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A New Paradigm on the Qualitative Behavior of Chua's Circuit

Chua Devresinin Kalitatif Davranışı Üzerine Yeni Bir Paradigma

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Abstract

Today, the qualitative behavior of dynamical systems is a very important subject of control theory. Based on this, we consider the stability and instability properties of the equilibrium points of Chua's circuit under suitable conditions by the Lyapunov direct method. This method gives us qualitative information directly without solving the given systems. From this circuit, we construct suitable energy or candidate Lyapunov function and then apply the method as a tool to investigate the global asymptotic stability and instability of the system. We also determine under which conditions the system behaves as a chaotic system or a memristor. In this study, we realized that an unforced dissipative dynamical system with bounded initial states has zero solution or motion at infinity. Some simulation results and examples are given to verify the obtained theoretical predictions.

Keywords: Chua's circuit, Lyapunov stability, Memristor, Nonlinear RLC circuit

Öz

Günümüzde dinamik sistemlerin niteliksel davranışı, kontrol teorisinin çok önemli bir konusudur. Buna dayanarak, Lyapunov direkt yöntemi ile Chua devresinin denge noktalarının uygun koşullar altında kararlılık ve kararsızlık özelliklerini ele alıyoruz. Bu yöntem, verilen sistemleri çözmeden bize doğrudan niteliksel bilgi verir. Bu devreden, uygun enerji veya aday Lyapunov fonksiyonunu oluşturuyoruz ve daha sonra yöntemi, sistemin global asimptotik kararlılığını ve kararsızlığını araştırmak için bir araç olarak uyguluyoruz. Ayrıca sistemin hangi koşullar altında kaotik bir sistem veya bir memristör gibi davrandığını da belirleriz. Bu çalışmada, sınırlı başlangıç durumları olan bir zorlamasız tüketen dinamik sistemin sonsuzda sıfır çözüme veya harekete sahip olduğunu fark ettik. Elde edilen teorik tahminleri doğrulamak için bazı simülasyon sonuçları ve örnekler verilmiştir.

Anahtar Kelimeler: Chua devresi, Lyapunov kararlılığı, Memristör, Doğrusal olmayan RLC devresi

1. Introduction

In this study the handled Chua-like circuit is a third-order nonlinear autonomous circuit with at least one nonlinear element (*NR*), one linear resistor and three linear energy storage elements (Adamatzky et al. 2013). In 1983, Chua realized that chaos could be produced in such third section- order circuits when shows symmetric piecewise–linear characteristics $(V_{NR} - I_{NR})$, but this is not necessary (Adamatzky et al. 2013), it can also be realized with a cubic characteristic (Chua 1992, Zhong 1994). Since Chua's circuit has become a reference circuit for studying chaos. Chaotic systems can be found in physics (the Lorentz actuator), in finance, in biology, or electrical circuits (Chua's electrical circuits) which have found interest in the literature (Sene 2021, Srisuchinwong et al. 2007). Here, we will study three different aspects: stability, instability and chaos properties of the circuit. We give the conditions (for *NR*) under which the present model exhibit stability, instability and chaos. We support our theatrical results with graphical representations in 3D. We turn out our discussion on the stability and instability of the defined circuit with the more general nonlinear characteristic of *NR*. We also discuss the chaotic behaviors of the circuit for symmetric piecewise–linear characteristics of $V_{NR} - I_{NR}$. Hence, understanding nonlinear phenomena will become more and more important as time goes (Johnsen 2012), since nature itself is nonlinear. Therefore, the qualitative behavior of

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many nonlinear systems (of any order) such as biological, electrical, mechanical and neural systems are analyzed today.

The term *stability* is a device of some sort that operates under certain general conditions (La Salle and Lefschetz 1961). Stability is the heart of control theory (Gil 2005). Therefore, the Lyapunov direct method is a very efficient tool to determine the qualitative behaviors of dynamical systems (Sugie and Amano 2004, Zhang and Yu 2013). The Lyapunov characterizations of the fractional differential equations are detailed in (Sene 2020), and fractional input stability is addressed in a recent paper (Sene 2019). Unlike these studies, we construct the Lyapunov function from the storage elements of the circuit that fortified the application of the method. For stability analysis of differential equations, the proposed approach improves some relevant studies such as (Sugie and Amano 2004) and (Tunc and Tunc, 2007). Because the derivative of the Lyapunov functions in these studies may be in the form of (2) at a blow, but they are not (for explanation see section 4). So, the proposed approach can apply to a fourth-order elliptic filter (Seidi et al. 2007).

In this connection, we shall cite some excellent studies in the relevant literature discussing qualitative behaviors of certain circuit systems (Chua et al. 1986, Kennedy 1994, Tchitnga et al. 2012, Kocamaz and Uyaroğlu 2014). Here, we applied the Lyapunov method to certain third-order nonlinear *RLC* circuit systems. Passive systems are stable, in this respect a novel passivity property of nonlinear *RLC* circuit investigated in (Jeltsema et al. 2003).

The rest of this paper is organized as follows. Section 2 presents some definitions and auxiliary results. Section 3 deals with three main results and simulations. Section 4 deals with the discussion. Section 5 closes the paper with a short conclusion.

Let's give the following fundamentals relevant to the subject before the main results.

2. Preliminaries

A dynamical system is investigated as a theoretic mathematical model that maps inputs (excitations, causes) into outputs (responses, effects) by a set of intermediate variables (state variables). In this investigation, we consider the nonlinear dynamical system of the form

$$\dot{x}(t) = f(x(t)), x(0) = x_0, \forall t \geq 0, \quad (1)$$

where $t \in \Re^+ (\Re^+ = [0,\infty))$ denotes time, $x \in \Re^n$ denotes the state of the system, $f\colon \Re^n \to \Re^n$ satisfies global Lipschitz condition. The state vector $x(t) \in D$, in which $D \subseteqq \Re^n$ is a domain that contains the origin $x = 0$. A constant vector $x_e \in D$ is said to be an equilibrium state of the system (1) if the condition $f(x_e) = 0$ is satisfied, where 0 is the null vector. The velocity vector $\dot{x}(t)$ vanishes at the equilibrium state x_e, and therefore the constant function $x(t) = x_e$ is a solution of (1). We assume that (1) is well posed, that is, there exists a unique solution $x\colon [0,\infty) \to \Re^n$ for every initial data $x(0) = x_0 \in \Re^n$, and x depends continuously on x_0 according to the normed topology on $\Re^n$. Let $f(0) = 0,\ f(x) \neq 0$ for $x \neq 0$. Let the measured output of (1) is $x(t)$. The energy or the Lyapunov function $E(t) = E(x) \in C^1(\Re^n, \Re_+)$ of (1) is positive definite function (pdf) and continuously differentiable along the motions of (1) such as

$$\frac{d}{dt}E(x(t)) = \dot{E}(t).$$

Now, we can define some properties of the energy (Lyapunov) functions. The following two definitions are from (Zhong et al. 2009).

Definition 1 A function $\alpha(\Re_+, \Re_+)$ is of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$ A class $\mathcal{K}_\infty$ function $\alpha(r)$ is a subset of class $\mathcal{K}$ if $\alpha(r) \to \infty$ as $r \to \infty$.

We extend Proposition 2.1 (Zhong et al. 2009) as the following definition:

Definition 2 A function $E(x) \in C^1(\Re^n, \Re_+)$ is said to be positive definite, decrescent and radially unbounded function if there exist functions α and β of class $\mathcal{K}$, and some pdf γ such that

(i) $\alpha(\|x\|) \leq \beta(\|x\|), \forall t \geq 0, \forall x \in \Re^n,$

(ii) $E(0) = 0,$

(iii) $\dot{E}(x) \leq -\gamma(\|x\|) \leq 0.$

For globally asymptotically stability of our discussion, we may give the following definition (Haykin 2009).

Definition 3 The equilibrium state x_e is said to be globally asymptotically stable if it is stable and all trajectories of the system (1) converge to x_e as time *t* approaches infinity.

Corollary 1 For any unforced dissipative dynamical system, the time derivative of the storage (Lyapunov) function $E(t)$ along the system trajectories is

$$\dot{E}(t) = -\sum_{i=1}^{n} R_i I_i^2 = -\sum_{i=1}^{n} \frac{1}{R_i} v_i^2 \leq 0 \quad (2)$$

where $R_i \geq 0$ is the resistance, I_i and v_i are the current and voltage of the *ith* component of the system. For a conservative or Hamiltonian system $(R_i = 0, \dot{E}(t) = 0)$. The physical significance of (2) may be found in a recent study (Ates 2021).

We are now in a position to state and proof our main results.

3. Main Results

Now with the above projection, we can easily discuss the qualitative behaviors of the following Chua's circuit (Adamatzky et al. 2013).

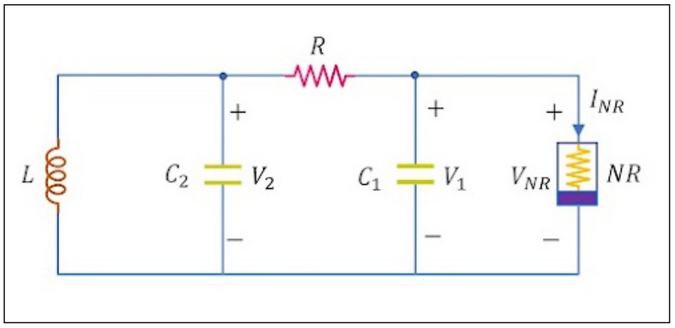

Figure 1: Chua's circuit with nonlinear resistor $NR, I_{NR} = p(x)$.

Here, I_{NR} for the stability and instabilitty analysis of the above circuit may be in the form $p(x) = \frac{1}{k}x^{2n+1} \pm x$, where $n \in \mathbb{N}, \mathbb{N} = \{1,2,3,\ldots\}$ and $k > 0$ is a constant. The circuit shown in Figure 1 generates the following system (with $x = v_{c1}, y = v_{c2}, z = i_L$)

$$\dot{x} = \frac{1}{C_1}\left[\frac{y-x}{R} - p(x)\right],$$
$$\dot{y} = \frac{1}{C_2}\left[\frac{x-y}{R} - z\right], \quad (3)$$
$$\dot{z} = \frac{1}{L}y.$$

Theorem 1 The invariant equilibrium point (0,0,0) of (3) is globally asymptotically stable or this point makes the system lossless if $p(x)$ satisfies the followings:

(i) $p(0) = 0$,

(ii) $\dot{p}(x) \geq 0$,

(iii) $xp(x) \geq 0$,

(iv) $p(x) \to \pm\infty$ **as** $x \to \pm\infty$.

Proof The storage energy function $E_1(t) = E_1(x,y,z)$ of the circuit in Figure 1 from power-energy relationship of the circuit theory is

$$E_1(t) = \frac{1}{2}C_1x^2 + \frac{1}{2}C_2y^2 + \frac{1}{2}Lz^2.$$

The energy function $(E_1: \Re^3 \to \Re^+)$ satisfies

(i) $E_1(0,0,0) = 0$,

(ii) $E_1(x,y,z) > 0, \forall x,y,z \in \Re - \{0,0,0\}$.

E_1 is confirmed by the Definitions 1 and 2. Thus, E_1 is a pdf. Then, we write

$$E_1(t) \geq \frac{1}{2}C_1x^2.$$

The time rate of change of $E_1(t)$ along trajectory (3) is given by

$$\dot{E}_1(t) = -R\left(\frac{x-y}{R}\right)^2 - \left(\frac{x}{p(x)}\right)p^2(x).$$

$\dot{E}_1(t)$ is in the form of (2). Thus, $\dot{E}_1(t) \leq 0$ on $\Re^3$, $E_1(\infty) = 0$ and $E_1(x) \to \infty$ as $x^2 + y^2 + z^2 \to \infty$. Hence, (3) have bounded motions. The set S where $\dot{E}_1 = 0$ is $\{0,0,0\}$. This implies that the origin is the only invariant subset of S. Then, the zero motion $\{0,0,0\}$ or the equilibrium motion of (3) is globally asymptotically stable. Hence, E_1 with the associated system satisfy the properrties of Definitions 1, 2 and 3. Consequently, one may conclude that (3) is lossless at infinity, that is, $x(\infty) = 0$. For convince, see Figures 2a and 2b with $C_1 = C_2 = 1F, L = 0.1H, R = 1$, and $p(x) = \frac{1}{30}x^3$.

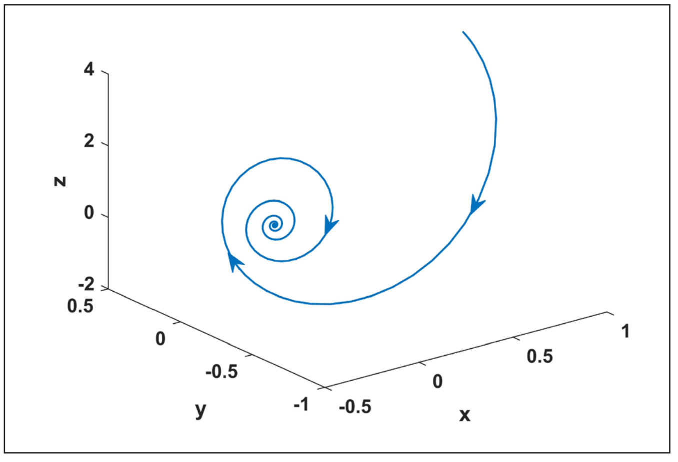

Figure 2A: Stable phase portrait of the system (3).

In addition, recall that $(0,0,0)$ is the only invariant equilibrium state of (3). For $p(x) = x^3 + x$, we have the following Jacobian matrix for (3) about equilibrium state (with $R = C_1 = C_2 = L = 1$)

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

which has the following eigenvalues:

-2.5214 + 0.0000i, -0.2393 + 0.8579i, -0.0.8579i.

The negative real part of the eigenvalues implies that system (3) is stable. Both the results obtained by the Lyapunov method and the Jacobian matrix method are compatible with Definition 3.

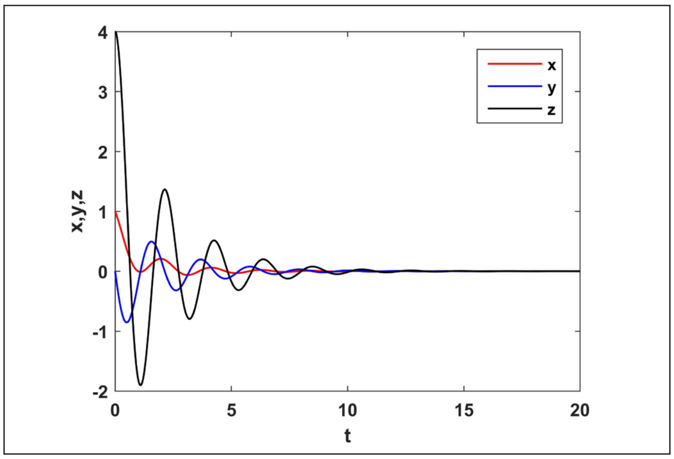

Figure 2b. Stable time series solutions of system (3).

Theorem 2 The system (3) is unstable if the (ii) condition of Theorem 1 is replaced by

$\dot{p}(0) < 0$.

Proof In this case, let $p(x) = \frac{1}{k}x^{2n+1} - x$. Now, the condition (iii) of Theorem 1 is not satisfied in the neighborhood of the origin, that is, $xp(x) < 0$. In this case, $\dot{E}_1(t) > 0$ implies that system (3) is unstable. The phase portrait in the neighborhood of the origin (0,0,0) blow-up which is a fundamental difficulty in the study of nonlinear systems and the time series solutions of (3) do not converge to the equilibrium point for large time *t*. This explanation matches with Figures 3a and 3b. Here, all the parameters of (3) are the same as defined in the above simulations, but only $p(x)$ is different. The explanation for the new $p(x)$ will be given in the following example.

Moreover, in this case, the Jacobian matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

verifies the instability of the system (3).

Example 1 For $n = 1, p(x) = \frac{1}{30}x^3 - x$ *third-order*, $xp(x) < 0$ (in the neighborhood of the equilibrium) which contradicts with (iii) of Theorem 1.

Theorem 3 The system (3) is chaotic if the nonlinear resistor *NR* in Chua's circuit represents symmetric piecewise-linear $(V_{NR} - I_{NR})$ characteristics with negative slopes.

Proof Now, *NR* can represent the working process of a memristor or Chua's diode such that

$$p(x) = \begin{cases} bx + (b-a)x_0, \textit{if } x < -x_0 \\ ax, \textit{if } -x_0 \le x \le x_0 \\ bx + (a-b)x_0, \textit{if } x < -x_0 \end{cases} \tag{4}$$

where $x_0 > 0, a < 0$, and $b < 0$ are constants. In this case, the system (3) can be represented by (4). With the following graphic (Figure 4) which has three equilibrium points: one at the origin where *NR* has a locally negative slope or conductance *a*, and the two others occur at $\pm x_0$, where *NR* has a locally negative conductance *b*. The new one may represent an affine system, that is, $p(0) \neq 0$. The Lyapunov function at the equilibrium points:

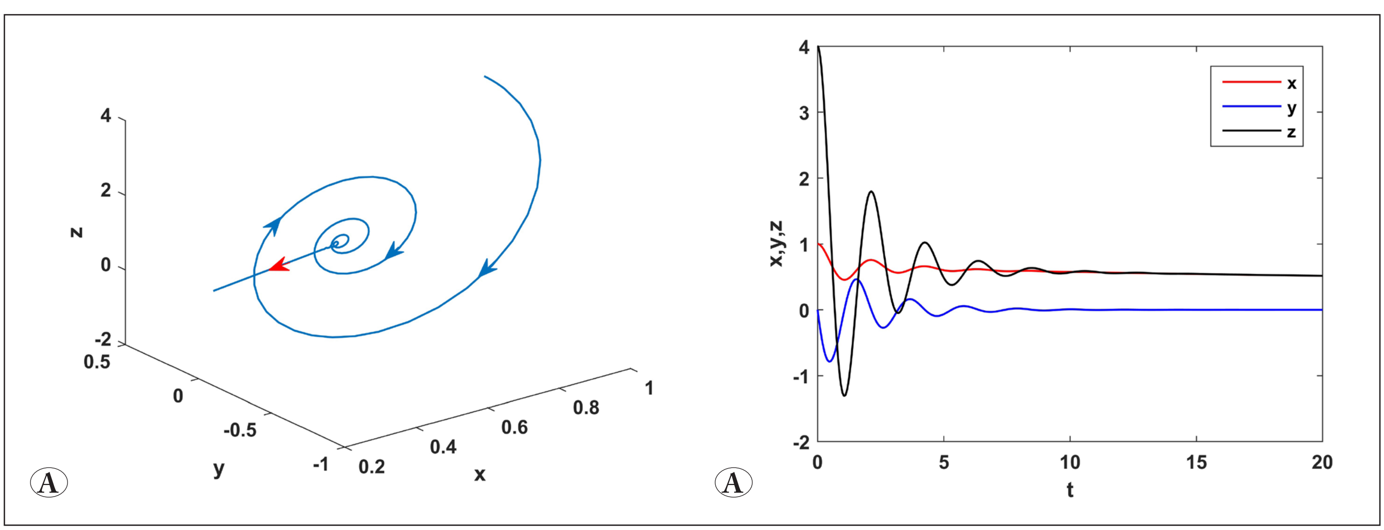

Figure 3: A) Unstable phase portrait of the system (3). **B)** Unstable time series solutions of system (3).

(i) $E_1(0,0,0) = 0$,

(ii) $E_1(\pm x_0, \pm y_0, \pm z_0) \neq 0$.

The second condition contradicts with the minimum energy configuration at the equilibrium point. The chaotic behavior of the system (3) with (4)

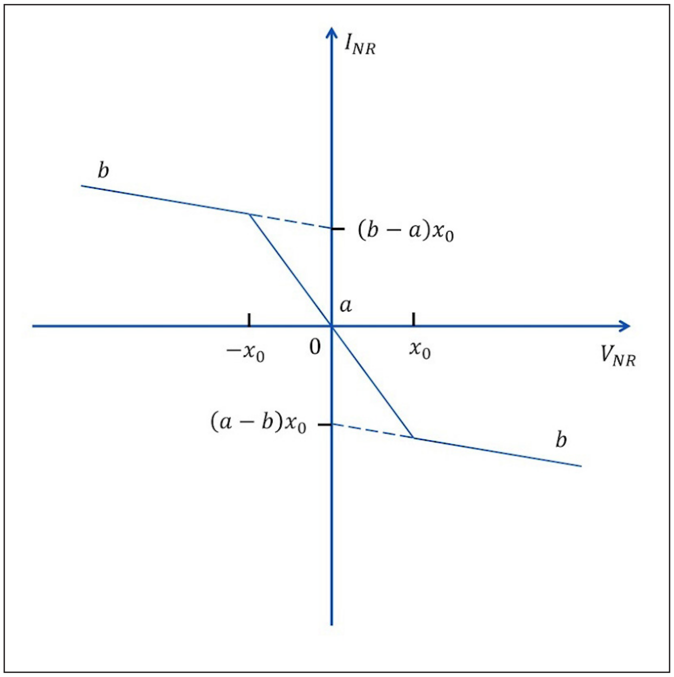

Figure 4: The driving point characteristic of the nonlinear resistor *NR* in Chua's circuit has breakpoints at $\pm x_0$. The driving point characteristic of the nonlinear resistor *NR* in Chua's circuit has breakpoints at $\pm x_0$ and slops *a* and *b*.

The chaotic behavior of the system (3) with (4) is depicted in Figure 5.

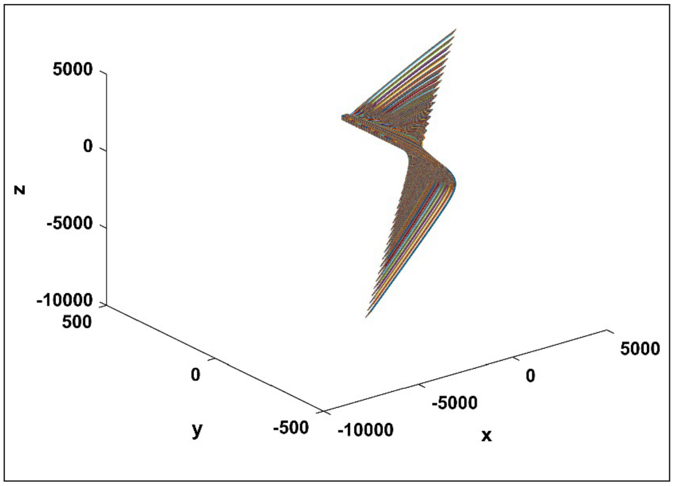

Figure 5. Phase portrait of the system (3) with (4).

4. Discussion

The proposed approach can also be used in the stability analysis of first, second and higher-order (more than 3rd order) differential systems, especially *RLC* circuits and elliptic filters (Saeidi 2007). In addition, the nonlinear element (*NR*) of Chua's circuit is treated as a cubic nonlinear element form $p(x) = ax + bx^3$ where $a, b > 0$ are constants in the relevant literature (Zhong 1994). But, in this study, we relaxed this condition to the most general characteristic of the element as defined above system (3). Moreover, according to the proposed approach the directional derivative of the Lyapunov function of an unforced system is equal to the negative value of the dissipated power. This approach improves some existing conclusions. For example, according to the proposed approach, the derivative of Lyapunov function for system (E^*) in (Sugie and Amano 2004) will be $\dot{V} = -y^2$ that is in the form of (2). But in (Sugie and Amano 2004) it is a long equation and it is not in the form of (2). This is just one example.

5. Conclusion

In this paper, the global asymptotic stability, instability and memristive performance of nonlinear third-order systems have been investigated. The paper presents a new idea to determine the construction and derivative of the Lyapunov function from the perspective of physical meaning. We precisely construct the suitable (real) Lyapunov function by using the power-energy relationship of basic circuit theory. Therefore, the stability and instability of the systems framed by *LRC* circuits. So, the obtained results can be unified with (2) and different from the existing literature. Symmetric piecewise-linear $(V_{NR} - I_{NR})$ characteristics with negative slopes of *NR* determine the memristive performance of Chua's circuit. The numerical examples have been presented to prove the applicability of the proposed theoretical results. The methodology handled here may be applicable for higher-order differential systems and impressive for future stability analysis.

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