Konuralp Journal of Mathematics, 10 (1) (2022) 149-158

Konuralp Journal of Mathematics

Research Paper

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath

e-ISSN: 2147-625X

Practical Stability in Terms of Two Measures with Initial Time Difference for Set Differential Equations involving Causal Operators

Coşkun Yakar[1*] **and Hazm Talab**[1]

[1]*Department of Mathematics, Faculty of Science, Gebze Technical University, Gebze, 141-41400 Kocaeli, Turkey*
[*]*Corresponding author*

Abstract

In this paper, we investigate generalized variational comparison results aimed to study the stability properties in terms of two measures for solutions of Set Differential Equations (SDEs) involving causal operators, taking into consideration the difference in initial conditions. Next, we employ these comparison results in proving the theorems that give sufficient conditions for practical stability in terms of two measures with initial time difference (ITD) for the solutions of perturbed SDEs involving causal operators in regard to their unperturbed ones.

Keywords: *Causal Operators; Initial Time Difference; Lyapunov function; Practical Stability; Set Differential Equations; Two Measures*
2010 Mathematics Subject Classification: *34D10; 34D20; 34D99*

1. Introduction

Many researchers were interested in studying set differential equations (SDEs) in the recent decades [1, 2, 6, 7, 8, 9, 11, 15, 18, 22, 31, 35, 47] due to their unifying properties. Lakshmikantham et al. highlighted these properties in one of the most important resources on this topic [15]. The comprehensiveness of the SDEs is driven from the fact that they encompass the conventional differential and integral equations when the Hukuhara difference and integrals defined on the SDEs are restricted to $\mathbb{R}$; whereas they give us vector differential equations when the restriction is done to $\mathbb{R}^n$ [3, 17, 20]. On the other hand, many well-known differential equations such as integro differential equations [25], impulsive differential equations [14], and differential equations with delay [34], are examples of differential equations involving causal operators. Many research papers dealt with those types of equations [5, 6, 7, 8, 9, 19, 37, 43].

SDEs with causal operators unifies the fundamental theory of SDEs, including various corresponding dynamical systems. Some relevant works can be found in [6, 7, 8, 9, 10, 11, 30, 31, 47].

Although it is never feasible to know the exact solutions of all dynamical systems in practice, their attributes may be determined through a variety of qualitative studies such as stability analysis [1, 2, 3, 12, 18, 20, 24, 31, 35], initial time difference (ITD) stability analysis [4, 26, 27, 32, 33, 36, 39, 40, 41, 43, 44, 45, 46, 47], practical stability analysis [21, 28, 42, 46], boundedness [1, 4, 10, 13, 29, 36, 39, 40, 41, 42], etc.

Many techniques have been used in this process, including the Lyapunov second method [20, 24, 32, 43, 44], variation of parameters [16, 29, 32], ”in terms of two measures” methodology [22, 23, 29, 31, 36, 40, 45, 46], and so on.

In this manuscript we study the practical stability in terms of two measures with ITD for SDEs involving causal operators.

2. Preliminaries

In what follows, we denote the set of all compact non-empty subsets of $\mathbb{R}^n$ by $K(\mathbb{R}^n)$, and the set of all compact and convex non-empty subsets of $\mathbb{R}^n$ by $K_c(\mathbb{R}^n)$.

Email addresses: cyakar@gtu.edu.tr (Coşkun Yakar), h.talab@gtu.edu.tr (Hazm Talab)

The Hausdorff metric between any bounded sets A and B in $\mathbb{R}^n$ is defined as

$$D(A,B)=\max\left[\sup_{x\in B} d(x,A)\ ,\ \sup_{y\in A} d(y,B)\right] \tag{2.1}$$

where

$$d(x,A)=\inf\ \{d(x,y)\ :\ y\in A\} \tag{2.2}$$

Each of $(K(\mathbb{R}^n),D)$ and $(K_c(\mathbb{R}^n),D)$ forms a complete metric space. The space $K_c(\mathbb{R}^n)$ equipped with the natural addition and non-negative scalar multiplication becomes a semi-linear metric space which can be embedded as a cone into a corresponding Banach space.

The Hausdorff metric satisfies the following properties:

$$\begin{aligned}&(1)\ D(A,B)=D(B,A)\\&(2)\ D(A+C,B+C)=D(A,B)\\&(3)\ D(kA,\ kB)=k\,D(A,B)\\&(4)\ D(A,B)\leq D(A,C)+D(C,B)\end{aligned} \tag{2.3}$$

for any $A,B,C\in K_c(\mathbb{R}^n)$ and $k\in\mathbb{R}_+$, where Minkowski addition of any two non-empty subsets A and B of $\mathbb{R}^n$ is defined by $A+B=\{a+b\ :\ a\in A,\ b\in B\}$ and where scalar multiplication of a value $k\in\mathbb{R}$ and a non-empty subset A of $\mathbb{R}^n$ is defined by $kA=\{ka\ :a\in A\}$. If $k=-1$, we get $-A=(-1)A=\{-a\ :a\in A\}$.

In general, $A+(-A)\neq\{0\}$ (unless $A=\{a\}$ is a singleton). To overcome with this implication of Minkowski difference, i.e.

$$A-B=A+(-1)B=\{a-b\ :a\in A,\ \ b\in B\} \tag{2.4}$$

Hukuhara difference between two sets $A,B\in K_c(\mathbb{R}^n)$ is defined as follows:

If there exists a set $C\in K_c(\mathbb{R}^n)$ such that $C+B=A$, then Hukuhara difference exists and we denote it by $A\ominus B$, or simply $A-B$ when there is no confusion with Minkowski difference. i.e. $A\ominus B=C\Leftrightarrow C+B=A$.

An important property of Hukuhara difference is $A-A=\{0\}$ for $A\in K_c(\mathbb{R}^n)$.

Let $U\ :I\rightarrow K_c(\mathbb{R}^n)$ be a given multifunction, where I is an interval of real numbers.U is said to be Hukuhara differentiable at a point $t_0\in I$, if there exists an element $D_HU(t_0)\in K_c(\mathbb{R}^n)$ such that the limits

$$\lim_{h\to 0^+}\frac{U(t_0+h)-U(t_0)}{h}\quad\text{and}\quad\lim_{h\to 0^+}\frac{U(t_0)-U(t_0-h)}{h} \tag{2.5}$$

both exist in the topology of $K_c(\mathbb{R}^n)$ and are equal to $D_HU(t_0)$.

It is implicit in the definition of $D_HU(t_0)$ the exitance of the two differences $U(t_0+h)-U(t_0)$ and $U(t_0)-U(t_0-h)$, for sufficiently small $h>0$.

By embedding $K_c(\mathbb{R}^n)$ as a complete cone in a corresponding Banach space and taking into account the result on differentiation of Bochner integral, we find that if

$$G(t)=G(t_0)+\int_{t_0}^{t}F(s)\,ds,\ \ t\in I \tag{2.6}$$

where $F\ :I\rightarrow K_c(\mathbb{R}^n)$ is integrable in the sense of Bochner, then G is Hukuhara differentiable, i. e. $D_HG(t)$ exits, and the equality $D_HG(t)=F(t)$, a. e. on I, holds.

Also, the Hukuhara integral

$$\int_I F(s)ds=\left[\int_I f(s)ds\ :\ f\ is\ a\ continuous\ selector\ of\ F\right] \tag{2.7}$$

for any compact set $I\subset\mathbb{R}_+$.

Let $E=C[[t_0,\infty),K_c(\mathbb{R}^n)]$ with norm

$$\sup_{t\in[t_0,\infty)}\frac{D[U(t),\theta]}{h(t)}\ <\infty \tag{2.8}$$

where $U\in E$, θ is the zero element of $\mathbb{R}^n$, which is regarded as a point set; and $h:[t_0,\infty)\rightarrow\mathbb{R}_+$ is a continuous map. E equipped with such a norm is a Banach space.

Let $Q \in C[E,E]$. Q is said to be a causal map if $U(s) = V(s)$, $t_0 \le s \le t < \infty$, and $U, V \in E$ then

$$(QU)(s) = (QV)(s), \quad t_0 \le s \le t < \infty. \tag{2.9}$$

Let us consider the following differential equations

$$D_H U = (QU)(t), \quad U(t_0) = U_0 \ \ for \ \ U_0 \in K_c(\mathbb{R}^n) \ \ and \ \ t \ge t_0 \ge 0, \tag{2.10}$$

$$D_H U = (QU)(t), \quad U(\tau_0) = V_0 \ \ for \ \ V_0 \in K_c(\mathbb{R}^n) \ \ and \ \ t \ge \tau_0 \ge 0 \tag{2.11}$$

$$D_H V = (PV)(t), \quad V(\tau_0) = V_0 \ \ for \ \ V_0 \in K_c(\mathbb{R}^n) \ \ and \ \ t \ge \tau_0 \tag{2.12}$$

$$D_H W = (SW)(t), \ W(\tau_0) = V_0 - U_0 \ \ for \ \ W(\tau_0) = W_0 \in K_c(\mathbb{R}^n) \ \ and \ \ t \ge \tau_0 \tag{2.13}$$

where $Q, P, S : E \to E$ are causal operators, and satisfy a local Lipschitz condition on $\mathbb{R}_+ \times S_\rho$ where $S_\rho = \left\{ U \in K_c(\mathbb{R}^n) : D\left[U, \tilde{0}\right] < \rho < \infty \right\}$. It is clear that (2.10) and (2.11) are different in the initial time and position. Moreover, if $(PV)(t)$ in (2.12) is written as $(PV)(t) = (QV)(t) + (RV)(t)$; Then, we consider (2.12) as the perturbed form corresponding to the unperturbed equation (2.11) with the perturbation term $(RV)(t)$.

Assuming that $\left(Q\tilde{0}\right)(t) \equiv \tilde{0}$ for $t \ge 0$, and assuming the necessary smoothness of P, Q and R to guarantee the existence and uniqueness of the solution $U(t) = U(t, t_0, U_0)$ of (2.10) through (t_0, U_0) for all $t \ge t_0$, and those of the solution $V(t) = V(t, \tau_0, V_0)$ of (2.12) through (τ_0, V_0) for all $t \ge \tau_0$, in addition to their continuous dependence on the initial conditions.

If $U \in C^1[J_1, K_c(\mathbb{R}^n)]$ on $J_1 = [t_0, t_0 + T_1]$, then it is said to be a solution of (2.10) on J_1 if it satisfies (2.10) on J_1. If U, V and $W \in C^1[J_2, K_c(\mathbb{R}^n)]$ on $J_2 = [t_0, t_0 + T_2]$, then these are said to be solutions of (2.11), (2.12), (2.13) on J_2 provided that they satisfy (2.11), (2.12), (2.13) on J_2, respectively.

Now let us define a partial order in the metric space $(K_c(\mathbb{R}^n), D)$. First, we start by defining a cone in $K_c(\mathbb{R}^n)$.

Definition 2.1. *The subfamily $K \subset K_c(\mathbb{R}^n)$ is said to be a cone in $K_c(\mathbb{R}^n)$ if it consists of sets $U \in K_c(\mathbb{R}^n)$ such that any $u \in U$ is a non-negative n-component vector $u = (u_1, u_2, \ldots, u_n)$ satisfying $u_i \ge 0$ for $i = 1 \ldots n$. The subfamily $K^0 \subset K_c(\mathbb{R}^n)$, that consists of sets $U \in K_c(\mathbb{R}^n)$ such that any $u \in U$ is a positive n-component vector $u = (u_1, u_2, \ldots, u_n)$ satisfying $u_i > 0$ for $i = 1 \ldots n$, is the nonempty interior of the cone K.*

Definition 2.2. *For any $U, V \in K_c(\mathbb{R}^n)$, if there exists $Z \in K_c(\mathbb{R}^n)$ such that $Z \in K$ and $U = V + Z$ then we say that $U \ge V$ or $V \le U$. Similarly, if there exists $Z \in K_c(\mathbb{R}^n)$ such that $Z \in K^0$ and $U = V + Z$ then we say that $U > V$ or $V < U$.*

We present below some needed classes to develop the stability results in terms of two measures.

$$\mathbb{K} = \{a \in C[\mathbb{R}_+, \mathbb{R}_+] \ : \ a(u) \ \ is \ strictly \ increasing \ in \ u \ \ and \ \ a(0) = 0\} \tag{2.14}$$

$$\mathbb{L} = \left\{ \sigma \in C[\mathbb{R}_+, \mathbb{R}_+] \ : \sigma(u) \ \ is \ strictly \ decreasing \ in \ u \ \ and \ \ \lim_{u \to \infty} \sigma(u) = 0 \right\} \tag{2.15}$$

$$\mathbb{CK} = \left\{ \begin{array}{c} a \in C\left[\mathbb{R}_+^2, \mathbb{R}_+\right] \ : \ a(t,s) \in \mathbb{K} \ for \ each \ t \\ and \ \ a(t,s) \ is \ continuous \ for \ each \ s \end{array} \right\} \tag{2.16}$$

$$\Gamma = \left\{ h \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+] : \inf_{(t,U)} h(t,U) = 0 \right\} \tag{2.17}$$

$$\Gamma_0 = \left\{ h \in \Gamma : \inf_{U} h(t,U) = 0 \ , \ \ for \ each \ t \in \mathbb{R}_+ \right\} \tag{2.18}$$

Next, to introduce a Lyapunov-like function, we present some required definitions when dealing with the concept of ”two measures”.

Definition 2.3. *Let $L\in C\left[\mathbb{R}_{+}\times K_{c}\left(\mathbb{R}^{n}\right),\mathbb{R}_{+}\right]$, then L is said to be*
(i) h-positive definite if there exists a $\rho>0$ and a $b\in\mathbb{K}$ such that

$$h(t,U)<\rho \ \text{ implies } \ b(h(t,U))\leq L(t,U) \tag{2.19}$$

(ii) h-decrescent if there exists a $\rho>0$ and a function $a\in\mathbb{K}$ such that

$$h(t,U)<\rho \ \text{ implies } \ L(t,U)\leq a(h(t,U)) \tag{2.20}$$

(iii) h-weakly decrescent if there exists a $\rho>0$ and a function $a\in\mathbb{CK}$ such that

$$h(t,U)<\rho \ \text{ implies } \ L(t,U)\leq a(t,\ h(t,U)) \tag{2.21}$$

Definition 2.4. *Let $h_0,h\in\Gamma$, then we say that h_0 is finer than h if there exists a $\rho>0$ and a function $\phi\in\mathbb{CK}$ such that*

$$h_{0}(t,U)\leq\rho \ \text{ implies } \ h(t,U)\leq\phi(t,h_{0}(t,U)) \tag{2.22}$$

h_0 is uniformly finer than h if the function ϕ in the above definition is independent of t.

Now, let us introduce the definitions of generalized Dini-like derivatives of L.

Definition 2.5. *We define the generalized derivative (Dini-like derivatives) for a real-valued function*
$L\in C\left[\mathbb{R}_{+}\times K_{c}\left(\mathbb{R}^{n}\right),\mathbb{R}_{+}\right]$ as follows:

$$D_{*}^{+}L(t,s,U)=\lim_{h\to0^{+}}\sup\frac{1}{h}\ \left[L\left(s+h,V\left(t,s+h,U+h\left(\tilde{Q}\tilde{U}\right)(s)\right)\right)-L(s,V(t,s,U))\right] \tag{2.23}$$

$$D_{*-}L(t,s,U)=\lim_{h\to0^{-}} \inf\frac{1}{h}\ \left[L\left(s+h,V\left(t,s+h,U+h\left(\tilde{Q}\tilde{U}\right)(s)\right)\right)-L(s,V(t,s,U))\right] \tag{2.24}$$

for $t,s\in\mathbb{R}_{+}$ and $U\in K_{c}\left(\mathbb{R}^{n}\right)$.

Next, let us introduce the definitions of initial time difference (ITD) practically stability in terms of two measures, before proceeding with the stability results.

Definition 2.6. *Let $U(t,t_0,U_0)$ be any solution of (2.10) for $t\geq t_0\geq 0$, and let $\tilde{U}(t,\tau_0,U_0)=U(t-\eta,t_0,U_0)$, for $\eta=\tau_0-t_0$. The solution $V(t,\tau_0,V_0)$ of (2.12) for $t\geq\tau_0$ is said to be*

(i) ITD (h_0,h)-practically stable with respect to the solution $\tilde{U}$ if and only if given any (λ,A) with $0<\lambda<A$ and for some $\tau_0\in\mathbb{R}_{+}$ such that $h_0(\tau_0,V_0-U_0)\leq\lambda$ implies

$$h(t,V(t,\tau_{0},V_{0})-U(t-\eta,t_{0},U_{0}))\leq A,\ \ t\geq\tau_{0} \tag{2.25}$$

(ii) ITD (h_0,h)-uniformly practically stable with respect to the solution $\tilde{U}$ if the previous implication in (i) holds for every $\tau_0\in\mathbb{R}_{+}$.

(iii) ITD (h_0,h)-practically quasi-stable with respect to the solution $\tilde{U}$, if and only if given any $(\lambda,B,T)>0$ with $0<\lambda<B$ and for some $\tau_0\in\mathbb{R}_{+}$ such that $h_0(\tau_0,V_0-U_0)\leq\lambda$ implies

$$h(t,V(t,\tau_{0},V_{0})-U(t-\eta,t_{0},U_{0}))\leq B,\ \ t\geq\tau_{0}+T \tag{2.26}$$

(iv) ITD (h_0,h)-uniformly practically quasi-stable with respect to the solution $\tilde{U}$ if the previous implication in (iii) holds for every $\tau_0\in\mathbb{R}_{+}$.

3. ITD Stability Results in Terms of Two Measures

3.1. ITD Variational Comparison Results

In what follows, let us present generalized variational comparison results aimed to study the stability properties in terms of two measures for solutions of SDEs involving causal operators, taking into consideration the difference in the initial conditions.

Theorem 3.1. *Assume that*

(i) Both $L(t,\Omega)\in C\left[\mathbb{R}_{+}\times K_{c}\left(\mathbb{R}^{n}\right),\mathbb{R}_{+}^{N}\right]$ and $\|W(t,s,\Omega)\|$ satisfy a local Lipschitz condition in Ω for any t,s; where $W(t)=W(t,\tau_0,V_0-U_0)$ is the solution of (2.13) for $t\geq\tau_0$, $\tilde{U}(t,\tau_0,U_0)=U(t-\eta,t_0,U_0)$, for $\eta=\tau_0-t_0$, $U(t,t_0,U_0)$ is any solution of (2.10) for $t\geq t_0$, and $V(t)=V(t,\tau_0,V_0)$ is the solution of (2.12) for $t\geq\tau_0$; and let $\Omega(t)=V(t)-\tilde{U}(t)$.

(ii)

$$D_{*-}L(t,s,\Omega)\leq g(t,s,L(s,W(t,s,\Omega))) \tag{3.1}$$

where

$$D_{*-}L(t,s,\Omega)=\lim_{\delta\to0^{-}}\inf\frac{1}{\delta}\left(L\left(s+\delta,W\left(t,s+\delta,\Omega+\delta\left((PV)(s)-\left(\tilde{Q}\tilde{U}\right)(s)\right)\right)\right)-L(s,W(t,s,\Omega))\right) \tag{3.2}$$

(iii) $g \in C\left[\mathbb{R}_+ \times \mathbb{R}_+^N, \mathbb{R}^N\right]$, $g(t,s,u)$ *is quasi-monotone non-decreasing in* u *for any* t,s*; that is, if* $u \le v$, $u_i = v_i$ *for some* i *such that* $1 \le i \le N$*, then* $g_i(t,s,u) \le g_i(t,s,v)$*, for* $t,s \in \mathbb{R}_+$ *(In this context, the inequality symbol used in the vectorial inequalities is understood to denote component-wise inequality [38]);*

and $r(t,s,\tau_0,V_0)$ *is the maximal solution of*

$$\frac{du(s)}{ds} = g(t,s,u(s)), \quad u(\tau_0) = u_0 \ge 0 \tag{3.3}$$

existing for $\tau_0 \le s \le t < \infty$.

Then, $L(\tau_0, W(t,\tau_0,V_0 - U_0)) = u_0$ *implies*

$$L(t,\Omega(t,\tau_0,V_0-U_0)) \le r_0(t,\tau_0,L(\tau_0,W(t,\tau_0,V_0-U_0))) \tag{3.4}$$

where $r_0(t,\tau_0,u_0) = r(t,t,\tau_0,u_0)$.

Proof. Let us set

$$m(t,s) = L(s,W(t,s,\Omega(s))) \quad for \quad \tau_0 \le s \le t \tag{3.5}$$

Then, we have

$$m(t,\tau_0) = L(\tau_0,W(t,\tau_0,\Omega(\tau_0))) = L\left(\tau_0,W\left(t,\tau_0,V(\tau_0)-\tilde{U}(\tau_0)\right)\right) = L(\tau_0,W(t,\tau_0,V_0-U_0)) = u_0 \tag{3.6}$$

For a sufficiently small positive value δ, we have

$$\begin{aligned} &m(t,s+\delta) - m(t,s) \\ &= L(s+\delta,W(t,s+\delta,\Omega(s+\delta))) - L(s,W(t,s,\Omega(s))) \\ &= L(s+\delta,W(t,s,\Omega(s)) + \delta(SW(t,s,\Omega(s)))(s) + \varepsilon(\delta)) - L(s,W(t,s,\Omega(s))) \end{aligned} \tag{3.7}$$

where ε stands for error and $\lim_{\delta\to 0^-} \frac{\varepsilon(\delta)}{\delta} = 0$.

Taking into consideration the assumptions in (i) regarding the locally Lipschitz property of $L(t,\Omega)$ and $\|W(t,s,\Omega)\|$ in Ω, it is seen that

$$\begin{aligned} m(t,s+\delta) - m(t,s) \le{}& k(\varepsilon_1(\delta) - \varepsilon_2(\delta)) + L\left(s+\delta, W\left(t,s,V(s)-\tilde{U}(s)\right) + \delta\left((PV)(s) - \left(\tilde{Q}\tilde{U}\right)(s)\right)\right) \\ &- L\left(s,W\left(t,s,V(s)-\tilde{U}(s)\right)\right) \end{aligned} \tag{3.8}$$

where $\varepsilon_1, \varepsilon_2$ stand for errors, k stands for Lipschitz constant.

The inequality in the assumption (ii) gives us the following estimation regarding the Dini derivative of $m(t,s)$

$$\begin{aligned} &D_{*-}m(t,s) \\ &\le \lim_{\delta\to 0^-} \inf \frac{1}{\delta} K(\varepsilon_1(\delta) - \varepsilon_2(\delta)) + \lim_{\delta\to 0^-} \inf \frac{1}{\delta} \; L\left(s+\delta, W\left(t,s,V(s)-\tilde{U}(s)\right) + \delta\left((PV)(s) - \left(\tilde{Q}\tilde{U}\right)(s)\right)\right) \\ &- \lim_{\delta\to 0^-} \inf \frac{1}{\delta} \; L\left(s,W\left(t,s,V(s)-\tilde{U}(s)\right)\right) \le g\left(t,s,L\left(s,W\left(t,s,V(s)-\tilde{U}(s)\right)\right)\right) \\ &= g(t,s,L(s,W(t,s,\Omega(s)))) = g(t,s,m(t,s)) \end{aligned} \tag{3.9}$$

for $\tau_0 \le s \le t < \infty$.

A comparison result [Theorem 1.7.1] from [17] gives us the following inequality

$$m(t,s) \le r(t,s,\tau_0,L(\tau_0,W(t,\tau_0,V_0-U_0))) \quad for \quad \tau_0 \le s \le t \tag{3.10}$$

Choosing $s = t$ in the right-hand side of the previous inequality, we get

$$m(t,s) \le r(t,t,\tau_0,L(\tau_0,W(t,\tau_0,V_0-U_0))) = r_0(t,\tau_0,L(\tau_0,W(t,\tau_0,V_0-U_0))) \tag{3.11}$$

which yields the desired estimation in (3.4) completing the proof. □

Theorem 3.2. *Under the assumptions of Theorem 1 with* $N = 1$ *and* $g(t,s,u) \equiv 0$*, we have*

$$L(t,\Omega(t,\tau_0,V_0-U_0)) \le L(\tau_0,W(t,\tau_0,V_0-U_0)), \quad t \ge \tau_0 \tag{3.12}$$

Furthermore, we assume

$$D_{*-}L(t,s,\Omega) \le -c(h(s,W(t,s,\Omega))), \quad \tau_0 \le s \le t < \infty \tag{3.13}$$

where $c \in \mathbb{K} = \{\phi \in C[\mathbb{R}_+,\mathbb{R}_+]$ *such that* $\phi(0) = 0$ *and* $\phi(s)$ *is increasing in* $s\}$ *and* $h \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$.

Then, for $t \ge \tau_0$

$$L(t,\Omega(t,\tau_0,V_0-U_0)) \le L(\tau_0,W(t,\tau_0,V_0-U_0)) - \int_{\tau_0}^{t} c(h(s,W(t,s,\Omega(s))))\,ds \tag{3.14}$$

Proof. Starting from the statement (3.9) in the proof of Theorem 1,

$$D_{*-}m(t,s) \le g(t,s,m(t,s)) \quad for \quad \tau_0 \le s \le t < \infty \tag{3.15}$$

Then, since $g(t,s,u) \equiv 0$, we get by integrating the two sides of the previous inequality (3.15), for $s \in [\tau_0, t]$,

$$\int_{\tau_0}^{t} D_{*-}m(t,s)ds = L(t, W(t,t,\Omega(t))) - L(\tau_0, W(t,\tau_0,\Omega(\tau_0))) \le 0 \tag{3.16}$$

Hence, we have

$$L(t,\Omega(t,\tau_0,V_0-U_0)) \le L(\tau_0, W(t,\tau_0, \mathrm{V}_0 - U_0)) \quad for \quad t \ge \tau_0 \tag{3.17}$$

Now, let us set

$$M(s,W(t,s,\Omega(s))) \equiv L(s,W(t,s,\Omega(s))) + \int_{\tau_0}^{s} c(h(\xi, W(t,\xi,\Omega(\xi))))d\xi \tag{3.18}$$

Then, by taking Dini derivatives of both sides and by assumption (3.13), we have

$$\begin{aligned} D_{*-}M(t,s,\Omega(s)) &= D_{*-}L(t,s,\Omega(s)) + c(h(s,W(t,s,\Omega(s)))) - c(h(\tau_0, W(t,\tau_0,\Omega(\tau_0)))) \\ &\le D_{*-}L(t,s,\Omega(s)) + c(h(s,W(t,s,\Omega(s)))) \\ &\le -c(h(s,W(t,s,\Omega(\mathrm{s})))) + c(h(s,W(t,s,\Omega(s)))) = 0 \end{aligned} \tag{3.19}$$

Thus, $D_{*-}M(t,s,\Omega(s)) \le 0$, in view of (3.17), gives us for $t \ge \tau_0$,

$$M(t,\Omega(t,\tau_0,V_0-U_0)) \le M(\tau_0, W(t,\tau_0,V_0-U_0)) \tag{3.20}$$

By the definition of M, this implies, for $t \ge \tau_0$,

$$\begin{aligned} L(t,\Omega(t,\tau_0,V_0-U_0)) &+ \int_{\tau_0}^{t} c(h(\xi,W(t,\xi,\Omega(\xi))))d\xi \\ &\le L(\tau_0,W(t,\tau_0,V_0-U_0)) + \int_{\tau_0}^{\tau_0} c(h(\xi,W(t,\xi,\Omega(\xi))))d\xi \end{aligned} \tag{3.21}$$

$$L(t,\Omega(t,\tau_0,V_0-U_0)) + \int_{\tau_0}^{t} c(h(\xi,W(t,\xi,\Omega(\xi))))d\xi \le L(\tau_0,W(t,\tau_0,V_0-U_0)) \tag{3.22}$$

Moving the integral term to the right-hand side gives us the desired estimation (3.14) and this completes the proof. □

3.2. Main ITD Practical Stability Results in Terms of Two Measures

Now, let us employ the comparison results in section 3.1 to prove the following theorems giving sufficient conditions for ITD practical stability in terms of two measures for the solutions of perturbed SDEs involving causal operators in regard to their unperturbed ones.

The following theorem gives sufficient conditions for ITD (h_0,h)-practically stability of the solution $V(t,\tau_0,V_0)$ of (2.12) for $t \ge \tau_0$ through (τ_0,V_0) with respect to the solution $\tilde{U}(t,\tau_0,U_0) = U(t-\eta,t_0,U_0)$, for $\eta = \tau_0 - t_0$, where $U(t) = U(t,t_0,U_0)$ is the solution of (2.10) through (t_0,U_0) for $t \ge t_0$.

Theorem 3.3. *Assume that*

(i) $0 < \lambda < A$;

(ii) $h_0, h \in \Gamma$ *and* h_0 *is uniformly finer than* h*, that is, there exists a function* $\phi \in \mathbb{K}$ *such that*

$$h_0(t,\Omega) \le \lambda \quad implies \quad h(t,\Omega) \le \phi(h_0(t,\Omega)) \tag{3.23}$$

(iii) $L(t,\Omega) \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$ *satisfies a local Lipschitz condition in* Ω *and*

$$h(t,\Omega) \le A \quad implies \quad b(h(t,\Omega)) \le L(t,\Omega), \quad b \in \mathbb{K} \tag{3.24}$$

$$h_0(t,\Omega) \le \lambda \quad implies \quad L(t,\Omega) \le a(h_0(t,\Omega)), \quad a \in \mathbb{K} \tag{3.25}$$

and the inequality

$$D^+L(t,\Omega) \le g(t, L(t, W(t,s,\Omega))), \quad (t,\Omega) \in S(h,A) \tag{3.26}$$

where $g \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}]$*, and* $W(t) = W(t,\tau_0,V_0-U_0)$ *is the solution of (2.13) and*

$$\Omega(t,\tau_0,V_0-U_0) = V(t) - \tilde{U}(t) \quad for \quad t \ge \tau_0 \tag{3.27}$$

(iv) $\varphi(\lambda) < A$ *and* $a(\lambda) < b(A)$ *hold.*

Then, the practical stability of the solution of the equation

$$\frac{du(s)}{ds} = g(t,s,u(s)), \ \ u(\tau_0) = u_0 \geq 0 \tag{3.28}$$

for $\tau_0 \leq s \leq t < \infty$, *with* $g(t,0) = 0$, *implies that the corresponding ITD* (h_0,h)*-practically stability property of the solution* $V(t,\tau_0,V_0)$ *of (2.12) for* $t \geq \tau_0$ *with respect to the solution* $\tilde{U}(t,\tau_0,U_0) = U(t-\eta,t_0,U_0)$, *for* $\eta = \tau_0 - t_0$, *where* $U(t) = U(t,t_0,U_0)$ *is the solution of (2.10) through* (t_0,U_0) *for* $t \geq t_0$.

Proof. Consider the practical stability of the solution $u(t,\tau_0,u_0)$ of (3.28) for $t \geq \tau_0$, with the assumption in (iv), that $a(\lambda) < b(A)$, we may write, for $t \geq \tau_0$,

$$u_0 < a(\lambda) \quad implies \quad u(t,\tau_0,u_0) < b(A) \tag{3.29}$$

We shall prove that the solution $V(t,\tau_0,V_0)$ of (2.12) for $t \geq \tau_0$ is ITD (h_0,h)-practically stable with respect to the solution $\tilde{U}$ corresponding to (λ,A).

If we assume this assertion is not true, then there would exist a $t_1 > \tau_0$ and a solution $W(t) = W(t,\tau_0,V_0-U_0)$ of (2.13) for $t \geq \tau_0$ such that

$$\begin{aligned} &h_0(\tau_0,V_0-U_0) < \lambda, \\ &h(t_1,V(t_1,\tau_0,V_0) - U(t_1-\eta,t_0,U_0)) = A, \\ &h(t,V(t,\tau_0,V_0) - U(t-\eta,t_0,U_0)) < A \ \ for \ \ \tau_0 \leq t < t_1 \end{aligned} \tag{3.30}$$

Since $\phi \in \mathbb{K}$ is strictly monotone increasing, the assumptions (ii) and (iv) give us

$$h(\tau_0,V_0-U_0) \leq \phi(h_0(\tau_0,V_0-U_0)) < \phi(\lambda) < A \tag{3.31}$$

Thus, in view of (3.26), and by using [Theorem 1.3.1] in [23], we have the following inequality

$$L(t,V(t,\tau_0,V_0) - U(t-\eta,t_0,U_0)) \leq \gamma(t,\tau_0,u_0), \ \ \tau_0 \leq t \leq t_1 \tag{3.32}$$

where $\gamma(t,\tau_0,u_0)$ is the maximal solution of (3.28) for $t \geq \tau_0$ and $u_0 = L(\tau_0,V_0-U_0)$.

Since $a \in \mathbb{K}$ is strictly monotone increasing, the assumptions (iii) and (iv) give us

$$u_0 = L(\tau_0,V_0-U_0) \leq a(h_0(\tau_0,V_0-U_0)) < a(\lambda) \tag{3.33}$$

Considering that (3.29) holds for any solution $u(t,\tau_0,u_0)$ of (3.28) for $t \geq \tau_0$, so it holds for the maximal solution $\gamma(t,\tau_0,u_0)$ of (3.28) for $t \geq \tau_0$, this yield

$$\gamma(t,\tau_0,u_0) < b(A), \ \ t \geq \tau_0 \tag{3.34}$$

The last inequality holds for any $t \geq \tau_0$ so it holds for $t = t_1 \geq \tau_0$.

Thus, employing (iii), (3.30), (3.32) and (3.34), we have

$$b(A) = b(h(t_1,V(t_1,\tau_0,V_0) - U(t_1-\eta,t_0,U_0))) \leq L(t_1,V(t_1,\tau_0,V_0) - U(t_1-\eta,t_0,U_0)) \leq \gamma(t_1,\tau_0,u_0) < b(A) \tag{3.35}$$

This contradiction proves the assertion. Hence, the solution $V(t,\tau_0,V_0)$ of (2.12) for $t \geq \tau_0$ is ITD (h_0,h)-practically stable with respect to the solution $\tilde{U}$. □

The next theorem gives sufficient conditions to the ITD (h_0,h)-practically stability of the solution $V(t,\tau_0,V_0)$ of (2.12) through (τ_0,V_0) for $t \geq \tau_0$ with respect to the solution $\tilde{U}(t) = U(t-\eta,t_0,U_0)$ for $t \geq t_0$, where $U(t) = U(t,t_0,U_0)$ is the solution of (2.10) through (t_0,U_0) for $t \geq t_0$; providing that the solution $V(t,\tau_0,V_0)$ of (2.12) is ITD (h_0,h_0)-practically stable with respect to $\tilde{U}$.

Theorem 3.4. *Assume that*

(i) Both $L(t,\Omega) \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$ *and* $\|W(t,s,\Omega)\|$ *satisfy a local Lipschitz condition in* Ω *for any* t,s; *where* $W(t) = W(t,\tau_0,V_0-U_0)$ *is the solution of (2.13) for* $t \geq \tau_0$ *and*

$$\Omega(t,\tau_0,V_0-U_0) = V(t) - \tilde{U}(t) \ \ for \ \ t \geq \tau_0 \tag{3.36}$$

(ii)

$$D_{*-}L(t,s,\Omega) \leq 0 \ \ in \ \ S(h,M) \tag{3.37}$$

where

$$S(h,M) = \{(t,\Omega) \ : h(t,\Omega) < M \text{ for some } h \in \Gamma \text{ and } M > 0\} \tag{3.38}$$

and

$$D_{*-}L(t,s,\Omega) = \lim_{\delta \to 0^-} \inf \frac{1}{\delta}\left(L\left(s+\delta, W\left(t,s+\delta,\Omega+\delta\left((PV)(s) - \left(\tilde{Q}\tilde{U}\right)(s)\right)\right)\right) - L(s,W(t,s,\Omega))\right) \tag{3.39}$$

(iii) For $b \in \mathbb{K}$ and $a_1, a_0 \in \mathbb{CK}$,

$$b(h(t,\Omega)) \leq L(t,\Omega) \;\; in \;\; S(h,M) \;\; and \;\; L(t,\Omega) \leq a_1(t,h(t,\Omega)) + a_0(t,h_0(t,\Omega)) \;\; in \;\; S(h,M) \cap S(h_0,M) \tag{3.40}$$

(iv) h_0 is finer that h, that is, there exists a function $\phi \in \mathbb{K}$ such that

$$h_0(t,\Omega) \leq M_0 \;\; implies \;\; h(t,\Omega) \leq \phi(h_0(t,\Omega)) \tag{3.41}$$

for some M_0 with $\phi(M_0) \leq M$;

(v) The solution $V(t,\tau_0,V_0)$ of (2.12) for $t \geq \tau_0$ is ITD (h_0,h_0)-practically stable with respect to the solution $\tilde{U}(t,\tau_0,U_0) = U(t-\eta,t_0,U_0)$, for $\eta = \tau_0 - t_0$.

Then, this implies the ITD (h_0,h)-practically stability of the solution $V(t,\tau_0,V_0)$ of (2.12) for $t \geq \tau_0$ with respect to the solution $\tilde{U}$.

Proof. We shall show that the solution $V(t,\tau_0,V_0)$ of (2.12) for $t \geq \tau_0$ is ITD (h_0,h)-practically stable with respect to the solution $\tilde{U}$, that is, given any (λ,A) with $0 < \lambda < A$ and for some $\tau_0 \in \mathbb{R}_+$, we have

$$h_0(\tau_0,V_0-U_0) < \lambda \;\; implies \;\; h(t,\Omega(t)) < A \;\; for \;\; t \geq \tau_0 \tag{3.42}$$

If (3.42) is not true, then there exist solutions $\tilde{U}(t) = U(t-\eta,t_0,U_0)$, where $U(t,t_0,U_0)$ is the solution of (2.10) for $t \geq t_0$; and $V(t) = V(t,\tau_0,V_0)$ of (2.12) for $t \geq \tau_0$, and $t_1 > \tau_0$ such that

$$h_0(\tau_0,V_0-U_0) < \lambda, \;\; h(t_1,\Omega(t_1)) = A \;\; and \;\; h(t,\Omega(t)) \leq A, \;\; for \;\; \tau_0 \leq t \leq t_1 \tag{3.43}$$

where $\Omega(t) = V(t) - \tilde{U}(t)$ for $t \geq \tau_0$.

By Theorem 3.2, we have

$$L(t,\Omega(t)) \leq L(\tau_0,W(t,\tau_0,V_0-U_0)), \;\; for \;\; \tau_0 \leq t \leq t_1 \tag{3.44}$$

Starting from the statement of $b(A)$ and using the assumptions (iii), (3.43) and (3.44), we obtain

$$\begin{aligned} b(A) &= b(h(t_1,\Omega(t_1))) \leq L(t_1,\Omega(t_1)) \leq L(\tau_0,W(t_1,\tau_0,V_0-U_0)) \\ &\leq a_1(\tau_0,h(\tau_0,W(t_1,\tau_0,V_0-U_0))) + a_0(\tau_0,h_0(\tau_0,W(t_1,\tau_0,V_0-U_0))) \end{aligned} \tag{3.45}$$

We aim to reach a contradiction to conclude the proof of the theorem. We will use the assumption (v) for this purpose.

Given $0 < A < M$ and that there exists a M_0 with $\phi(M_0) \leq M$. Choosing $N_1 = N_1(\tau_0,A)$ such that $0 < N_1(\tau_0,A) < M_0$, and

$$h_0(t,\Omega(t)) < N_1 \;\; implies \;\; a_0(t,h_0(t,\Omega(t))) < \frac{b(A)}{2} \;\; for \;\; t \geq \tau_0 \tag{3.46}$$

By assumption (v), corresponding to this N_1, there exists a $\lambda_1 = \lambda_1(\tau_0,N_1)$ such that

$$h_0(\tau_0,V_0-U_0) < \lambda_1 \;\; implies \;\; h_0(t,\Omega(t)) < N_1 \;\; for \;\; t \geq \tau_0 \tag{3.47}$$

Thus (3.46) and (3.47) give us

$$h_0(\tau_0,V_0-U_0) < \lambda_1 \;\; implies \;\; a_0(t,h_0(t,\Omega(t))) < \frac{b(A)}{2} \;\; for \;\; t \geq \tau_0 \tag{3.48}$$

Similarly, we choose $N_2 = N_2(\tau_0,A)$ such that $0 < N_2(\tau_0,A) < M_0$ and

$$h(t,\Omega(t)) < N_2 \;\; implies \;\; a_1(t,h(t,\Omega(t))) < \frac{b(A)}{2} \;\; for \;\; t \geq \tau_0 \tag{3.49}$$

By the assumptions (iv) and (v) also, corresponding to $\phi^{-1}(N_2)$, there exists a $\lambda_2 = \lambda_2(\tau_0,N_2)$ such that

$$h_0(\tau_0,V_0-U_0) < \lambda_2 \;\; implies \;\; h_0(t,\Omega(t)) < \phi^{-1}(N_2) \;\; for \;\; t \geq \tau_0 \tag{3.50}$$

Since $\phi \in \mathbb{K}$ is strictly monotone increasing; then, we have by taking the composition of ϕ of both sides of the inequality $h_0(t,\Omega(t)) < \phi^{-1}(N_2)$ in (3.50), with considering (3.41),

$$h_0(\tau_0,V_0-U_0) < \lambda_2 \;\; implies \;\; h(t,\Omega(\mathrm{t})) \leq \phi(h_0(t,\Omega(t))) < \phi\left(\phi^{-1}(N_2)\right) = N_2 \;\; for \;\; t \geq \tau_0 \tag{3.51}$$

So, (3.49) and (3.51) give us, for $t \geq \tau_0$,

$$h_0(\tau_0,V_0-U_0) < \lambda_2 \;\; implies \;\; a_1(t,h(t,\Omega(t))) < \frac{b(A)}{2} \tag{3.52}$$

Let $\lambda = \min\{\lambda_1,\lambda_2\}$, then with this λ the following statement holds.

$$h_0(\tau_0,V_0-U_0) < \lambda \;\; implies \;\; a_0(t,h_0(t,\Omega(t))) < \frac{b(A)}{2} \;\; and \;\; a_1(t,h(t,\Omega(t))) < \frac{b(A)}{2} \;\; for \;\; t \geq \tau_0 \tag{3.53}$$

Hence, when $t = t_1$, using (3.53), the statement (3.45) can be written as

$$\begin{aligned} b(A) &= b(h(t_1,\Omega(t_1))) \leq L(t_1,\Omega(t_1)) \leq L(\tau_0,W(t_1,\tau_0,V_0-U_0)) \\ &\leq a_1(\tau_0,h(\tau_0,W(t_1,\tau_0,V_0-U_0))) + a_0(\tau_0,h_0(\tau_0,W(t_1,\tau_0,V_0-U_0))) < \frac{b(A)}{2} + \frac{b(A)}{2} = b(A) \end{aligned} \tag{3.54}$$

This contradiction proves that the solution $V(t,\tau_0,V_0)$ of (2.12) through (τ_0,V_0) for $t \geq \tau_0$ is ITD (h_0,h)-practically stable with respect to the solution $\tilde{U}$.

□

4. Conclusion

In this manuscript, we have presented sufficient conditions for ITD practical stability in terms of two measures for the solutions of perturbed SDEs involving causal operators in regard to their unperturbed ones, and proved the sufficincy of these conditions using ITD variational comparison results.

Acknowledgement

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

References

[1] T. G. Bhaskar and J. V. Devi, Nonuniform stability and boundedness criteria for set differential equations, Applicable Analysis, 84(2), (2005), 131-143. doi: 10.1080/00036810410001724346
[2] T. G. Bhaskar and J. V. Devi, Stability criteria for set differential equations, Mathematical and Computer Modelling, 41(11-12), (2005) 1371-1378.
[3] F. Brauer and J. A. Nohel, The qualitative theory of ordinary differential equations: an introduction, Dover, NY, USA, 1989.
[4] M. Çiçek, C. Yakar and B. Oğur, Stability, boundedness, and Lagrange stability of fractional differential equations with initial time difference, The Scientific World Journal, Volume 2014, Article ID 939027, (2014), 1-7. doi: 10.1155/2014/939027
[5] C. Corduneanu, Functional equations with causal operators, CRC Press, 2002. doi: 10.1201/9780203166376
[6] J. V. Devi, Existence, uniqueness of solutions for set differential equations involving causal operators with memory, European Journal of Pure and Applied Mathematics, 3(4), (2010), 737-747.
[7] J. V. Devi, Comparison theorems and existence results for set causal operators with memory, Nonlinear Studies, 18(4), (2011), 603-610.
[8] J. V. Devi, Generalized monotone iterative technique for set differential equations involving causal operators with memory, International Journal of Advances in Engineering Sciences and Applied Mathematics, 3(1-4), (2011), 74–83. doi: 10.1007/s12572-011-0031-1
[9] J. V. Devi and Ch. A. Naidu, Stability results for set differential equations involving causal operators with memory, European Journal of Pure and Applied Mathematics, 5(2), (2012), 187-196.
[10] J. V. Devi and Ch. A. Naidu, Boundedness results for impulsive set differential equations involving causal operators with memory, Communications in Applied Analysis, 17(1), (2013), 9–19.
[11] Z. Drici, F. A. Mcrae and J. V. Devi, Stability results for set differential equations with causal maps, Dynamic Systems and Applications, 15(3), (2006), 451-464.
[12] M. B. Gücen and C. Yakar, Strict stability of fuzzy differential equations by Lyapunov functions, World Academy of Science, Engineering and Technology International Journal of Computer and Information Engineering, 12(5), (2018), 315-319. doi: 10.5281/zenodo.1316718
[13] V. Lakshmikantham, On the stability and boundedness of differential systems, Mathematical Proceedings of the Cambridge Philosophical Society, 58(3), (1962), 492-496. doi: 10.1017/S030500410003677X
[14] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, Theory of impulsive differential equations. World Scientific, Singapore, 1989. doi: 10.1142/0906
[15] V. Lakshmikantham, T. G. Bhaskar and J. V. Devi, Theory of set differential equations in metric spaces, Cottenham, Cambridge, Cambridge Scientific Publishers, 2006.
[16] V. Lakshmikantham and S. G. Deo, Method of variation of parameters for dynamic systems, Amsterdam, Netherlands, Gordon and Breach Science Publishers, 1998.
[17] V. Lakshmikantham and S. Leela, Differential and integral inequalities: theory and applications, New York, USA, Academic Press, 1969.
[18] V. Lakshmikantham, S. Leela and J. V. Devi, Stability theory for set differential equations, Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis, 11(2-3), (2004), 181-190.
[19] V. Lakshmikantham, S. Leela, Z. Drici and F. A. McRae, Theory of causal differential equations, Atlantis Studies in Mathematics for Engineering and Science, 2010. doi: 10.2991/978-94-91216-25-1
[20] V. Lakshmikantham, S. Leela and A. A. Martynyuk, Stability analysis of nonlinear systems, NY, USA: M. Dekker, 1989.
[21] V. Lakshmikantham, S. Leela and A. A. Martynyuk, Practical stability of nonlinear systems, World Scientific, Singapore, 1991. doi: 10.1142/1192
[22] V. Lakshmikantham, S. Leela and A. S. Vatsala, Setvalued hybrid differential equations and stability in terms of two measures, International Journal of Hybrid Systems, 2(2), (2002), 169-188.
[23] V. Lakshmikantham and X. Z. Liu, Stability analysis in terms of two measures, World Scientific, Singapore, 1993. doi: 10.1142/2018
[24] V. Lakshmikantham, V. M. Matrosov and S. Sivasundaram, Vector Lyapunov functions and stability analysis of nonlinear systems, Dordrecht, Boston, USA, Kluwer Academic Publishers, 1991.
[25] V. Lakshmikantham and M. Rama Mohana Rao, Theory of integro-differential equations, Lausanne, Switzerland, Gordon and Breach Science Publishers, 1995.
[26] V. Lakshmikantham and A. S. Vatsala, Theory of differential and integral inequalities with initial time difference and applications, In: Rassias T.M., Srivastava H.M. (eds) Analytic and Geometric Inequalities and Applications, Mathematics and Its Applications, Vol 478, Dordrecht, Netherlands, Springer, 1999. doi:10.1007/978-94-011-4577-0_12
[27] V. Lakshmikantham and A. S. Vatsala, Differential inequalities with initial time difference and applications, Journal of Inequalities and Applications, 3(3), (1999), 233-244. doi: 10.1155/S1025583499000156
[28] J. LaSalle and S. Lefschetz, Stability by Liapunov's direct methods with applications, Mathematics in Science and Engineering, V. 4. New York, Academic Press, 1961.
[29] X. Liu and M. D. Shaw, Boundedness in terms of two measures for perturbed systems by generalized variation of parameters, Communications in Applied Analysis, 5(4), (2001), 435-444.
[30] Ch. A. Naidu, Stability results for impulsive set differential equations involving causal operators with memory, Global Journal of Mathematical Sciences: Theory & Practical, 2(2), (2014), 49-53.
[31] Ch. A. Naidu, D. B. Dhaigude and J. V. Devi, Stability results in terms of two measures for set differential equations involving causal operators, European Journal of Pure and Applied Mathematics, 10(4), (2017), 645-654.
[32] M. D. Shaw and C. Yakar, Generalized variation of parameters with initial time difference and a comparison result in terms of Lyapunov-like functions, International Journal of Nonlinear Differential Equations Theory-Methods and Applications, 5(1-2), (1999), 86-108.
[33] M. D. Shaw and C. Yakar, Stability criteria and slowly growing motions with initial time difference, Problems of Nonlinear Analysis in Engineering Systems, 6, (2000), 50–66.
[34] H. Smith, An introduction to delay differential equations with applications to the life sciences, New York, USA, Springer, 2011. doi: 10.1007/978-1-4419-7646-8
[35] N. N. Tu and T. T. Tung, Stability of set differential equations and applications, Nonlinear Analysis: Theory, Methods & Applications, 71(5-6), (2009), 1526-1533. doi: 10.1016/j.na.2008.12.045
[36] C. Yakar, Boundedness criteria in terms of two measures with initial time difference, Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis, Watam Press. Waterloo, (2007), pp 270–275. DCDIS 14 (S2) 1–305.
[37] C. Yakar and M. Arslan, Terminal value problems with causal operators, Hacettepe Journal of Mathematics and Statistics, 47(4), (2018), 897-907. doi: 10.15672/HJMS.2018.566
[38] C. Yakar, B. Bal and A. Yakar, Monotone technique in terms of two monotone functions in finite system, Journal of Concrete and Applicable Mathematics, 9(3), (2011), 233-239.
[39] C. Yakar and M. Çiçek, Initial time difference boundedness criteria and Lagrange stability, Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis, 18(6), (2011), 797-811.
[40] C. Yakar and M. Çiçek, Theory, methods and applications of initial time difference, boundedness and Lagrange stability in terms of two measures for nonlinear systems, Hacettepe Journal of Mathematics and Statistics, 40(2), (2011), 305-330.
[41] C. Yakar, M. Çiçek and M. B. Gücen, Boundedness and Lagrange stability of fractional order perturbed system related to unperturbed systems with initial time difference in Caputo's sense, Advances in Difference Equations, 54, (2011), 1-14. doi: 10.1186/1687-1847-2011-54

[42] C. Yakar, M. Çiçek and M. B. Gücen, Practical stability, boundedness criteria and Lagrange stability of fuzzy differential systems, Computers & Mathematics with Applications, 64(6), (2012), 2118-2127. doi: 10.1016/j.camwa.2012.04.008
[43] C. Yakar and M. B. Gücen, Initial time difference stability of causal differential systems in terms of Lyapunov functions and Lyapunov functionals, Journal of Applied Mathematics, Volume 2014, Article ID 832015, (2014), 1-7. doi: 10.1155/2014/832015
[44] C. Yakar and M. D. Shaw, A comparison result and Lyapunov stability criteria with initial time difference, Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis, 12(6), (2005), 731-737.
[45] C. Yakar and M. D. Shaw, Initial time difference stability in terms of two measures and variational comparison result, Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis, 15(3), (2008), 417-425.
[46] C. Yakar and M. D. Shaw, Practical stability in terms of two measures with initial time difference, Nonlinear Analysis: Theory, Methods & Applications, 71(12), (2009), e781-e785. doi: 10.1016/j.na.2008.11.039
[47] C. Yakar and H. Talab, Stability of perturbed set differential equations involving causal operators in regard to their unperturbed ones considering difference in initial conditions, Advances in Mathematical Physics, Volume 2021, Article ID 9794959, (2021), 1-12. doi: 10.1155/2021/9794959.