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A Short Proof of the Size of Edge-Extremal Chordal Graphs

Mordechai Shalom

Department of Computer Engineering. Işık University, Istanbul, Turkey

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Abstract

Blair et. al. [3] have recently determined the maximum number of edges of a chordal graph with a maximum degree less than d and the matching number at most ν by exhibiting a family of chordal graphs achieving this bound. We provide simple proof of their result.

1. Introduction

Consider a graph $G=(V,E)$ with maximum degree $\Delta(G)<d$ and matching number ν. Vizing's theorem states that there exists a coloring of E using at most $\Delta(G)+1\leq d$ colors. Each color class contains at most ν edges, since it constitutes a matching. Therefore, G has at most $d\cdot\nu$ edges, i.e., bounding both the matching number and the maximum degree of a graph bounds the number of its edges. We want to note that none of the parameters d and ν alone is sufficient to bound the number of edges of G, as the following examples show. The graph mK_2 that is a matching with m vertices has maximum degree 1 and an unbounded number of edges. On the other hand, the graph $K_{1,m}$ which is a star with m leaves has matching number 1 and an unbounded number of edges.
This observation gives rise to the following two questions

- What is the maximum number $m(d,\nu)$ of edges of a graph with matching number at most ν and maximum degree less than d?
- What is the set $\mathscr{M}(d,\nu)$ graphs with maximum degree less than d and matching number at most ν that contain (exactly) $m(d,\nu)$ edges ?

The first question is resolved in the work [1] and the second is resolved later in the work [2] that provided a constructive proof.
The same questions can be posed by confining ourselves to any graph class $\mathscr{C}$, therefore defining:

- $m_{\mathscr{C}}(d,\nu)$ as the maximum number of edges of a graph $G\in\mathscr{C}$ with maximum degree $\Delta(G)<d$ and matching number at most ν, and
- $\mathscr{M}_{\mathscr{C}}(d,\nu)$ the set of graphs $G\in\mathscr{C}$ with maximum degree $\Delta(G)<d$, matching number at most ν having $m_{\mathscr{C}}(d,\nu)$ edges.

A graph $G\in\mathscr{M}(d,\nu)$ (resp. $G\in\mathscr{M}_{\mathscr{C}}(d,\nu)$) is said to be *edge-extremal* (resp. *edge-extremal-*$\mathscr{C}$).
The authors of [3] consider the class of chordal graphs, and determine the number $m_{\text{CHORDAL}}(d,\nu)$ by exhibiting a set of edge-extremal-chordal graphs. In this work we provide a short proof of their following result.

Theorem 3.3. *[3] There exists an edge-extremal graph in $\mathscr{M}_{\text{CHORDAL}}(d,\nu)$ that is a disjoint union of cliques and stars.*

The result is obtained by showing that all the minimal elements of a carefully chosen preorder on the set of minimal representations of the graphs in $\mathscr{M}_{\text{CHORDAL}}(d,\nu)$ have this property. Namely, they are disjoint unions of cliques and stars.

2. Preliminaries

A vertex v of a graph G is *simplicial* if its neighbourhood is a clique and *universal* if its closed neighbourhood is the entire graph. A *star* is a tree with at most one non-leaf vertex. A d-star is a star with maximum degree d. Any total order on a set A defines a corresponding *lexicographic* order on the set A^* of all sequences over the elements of A. In a way similar to a dictionary, the order between two distinct elements a,b of A^* in the lexicographic order is determined by the order of the entries $a_i,b_i\in A$ where i is the lowest index such that $a_i\neq b_i$.

Email address and ORCID number: **cmshalom@gmail.com, 0000-0002-2688-5703 (M. Shalom)**

Observation 2.1. *A simplicial vertex of a graph G is of maximum degree if and only if G is a complete graph.*

A graph G is *factor-critical* if every subgraph obtained by the removal of a single vertex from G admits a perfect matching. It is easy to see that a factor-critical graph is odd and connected.

Definition 2.2. *A graph class $\mathscr{C}$ is* special hereditary *if*

- *$\mathscr{C}$ is closed under the vertex deletion and disjoint union operations, and*
- *$\mathscr{C}$ contains all stars and cliques.*

We will use the following theorem proven in [2].

Theorem 2.3. *[2] Let $\mathscr{C}$ be a special hereditary graph class. Let $G \in \mathscr{C}$ be an edge-extremal graph having the maximum possible number of connected components that are stars . Then every other connected component of G is factor-critical.*

Chordal graphs and subtree representations: A *hole* of a graph is an induced cycle of at least four vertices. A graph is *chordal* if it does not contain a hole.
Consider a forest T and a set $\mathscr{T} = \{T_1, \ldots, T_n\}$ of n subtrees of T. Without loss of generality we assume that every edge of T is used by at least one tree in $\mathscr{T}$. In other words, T is the union of the trees in $\mathscr{T}$. We denote by $G(\mathscr{T})$ *the intersection graph* of these subtrees, i.e., the graph with vertex set $[n] = \{1, 2, \ldots n\}$ such that two vertices $i, j \in [n]$ of G are adjacent if and only if T_i and T_j intersect (in at at least one vertex of T). Given a graph G, a set $\mathscr{T}$ of subtrees such that $G(\mathscr{T}) = G$ is termed a *subtree intersection representation* of G. In the rest of this work we refer to the vertices of T as *nodes* to distinguish them from the vertices of G. It is well known that a graph is chordal if and only if it has a subtree intersection representation [4]. Note that the set of trees of the forest T is in one-to-one correspondence with the connected components of $G(\mathscr{T})$.
Minimal representations and maximal cliques: For a node v of T, let $\mathscr{T}_v \subseteq \mathscr{T}$ be the set of subtrees in $\mathscr{T}$ that contain the node v, and let K_v be the set of vertices of G that correspond to the subtrees $\mathscr{T}_v$. It follows from the definitions that K_v is a clique. Moreover, it is known that a chordal graph G has a subtree representation $\mathscr{T}$ in which the nodes of T are in one-to-one correspondence with the maximal cliques of G. Such a representation is termed *minimal* (see also [5]) and the forest T is termed *a clique forest* of G. By definition, $K_u \setminus K_v \neq \emptyset$ and $K_v \setminus K_u \neq \emptyset$ for any two maximal cliques K_u and K_v of a graph G. In particular, this holds whenever G is chordal and uv is an edge of a clique forest T of G.
Let uv be an edge of T where u is a leaf. From the above definitions and facts, it follows that every vertex in $K_u \setminus K_v \neq \emptyset$ is simplicial. We term such a vertex as *leaf-simplicial* vertex of $\mathscr{T}$.

3. The Short Proof

We start with definitions that are needed for our proof.
Given a minimal representation $\mathscr{T}$ of a chordal graph G with T being the union of the subtrees in $\mathscr{T}$ we denote:

- by $d2(\mathscr{T})$ the number of degree-two nodes of T,
- by $L(\mathscr{T})$ the set of leaves of T,
- by $\ell(\mathscr{T}) \stackrel{def}{=} |L(\mathscr{T})|$ the number of leaves of T,
- by $k(\mathscr{T}) \stackrel{def}{=} \max_{u \in L(\mathscr{T})} |K_u|$, the maximum size of a clique of G that corresponds to a leaf of T, and
- by $s(\mathscr{T})$ the number of leaf-simplicial vertices of $\mathscr{T}$.

We associate with every minimal representation $\mathscr{T}$ a quadruple $Q(\mathscr{T}) \stackrel{def}{=} (\ell(\mathscr{T}), -k(\mathscr{T}), -d2(\mathscr{T}), s(\mathscr{T}))$. Denote by $\prec_{LEX}$ the lexicographic order on $\mathbb{Z}^4$ and by $\preceq_{LEX}$ its reflexive closure. We write $\mathscr{T} \prec_{LEX} \mathscr{T}'$ (resp. $\mathscr{T} \preceq_{LEX} \mathscr{T}'$) as a shorthand for $Q(\mathscr{T}) \preceq_{LEX} Q(\mathscr{T}')$ (resp. $Q(\mathscr{T}) \preceq_{LEX} Q(\mathscr{T}')$).

Lemma 3.1. *Let d, ν be two integers. If all the graphs in $\mathscr{M}_{\text{CHORDAL}}(d, \nu)$ are factor-critical then $K_{2\nu+1} \in \mathscr{M}_{\text{CHORDAL}}(d, \nu)$.*

Proof. Among all minimal representations of graphs in $\mathscr{M}_{\text{CHORDAL}}(d, \nu)$ let $\mathscr{T}$ be one such that $Q(\mathscr{T})$ is minimum in $\preceq_{LEX}$. Let $G = G(\mathscr{T})$ and let T be the union of the subtrees in $\mathscr{T}$. By the assumption of the lemma G is factor-critical, thus contains $n = 2\nu + 1$ vertices.
If T consists of one node then G has one maximal clique, i.e., G is a clique and the proof is completed. If T has exactly two nodes, then they are necessarily adjacent, i.e., G consists of two maximal cliques with at least one common vertex. Then this vertex is universal and has degree at most $d-1$. Therefore, $n-1 < d$, i.e., $n \leq d$. Then, the clique K_n on n vertices is a chordal graph with matching number ν, maximum degree less than d and more edges than G contradicting the assumption that $G \in \mathscr{M}_{\text{CHORDAL}}(d, \nu)$. In the rest of the proof we assume that T has at least three nodes.
We will now present two successive transformations on $\mathscr{T}$ by which we obtain two minimal representations $\mathscr{T}'$ and $\mathscr{T}''$ such that

$$\mathscr{T}'' \preceq_{LEX} \mathscr{T}' \prec_{LEX} \mathscr{T}. \tag{3.1}$$

Denote $G' = G(\mathscr{T}')$, $G'' = G(\mathscr{T}'')$. The transformations will preserve the number of subtrees, thus the number of vertices of the graphs. Therefore, the graphs G' and G'' will be chordal graphs on $n = 2\nu + 1$ vertices. As such, their matching numbers are at most ν.
The transformations ensure that G' is obtained by adding one edge ij to G where j is a simplicial vertex of G, and G'' is obtained from G' by removing one edge ij'. The only vertex whose degree increases after these transformations is j. Since j is simplicial in G it does not have maximum degree. Therefore, $\Delta(G'') \leq \Delta(G) < d$. Clearly, G and G' have the same number of edges. Then $G'' \in \mathscr{M}_{\text{CHORDAL}}(d, \nu)$. Since $\mathscr{T}'' \prec_{LEX} \mathscr{T}$, this is a contradiction to the way $\mathscr{T}$ is chosen.
We now describe the first transformation: Let $u \in L(\mathscr{T})$ be a leaf of T such that $|K_u| = k(\mathscr{T})$ and let v be the unique neighbour of u in T. Let also $\bar{T} = T \setminus \{u, v\}$ be the forest obtained by removing the nodes u and v from T. If K_v contains a simplicial vertex i then it is not of maximum degree. Then adding the edge ij to G will not violate the degree restriction, contradicting the fact that $G \in \mathscr{M}_{\text{CHORDAL}}(d, \nu)$.

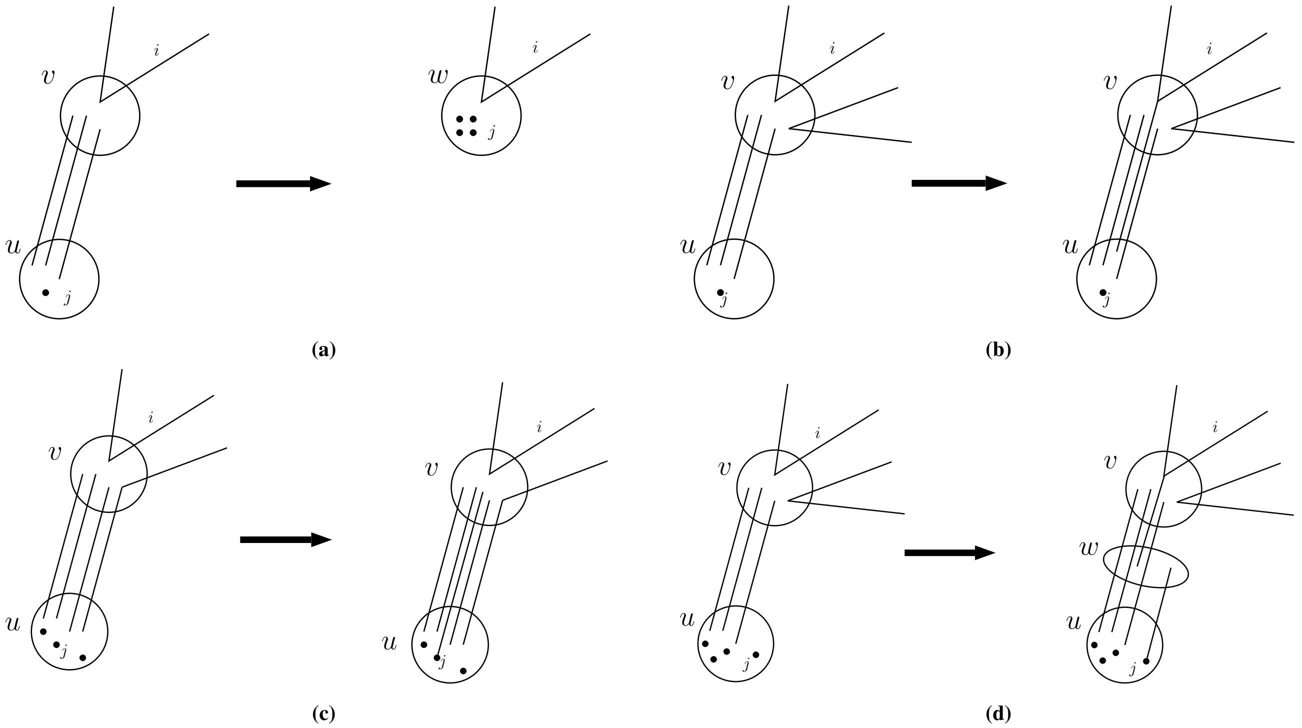


Figure 3.1: The first transformation

Therefore, K_v does not contain simplicial vertices. Consider a vertex $i \in K_v \setminus K_u$. Since i is not simplicial, it has at least one neighbour in $G \setminus K_u \setminus K_v$ In other words, the subtree $T_i \in \mathscr{T}$ that corresponds to vertex i has a non-empty intersection with the forest $\bar{T}$.
We consider four disjoint and complementing cases. Consult Figure 3.1 for illustrations.

(a) $K_u \setminus K_v = \{j\}$ and $K_v \setminus K_u = \{i\}$: In this case we contract the edge uv to obtain a node w and set $K_w = K_u \cup K_v = K_u \cup \{j\}$. If w is not a leaf then $\ell(\mathscr{T})$ decreases. Otherwise, w is a leaf and $|K_w| = |K_v| + 1$, i.e., $\ell(\mathscr{T})$ remains intact and $k(\mathscr{T})$ increases.
(b) $K_u \setminus K_v = \{j\}$ and $K_v \setminus K_u \supsetneq \{i\}$: In this case we add i to K_u, leaving the number of leaves intact and increasing $k(\mathscr{T})$ by one.
(c) $K_u \setminus K_v \supsetneq \{j\}$ and $K_v \setminus K_u = \{i\}$: In this case we add j to K_v decreasing $s(\mathscr{T})$ and leaving the rest of the parameters intact.
(d) $K_u \setminus K_v \supsetneq \{j\}$ and $K_v \setminus K_u \supsetneq \{i\}$: In this case subdivide the edge uv by adding a new node w and set $K_w = (K_u \cap K_v) \cup \{i, j\}$. This does not affect $\ell(\mathscr{T})$ and $k(\mathscr{T})$ and increases $d2(\mathscr{T})$ by one.

In all the above cases we have $\mathscr{T}' \prec_{LEX} \mathscr{T}$ as required.
We now proceed with the second transformation. Let u' be a leaf of $T_i \cap \bar{T}$ that is most distant from v. Let v' be the unique neighbour of u' in $T_i \cap \bar{T}$ (possibly $v' = v$) and let j' be a vertex of $K_{u'} \setminus K_{v'}$. By definition $i \in K_{u'} \cap K_{v'}$. We consider two disjoint and complementing cases. Consult Figure 3.2 for illustrations.

(a) $K_{u'} \setminus K_{v'} = \{j'\}$: In this case we remove i from $K_{u'}$, effectively removing the edge ij' from G'. Note that this transformation does not disconnect G' since we assume that all the graphs in $\mathscr{M}_{\text{CHORDAL}}(d, \nu)$ are factor-critical, thus connected. Therefore, T is not affected by the transformation, leaving $\ell(\mathscr{T})$ and $d2(\mathscr{T}')$ intact. Since i is not simplicial, $s(G)$ is left intact too.
(b) $K_{u'} \setminus K_{v'} \supsetneq \{j'\}$: In this case we subdivide the edge $u'v'$ by adding a new node w' and set $K_{w'} = K_{u'} \setminus \{i\}$. As in the previous case this modification does not disconnect G'. The transformation leaves $\ell(\mathscr{T}')$ intact and increases $d2(\mathscr{T}')$.

Since the transformation does not modify K_u and $|K_u| = k(\mathscr{T}')$ does not decrease. In both of the cases above we have $\mathscr{T}'' \preceq_{LEX} \mathscr{T}'$ as required. □

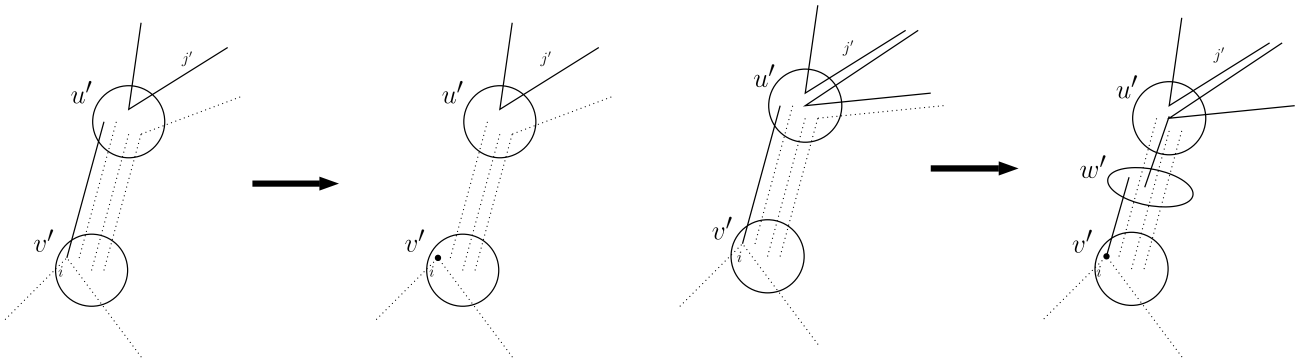
Observation 3.2. *Let $\mathscr{C}$ be a special hereditary graph class, and d, ν two positive integers, and let G be a graph of $\mathscr{M}_{\mathscr{C}}(d, \nu)$ with maximum number of connected components that are stars and maximum number of connected components subject to this constraint. Let $\nu' > 1$ be the matching number of a connected component G' of G. Then all the graphs in $\mathscr{M}_{\mathscr{C}}(d, \nu')$ are factor-critical.*

Proof. Suppose that $\mathscr{M}_{\mathscr{C}}(d, \nu')$ contains a graph G'' that is not factor-critical. By replacing G' by G'' in G we obtain a graph in $\mathscr{M}_{\mathscr{C}}(d, \nu)$. If G'' contains a connected component that is a star then the resulting graph has one star more than G. If G'' is not connected then the resulting graph has one more connected component than G. If G'' is connected it contradicts Theorem 2.3. □

We are now ready to prove the main result.

Theorem 3.3. *There exists a graph $G \in \mathscr{M}_{\text{CHORDAL}}(d, \nu)$ that is the disjoint union of $(d-1)$-stars and odd cliques.*

Proof. Let G be a graph in $\mathscr{M}_{\text{CHORDAL}}(d, \nu)$ with maximum number of stars and maximum number of connected components subject to this condition. Clearly, every connected component of G that is a star, is a $(d-1)$-star, since otherwise we can add at least one edge to G. Let $G_1, \ldots, G_k$ be the connected components of G that are not stars, and let ν_i be the matching number of G_i for every $i \in [k]$. It is easy to verify that the class of chordal graphs is special hereditary. By Observation 3.2, all the graphs in $\mathscr{M}_{\text{CHORDAL}}(d, \nu_i)$ are factor-critical. By Lemma 3.1, G_i can be replaced by a $K_{2\nu_i+1}$. □



(a) **(b)**

Figure 3.2: The second transformation

4. Conclusion

We have presented a short proof of the number of edges of an edge-extremal chordal graph. The simplicity of our technique opens room for further improvements. We believe that this proof may be further enhanced to characterize the edge-extremal chordal graphs.

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Author's contributions

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