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Solution of a Solvable System of Difference Equations

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Abstract — In this study we give solutions for the following difference equation sytem

$$x_{n+1}=\frac{ax_ny_{n-3}}{y_{n-2}-\alpha}+\beta,\ \ y_{n+1}=\frac{bx_{n-3}y_n}{x_{n-2}-\beta}+\alpha,\ \ n\in\mathbb{N}_0$$

where the parameters a, b, α, β and initial values $x_{-i},\ y_{-i},\ i=0,1,2,3$ are non-zero real numbers. We show the asymptotic behavior of the system of equation.

Subject Classification (2020): 39A10.

1. Introduction

Although difference equations are look simple, it is very difficult fully understand the behaviors of their solutions. Continuous events in engineering, finance, physics, biology etc., is modelled by using differential equations. But, discontinuous events can be formed by a difference equations. Also, difference equations can be used to solve differential equations numerically. So, there is a great recent interest in difference equations(see [1-27]). In this study we investigate following system of the equation which is motivated by Haddad at al. [5].

$$x_{n+1}=\frac{ax_ny_{n-3}}{y_{n-2}-\alpha}+\beta,\ \ y_{n+1}=\frac{bx_{n-3}y_n}{x_{n-2}-\beta}+\alpha,\ \ n\in\mathbb{N}_0 \tag{1.1}$$

where the parameters a, b, α, β and initial values $x_{-i},\ y_{-i},\ i=0,1,2,3$ are non-zero real numbers.

Let's give following well known lemma which be used to prove our theorems.

Lemma 1.1. *Let $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ be two sequences of real numbers and consider the linear difference equation $a_n\neq 0$ for $\forall n\in\mathbb{N}_0$*

$$y_{n+1}=a_ny_n+b_n.$$

Then

$$y_n=(\prod_{i=0}^{n-1}a_i)y_0+\sum_{r=0}^{n-1}(\prod_{i=r+1}^{n-1}a_i)b_r.$$

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Moreover, if $(a_n)_{n\in\mathbb{N}_0}$ *and* $(b_n)_{n\in\mathbb{N}_0}$ *are constant (i.e* $a_n = a$ *and* $b_n = b$ *for some real numbers* a *and* b *for all* $n \in \mathbb{N}_0$*), then*

$$y_n = \begin{cases} y_0 + bn, & a = 1, \\ a^n y_0 + (\frac{a^n-1}{a-1})b, & a \neq 1 \end{cases} \quad n \in \mathbb{N}_0.$$

2. Solutions of the system

In this section we give well-defined solutions of the system

Theorem 2.1. *Let* (x_n, y_n) *be a well-defined solution of the system* (1.1)*. Then for* $n \in \mathbb{N}_0$

$$x_{6n+i} = \left(\prod_{t=0}^{n-1} \left(\frac{a}{b}\right)^{6t+i} d\right) x_i + \sum_{r=0}^{n-1} \left(\prod_{t=r+1}^{n-1} \left(\frac{a}{b}\right)^{6t+i} d\right) F_i^r(u, \frac{a}{b})\beta,$$

$$y_{6n+i} = \left(\prod_{t=0}^{n-1} \left(\frac{b}{a}\right)^{6t+i} e\right) y_i + \sum_{r=0}^{n-1} \left(\prod_{t=r+1}^{n-1} \left(\frac{b}{a}\right)^{6t+i} e\right) F_i^r(v, \frac{b}{a})\alpha,$$

where $u_i = \frac{x_{i+1}-\beta}{x_i}$, $v_i = \frac{y_{i+1}-\alpha}{y_i}$, $i \in \{0,1,2,3,4,5\}$, $d = u_5u_4u_3u_2u_1u_0$, $e = v_5v_4v_3v_2v_1v_0$ *and*

$$F_0^n(x,y) = y^{5n}x_5x_4x_3x_2x_1 + y^{4n}x_5x_4x_3x_2 + y^{3n}x_5x_4x_3 + y^{2n}x_5x_4 + y^n x_5 + 1$$

$$F_1^n(x,y) = y^{5n+1}x_5x_4x_3x_2x_0 + y^{4n+1}x_5x_4x_3x_0 + y^{3n+1}x_5x_4x_0 + y^{2n+1}x_5x_0 + y^{n+1}x_0 + 1$$

$$F_2^n(x,y) = y^{5n+2}x_5x_4x_3x_1x_0 + y^{4n+2}x_5x_4x_1x_0 + y^{3n+2}x_5x_1x_0 + y^{2n+2}x_1x_0 + y^{n+1}x_1 + 1$$

$$F_3^n(x,y) = y^{5n+3}x_5x_4x_2x_1x_0 + y^{4n+3}x_5x_2x_1x_0 + y^{3n+3}x_2x_1x_0 + y^{2n+2}x_2x_1 + y^{n+1}x_2 + 1$$

$$F_4^n(x,y) = y^{5n+4}x_5x_3x_2x_1x_0 + y^{4n+4}x_3x_2x_1x_0 + y^{3n+3}x_3x_2x_1 + y^{2n+2}x_3x_2 + y^{n+1}x_3 + 1$$

$$F_5^n(x,y) = y^{5n+5}x_4x_3x_2x_1x_0 + y^{4n+4}x_4x_3x_2x_1 + y^{3n+3}x_4x_3x_2 + y^{2n+2}x_4x_3 + y^{n+1}x_4 + 1.$$

Proof.
Firstly let's write system (1.1) as

$$\frac{x_{n+1}-\beta}{x_n} = \frac{ay_{n-3}}{y_{n-2}-\alpha}, \quad \frac{y_{n+1}-\alpha}{y_n} = \frac{bx_{n-3}}{x_{n-2}-\beta}.$$

Assume

$$u_n = \frac{x_{n+1}-\beta}{x_n}, \; v_n = \frac{y_{n+1}-\alpha}{y_n}, \quad n \in \mathbb{N}_0 \tag{2.1}$$

then

$$u_n = \frac{a}{v_{n-3}}, \; v_n = \frac{b}{u_{n-3}} \Rightarrow u_{n+3} = \frac{a}{v_n}, \; v_{n+3} = \frac{b}{u_n}, \quad n \in \mathbb{N}_0 \tag{2.2}$$

and

$$u_{n+6} = \frac{a}{b}u_n, \; v_{n+6} = \frac{b}{a}v_n, \quad n \in \mathbb{N}_0. \tag{2.3}$$

Hence for $n \in \mathbb{N}_0$

$$u_{6n+i} = \left(\frac{a}{b}\right)^n u_i, \; v_{6n+i} = \left(\frac{b}{a}\right)^n v_i, \; i = 0,1,2,3,4,5. \tag{2.4}$$

Rearranging equation (2.1), we get

$$x_{n+1} = u_n x_n + \beta \tag{2.5}$$

$$y_{n+1} = v_n y_n + \alpha \tag{2.6}$$

Replacing n by $6n+i$ for $i \in \{0,1,2,3,4,5\}$

$$x_{6n+i+1} = u_{6n+i} x_{6n+i} + \beta = \left(\frac{a}{b}\right)^n u_i x_{6n+i} + \beta, \quad n \in \mathbb{N}_0, \ i \in \{0,1,2,3,4,5\},$$

$$y_{6n+i+1} = v_{6n+i} y_{6n+i} + \alpha = \left(\frac{b}{a}\right)^n v_i y_{6n+i} + \alpha, \quad n \in \mathbb{N}_0, \ i \in \{0,1,2,3,4,5\}.$$

$$\begin{aligned} x_{6n+6} = {} & \left(\frac{a}{b}\right)^{6n} u_5 u_4 u_3 u_2 u_1 u_0 x_{6n} + \beta \Bigg(\left(\frac{a}{b}\right)^{5n} u_5 u_4 u_3 u_2 u_1 + \left(\frac{a}{b}\right)^{4n} u_5 u_4 u_3 u_2 \\ & + \left(\frac{a}{b}\right)^{3n} u_5 u_4 u_3 \left(\frac{a}{b}\right)^{2n} u_5 u_4 + \left(\frac{a}{b}\right)^{n} u_5 + 1 \Bigg), \end{aligned} \tag{2.7}$$

$$\begin{aligned} x_{6n+7} = {} & \left(\frac{a}{b}\right)^{6n+1} u_5 u_4 u_3 u_2 u_1 u_0 x_{6n+1} + \beta \Bigg(\left(\frac{a}{b}\right)^{5n+1} u_5 u_4 u_3 u_2 u_0 + \left(\frac{a}{b}\right)^{4n+1} u_5 u_4 u_3 u_0 \\ & + \left(\frac{a}{b}\right)^{3n+1} u_5 u_4 u_0 + \left(\frac{a}{b}\right)^{2n+1} u_5 u_0 + \left(\frac{a}{b}\right)^{n+1} u_0 + 1 \Bigg), \quad n \in \mathbb{N}_0 \end{aligned} \tag{2.8}$$

$$\begin{aligned} x_{6n+8} = {} & \left(\frac{a}{b}\right)^{6n+2} u_5 u_4 u_3 u_2 u_1 u_0 x_{6n+2} + \beta \Bigg(\left(\frac{a}{b}\right)^{5n+2} u_5 u_4 u_3 u_1 u_0 + \left(\frac{a}{b}\right)^{4n+2} u_5 u_4 u_1 u_0 \\ & + \left(\frac{a}{b}\right)^{3n+2} u_5 u_1 u_0 + \left(\frac{a}{b}\right)^{2n+2} u_1 u_0 + \left(\frac{a}{b}\right)^{n+1} u_1 + 1 \Bigg), \quad n \in \mathbb{N}_0 \end{aligned} \tag{2.9}$$

$$\begin{aligned} x_{6n+9} = {} & \left(\frac{a}{b}\right)^{6n+3} u_5 u_4 u_3 u_2 u_1 u_0 x_{6n+3} + \beta \Bigg(\left(\frac{a}{b}\right)^{5n+3} u_5 u_4 u_2 u_1 u_0 + \left(\frac{a}{b}\right)^{4n+3} u_5 u_2 u_1 u_0 \\ & + \left(\frac{a}{b}\right)^{3n+3} u_2 u_1 u_0 + \left(\frac{a}{b}\right)^{2n+2} u_2 u_1 + \left(\frac{a}{b}\right)^{n+1} u_2 + 1 \Bigg), \quad n \in \mathbb{N}_0, \end{aligned} \tag{2.10}$$

$$\begin{aligned} x_{6n+10} = {} & \left(\frac{a}{b}\right)^{6n+4} u_5 u_4 u_3 u_2 u_1 u_0 x_{6n+4} + \beta \Bigg(\left(\frac{a}{b}\right)^{5n+4} u_5 u_3 u_2 u_1 u_0 + \left(\frac{a}{b}\right)^{4n+4} u_3 u_2 u_1 u_0 \\ & + \left(\frac{a}{b}\right)^{3n+3} u_3 u_2 u_1 + \left(\frac{a}{b}\right)^{2n+2} u_3 u_2 + \left(\frac{a}{b}\right)^{n+1} u_3 + 1 \Bigg), \quad n \in \mathbb{N}_0, \end{aligned} \tag{2.11}$$

$$\begin{aligned} x_{6n+11} = {} & \left(\frac{a}{b}\right)^{6n+5} u_5 u_4 u_3 u_2 u_1 u_0 x_{6n+5} + \beta \Bigg(\left(\frac{a}{b}\right)^{5n+5} u_4 u_3 u_2 u_1 u_0 + \left(\frac{a}{b}\right)^{4n+4} u_4 u_3 u_2 u_1 \\ & + \left(\frac{a}{b}\right)^{3n+3} u_4 u_3 u_2 + \left(\frac{a}{b}\right)^{2n+2} u_4 u_3 + \left(\frac{a}{b}\right)^{n+1} u_4 + 1 \Bigg), \quad n \in \mathbb{N}_0, \end{aligned} \tag{2.12}$$

$$y_{6n+6} = \left(\frac{b}{a}\right)^{6n} v_5 v_4 v_3 v_2 v_1 v_0 y_{6n} + \alpha \left(\left(\frac{b}{a}\right)^{5n} v_5 v_4 v_3 v_2 v_1 + \left(\frac{b}{a}\right)^{4n} v_5 v_4 v_3 v_2 + \left(\frac{b}{a}\right)^{3n} v_5 v_4 v_3 + \left(\frac{b}{a}\right)^{2n} v_5 v_4 + \left(\frac{b}{a}\right)^{n} v_5 + 1 \right), \quad n \in \mathbb{N}_0, \tag{2.13}$$

$$y_{6n+7} = \left(\frac{b}{a}\right)^{6n+1} v_5 v_4 v_3 v_2 v_1 v_0 y_{6n+1} + \alpha \left(\left(\frac{b}{a}\right)^{5n+1} v_5 v_4 v_3 v_2 v_0 + \left(\frac{b}{a}\right)^{4n+1} v_5 v_4 v_3 v_0 + \left(\frac{b}{a}\right)^{3n+1} v_5 v_4 v_0 + \left(\frac{b}{a}\right)^{2n+1} v_5 v_0 + \left(\frac{b}{a}\right)^{n+1} v_0 + 1 \right), \quad n \in \mathbb{N}_0, \tag{2.14}$$

$$y_{6n+8} = \left(\frac{b}{a}\right)^{6n+2} v_5 v_4 v_3 v_2 v_1 v_0 y_{6n+2} + \alpha \left(\left(\frac{b}{a}\right)^{5n+2} v_5 v_4 v_3 v_1 v_0 + \left(\frac{b}{a}\right)^{4n+2} v_5 v_4 v_1 v_0 + \left(\frac{b}{a}\right)^{3n+2} v_5 v_1 v_0 + \left(\frac{b}{a}\right)^{2n+2} v_1 v_0 + \left(\frac{b}{a}\right)^{n+1} v_1 + 1 \right), \quad n \in \mathbb{N}_0, \tag{2.15}$$

$$y_{6n+9} = \left(\frac{b}{a}\right)^{6n+3} v_5 v_4 v_3 v_2 v_1 v_0 y_{6n+3} + \alpha \left(\left(\frac{b}{a}\right)^{5n+3} v_5 v_4 v_2 v_1 v_0 + \left(\frac{b}{a}\right)^{4n+3} v_5 v_2 v_1 v_0 + \left(\frac{b}{a}\right)^{3n+3} v_2 v_1 v_0 + \left(\frac{b}{a}\right)^{2n+2} v_2 v_1 + \left(\frac{b}{a}\right)^{n+1} v_2 + 1 \right), \quad n \in \mathbb{N}_0, \tag{2.16}$$

$$y_{6n+10} = \left(\frac{b}{a}\right)^{6n+4} v_5 v_4 v_3 v_2 v_1 v_0 y_{6n+4} + \alpha \left(\left(\frac{b}{a}\right)^{5n+4} v_5 v_3 v_2 v_1 v_0 + \left(\frac{b}{a}\right)^{4n+4} u_3 u_2 u_1 u_0 + \left(\frac{b}{a}\right)^{3n+3} v_3 v_2 v_1 + \left(\frac{b}{a}\right)^{2n+2} v_3 v_2 + \left(\frac{b}{a}\right)^{n+1} v_3 + 1 \right), \quad n \in \mathbb{N}_0, \tag{2.17}$$

$$y_{6n+11} = \left(\frac{b}{a}\right)^{6n+5} v_5 v_4 v_3 v_2 v_1 v_0 y_{6n+5} + \alpha \left(\left(\frac{b}{a}\right)^{5n+5} v_4 v_3 v_2 v_1 v_0 + \left(\frac{b}{a}\right)^{4n+4} u_4 u_3 u_2 u_1 + \left(\frac{b}{a}\right)^{3n+3} v_4 v_3 v_2 + \left(\frac{b}{a}\right)^{2n+2} v_4 v_3 + \left(\frac{b}{a}\right)^{n+1} v_4 + 1 \right), \quad n \in \mathbb{N}_0. \tag{2.18}$$

Let

$$x_{6n} = K_n^0,\ x_{6n+1} = K_n^1,\ x_{6n+2} = K_n^2,\ x_{6n+3} = K_n^3,\ x_{6n+4} = K_n^4,\ x_{6n+5} = K_n^5,$$

$$y_{6n} = L_n^0,\ y_{6n+1} = L_n^1,\ y_{6n+2} = L_n^2,\ y_{6n+3} = L_n^3,\ y_{6n+4} = L_n^4,\ y_{6n+5} = L_n^5,\ n \in \mathbb{N}_0.$$

$$F_0^n(x,y) = y^{5n} x_5 x_4 x_3 x_2 x_1 + y^{4n} x_5 x_4 x_3 x_2 + y^{3n} x_5 x_4 x_3 + y^{2n} x_5 x_4 + y^n x_5 + 1,$$

$$F_1^n(x,y) = y^{5n+1} x_5 x_4 x_3 x_2 x_0 + y^{4n+1} x_5 x_4 x_3 x_0 + y^{3n+1} x_5 x_4 x_0 + y^{2n+1} x_5 x_0 + y^{n+1} x_0 + 1,$$

$$F_2^n(x,y) = y^{5n+2} x_5 x_4 x_3 x_1 x_0 + y^{4n+2} x_5 x_4 x_1 x_0 + y^{3n+2} x_5 x_1 x_0 + y^{2n+2} x_1 x_0 + y^{n+1} x_1 + 1,$$

$$F_3^n(x,y) = y^{5n+3} x_5 x_4 x_2 x_1 x_0 + y^{4n+3} x_5 x_2 x_1 x_0 + y^{3n+3} x_2 x_1 x_0 + y^{2n+2} x_2 x_1 + y^{n+1} x_2 + 1,$$

$$F_4^n(x,y) = y^{5n+4}x_5x_3x_2x_1x_0 + y^{4n+4}x_3x_2x_1x_0 + y^{3n+3}x_3x_2x_1 + y^{2n+2}x_3x_2 + y^{n+1}x_3 + 1,$$

$$F_5^n(x,y) = y^{5n+5}x_4x_3x_2x_1x_0 + y^{4n+4}x_4x_3x_2x_1 + y^{3n+3}x_4x_3x_2 + y^{2n+2}x_4x_3 + y^{n+1}x_4 + 1.$$

Then, from (2.7)-(2.18) we get

$$K_{n+1}^i = \left(\frac{a}{b}\right)^{6n} u_5u_4u_3u_2u_1u_0K_n^i + \beta F_i^n(u,\frac{a}{b}), \quad n \in \mathbb{N}_0,\ i \in \{0,1,2,3,4,5\},$$

$$L_{n+1}^i = \left(\frac{b}{a}\right)^{6n} v_5v_4v_3v_2v_1v_0L_n^i + \alpha F_i^n(v,\frac{b}{a}), \quad n \in \mathbb{N}_0,\ i \in \{0,1,2,3,4,5\}.$$

From Lemma 1.1 we have

$$x_{6n+i} = \left(\prod_{t=0}^{n-1}\left(\frac{a}{b}\right)^{6t+i} u_5u_4u_3u_2u_1u_0\right)x_i + \sum_{r=0}^{n-1}\left(\prod_{t=r+1}^{n-1}\left(\frac{a}{b}\right)^{6t+i} u_5u_4u_3u_2u_1u_0\right)F_i^r(u,\frac{a}{b})\beta,$$

$$y_{6n+i} = \left(\prod_{t=0}^{n-1}\left(\frac{b}{a}\right)^{6t+i} v_5v_4v_3v_2v_1v_0\right)y_i + \sum_{r=0}^{n-1}\left(\prod_{t=r+1}^{n-1}\left(\frac{b}{a}\right)^{6t+i} v_5v_4v_3v_2v_1v_0\right)F_i^r(v,\frac{b}{a})\alpha,$$

this ends the proof.

3. Asymptotic behaviour of (1.1) for $a = b$

We study here asymptotic behavior and periodicity the case when $a = b$ of system (1.1). Let's give the following corollary, which is a direct result of Theorem 2.1.

Corollary 3.1. *Let (x_n, y_n) be a well-defined solution of the system (1.1) with $a = b$. Then, for $n \in \mathbb{N}_0$, $i \in \{0,1,2,3,4,5\}$*

$$x_{6n+i} = \begin{cases} x_i + F_i^r(u,1)\beta n, & d = 1 \\ d^n x_i + \left(\frac{d^n-1}{d-1}\right)F_i^r(u,1)\beta, & otherwise \end{cases} \tag{3.1}$$

$$y_{6n+i} = \begin{cases} y_i + F_i^r(v,1)\beta n, & e = 1 \\ e^n y_i + \left(\frac{e^n-1}{e-1}\right)F_i^r(v,1)\beta, & otherwise \end{cases} \tag{3.2}$$

Now we study the limits of solutions of system (1.1).

Theorem 3.2. *Let (x_n, y_n) be a well-defined solution of the system (1.1) with a=b. Then, the following statements are true.*

a) *Let's assume $d = 1$. When $F_i^r(u,1) \neq 0$ then $|x_{6n+i}| \to \infty$ as $n \to \infty$ for $i \in \{0,1,2,3,4,5\}$. When $F_i^r(u,1) = 0$ then $x_{6n+i} = x_i$ for all $n \in \mathbb{N}_0$ and $i \in \{0,1,2,3,4,5\}$.*

b) *When $(d-1)x_i + F_i^r(u,1)\beta \neq 0$ then*

$$\lim_{n\to\infty}|x_{6n+i}| = \begin{cases} \left|\frac{F_i^r(u,1)\beta}{d-1}\right|, & |d| < 1, \\ \infty, & |d| > 1. \end{cases}$$

When $(d-1)x_i + F_i^r(u,1)\beta = 0$ then $x_{6n+i} = x_i$ for all $n \in \mathbb{N}_0$ and $i \in \{0,1,2,3,4,5\}$.

c) *Let's assume* $e = 1$. *If* $F_i^r(v,1) \neq 0$ *then* $|y_{6n+i}| \to \infty$ *as* $n \to \infty$ *for* $i \in \{0,1,2,3,4,5\}$. *Otherwise, if* $F_i^r(v,1) = 0$ *then* $y_{6n+i} = y_i$ *for* $i \in \{0,1,2,3,4,5\}$.

d) *If* $(e-1)\, y_i + F_i^r(v,1)\alpha \neq 0$ *then*

$$\lim_{n\to\infty} |y_{6n+i}| = \begin{cases} \left|\frac{F_i^r(v,1)\alpha}{d-1}\right|, & |d| < 1, \\ \infty, & |d| > 1. \end{cases}$$

Else $(e-1)\, y_i + F_i^r(v,1)\alpha = 0$ *then* $y_{6n+i} = y_i$ *for all* $n \in \mathbb{N}_0$ *and* $i \in \{0,1,2,3,4,5\}$.

Proof.

We are going to prove a) and b) rest can be done with same manner.

a) Assuming $d = 1$ and $F_i^r(u,1) \neq 0$ we have from (3.1)

$$x_{6n+i} = x_i + F_i^r(u,1)\beta n \neq 0,$$

when $n \to \infty$ in this equation $|x_{6n+i}| \to \infty$. If $F_i^r(u,1) = 0$ then

$$x_{6n+i} = x_i + 0.\beta n = x_i$$

for all $n \in \mathbb{N}_0$ and $i \in \{0,1,2,3,4,5\}$.

b) Suppose that $(d-1)\, x_i + F_i^r(u,1)\beta \neq 0$ for $i \in \{0,1,2,3,4,5\}$. Then, it shows $x_{6n+i} \neq 0$. From (3.1)

$$\begin{aligned} \lim_{n\to\infty} |x_{6n+i}| &= \lim_{n\to\infty} \left| \frac{(d-1)x_i + F_i^r(u,1)\beta}{d-1} d^n + \frac{F_i^r(u,1)\beta}{1-d} \right| \\ &= \left| \frac{(d-1)x_i + F_i^r(u,1)\beta}{d-1} \lim_{n\to\infty} d^n + \lim_{n\to\infty} \frac{F_i^r(u,1)\beta}{1-d} \right| \\ &= \begin{cases} \left| \frac{F_i^r(u,1)\beta}{d-1} \right|, & |d| < 1, \\ \infty, & |d| > 1. \end{cases} \end{aligned}$$

If $(d-1)\, x_i + F_i^r(u,1)\beta = 0$ and $d \neq 1$. Then

$$\begin{aligned} x_{6n+i} &= d^n x_i + \left(\frac{d^n-1}{d-1}\right) F_i^r(u,1)\beta = d^n x_i + \left(\frac{d^n-1}{d-1}\right)(-(d-1)\, x_i) \\ &= d^n x_i - (d^n - 1)\, x_i = x_i, \quad \forall n \in \mathbb{N}_0,\ i \in \{0,1,2,3,4,5\}. \end{aligned}$$

this ends the proof.

Corollary 3.3. *Let* (x_n, y_n) *be a well-defined solution of the system* (1.1) *with a=b. Then, the following statements are true.*

a) If $d = -1$ then for all $n \in \mathbb{N}_0$ and $i \in \{0,1,2,3,4,5\}$,

$$\begin{cases} x_{12n+i} = x_i, \\ x_{12n+6+i} = -x_i + F_i^r(u,1)\beta. \end{cases}$$

b) If $e=-1$ then for all $n\in\mathbb{N}_0$ and $i\in\{0,1,2,3,4,5\}$,

$$\begin{cases} y_{12n+i}=y_i, \\ y_{12n+6+i}=-y_i+F_i^r(v,1)\alpha. \end{cases}$$

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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