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Generalized Qi's Integral Inequality

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Abstract. In [14], Qi presents an open problem and many authors tried to solve this problem. They made efforts to show the validity of solutions on what conditions [1], [10]-[15].

In this work, we generalized the Qi type inequalties which are derived from [11], [12] and [14].

Keywords: Integral inequalities

Genelleştirilmiş Qi İntegral Eşitsizliği

Özet. [14]' deki makelede, Qi açık bir problem vermiş ve birçok yazar bu problemi çözmeye uğraşmıştır. Yazarlar çözümün varlığının hangi koşullar altında sağlandığını göstermeye çalışmıştır. Bu çalışmada, çeşitli çalışmalardan elde edilen Qi tipli eşitsizlikler genelleştirilmiştir.

Anahtar Kelimeler: İntegral eşitsizlikleri

1. INTRODUCTION

Integral inequalities have been frequently employed in the theory of applied sciences, differential equations, and functional analysis. In the last two decades, they have been the focus of attention in [1]-[15]. Recently, especially Qi inequality, one of the integral inequalities, has been studied by many authors.

The following Qi inequality (1.1) has been obtained in [14]: Suppose that f has continuous $n-th$ order derivative on $[a,b]$, and $f^{(i)}(a)\geq 0$ and $f^{(n)}(x)\geq n!$; $0\leq i\leq n-1$, then the following inequality

$$\int_a^b f^{n+2}(x)dx \geq \left(\int_a^b f(x)dx\right)^{n+1} \tag{1.1}$$

holds for $x\in[a,b]$.

This inequality (1.1) posed the following open problem in [14].

What are the conditions of validity of the inequality

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$$\int_a^b f^t(x)dx \geq \left(\int_a^b f(x)dx\right)^{t-1} \tag{1.2}$$

for $t > 1$?

On the basis of this integral inequality an open problem (1.2) has been established and the conditions for its validity has been investigated in [12], [11]:

This open problem was studied by several authors and many valuable results have been established such as [12], [11].

It has been shown that the (1.2) inequality was valid when $f \in C^1[a,b]$, $f(a) \geq 0$, $f'(x) \geq (t-2)(x-a)^{t-3}$ for $x \in [a,b]$, and $t \geq 3$ in [12].

In [11], Ngŏ et al. gave the following inequality which is one of the open problem's solution.

Theorem 1 *Let* $f \in C[0,1]$ *and* $f(x) \geq 0$ *for every* $x \in [0,1]$*. If*

$$\int_x^1 f(t)dt \geq \frac{1-x^2}{2}, \ \forall x \in [0,1] \tag{1.3}$$

then, for every $n \in \mathbb{N}$

$$\int_0^1 f^{n+1}(x)dx \geq \int_0^1 x^n f(x)dx \tag{1.4}$$

holds.

In this paper, we will make a generalization benefiting from (1.1)-(1.4).

Now we prove the following auxiliary result which plays a key role in proving our main results.

Lemma 1 *Let* f *and* h *be a continuous function on* $[0,1]$*, and* $f(x) \geq h(x) \geq 0$*. Also let* $h(x)$ *be an increasing and positive monotone function on* $(0,1]$*, having a continuous derivative* $h'(x)$ *on* $(0,1)$*. The function* h *on* $[0,1]$ *is defined by [6]. If*

$$\int_x^1 f(t)h'(t)dt \geq \frac{h^2(1)-h^2(x)}{2} \geq 0, \ \forall x \in [0,1] \tag{1.5}$$

holds then we have following inequality for $h(0)=0$,

$$\int_0^1 f^2(x)h'(x)dx \geq \int_0^1 h(x)f(x)h'(x)dx. \tag{1.6}$$

Proof. By using hypothesis, we have

$$
\begin{aligned}
0 \le \int_0^1 (f(x)-h(x))^2 h'(x)dx &= \int_0^1 \left(f^2(x)-2h(x)f(x)+h^2(x)\right)h'(x)dx \\
&= \int_0^1 f^2(x)h'(x)dx - 2\int_0^1 h(x)f(x)h'(x)dx \\
&+ \int_0^1 h^2(x)h'(x)dx \\
&= \int_0^1 f^2(x)h'(x)dx - 2\int_0^1 h(x)f(x)h'(x)dx + \frac{h^3(1)}{3}.
\end{aligned}
$$

Where

$$
\int_0^1 f^2(x)h'(x)dx \ge 2\int_0^1 h(x)f(x)h'(x)dx - \frac{h^3(1)}{3}. \tag{1.7}
$$

Let $A := \int_0^1 (\int_x^1 f(t)dt)h'(x)dx$. By using our assumption we have

$$
A = \int_0^1 \left(\int_x^1 f(t)h'(t)dt\right)h'(x)dx = \int_0^1 h(x)f(x)h'(x)dx. \tag{1.8}
$$

Also from (1.5) and (1.8), we obtain

$$
\begin{aligned}
\int_0^1 h(x)f(x)h'(x)dx &= \int_0^1 \left(\int_x^1 f(t)h'(t)dt\right)h'(x)dx \\
&\ge \int_0^1 \left(\frac{h^2(1)-h^2(x)}{2}\right)h'(x)dx \qquad (1.9) \\
&= \frac{h^3(1)}{3}.
\end{aligned}
$$

By using (1.7) and (1.9) we also get

$$
\begin{aligned}
\int_0^1 f^2(x)h'(x)dx &\ge 2\int_0^1 h(x)f(x)h'(x)dx - \frac{h^3(1)}{3} \\
&\ge \int_0^1 h(x)f(x)h'(x)dx + \frac{h^3(1)}{3} - \frac{h^3(1)}{3} \\
&= \int_0^1 h(x)f(x)h'(x)dx,
\end{aligned}
$$

thus

$$
\int_0^1 f^2(x)h'(x)dx \ge \int_0^1 h(x)f(x)h'(x)dx
$$

which gives the conclusion.

Lemma 2 *Under the hypothesis of Lemma1, the following inequality holds*

$$\int_0^1 h^{n+1}(x) f(x) h'(x) dx \geq \frac{1}{n+3} h^{n+3}(1),$$

for all $n \in \mathbb{N}$.

Proof. We have

$$\int_0^1 h^n(x)\left(\int_x^1 f(t) h'(t) dt\right) h'(x) dx \tag{1.10}$$

In (1.10) , with partial integration we have

$$\begin{aligned}\int_0^1 h^n(x)\left(\int_x^1 f(t) h'(t) dt\right) h'(x) dx \quad &= \frac{h^{n+1}(x)}{n+1}\int_x^1 f(t) h'(t) dt\Big|_0^1 \\ &+ \frac{1}{n+1}\int_0^1 h^{n+1}(x) f(x) h'(x) dx\end{aligned}$$

which yields

$$(n+1)\int_0^1 h^n(x)\left(\int_x^1 f(t) h'(t) dt\right) h'(x) dx = \int_0^1 h^{n+1}(x) f(x) h'(x) dx.$$

By using (1.5),

$$\begin{aligned}\int_0^1 h^{n+1}(x) f(x) h'(x) dx \quad &= (n+1)\int_0^1 h^n(x)\left(\int_x^1 f(t) h'(t) dt\right) h'(x) dx \\ &\geq (n+1)\int_0^1 h^n(x)\left(\frac{h^2(1) - h^2(x)}{2}\right) h'(x) dx \\ &= \frac{1}{n+3} h^{n+3}(1)\end{aligned}$$

and thus we get the result.

Remark 1 *Taking* $h(x) = x$ *for* $Lemma\,1$ *and* $Lemma\,2$, *we get conclusions of [11].*

2. MAIN RESULTS

In this section, we will make a generelazition to [11] and [12] which is one of answer Qi's open problem.

Theorem 2 *Let* f *and* h *be a continuous function on* $[0,1]$, *and* $f(x) \geq h(x) \geq 0$. *Also let* $h(x)$ *be an increasing and positive monotone function on* $(0,1]$, *having a continuous derivative* $h'(x)$ *on* $(0,1)$. *Then the following inequality*

$$\int_0^1 f^{n+1}(x)h'(x)dx \geq \int_0^1 h^n(x)f(x)h'(x)dx, \; n \in \mathbb{N} \tag{2.1}$$

holds for $h(0) = 0$.

Proof. From Cauchy inequality, we have

$$\frac{1}{n+1} f^{n+1}(x) + \frac{n}{n+1} h^{n+1}(x) \geq f(x)h^n(x). \tag{2.2}$$

Multiplying both sides of inequality (2.2) by $h'(x)$ and integrating with respect to x from 0 to 1, we have

$$\int_0^1 f^{n+1}(x)h'(x)dx + n\int_0^1 h^{n+1}(x)h'(x)dx \geq (n+1)\int_0^1 f(x)h^n(x)h'(x)dx.$$

Moreover, by using Lemma 2, we get

$$\begin{aligned}\int_0^1 f^{n+1}(x)h'(x)dx + \frac{n}{n+2}h^{n+2}(1) &\geq (n+1)\int_0^1 f(x)h^n(x)h'(x)dx \\ &= n\int_0^1 f(x)h^n(x)h'(x)dx + \int_0^1 f(x)h^n(x)h'(x)dx \\ &\geq \frac{n}{n+2}h^{n+2}(1) + \int_0^1 f(x)h^n(x)h'(x)dx\end{aligned}$$

that is

$$\int_0^1 f^{n+1}(x)h'(x)dx \geq \int_0^1 h^n(x)f(x)h'(x)dx,$$

which completes this proof.

Theorem 3 *Under the hypothesis of Lemma1. Then*

$$\int_0^1 f^{n+1}(x)h'(x)dx \geq \int_0^1 f^n(x)h(x)h'(x)dx,$$

holds *for* every $n \in \mathbb{N}$.

Proof. From Lemma1's hypothesis, we can obtain $f(x) \ge h(x) \ge 0$, thus

$$(f^n(x) - h^n(x))(f(x) - h(x)) \ge 0, \ \forall x \in [0,1].$$

That is

$$f^{n+1}(x) + h^{n+1}(x) \ge f^n(x)h(x) + f(x)h^n(x), \ \forall x \in [0,1]. \tag{2.3}$$

Multiplying both sides of inequality (2.3) by $h^{'}(x)$ and integrating with respect to x from 0 to 1. With some simple calculation, we conclude that

$$\int_0^1 f^{n+1}(x)h^{'}(x)dx + \int_0^1 h^{n+1}(x)h^{'}(x)dx \ge \int_0^1 f^n(x)h(x)h^{'}(x)dx + \int_0^1 f(x)h^n(x)h^{'}(x)dx.$$

Once again, by Lemma2, we obtain

$$\int_0^1 f^{n+1}(x)h^{'}(x)dx + \frac{1}{n+2}h^{n+2}(1) \ge \frac{1}{n+2}h^{n+2}(1) + \int_0^1 f^n(x)h(x)h^{'}(x)dx,$$

thus

$$\int_0^1 f^{n+1}(x)h^{'}(x)dx \ge \int_0^1 f^n(x)h(x)h^{'}(x)dx$$

which gives the conclusion.

Remark 2 *By taking* $h(x) = x$ *for Theorem 2 and Theorem 3, we get conclusions of [11].*

Theorem 4 *Let* f *and* h *satisfy the hypothesis in Lemma1 by taking* $[a,b]$ *instead of interval* $[0,1]$. *If* $f \in C^1[(a,b)]$ *and* $f(a) \ge 0$, $f^{'}(x) \ge (t-2)(h(x) - h(a))^{t-3}h^{'}(x)$ *satisfy, then the following inequality holds*

$$\int_a^b f^t(x)h^{'}(x)dx \ge \left(\int_a^b f(x)h^{'}(x)dx\right)^{t-1} \tag{2.4}$$

for $x \in [a,b]$ and $t \ge 3$. Only when $a = b$ *or* $f(x) = x - a$ *and* $t = 3$ *the equality holds*.

Proof. Under conditions of Theorem4, f is increasing because $f^{'}(x) > 0$ for $x \in (a,b]$, and

$$f(\xi) \le f(x) \Rightarrow \int_a^x f(\xi)h^{'}(\xi)d\xi \le f(x)\int_a^x h^{'}(\xi)d\xi$$

for $\xi \in [a,x]$, then we have

$$f(x)(h(x)-h(a)) \geq \int_a^x f(\xi) h'(\xi) d\xi \tag{2.5}$$

for all $x \in [a,b]$.

Let us define a function

$$F(x) := \int_a^x f^t(\xi) h'(\xi) d\xi - \left(\int_a^x f(\xi) h'(\xi) d\xi \right)^{t-1} \geq 0.$$

Then we obtain $F(a) = 0$ and $F'(x) = f(x) h'(x) G(x)$, where

$$G(x) = f^{t-1}(x) - (t-1) \left(\int_a^x f(\xi) h'(\xi) d\xi \right)^{t-2}.$$

Taking the derivative of $G(x)$ with respect to x

$$\begin{aligned} G'(x) &= (t-1) f^{t-2}(x) f'(x) \\ &\quad - (t-1)(t-2) f(x) h'(x) \left(\int_a^x f(\xi) h'(\xi) d\xi \right)^{t-3} \\ &= (t-1) f(x) \left[f^{t-3}(x) f'(x) - (t-2) h'(x) \left(\int_a^x f(\xi) h'(\xi) d\xi \right)^{t-3} \right]. \end{aligned} \tag{2.6}$$

From (2.5) and hypothesis reduce to the following form:

$$\begin{aligned} f^{t-3}(x) f'(x) &\geq f^{t-3}(x)(t-2)(h(x)-h(a))^{t-3} h'(x) \\ &= (t-2)[f(x)(h(x)-h(a))]^{t-3} h'(x) \\ &\geq (t-2) \left(\int_a^x f(\xi) h'(\xi) d\xi \right)^{t-3} h'(x). \end{aligned} \tag{2.7}$$

Thus $G'(x) \geq 0$, so with $G(a) = f^{t-1}(a) \geq 0$ and clearly we can obtain $G(x) \geq 0$.

Since $F(a) = 0$ and $F'(x) = f(x) h'(x) G(x) \geq 0$ it follows that $F(x) \geq 0$ for all $x \in [a,b]$. For the case $x = b$, we obtain

$$F(b) = \int_a^b f^t(\xi) h'(\xi) d\xi - \left(\int_a^b f(\xi) h'(\xi) d\xi \right)^{t-1} \geq 0.$$

(2.4) equality holds only if $F^{'}(x)=0$ for all $x\in[a,b]$ which is equivalent to $f(a)=0$ and $G^{'}(x)=0.$ From (2.7), if $t>3$, this holds only for $f(a)=0$, where f^{t-3} and f constant on $[a,b]$. For $b\neq a$, it is not certain for the last two conditions to hold. If $f(a)=0$ and $t=3,$ then the other possible condition of equility holds. In that case (2.7) implies that $f^{'}(x)=1$ on $[a,b]$ so $f(x)=x-a.$

Remark 3 By taking $h(x)=x$ for *Theorem* 4, we get conclusions of [12].

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