T}}{\pi}\left(\int_0^t\left(\int_0^{\pi}\frac{h^2(\xi,\tau)}{s^2(\tau)}d\xi\right)d\tau\right)^{\frac{1}{2}}$$

$$+\frac{1}{2\sqrt{2}\pi}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}\left(\frac{|h(x,t)|}{|s(\tau)|}\right)^2\right)^{\frac{1}{2}}$$

$$+\frac{1}{2\sqrt{2}\pi}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}\left(\frac{|h(x,t)|}{|s(\tau)|}\right)^2\right)^{\frac{1}{2}}$$

$$|\omega(x,t)| \le \frac{|\vartheta_0|}{2} + \sum_{k=1}^{\infty}\left(|\vartheta_{ck}(t)|+|\vartheta_{sk}(t)|\right)$$

$$+\frac{\sqrt{T}}{\pi}\frac{\|h(x,t)\|}{\|s(\tau)\|}$$

$$+\frac{1}{4\sqrt{3}}\frac{\|h(x,t)\|}{\|s(\tau)\|}+\frac{1}{4\sqrt{3}}\frac{\|h(x,t)\|}{\|s(\tau)\|}.$$

$$|\omega(x,t)| \le \frac{|\vartheta_0|}{2} + \sum_{k=1}^{\infty}\left(|\vartheta_{ck}(t)|+|\vartheta_{sk}(t)|\right)$$

$$+\frac{1}{4\sqrt{3}}\frac{M}{\|s(\tau)\|}+\frac{1}{4\sqrt{3}}\frac{\mathrm{M}}{\|\mathrm{s}(\tau)\|} \qquad (8)$$

$\omega(x,t)\in C^{2,2}(W)\cap C^{1,0}(\overline{W})$.Since (8) is limited, this series is uniformly convergent according to the Weierstrass theorem.

$$\omega_x(x,t)=\sum_{k=1}^{\infty}(-2k)\vartheta_{ck}e^{-(2k)^2 t}\sin 2kx$$
$$+\sum_{k=1}^{\infty}(-2k)\sin 2kx\int_0^t \frac{1}{s(\tau)}h_{sk}(\tau)e^{-(2k)^2(t-\tau)}d\tau \quad (9)$$
$$+\sum_{k=1}^{\infty}(2k)\vartheta_{sk}e^{-(2k)^2 t}\cos 2kx$$
$$+\sum_{k=1}^{\infty}(2k)\cos 2kx\int_0^t \frac{1}{s(\tau)}h_{ck}(\tau)e^{-(2k)^2(t-\tau)}d\tau$$

Let's take the absolute value of both sides of (9), we have

$$|\omega_x(x,t)|\le\sum_{k=1}^{\infty}2k\left(|\vartheta_{ck}(t)|+|\vartheta_{sk}(t)|\right)$$
$$+\sum_{k=1}^{\infty}\left|(2k)\int_0^t \frac{1}{s(\tau)}h_{sk}(\tau)e^{-(2k)^2(t-\tau)}d\tau\right|$$
$$+\sum_{k=1}^{\infty}\left|(2k)\int_0^t \frac{1}{s(\tau)}h_{sk}(\tau)e^{-(2k)^2(t-\tau)}d\tau\right|$$

Applying Cauchy inequality both of side to the previous equation,we have

$$|\omega_x(x,t)|\le\sum_{k=1}^{\infty}\left(\left|\vartheta'_{sk}(t)\right|+\left|\vartheta'_{ck}(t)\right|\right)$$
$$+\sum_{k=1}^{\infty}2k\left(\left(\int_0^t e^{-2(2k)^2(t-\tau)}d\tau\right)^{\frac{1}{2}}x\left(\int_0^t\left(\int_0^{\pi}\frac{h^2(\xi,\tau)}{s^2(\tau)}\cos 2k\xi d\xi\right)^{\frac{1}{2}}d\tau\right)\right)$$
$$+\sum_{k=1}^{\infty}2k\left(\left(\int_0^t e^{-2(2k)^2(t-\tau)}d\tau\right)^{\frac{1}{2}}x\left(\int_0^t\left(\int_0^{\pi}\frac{h^2(\xi,\tau)}{s^2(\tau)}\sin 2k\xi d\xi\right)^{\frac{1}{2}}d\tau\right)\right)$$

$$|\omega_x(x,t)|\le\sum_{k=1}^{\infty}\left(\left|\vartheta'_{sk}(t)\right|+\left|\vartheta'_{ck}(t)\right|\right)$$
$$+\sum_{k=1}^{\infty}\frac{1}{\sqrt{2}}\left(\int_0^t\left(\int_0^{\pi}\frac{h^2(\xi,\tau)}{s^2(\tau)}\cos 2k\xi d\xi\right)^{\frac{1}{2}}d\tau\right)$$
$$+\sum_{k=1}^{\infty}\frac{1}{\sqrt{2}}\left(\int_0^t\left(\int_0^{\pi}\frac{h^2(\xi,\tau)}{s^2(\tau)}\sin 2k\xi d\xi\right)^{\frac{1}{2}}d\tau\right)$$

$$|\omega_x(x,t)|\le\sum_{k=1}^{\infty}\left(\left\|\vartheta'_{ck}(t)\right\|+\left\|\vartheta'_{sk}(t)\right\|\right)$$
$$+\sum_{k=1}^{\infty}\frac{1}{\sqrt{2}}\frac{M}{\|s(\tau)\|}+\sum_{k=1}^{\infty}\frac{1}{\sqrt{2}}\frac{M}{\|s(\tau)\|}$$

Since sum of x-partial derivative series($\omega_x(x,t)$) is limited, this series is uniformly convergent according to the Weierstrass theorem.

$$\omega_{xx}(x,t)=\sum_{k=1}^{\infty}(-2k)^2\vartheta_{ck}e^{-(2k)^2 t}\cos 2kx$$
$$\sum_{k=1}^{\infty}(-2k)^2\int_0^t \frac{1}{s(\tau)}h_{ck}(\tau)e^{-(2k)^2(t-\tau)}\cos 2kxd\tau \quad (10)$$

$$+\sum_{k=1}^{\infty}(-2k)^2\vartheta_{sk}e^{-(2k)^2 t}\sin 2kx$$

$$\sum_{k=1}^{\infty}(-2k)^2\int_0^t\frac{1}{s(\tau)}h_{sk}(\tau)e^{-(2k)^2(t-\tau)}\sin 2kxd\tau$$

Let's take the absolute value of both sides of (10),we have

$$|\omega_{xx}(x,t)|\le\sum_{k=1}^{\infty}\left(\left\|\vartheta_{ck}''(t)\right\|+\left\|\vartheta_{sk}''(t)\right\|\right)$$

$$+\sum_{k=1}^{\infty}(-2k)^2\left(\int_0^t e^{-2(2k)^2(t-\tau)}d\tau\right)^{\frac{1}{2}}$$

$$\left(\int_0^t\left(\int_0^\pi\frac{h'(\xi,\tau)}{s(\tau)}\sin 2k\xi d\xi\right)^2 d\tau\right)^{\frac{1}{2}}$$

$$+\sum_{k=1}^{\infty}(-2k)^2\left(\int_0^t e^{-2(2k)^2(t-\tau)}d\tau\right)^{\frac{1}{2}}$$

$$\left(\int_0^t\left(\int_0^\pi\frac{h'(\xi,\tau)}{s(\tau)}\sin 2k\xi d\xi\right)^2 d\tau\right)^{\frac{1}{2}}$$

$$|\omega_{xx}(x,t)|\le\sum_{k=1}^{\infty}\left(\left\|\vartheta_{ck}''(t)\right\|+\left\|\vartheta_{sk}''(t)\right\|\right)$$

$$+\sum_{k=1}^{\infty}\frac{1}{\sqrt{2}}\frac{M}{\|s(\tau)\|}+\sum_{k=1}^{\infty}\frac{1}{\sqrt{2}}\frac{M}{\|s(\tau)\|}$$

Since sum of xx-partial derivative series($\omega_{xx}(x,t)$) is limited, this series is uniformly convergent according to the Weierstrass theorem. So, Fourier series $\omega(x,t)$ has been the unique solution.

From the second kind of Volterra equation :

$$q(t)=F(t)+\int_0^t K(t,\tau)q(\tau)d\tau, t\varepsilon[0,T] \quad (11)$$

where

$$q(t)=\frac{1}{s(t)},$$

$$F(t)=\frac{E'(t)-\pi\sum_{k=1}^{\infty}2k\varphi_{sk}e^{-(2k)^2 t}}{\pi\int_0^\pi h(\xi,t)d\xi-\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k}\int_0^\pi h(\xi,t)\sin 2k\xi d\xi}, \quad (12)$$

$$K(t,\tau)=\frac{-\pi\sum_{k=1}^{\infty}2k\left(\int_0^t\frac{1}{s(\tau)}h_{sk}(\tau)e^{-(2k)^2(t-\tau)}d\tau\right)}{\pi\int_0^\pi h(\xi,t)d\xi-\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k}\int_0^\pi h(\xi,t)\sin 2k\xi d\xi} \quad (13)$$

where

$$\pi\int_0^\pi h(\xi,t)d\xi-\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k}\int_0^\pi h(\xi,t)\sin 2k\xi d\xi\ne 0,$$

$\forall t\varepsilon[0,T]$.

Now,let show the $F(t)$ and $K(t,\tau)$ are continuous function. Let's take the absolute value of both sides of (13),we have

$$|F(t)|\le\frac{|L'(t)|+\left|\pi\sum_{k=1}^{\infty}\frac{2k}{2k}\varphi_{ck}'e^{-(2k)^2 t}\right|}{\left|\pi\int_0^\pi h(\xi,t)d\xi-\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k}\int_0^\pi h(\xi,t)\sin 2k\xi d\xi\right|}$$

$$\|F(t)\|\le\frac{\|L'(t)\|+\pi\sum_{k=1}^{\infty}\left\|\varphi_{ck}'\right\|}{M}$$

$$|K(t,\tau)| \le \frac{\left|\pi\sum_{k=1}^{\infty}\frac{2k}{2k}\left(\int_0^t q(\tau)h'_{sk}(\tau)e^{-(2k)^2(t-\tau)}d\tau\right)\right|}{\left|\pi\int_0^{\pi}h(\xi,t)d\xi-\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\pi}h(\xi,t)\sin 2k\xi d\xi\right|} \tag{14}$$

$$\|K(t,\tau)\| \le \frac{\pi^2\|q(\tau)\|\|h(x,t)\|}{M}$$

According to the assumption (S1)-(S2) the $F(t)$ and the $K(t,\tau)$ (kernel) are continuous on $[0,T]$.

For the uniqueness, let's assume the opposite is the case. Let the problem has two solutions (u,s_1), (v,s_2) .

$$\omega(x,t)-v(x,t)=\frac{1}{2}\int_0^t\left(\frac{1}{s_1(t)}-\frac{1}{s_2(t)}\right)h_0(\tau)d\tau$$

$$+\sum_{k=1}^{\infty}\cos 2kx\int_0^t\left(\frac{1}{s_2(t)}-\frac{1}{s_2(t)}\right)h_{ck}(\tau)e^{-(2k)^2(t-\tau)}d\tau$$

$$+\sum_{k=1}^{\infty}\sin 2kx\int_0^t\left(\frac{1}{s_2(t)}-\frac{1}{s_2(t)}\right)h_{sk}(\tau)e^{-(2k)^2(t-\tau)}d\tau$$

$$\omega(x,t)-v(x,t)=\frac{1}{2}\int_0^t(q(t)-r(t))h_0(\tau)d\tau$$

$$+\sum_{k=1}^{\infty}\cos 2kx\int_0^t(q(t)-r(t))h_{ck}(\tau)e^{-(2k)^2(t-\tau)}d\tau$$

$$+\sum_{k=1}^{\infty}\sin 2kx\int_0^t(q(t)-r(t))hf_{sk}(\tau)e^{-(2k)^2(t-\tau)}d\tau$$

where

$$q(t)=\frac{1}{s_1(t)} \text{ and } r(t)=\frac{1}{s_2(t)}$$

$$|\omega-v| \le \left(\int_0^t\int_0^{\pi}h^2(x,\tau)(q(t)-r(t))^2d\tau\right)^{\frac{1}{2}}$$

$$+\sum_{k=1}^{\infty}\cos 2kx\left(\int_0^t e^{-2(2k)^2(t-\tau)}d\tau\right)^{\frac{1}{2}}$$

$$\left(\int_0^t\int_0^{\pi}h^2(x,\tau)(q(t)-r(t))^2\cos^2 2kxdxd\tau\right)^{\frac{1}{2}}$$

$$+\sum_{k=1}^{\infty}\sin 2kx\left(\int_0^t e^{-2(2k)^2(t-\tau)}d\tau\right)^{\frac{1}{2}}$$

$$\left(\int_0^t\int_0^{\pi}h^2(x,\tau)(q(t)-r(t))^2\sin^2 2kxdxd\tau\right)^{\frac{1}{2}}$$

$$\|\omega-v\| \le \frac{1}{2\sqrt{3}}M\|r-q\| \tag{15}$$

$$r-q=\frac{-\pi\sum_{k=1}^{\infty}(2k)\int_0^t\int_0^{\pi}h(r-q)\sin 2kxe^{-2(2k)^2(t-\tau)}d\xi d\tau}{\pi\int_0^{\pi}hd\xi-\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\pi}h\sin 2k\xi d\xi}$$

$$\|\mathrm{r(t)}-\mathrm{q(t)}\| \le \frac{\sqrt{2}}{(\sqrt{2}-\pi)}$$

$$\|\omega - v\| \le \frac{M}{2\sqrt{3}} \frac{\sqrt{2}}{\left(\sqrt{2}-\pi\right)} \tag{16}$$

$\omega(t) = v(t)$. Hence $r(t) = q(t)$.

3. Analysis of Stability of the Solution

Theorem 2. *When the assumptions (S1)-(S3) be provided, the solution of the problem(1)-(4) constantly connected the* h, φ, ψ, E.

Proof. Let $E = \{\varphi, L\}$ and $\overline{E} = \{\overline{\varphi}, \overline{L}\}$ be two sets of the data. $M_i,\ i = 1,2,3,4$ such that

$$\|h\|_{C^1(D)} \le M, \|\vartheta\|_{C^2[0,\pi]} \le M_1, \|\overline{\vartheta}\|_{C^2[0,\pi]} \le M_1,$$

$$\|L\|_{C^2[0,T]} \le M_2, \|\overline{L}\|_{C^2[0,T]} \le M_2.$$

Let us denote $\|E\| = (\|L\|_{C^1[0,T]} + \|\vartheta\|_{C^2[0,\pi]})$. Let (ω, q) and $(\overline{\omega}, \overline{q})$ be solutions of the problems (1)-(4).

$$\omega(x,t) - \overline{\omega}(x,t) = \frac{1}{2}\left(\vartheta_0 - \overline{\vartheta_0}\right)$$

$$+ \sum_{k=1}^{\infty} \cos 2kx \left[\left(\vartheta_{ck} - \overline{\vartheta_{ck}}\right) e^{-(2k)^2 t}\right]$$

$$+ \sum_{k=1}^{\infty} \sin 2kx \left[\left(\vartheta_{sk} - \overline{\vartheta_{sk}}\right) e^{-(2k)^2 t}\right]$$

$$+ \sum_{k=1}^{\infty} \cos 2kx \int_0^t \left(q(\tau) - \overline{q(\tau)}\right) h_{ck}(\tau) e^{-2(2k)^2 (t-\tau)} d\tau$$

$$+ \sum_{k=1}^{\infty} \cos 2kx \int_0^t \left(h_{ck}(\tau) - \overline{h_{ck}(\tau)}\right) \overline{q(\tau)} e^{-2(2k)^2 (t-\tau)} d\tau$$

$$+ \sum_{k=1}^{\infty} \sin 2kx \int_0^t \left(q(\tau) - \overline{q(\tau)}\right) h_{sk}(\tau) e^{-2(2k)^2 (t-\tau)} d\tau$$

$$+ \sum_{k=1}^{\infty} \sin 2kx \int_0^t \left(h_{sk}(\tau) - \overline{h_{sk}(\tau)}\right) \overline{q(\tau)} e^{-2(2k)^2 (t-\tau)} d\tau,$$

$$\|\omega - \overline{\omega}\| \le \frac{1}{2}\|\vartheta_0 - \overline{\vartheta_0}\|$$

$$+ \sum_{k=1}^{\infty} \|\vartheta_{ck} - \overline{\vartheta_{ck}}\| + \|\vartheta_{sk} - \overline{\vartheta_{sk}}\|$$

$$+ \left(\frac{\sqrt{T}}{\pi} + \frac{1}{4\sqrt{3}}\right) M \|q - \overline{q}\|$$

$$+ \left(\frac{\sqrt{T}}{\pi} + \frac{1}{4\sqrt{3}}\right) M \|h - \overline{h}\|$$

$$F(t) - \overline{F(t)}$$

$$= \frac{L'(t) - \pi \sum_{k=1}^{\infty} 2k e^{-(2k)^2 t} \varphi_{sk}}{\pi \int_0^{\pi} h(\xi,t) d\xi - \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{1}{k} \int_0^{\pi} h(\xi,t) \sin 2k\xi d\xi}$$

$$- \frac{\overline{L'(t)} - \pi \sum_{k=1}^{\infty} 2k e^{-(2k)^2 t} \overline{\varphi_{sk}}}{\pi \int_0^{\pi} h(\xi,t) d\xi - \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{1}{k} \int_0^{\pi} h(\xi,t) \sin 2k\xi d\xi}$$

Applying Cauchy, Bessel inequality and taking maksimum of $F - \overline{F}$,

$$\left\|F-\overline{F}\right\| \leq \frac{1}{M}\left\|L'(t)-\overline{L'(t)}\right\|$$

$$+\frac{\pi}{M}\sum_{k=1}^{\infty}\left\|\vartheta_{sk}-\overline{\vartheta_{sk}}\right\|$$

$$K(t,\tau)-\overline{K}(t,\tau)$$

$$=\frac{\left(-\pi\sum_{k=1}^{\infty}(2k)\frac{1}{2k}\int_0^t q(\tau)h_{sk}^{'}(\tau)e^{-2(2k)^2(t-\tau)}d\tau\right)}{\pi\int_0^{\pi}h(\xi,t)d\xi-\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\pi}h(\xi,t)\sin 2k\xi d\xi}$$

$$-\frac{\left(-\pi\sum_{k=1}^{\infty}(2k)\frac{1}{2k}\int_0^t \overline{q(\tau)}h_{sk}^{'}(\tau)e^{-2(2k)^2(t-\tau)}d\tau\right)}{\pi\int_0^{\pi}h(\xi,t)d\xi-\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\pi}h(\xi,t)\sin 2k\xi d\xi}$$

$$\left\|K-\overline{K}\right\| \leq \frac{\pi}{M}\left\|q-\overline{q}\right\|,$$

$$\left\|r-\overline{r}\right\| \leq \left\|F-\overline{F}\right\|+\left\|K-\overline{K}\right\|\left\|\overline{q}\right\|+\left\|q-\overline{q}\right\|\left\|K\right\|$$

$$\left\|\omega-\overline{\omega}\right\| \leq \frac{1}{2}\left\|\vartheta_0-\overline{\vartheta_0}\right\|$$

$$+\frac{\pi}{M}\sum_{k=1}^{\infty}\left\|\vartheta_{ck}-\overline{\vartheta_{ck}}\right\|+\left\|\vartheta_{sk}-\overline{\vartheta_{sk}}\right\|$$

$$+\frac{1}{M}\left\|L^{'}(t)-\overline{L^{'}(t)}\right\|$$

$$+\frac{\pi}{M}\left\|q-\overline{q}\right\|\left\|\overline{q}\right\|+\left\|q-\overline{q}\right\|\left\|K\right\|.$$

$$\left\|\vartheta-\overline{\vartheta}\right\| = \frac{1}{2}\left\|\vartheta_0-\overline{\vartheta_0}\right\|$$

$$+\frac{\pi}{M}\sum_{k=1}^{\infty}\left\|\vartheta_{ck}-\overline{\vartheta_{ck}}\right\|+\left\|\vartheta_{sk}-\overline{\vartheta_{sk}}\right\|$$

$$M_3=\max\left\{\frac{1}{2},\frac{\pi}{M}\right\}.$$

$$\left\|E-\overline{E}\right\| \leq M_3\left\|\vartheta-\overline{\vartheta}\right\|+M_4\left\|L^{''}-\overline{L^{''}}\right\|$$

$$\left\|\omega-\overline{\omega}\right\| \leq M_5\left\|E-\overline{E}\right\|$$

where

$$M_5=\max\{M_3,M_4\}.$$

For $E \to \overline{E}$ then $\omega \to \overline{\omega}$.

4. The Finite Difference Approximation

Using the implicit formula in (1)-(4)

$$q^j\frac{\omega_i^{j+1}-\omega_i^j}{\tau}=\frac{\omega_{i-1}^{j+1}-2\omega_i^{j+1}+\omega_{i+1}^{j+1}}{h^2}+h_i^j, \tag{17}$$

$$\omega_i^1=\vartheta_i, \tag{18}$$

$$\omega_0^j=\omega_N^j, \tag{19}$$

$$\frac{\omega_1^j+\omega_{N-1}^j}{2}=\omega_N^j, \tag{20}$$

where $1 \le i \le N$ and $0 \le j \le M$. Equal lengths $h = \frac{\pi}{M}$ and $\tau = \frac{T}{N}$,

$$q(t) = \frac{L'(t) - \pi u_x(\pi, t)}{\int_0^\pi x f(x,t)dx}.$$

$$q^j = \frac{-\left(\left(L^{j+1} - L^j\right)/\tau\right) - \pi\left(u_N^j - u_{N-1}^j\right)/h}{(fin)^j},$$

where $L^j = L(t_j)$, $(fin)^j = \int_0^1 x f(x, t_j)dx$, $j = 0,1,\ldots,M$, q^j, u_i^j at the j-th iteration step. Finally,with the Gaussian elimination method, u_i^{j+1} and q^j can be solved.

5. Conclusions

This problem has been studied with periodic and integral conditions. This inverse problem is theoretically proved using the fourier method. Also, a finite difference scheme is made.

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Declaration of Ethical Standards

The authors of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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