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Generalized Dual Quaternions and Screw Motion in Generalized Space

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Abstract

In this paper, we showed that the set of displacements of generalized space is a group under the composite operation. We obtained this screw axis of displacement in generalized space. Using this screw axis, we obtained the Rodrigues equation in terms of spatial displacement in this space. Finally, the components of a dual generalized quaternion and the dual orthogonal matrix were obtained using Euler parameters in generalized space.

Keywords: *Generalized space; rigid motion; kinematics; matrix groups*

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1. Introduction

Dual quaternions are useful tools for describing rigid body motions. Dual numbers were initially introduced by Clifford in 1873 [7]. Their first applications to kinematics were held by Kotel'nikov in 1895, and Study in 1903 [14, 20]. Bottema and Roth used dual numbers in theoretical kinematics [6]. Hiller and Woernle in 1984 used dual matrices and dual-vectors for the representation of spatial displacement [11]. Agrawal in 1987 applied dual quaternions to spatial kinematics [1]. Herve in 2008 employed Lie's theory of groups to describe displacement sets of rigid bodies and their connections [10]. Kula and Yaylı in 2019 obtained screw motion by means of Hamilton operators in $3-$dimensional Lorentzian space L^3 [13] (see details [2, 6, 15, 16, 20]).

Vector spaces with a quadratic form generate an unital associative algebra called Clifford algebra. The quaternion algebra, subalgebra of Clifford algebra, is isomorphic to the algebra in which the set of rotations under composite operation. (see for details [1, 3, 6, 7, 15, 16, 19]). The generalized quaternions $E^3(\alpha,\beta)$ are four-dimensional algebra which is associative but not commutative. The algebra of generalized quaternion is a natural generalization of quaternion algebra $\mathbf{H}$. Jafari and Yaylı in 2011 described a rotational motion in generalized space using quaternions and studied the algebraic properties of generalized quaternions [12]). Ata and Savci in 2021obtained that generalized Cayley formula, Rodrigues equation and Eulerparameters of a rotation in space $E^3(\alpha,\beta)$ [5]. (see for details[4, 5, 9, 12]). Özkaldı and Gündoğan in 2011 introduce the screw axis of displacement and Rodrigues equation for a spatial displacement in 3-dimensional Lorentzian space L^3 [18].

In this study, we will obtain that the screw axis of displacement in $E^3(\alpha,\beta)$. Rodrigues equation for a spatial displacement will be obtained by using the screw axis in $E^3(\alpha,\beta)$. The components of a dual generalized quaternion, corresponding to a coordinate transformation, will be found using the generalized dual Euler parameters. Finally, we will get that the generalized dual orthogonal matrix via those components $E^3(\alpha,\beta)$ which includes the Euclidean and Lorentzian spaces, for every possible value of α and β except $\alpha,\beta=0$.

2. Introduction

Definition 2.1: Let $r=(r_1,r_2,r_3)$, $s=(s_1,s_2,s_3)$ be two vectors in $\mathbb{R}^3$ and $\alpha,\beta\in\mathbb{R}$. The generalized scalar product is defined by

$$<r,s>_G=\alpha r_1 s_1+\beta r_2 s_2+\alpha\beta r_3 s_3.$$

The vector space $\mathbb{R}^3$ with the generalized scalar product defined over it is called $3-$dimensional generalized space and is denoted by $E^3(\alpha,\beta)=(R^3,<,>_G)$. This space includes the Euclidean and semi-Euclidean spaces i.e;

If $\alpha=\beta=1$, then $E^3(1,1)=E^3$ 3-dimensional Euclidean space.

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If $\alpha = 1$ and $\beta = -1$, then $E^3(1,-1) = E_1^3$ 3-dimensional semi-Euclidean space. The generalized vectorial product is defined as follow;

$$r \wedge_G s = \beta(r_2 s_3 - r_3 s_2)i - \alpha(r_1 s_3 - s_3 r_2)j + (r_1 s_2 - r_2 s_1)k,$$

here $i \wedge_G j = k$, $j \wedge_G k = \beta i$, and $k \wedge_G i = -\alpha j$ [12].
Remark : For the sake of the short in the next part of our article, we will use notation G- instead of generalized such as generalized quaternions (G-quaternions).
Definition 2.2: Let $\widehat{q}$ be a G-dual quaternion, then Hamilton operators

$$H^+(\widehat{q}) = \begin{bmatrix} \widehat{q}_0 & -\alpha\widehat{q}_1 & -\beta\widehat{q}_2 & -\alpha\beta\widehat{q}_3 \\ \widehat{q}_1 & \widehat{q}_0 & -\beta\widehat{q}_3 & \beta\widehat{q}_2 \\ \widehat{q}_2 & \alpha\widehat{q}_3 & \widehat{q}_0 & -\alpha\widehat{q}_1 \\ \widehat{q}_3 & -\widehat{q}_2 & \widehat{q}_1 & \widehat{q}_0 \end{bmatrix}$$

and

$$H^-(\widehat{q}) = \begin{bmatrix} \widehat{q}_0 & -\alpha\widehat{q}_1 & -\beta\widehat{q}_2 & -\alpha\beta\widehat{q}_3 \\ \widehat{q}_1 & \widehat{q}_0 & \beta\widehat{q}_3 & -\beta\widehat{q}_2 \\ \widehat{q}_2 & -\alpha\widehat{q}_3 & \widehat{q}_0 & \alpha\widehat{q}_1 \\ \widehat{q}_3 & \widehat{q}_2 & -\widehat{q}_1 & \widehat{q}_0 \end{bmatrix},$$

are defined. The product of two quaternions $\widehat{q}$ and $\widehat{p}$ can be done in two different ways by using these operators;

$$\widehat{q}\widehat{p} = H^+(\widehat{q})\widehat{p} = H^-(\widehat{p})(\widehat{q})$$

[4, 12].
Definition 2.3: A matrix $S = \begin{bmatrix} 0 & \beta s_3 & \beta s_2 \\ \alpha s_3 & 0 & -\alpha s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}$ called a generalized skew-symmetric matrix if $S^T\varepsilon = -\varepsilon S$, where $\varepsilon = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha\beta \end{bmatrix}$

and $\alpha, \beta \in \mathbb{R} - \{0\}$ [4, 12].
Definition 2.4: Matrix A called a generalized orthogonal matrix if $A^T\varepsilon A = |A|\varepsilon$ where $\varepsilon = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha\beta \end{bmatrix}$ and $\alpha, \beta \in \mathbb{R} - \{0\}$. The set of all generalized orthogonal matrices with $detA = 1$ under the operation of matrix multiplication forms a group and denoted as $SO(\alpha,\beta)(3)$ called rotation group in $E^3(\alpha,\beta)$ [4, 12].
The rotation is an important part of the rigid transformation. To determine the rigid transformation in $E^3(\alpha,\beta)$, we will give necessary fundamental properties about the rotations.
i) Let $\alpha > 0$ and $\beta > 0$. Let the unit vector $s = (s_1, s_2, s_3)$. The G-orthogonal matrix B can be written as following

$$A = I + (\sin\theta)S + (1-\cos\theta)S^2. \tag{2.1}$$

ii) Let $\alpha > 0$ and $\beta < 0$. Let the vector a be a time-like vector. The G-orthogonal matrix B can be written as following

$$A = I + (\sinh\theta)S + (\cosh\theta - 1)S^2 \tag{2.2}$$

[5].
There are two different cases for all possible non zero values of α and β. First case: $\alpha, \beta > 0$ and second case: $\alpha > 0$ and $\beta < 0$. We will consider both cases separately in the end of the section 4.

3. Rigid Transformations in Generalized Space $E^3(\alpha,\beta)$

3.1. Coordinate Transformations

Interconnections between the coordinate frames enable to determination of the position of one body in relation to another. Let the coordinates of the moving body in frame M and fixed frame F be $y = (y_1, y_2, y_3)$ and $x = (x_1, x_2, x_3)$, respectively. Then, $D : F \to M$ coordinate transformation is as follows;

$$D(x) = y = A \cdot_G x + d$$

here A is an 3×3 matrix and d is an $3-$dimensional vector. If A is an 3×3 G-orthogonal matrix, then the transformation D is a rigid transformation. $D = (A,d)$, whose componenets are matrix and vector, defines a displacement in $3-$dimensional generalized space $E^3(\alpha,\beta)$.
Theorem: The set of displacements of $E^3(\alpha,\beta)$ under the composite operation forms group.
Proof:
1. If $D_1 : F \to M_1$ and $D_2 : M_1 \to M_2$ are displacements, then $D = D_1D_2 : F \to M_2$ is also a displacements. Indeed, $D = D_1D_2 = (A_1,d_1)(A_2,d_2) = (A_1 \cdot_G A_2, A_1 \cdot_G d_2 + d_1)$.
2. $I = (I_3, 0)$ is the unit element.
3. Since $A^T\varepsilon A = \varepsilon$ for $\alpha, \beta \neq 0$, every $D = (A,d)$ have an inverse such that $D^{-1} = (A^{-1}, -A^{-1} \cdot_G d)$. Consequently, this set in $E^3(\alpha,\beta)$ is a group and called as G-group.

4. Spatial Displacement in Generalized Space $E^3(\alpha,\beta)$

4.1. Coordinate Transformations

$y = A\cdot_G x + d$ represents the transformation of the relative position of rigid bodies that the coordinates $y = (y_1, y_2, y_3)$ in M and $x = (x_1, x_2, x_3)$ in F. A is an 3×3 G-orthogonal matrix and $d = (d_1, d_2, d_3)$ is the translation vector. Since $D = (A, d)$ does not change the distance between any two points, this transformation is called a spatial transformationin $E^3(\alpha,\beta)$.

4.2. Screw Axis of a Displacement

In this section, we investigate the points whose coordinates do not change after a spatial displacement in generalized space $E^3(\alpha,\beta)$. Let these points k be, then they those points satisfy the equation

$$k = A\cdot_G k + d$$

or

$$(I-A)\cdot_G k = d.$$

The solution of this equation

$$k = -(A-I)^{-1}\cdot_G d.$$

The $(A-I)$ matrix is not regular because one of the eigenvalues of the G-orthogonal matrix is 1

As a result; a spatial displacement has not fixed point. But, it has a fixed-line, the points on the line moving only on this line under the action of the displacement. This line called as screw axis in generalized space $E^3(\alpha,\beta)$. The direction of this axis is determined by the -G-Rodrigues- vector b, which is obtained from the rotation vector A. Assume d^* be projection of the translation vector d onto the a plane which perpendicular to vector b. Thus we can write follow equation;

$$(I-A)\cdot_G k = d^*.$$

This equation describes points k that remain constant after rotation around the vector b and translation on a plane perpendicular to the vector b. Therefore, position of this line can be determined as follows:

$$l = k + tb.$$

$ds = d - d^*$ is translation amount along the screw axis. This vector in direction of translation unit vector $s = \frac{b}{\|b\|}$.

4.3. Rodrigues Equation for Spatial Displacements

Let us obtain the the equivalent of the -G-Rodrigues equation- for spatial displacement using the screw axis $l = k + sb$. Let k be on the screw axis and take $x-k$ instead of x and $y-(dt+k)$ instead of y in G-Rodrigues equation. Thus, the equation

$$\begin{aligned} y-(ds+k)-(x-k) &= b\wedge_G (y-(ds+k)+x-k), \\ y-x-ds &= b\wedge_G (y+x-2ds-2k), \\ y-x &= b\wedge_G (y+x-2k)+ds. \end{aligned}$$

is obtained.

4.4. The Screw Matrix

In this section, let's do the opposite of what we have done so far, that is, let's determine the screw displacement $D = (A, d)$ using given displacement' angle θ, distance d and screw axis l.

Let's, the screw axis be $l = k + tb$ in F, s is a unit vector and let k such that $< k, s >_G = 0$. From the known the rotation angle θ we can easy get -G-Rodrigues vector- as $b = (\tan\frac{\theta}{2})$. The components of the vector b gives G-skew symmetric matrix B and from the -G-Cayley formula- we obtain G-rotation matrix A .

If the origin of the fixed frame F is on l, then $k = 0$, ie the translation vector $d = ds$. We get the displacement as $D' = (A, ds)$.

If the orijin of F fixed frame is not on l, $k \neq 0$. Let's take a reference frame F' whose origin is at point k, the position of this frame relative to the frame F i.e. $T : F \to F'$ can be determine by a simple translation $R = (I, k)$. If we change the coordinates of $D' = (A, ds)$ using $R = (I, k)$; such as $D = RD'R^{-1}$. Since $R^{-1} = (I, -k)$, we get

$$D = RD'R^{-1} = (I,k)(A,ds)(I,-k) = (A,\ ds - A\cdot_G k + k).$$

The last equation gives the vector d as;

$$d = ds + (I-A)\cdot_G k.$$

The resulting (A, d) is a 4×4 matrix that defines a displacement in the generalized space and is called a G-skew matrix.

4.5. Generalized Dual Quaternions

The transformation $T=(A,d)$ determines the position of M with respect to F can also be represented by G-dual quaternions $\widehat{q}=q+\varepsilon q^*$. The -G-Euler parameters- of the rotation matrix A gives $q=q_0+q_1i+q_2j+q_3k$, which is the real part of the G-dual quaternion $\widehat{q}$. We can use the following formula to find the dual part q^*;

$$q^*=\frac{1}{2}Dq,$$

here $D=d_1i+d_2j+d_3k$ is the G-quaternion obtained from the vector $d=(d_1,d_2,d_3)$.
d can be written as $d=ds+c-A\cdot_G k$ according to the screw parameters of the displacement in $E^3(\alpha,\beta)$. If these vectors are written the form of G-quaternions, then we get that

$$D=dS+K-qK\overline{q},$$

and

$$\begin{aligned} q^*&=\tfrac{1}{2}(dS+K-qK\overline{q})q\\ q^*&=\tfrac{1}{2}(dSq+Kq-qK). \end{aligned}$$

q^* can be written in matrix form as follows:

$$q^*=\frac{1}{2}(H^-(q)\cdot_G(dS+K)q-H^+(q)\cdot_G K) \tag{4.1}$$

Now, there are two cases according to value of α and β.
case1: Let $\alpha>0$ and $\beta>0$. Then equation (4.1) can be written;

$$H^-(q)\cdot_G(dS+K)=\begin{bmatrix} \cos\frac{\theta}{2} & -\alpha s_x\sin\frac{\theta}{2} & -\beta s_y\sin\frac{\theta}{2} & -\alpha\beta s_z\sin\frac{\theta}{2}\\ s_x\sin\frac{\theta}{2} & \cos\frac{\theta}{2} & \beta s_z\sin\frac{\theta}{2} & -\beta s_y\sin\frac{\theta}{2}\\ s_y\sin\frac{\theta}{2} & -\alpha s_z\sin\frac{\theta}{2} & \cos\frac{\theta}{2} & \alpha s_x\sin\frac{\theta}{2}\\ s_z\sin\frac{\theta}{2} & s_y\sin\frac{\theta}{2} & -s_x\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}\cdot_G \begin{bmatrix} 0\\ (\alpha d_1s_x+\beta d_2s_y+\alpha\beta d_3s_z)s_x+c_1\\ (\alpha d_1s_x+\beta d_2s_y+\alpha\beta d_3s_z)s_y+c_2\\ (\alpha d_1s_x+\beta d_2s_y+\alpha\beta d_3s_z)s_z+c_3 \end{bmatrix}$$

and

$$H^+(q)\cdot_G K=\begin{bmatrix} \cos\frac{\theta}{2} & -\alpha s_x\sin\frac{\theta}{2} & -\beta s_y\sin\frac{\theta}{2} & -\alpha\beta s_z\sin\frac{\theta}{2}\\ s_x\sin\frac{\theta}{2} & \cos\frac{\theta}{2} & -\beta s_z\sin\frac{\theta}{2} & \beta s_y\sin\frac{\theta}{2}\\ s_y\sin\frac{\theta}{2} & \alpha s_z\sin\frac{\theta}{2} & \cos\frac{\theta}{2} & -\alpha s_x\sin\frac{\theta}{2}\\ s_z\sin\frac{\theta}{2} & -s_y\sin\frac{\theta}{2} & s_x\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}\cdot_G \begin{bmatrix} 0\\ c_1\\ c_2\\ c_3 \end{bmatrix}.$$

By using the equation $d=<d,s>_G=\alpha d_1s_x+\beta d_2s_y+\alpha\beta d_3s_z$, we have that

$$q^*=\frac{1}{2}\begin{bmatrix} -\alpha s_x^2d\sin\frac{\theta}{2}-\alpha s_xc_1\sin\frac{\theta}{2}-\beta s_y^2d\sin\frac{\theta}{2}-\beta s_yc_2\sin\frac{\theta}{2}-\alpha\beta s_z^2d\sin\frac{\theta}{2}-\alpha\beta s_zc_3\sin\frac{\theta}{2}\\ s_xd\cos\frac{\theta}{2}+c_1\cos\frac{\theta}{2}+\beta s_zc_2\sin\frac{\theta}{2}-\beta s_yc_3\sin\frac{\theta}{2}\\ -\alpha s_zc_1\sin\frac{\theta}{2}+s_yd\cos\frac{\theta}{2}+c_2\cos\frac{\theta}{2}+\alpha s_xc_3\sin\frac{\theta}{2}\\ s_yc_1\sin\frac{\theta}{2}-s_xc_2\sin\frac{\theta}{2}+s_zd\cos\frac{\theta}{2}+c_3\cos\frac{\theta}{2} \end{bmatrix}$$

$$+\frac{1}{2}\begin{bmatrix} \alpha s_xc_1\sin\frac{\theta}{2}+\beta s_yc_2\sin\frac{\theta}{2}+\alpha\beta s_zc_3\sin\frac{\theta}{2}\\ -c_1\cos\frac{\theta}{2}+\beta s_zc_2\sin\frac{\theta}{2}-\beta s_yc_3\sin\frac{\theta}{2}\\ -\alpha s_zc_1\sin\frac{\theta}{2}-c_2\cos\frac{\theta}{2}+\alpha s_xc_3\sin\frac{\theta}{2}\\ s_yc_1\sin\frac{\theta}{2}-s_xc_2\sin\frac{\theta}{2}-c_3\cos\frac{\theta}{2} \end{bmatrix}.$$

if $s^*=(s_x^*,s_y^*,s_z^*)=k\wedge_G s$, we obtain that

$$q^*=\begin{bmatrix} -\frac{d}{2}\sin\frac{\theta}{2}\\ \frac{d}{2}s_x\cos\frac{\theta}{2}+\beta s_x^*\sin\frac{\theta}{2}\\ \frac{d}{2}s_y\cos\frac{\theta}{2}+\alpha s_y^*\sin\frac{\theta}{2}\\ \frac{d}{2}s_z\cos\frac{\theta}{2}+s_z^*\sin\frac{\theta}{2}, \end{bmatrix}.$$

Therefore, it is obtained that

$$\begin{aligned} q_0^*&=-\tfrac{d}{2}\sin\tfrac{\theta}{2}\\ q_1^*&=\tfrac{d}{2}s_x\cos\tfrac{\theta}{2}+\beta s_x^*\sin\tfrac{\theta}{2}\\ q_2^*&=\tfrac{d}{2}s_y\cos\tfrac{\theta}{2}+\alpha s_y^*\sin\tfrac{\theta}{2}\\ q_3^*&=\tfrac{d}{2}s_z\cos\tfrac{\theta}{2}+s_z^*\sin\tfrac{\theta}{2}. \end{aligned}$$

Let the G-axis is represented by G-dual vector $\widehat{s}=s+\varepsilon s^*$, the rotation along $\widehat{s}$ can be defined via the $\widehat{\theta}=\theta+\varepsilon d$;

$$\begin{aligned}\widehat{q}= \;& \cos\tfrac{\theta}{2}+s_x(\sin\tfrac{\theta}{2})i+s_y(\sin\tfrac{\theta}{2})j+s_z(\sin\tfrac{\theta}{2})k \\ &+\varepsilon[-\tfrac{d}{2}\sin\tfrac{\theta}{2}+(\tfrac{d}{2}s_x\cos\tfrac{\theta}{2}+\beta s_x^*\sin\tfrac{\theta}{2})i \\ &+(\tfrac{d}{2}s_y\cos\tfrac{\theta}{2}+\alpha s_y^*\sin\tfrac{\theta}{2})j+(\tfrac{d}{2}s_z\cos\tfrac{\theta}{2}+s_z^*\sin\tfrac{\theta}{2})k].\end{aligned}$$

Thus, we get that

$$\begin{aligned}\widehat{q}= \;& \cos\tfrac{\theta}{2}-\varepsilon\tfrac{d}{2}\sin\tfrac{\theta}{2}+s_x(\sin\tfrac{\theta}{2}+\varepsilon\tfrac{d}{2}\cos\tfrac{\theta}{2})i+\varepsilon\beta s_x^*\sin\tfrac{\theta}{2}i \\ &+s_y(\sin\tfrac{\theta}{2}+\varepsilon\tfrac{d}{2}\cos\tfrac{\theta}{2})j+\varepsilon\alpha s_y^*\sin\tfrac{\theta}{2}j \\ &+s_z(\sin\tfrac{\theta}{2}+\varepsilon\tfrac{d}{2}\cos\tfrac{\theta}{2})k+\varepsilon s_z^*\sin\tfrac{\theta}{2}k.\end{aligned}$$

Later, $\widehat{q}=q+\varepsilon q^*$ can be written as

$$\widehat{q}=\cos\frac{\widehat{\theta}}{2}+\widehat{s}_x(\sin\frac{\widehat{\theta}}{2})i+\widehat{s}_y(\sin\frac{\widehat{\theta}}{2})j+\widehat{s}_z(\sin\frac{\widehat{\theta}}{2})k.$$

which called as dual G-Euler parameters of the spatial displacement in $E^3(\alpha,\beta)$.
The G-dual orthogonal matrix can be represented by means of the dual G-Euler parameters as follows;

$$\widehat{A}=I+(\sin\widehat{\theta})\widehat{S}+(1-\cos\widehat{\theta})\widehat{S}^2,$$

we get a dual version of the equation (2.1).
case 2: Let $\alpha>0$ and $\beta<0$. We can write the equation (4.1) as follows, where s is a timelike vector

$$H^-(q)\cdot_G(dS+K)=\begin{bmatrix}\cosh\frac{\theta}{2} & -\alpha s_x\sinh\frac{\theta}{2} & -\beta s_y\sinh\frac{\theta}{2} & -\alpha\beta s_z\sinh\frac{\theta}{2}\\ s_x\sinh\frac{\theta}{2} & \cosh\frac{\theta}{2} & \beta s_z\sinh\frac{\theta}{2} & -\beta s_y\sinh\frac{\theta}{2}\\ s_y\sinh\frac{\theta}{2} & -\alpha s_z\sinh\frac{\theta}{2} & \cosh\frac{\theta}{2} & \alpha s_x\sinh\frac{\theta}{2}\\ s_z\sinh\frac{\theta}{2} & s_y\sinh\frac{\theta}{2} & -s_x\sinh\frac{\theta}{2} & \cosh\frac{\theta}{2}\end{bmatrix}\cdot_G \begin{bmatrix}0\\ (\alpha d_1s_x+\beta d_2s_y+\alpha\beta d_3s_z)s_x+c_1\\ (\alpha d_1s_x+\beta d_2s_y+\alpha\beta d_3s_z)s_y+c_2\\ (\alpha d_1s_x+\beta d_2s_y+\alpha\beta d_3s_z)s_z+c_3\end{bmatrix}$$

and

$$H^+(q)\cdot_G K=\begin{bmatrix}\cosh\frac{\theta}{2} & -\alpha s_x\sinh\frac{\theta}{2} & -\beta s_y\sinh\frac{\theta}{2} & -\alpha\beta s_z\sinh\frac{\theta}{2}\\ s_x\sinh\frac{\theta}{2} & \cosh\frac{\theta}{2} & -\beta s_z\sinh\frac{\theta}{2} & \beta s_y\sinh\frac{\theta}{2}\\ s_y\sinh\frac{\theta}{2} & \alpha s_z\sinh\frac{\theta}{2} & \cosh\frac{\theta}{2} & -\alpha s_x\sinh\frac{\theta}{2}\\ s_z\sinh\frac{\theta}{2} & -s_y\sinh\frac{\theta}{2} & s_x\sinh\frac{\theta}{2} & \cosh\frac{\theta}{2}\end{bmatrix}\cdot_G\begin{bmatrix}0\\c_1\\c_2\\c_3\end{bmatrix}$$

By letting $d=<d,s>_G=\alpha d_1s_x+\beta d_2s_y+\alpha\beta d_3s_z$, we have

$$q^*=\frac{1}{2}\begin{bmatrix}-\alpha s_x^2d\sinh\frac{\theta}{2}-\alpha s_xc_1\sinh\frac{\theta}{2}-\beta s_y^2d\sinh\frac{\theta}{2}-\beta s_yc_2\sinh\frac{\theta}{2}-\alpha\beta s_z^2d\sinh\frac{\theta}{2}-\alpha\beta s_zc_3\sinh\frac{\theta}{2}\\ s_xd\cosh\frac{\theta}{2}+c_1\cosh\frac{\theta}{2}+\beta s_zc_2\sinh\frac{\theta}{2}-\beta s_yc_3\sinh\frac{\theta}{2}\\ -\alpha s_zc_1\sinh\frac{\theta}{2}+s_yd\cosh\frac{\theta}{2}+c_2\cosh\frac{\theta}{2}+\alpha s_xc_3\sinh\frac{\theta}{2}\\ s_yc_1\sinh\frac{\theta}{2}-s_xc_2\sinh\frac{\theta}{2}+s_zd\cosh\frac{\theta}{2}+c_3\cosh\frac{\theta}{2}\end{bmatrix}$$

$$+\frac{1}{2}\begin{bmatrix}\alpha s_xc_1\sinh\frac{\theta}{2}+\beta s_yc_2\sinh\frac{\theta}{2}+\alpha\beta s_zc_3\sinh\frac{\theta}{2}\\ -c_1\cosh\frac{\theta}{2}+\beta s_zc_2\sinh\frac{\theta}{2}-\beta s_yc_3\sinh\frac{\theta}{2}\\ -\alpha s_zc_1\sinh\frac{\theta}{2}-c_2\cosh\frac{\theta}{2}+\alpha s_xc_3\sinh\frac{\theta}{2}\\ s_yc_1\sinh\frac{\theta}{2}-s_xc_2\sinh\frac{\theta}{2}-c_3\cosh\frac{\theta}{2}\end{bmatrix}.$$

if $s^*=(s_x^*,s_y^*,s_z^*)=k\wedge_G s$, and vector s is timelike vector, then we get that

$$q^*=\begin{bmatrix}\frac{d}{2}\sinh\frac{\theta}{2}\\ \frac{d}{2}s_x\cosh\frac{\theta}{2}+\beta s_x^*\sinh\frac{\theta}{2}\\ \frac{d}{2}s_y\cosh\frac{\theta}{2}+\alpha s_y^*\sinh\frac{\theta}{2}\\ \frac{d}{2}s_z\cosh\frac{\theta}{2}+s_z^*\sinh\frac{\theta}{2},\end{bmatrix}.$$

Thus, we have following equations:

$$\begin{aligned}q_0^*&=\tfrac{d}{2}\sinh\tfrac{\theta}{2}\\ q_1^*&=\tfrac{d}{2}s_x\cosh\tfrac{\theta}{2}+\beta s_x^*\sinh\tfrac{\theta}{2}\\ q_2^*&=\tfrac{d}{2}s_y\cosh\tfrac{\theta}{2}+\alpha s_y^*\sinh\tfrac{\theta}{2}\\ q_3^*&=\tfrac{d}{2}s_z\cosh\tfrac{\theta}{2}+s_z^*\sinh\tfrac{\theta}{2}.\end{aligned}$$

Let the G-axis is represented by G-dual vector $\widehat{s}=s+\varepsilon s^*$, the rotation along $\widehat{s}$ can be defined via the $\widehat{\theta}=\theta+\varepsilon d$;

$$\begin{aligned}\widehat{q}= \;& \cosh\tfrac{\theta}{2}+s_x(\sinh\tfrac{\theta}{2})i+s_y(\sinh\tfrac{\theta}{2})j+s_z(\sinh\tfrac{\theta}{2})k \\ &+\varepsilon[\tfrac{d}{2}\sinh\tfrac{\theta}{2}+(\tfrac{d}{2}s_x\cosh\tfrac{\theta}{2}+\beta s_x^*\sinh\tfrac{\theta}{2})i \\ &+(\tfrac{d}{2}s_y\cosh\tfrac{\theta}{2}+\alpha s_y^*\sinh\tfrac{\theta}{2})j+(\tfrac{d}{2}s_z\cosh\tfrac{\theta}{2}+s_z^*\sinh\tfrac{\theta}{2})k],\end{aligned}$$

and

$$\begin{aligned}\widehat{q}= \quad & \cosh\tfrac{\theta}{2}+\varepsilon\tfrac{d}{2}\sin\tfrac{\theta}{2}+s_x(\sinh\tfrac{\theta}{2}+\varepsilon\tfrac{d}{2}\cosh\tfrac{\theta}{2})i+\varepsilon\beta s_x^*\sinh\tfrac{\theta}{2}i\\ & +s_y(\sinh\tfrac{\theta}{2}+\varepsilon\tfrac{d}{2}\cosh\tfrac{\theta}{2})j+\varepsilon\alpha s_y^*\sinh\tfrac{\theta}{2}j\\ & +s_z(\sinh\tfrac{\theta}{2}+\varepsilon\tfrac{d}{2}\cosh\tfrac{\theta}{2})k+\varepsilon s_z^*\sinh\tfrac{\theta}{2}k.\end{aligned}$$

Therefore, $\widehat{q}=q+\varepsilon q^*$ can be written as

$$\widehat{q}=\cosh\frac{\widehat{\theta}}{2}+\widehat{s_x}(\sinh\frac{\widehat{\theta}}{2})i+\widehat{s_y}(\sinh\frac{\widehat{\theta}}{2})j+\widehat{s_z}(\sinh\frac{\widehat{\theta}}{2})k.$$

which called as dual G-Euler parameters of the spatial displacement in $E^3(\alpha,\beta)$.
The G-dual orthogonal matrix can be represented by means of the dual G-Euler parameters as follows;

$$\widehat{A}=I+(\sinh\widehat{\theta})\widehat{S}+(\cosh\widehat{\theta}-1)\widehat{S}^2$$

we get a dual version of the equation (2.2).
Conclusion: In this article, we obtained the screw axis of displacement and Rodrigues equation for a spatial displacement in generalized space. Finally, we find that, the components of a dual generalized quaternion, and the generalized dual orthogonal matrix from those components.
Consequently, we show that the coordinate transformation, which defines the position of a body relative to the fixed frame, can be expressed by using generalized dual quaternions. This result shows that generalized dual quaternion algebra can be used to determine the representation of the screw motion in generalized space.

References

[1] O.P. Agrawal, Hamilton Operators and Dual-number-quaternions in Spatial Kinematics, Mech. Mach. Theory, 22(1987), 569-575.
[2] B. Akyar, Dual Quaternions in Spatial Kinematics in an Algebraic Sense, Turk J. Math., 32(2008), 373-391.
[3] S.L. Altmann, Rotations, Quaternions, and Double Groups, Oxford University Press, Oxford, 1986.
[4] E. Ata, Y. Yıldırım, Different Polar Representation for Generalized and Generalized Dual Quaternions, Adv. Appl. Clifford Al., (28)(2010), 193-202.
[5] E. Ata, Ü.Z. Savci Spherical Kinematics in 3-Dimensional Generalized Space, International Journal of Geometric Methods in Modern Physics, 18(3)(2021), 2150033.
[6] O. Bottema, B. Roth, Theoretical Kinematics, North-Holland Press, New York 1979.
[7] W.K. Clifford, Preliminary sketch of bi-quaternions, Proc. London Math.Soc., 4(1873), 381-395.
[8] J. Cockle, On Systems of Algebra Involving More than One Imaginary, Philos. Mag. (series 3), 35(1849), 434-435.
[9] K. Erdmann, A. Skowronski, Algebras of generalized quaternion type, Advances in Mathematics, 349(2019), 1036-1116.
[10] J.M. Herve, The mathematical group structure of the set of displacements, Mech. Mach. Theory, 29(1994), 73-81.
[11] M. Hiller, C. Woernle, A Unified Representation of Spatial Displacements, Mech. Mach. Theory, 19(1984), 477-486.
[12] M. Jafari, Y. Yaylı, Generalızed Quaternions and Rotation in 3-Space $E^3_{\alpha\beta}$, TWMS J. Pure Appl. Math., 6(2)(2015), 224-232.
[13] L. Kula, Y. Yaylı, Dual Split Quaternions and Screw Motion in Minkowski $3-$space, Iranian Journal of Science & Technology, Transaction A, 30(2006), 245-258.
[14] A.P. Kotel'nikov, Screw calculus and some of its applications to geometry and mechanies, Annals of The Imperial University of Kazan, 1895.
[15] T.Y. Lam, Introduction to Quadratic Forms Over Fields, American Mathematical Society, USA 2005.
[16] J.M. McCarthy, An Introduction to Theoretical Kinematics, MIT Press, Cambridge, 1990.
[17] M. Ozdemir, A.A. Ergin, Rotations with unit timelike quaternions in Minkowski $3-$space, Journal of Geometry and Physics, 56(2006), 322-336.
[18] S. Ozkaldı, H. Gündoğan, Split Quaternions and Screw Motions in $3-$dimensional Lorentzian Space, Adv. Appl. Clifford Al., 21(2011), 193-202.
[19] H. Pottman, J. Wallner, Computational line geometry, Springer-Verlag Berlin Heidelberg, New York, 2000.
[20] E. Study, Geometrie der Dynamen, Leipzig, Germany, 1903.