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Research Article

Existence and extinction of solutions for parabolic equations with nonstandard growth nonlinearity

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Abstract

In this paper, we consider an initial boundary value problem for a class of $p(\cdot)$-Laplacian parabolic equation with nonstandard nonlinearity in a bounded domain. By using new approach, we obtain the global and decay of existence of the solutions. Moreover, the precise decay estimates of solutions before the occurrence of the extinction are derived.

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1. Introduction

In this paper, we discuss the global and decay of existence of solutions for the following parabolic equations involving the $p(\cdot)$

$$\begin{cases} u' - Lu = \lambda \left| u \right|^{q(x)-2} u \text{ in } \Omega \times (0,\infty), \\ u(x,t) = 0 \text{ on } \partial\Omega \times (0,\infty), \\ u(x,0) = u^0(x) \text{ in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with a smooth boundary $\partial\Omega$, $\lambda > 0$ is a real parameter, $u' = \partial u/\partial t$ and $Lu := -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left| \frac{\partial u}{\partial x_i} \right|^{p(x)-2} \frac{\partial u}{\partial x_i} \right)$ is the $p(x)$-Laplacian operator. Moreover, q is continuous and p is log-Hölder continuous (see [13]), that is, there exists a constant $C > 0$ such that, for any $x, y \in \overline{\Omega}$, we have

$$|p(x) - p(y)| \le \frac{C}{\log\left(e + |x-y|^{-1}\right)}. \tag{1.2}$$

Assume that

$$1 < p^- := \min_{\overline{\Omega}} p(\cdot) \le p(x) \le \max_{\overline{\Omega}} p(\cdot) := p^+ < +\infty. \tag{1.3}$$

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We denote by $P(\Omega)$ the set of all measurable real functions defined on Ω and $P_{\log}(\Omega)$ the set of all $p \in P(\Omega)$ satisfying the conditions (1.2) and (1.3).

The parabolic problem (1.1) can be regarded as the nonlinear diffusion equations which is well-known for the case of constant exponent $p(x) \equiv p$, where $u(x,t)$ represents density function and the diffusion coefficient depending on the gradient of density $\sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|^{p(\cdot)-2}$ by analogy with Fick's diffusion model. The equation of the type (1.1) also appears in the mathematical modeling of various physical phenomena such as the study of image restoration (see [12]) as well as in some model of electrorheological fluids (see [1,35]).

In mathematical point of view, equations of type (1.1) are usually referred to as equations with nonstandard growth conditions. Under certain conditions on the initial data and certain ranges of exponents, the existence, uniqueness and other qualitative properties of solutions for both elliptic and parabolic equations with variable nonlinearity have been studied by many authors (see [2,4–6,9,10,16,17,19,21,26,27,30,37] and references therein).

In the case $p(\cdot)$ and $q(\cdot)$ are constants, the problem (1.1) has been studied by many mathematicians. Such as Fujita [15], Kaplan [20], Komornik [22], Levine [24], Payne and Sattinger [32] and Ragusa [34]. Similar studies in case stationary studied by some authors (see [8,11,28,38,39]).

For the case of nonstandard growths, by using the Kaplan's method, Pinasco [33] established the global existence and nonexistence results for problem (1.1) in the case $p(\cdot) = 2$ and $q(\cdot)$ is a function. Antontsev and Shmarev [3] studied the property of existence and uniqueness of weak solutions in suitable Orlicz-Sobolev spaces, derive global and local in time L^{∞} bounds for the weak solutions.

Guo, Li and Gao in [18] considered the following $p(\cdot)$-parabolic problem:

$$u_t = \operatorname{div}(|\nabla u|^{p(x)-2}\, \nabla u) + |u|^{r-2}\, u, \;\; (x,t) \in \Omega \times (0,T),$$

subject to homogeneous Dirichlet boundary condition on $\partial\Omega \times (0,T)$, where constant exponent $r > 1$. In this paper the authors improved the regularity of weak solutions, and then proved that the weak solutions blow up in finite time for some positive initial energy and vanish in finite time by using the energy estimate method.

Antontsev, Chipot and Shmarev in [6] studied the homogeneous Dirichlet boundary problem for the doubly nonlinear parabolic equation with nonstandard growth conditions,

$$u_t = \operatorname{div}(a(x,t,u)\, |u|^{\alpha(x,t)}\, |\nabla u|^{p(x,t)-2}\, \nabla u) + f(x,t), \;\; (x,t) \in \Omega \times (0,T).$$

They established conditions on the data which guarantee the comparison principle and the uniqueness of bounded weak solutions in suitable function spaces of Orlicz-Sobolev type subject to some additional restrictions but under weaker conditions on the existence of weak solutions. In [31], Nhan, Chuong, Truong established the global existence and nonexistence results for problem (1.1). In the first case, they also showed the functional impairment properties of energy. Finally, they achieved the results of non-global assets with initial high-energy data.

In [29], Lourêdo et al considered the following $p(\cdot)$-parabolic problem:

$$\begin{cases} u' - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p(x)-2} \frac{\partial u}{\partial x_i}\right) + |u|^{\sigma(x)} & = 0 \text{ in } \Omega \times (0,\infty), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u^0(x) & \text{in } \Omega, \end{cases}$$

where $\sigma^- > 1$ and $2 \le p^- \le p(x) \le p^+ < \infty$ with $p^+ < \sigma^- + 1 \le \sigma^+ + 1 < Np(x)/(N - p(x))$, $\forall x \in \overline{\Omega}$ and $(p^+ - p^-)\, N < p^+ p^-$. Because the nonlinear perturbation leads to difficulties in obtaining a priori estimates in the energy method, the authors had to significantly modify the Tartar method. As a result, they could prove the existence of global solutions at least for small initial data.

In our this article, we use a new approach to prove the existence of a global weak solution and decay of existence of the solutions of the initial-boundary-value problem (1.1) with a restriction on the initial data u_0. Moreover, the precise decay estimates of solutions before the occurrence of the extinction are derived.

Let $q \in C_+(\overline{\Omega}) := \left\{ q \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} q(x) > 1 \right\}$. We define the Lebesgue space with variable exponent as

$$L^{q(\cdot)}(\Omega) := \left\{ u : u \in P(\Omega), \int_\Omega |u(x)|^{q(x)}\, dx < \infty \right\}.$$

The set $L^{q(\cdot)}(\Omega)$, equipped with the Luxemburg norm

$$\|u\|_{L^{q(\cdot)}(\Omega)} := \|u\|_{q(\cdot)} = \inf \left\{ \gamma > 0 : \int_\Omega \left| \frac{u(x)}{\gamma} \right|^{q(x)} dx \leq 1 \right\},$$

is a Banach space. The modular of $L^{q(\cdot)}(\Omega)$, which is the mapping $\rho_{q(\cdot)}(u) : L^{q(\cdot)}(\Omega) \to \mathbb{R}$, defined by

$$\rho_{q(\cdot)}(u) := \int_\Omega |u(x)|^{q(x)}\, dx.$$

For $p \in C_+(\overline{\Omega})$, we define the Sobolev space with variable exponent, $W^{1,p(\cdot)}(\Omega)$, as the space of functions $u \in L^{p(\cdot)}(\Omega)$, such that $\frac{\partial u}{\partial x_i} \in L^{p(\cdot)}(\Omega)$, $i = 1, ..., N$, equipped with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{p(\cdot)}, \ \forall u \in W^{1,p(\cdot)}(\Omega).$$

We denote $W_0^{1,p(\cdot)}(\Omega) := \overline{C_0^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)}$. Furthermore, for all $u \in W_0^{1,p(\cdot)}(\Omega)$, we can define an equivalent norm $\|u\|_{W_0^{1,p(\cdot)}(\Omega)}$ such that

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{p(\cdot)},$$

since Ω is bounded.

According to the characterization of linear and continuous functionals on $W_0^{1,p(\cdot)}(\Omega)$ given by [13], we deduce from this, that for all $u \in W_0^{1,p(\cdot)}(\Omega)$ and $\left|\frac{\partial u}{\partial x_i}\right|^{p(x)-2} \frac{\partial u}{\partial x_i} \in L^{p'(\cdot)}(\Omega)$, $i = 1, 2, ..., N$, the operator

$$u \to Lu = \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p(x)-2} \frac{\partial u}{\partial x_i} \right)$$

is well defined from $W_0^{1,p(\cdot)}(\Omega)$ into its dual $\left(W_0^{1,p(\cdot)}(\Omega)\right)^* := W_0^{-1,p'(\cdot)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1, \forall x \in \overline{\Omega}$ and defined by

$$\langle Lu, v\rangle = \sum_{i=1}^N \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p(x)-2} \frac{\partial u}{\partial x_i} . \frac{\partial v}{\partial x_i} dx$$

for all $v \in W_0^{1,p(\cdot)}(\Omega)$.

The operator L takes from bounded subsets of $W_0^{1,p(\cdot)}(\Omega)$ into bounded subsets of $W_0^{-1,p'(\cdot)}(\Omega)$. It is known that L is monotone and hemicontinuous. Moreover, it is well known that if $1 < q^- \leq q^+ < \infty$, $1 < p^- \leq p^+ < \infty$, then the spaces $\left(L^{q(\cdot)}(\Omega), \|\cdot\|_{q(\cdot)}\right)$, $\left(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)}\right)$ and $\left(W_0^{1,p(\cdot)}(\Omega), \|\cdot\|_{W_0^{1,p(\cdot)}(\Omega)}\right)$ are separable and reflexive Banach spaces. We refer to [13, 23] for further properties of variable exponent Lebesgue-Sobolev spaces.

Proposition 1.1. (see [13,23]) *Let $u \in L^{q(\cdot)}(\Omega)$. Then the following statements are true:*

(*i*) *if* $\|u\|_{q(\cdot)} > 1$, *then* $\|u\|_{q(\cdot)}^{q-} \le \rho_{q(\cdot)}(u) \le \|u\|_{q(\cdot)}^{q+}$;

(*ii*) *if* $\|u\|_{q(\cdot)} \le 1$, *then* $\|u\|_{q(\cdot)}^{q+} \le \rho_{q(\cdot)}(u) \le \|u\|_{q(\cdot)}^{q-}$;

(*iii*) $\min\left\{\|u\|_{q(\cdot)}^{q^-}, \|u\|_{q(\cdot)}^{q^+}\right\} \le \rho_{q(\cdot)}(u) \le \max\left\{\|u\|_{q(\cdot)}^{q^-}, \|u\|_{q(\cdot)}^{q^+}\right\}$.

Proposition 1.2. (Hölder-type inequality, see [13,23]) *Let $h \in L_+^{\infty}(\Omega)$. (i) The conjugate space to $L^{h(\cdot)}(\Omega)$ is $L^{h'(\cdot)}(\Omega)$, where $1/h(x) + 1/h'(x) = 1$ for almost every (a.e.) $x \in \Omega$. Moreover, the following inequality hold*

$$\left|\int_\Omega u(x)v(x)dx\right| \le 2\|u\|_{h(\cdot)}\|v\|_{h'(\cdot)}$$

for all $u \in L^{h(\cdot)}(\Omega)$ and $v \in L^{h'(\cdot)}(\Omega)$.

(ii) If $p_1, p_2 \in C_+(\overline{\Omega})$ and $p_1(x) \le p_2(x)$ for any $x \in \overline{\Omega}$, then there exists the continuous embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$.

Proposition 1.3. (see [13, 14, 23]) *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $p \in P_{\log}(\Omega)$. Let $q : \Omega \to [1, +\infty)$ be a measurable and bounded function and suppose that $q(x) \le p^*(x) = Np(x)/(N - p(x))_+$ for a.e. $x \in \Omega$. Then $W^{1,p(\cdot)}(\Omega)$ is continuously embedded in $L^{q(\cdot)}(\Omega)$. In addition, assume that $\operatorname*{ess\,inf}_{x\in\Omega}\{p^*(x) - q(x)\} > 0$. Then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact.*

In particular, if $p^- \ge \frac{2N}{N+2}$, then there exists a positive constant S such that

$$\|u\|_2 \le S\|u\|_{W_0^{1,p(\cdot)}(\Omega)}, \ \forall u \in W_0^{1,p(\cdot)}(\Omega). \tag{1.4}$$

We introduce the following auxiliary lemma:

Lemma 1.4. *Assume that $2 \le p^- \le p^+ < q^- \le q^+$ and $\xi_1, ..., \xi_N \ge 0$ with $\sum_{i=1}^N \xi_i < 1$ hold. Then, we have*

$$\left(\sum_{i=1}^N \xi_i^{p^-}\right)^{\frac{1}{2}} \le \sum_{i=1}^N \xi_i \le N^{\frac{q^- -1}{q^-}}\left(\sum_{i=1}^N \xi_i^{p^+}\right)^{\frac{1}{q^-}}, \tag{1.5}$$

and

$$\left(\sum_{i=1}^N \xi_i^{p^+}\right)^{\frac{1}{p^-}} \le \sum_{i=1}^N \xi_i \le N^{\frac{p^+ -1}{p^+}}\left(\sum_{i=1}^N \xi_i^{p^-}\right)^{\frac{1}{p^+}}. \tag{1.6}$$

Proof. For $\xi_i < 1$ $(i = 1, ..., N)$, we have

$$\left(\xi_i^{p^-}\right)^{\frac{1}{2}} = \xi_i^{\frac{p^-}{2}} \le \xi_i \le \xi_i^{\frac{p^+}{q^-}} = \left(\xi_i^{p^+}\right)^{\frac{1}{q^-}}. \tag{1.7}$$

Summing (1.7) over i, we easily get

$$\sum_{i=1}^N \xi_i \ge \sum_{i=1}^N \left(\xi_i^{p^-}\right)^{\frac{1}{2}} \ge \left(\sum_{i=1}^N \xi_i^{p^-}\right)^{\frac{1}{2}},$$

and

$$\sum_{i=1}^N \xi_i \le \sum_{i=1}^N \left(\xi_i^{p^+}\right)^{\frac{1}{q^-}}.$$

Since $\varphi(t) = t^{q^-}$ is convex, by applying Jensen's inequality we have

$$\left(\frac{\sum_{i=1}^N \left(\xi_i^{p^+}\right)^{\frac{1}{q^-}}}{N}\right)^{q^-} = \varphi\left(\frac{\sum_{i=1}^N \left(\xi_i^{p^+}\right)^{\frac{1}{q^-}}}{N}\right) \le \frac{\sum_{i=1}^N \varphi\left(\xi_i^{p^+}\right)^{\frac{1}{q^-}}}{N}.$$

Then

$$\frac{\sum_{i=1}^{N}\left(\xi_i^{p^+}\right)^{\frac{1}{q^-}}}{N} \le \left(\frac{\sum_{i=1}^{N}\xi_i^{p^+}}{N}\right)^{\frac{1}{q^-}}.$$

That is

$$\sum_{i=1}^{N}\left(\xi_i^{p^+}\right)^{\frac{1}{q^-}} \le N^{\frac{q^--1}{q^-}}\left(\sum_{i=1}^{N}\xi_i^{p^+}\right)^{\frac{1}{q^-}}.$$

Thus

$$\sum_{i=1}^{N}\xi_i \le \sum_{i=1}^{N}\left(\xi_i^{p^+}\right)^{\frac{1}{q^-}} \le N^{\frac{q^--1}{q^-}}\left(\sum_{i=1}^{N}\xi_i^{p^+}\right)^{\frac{1}{q^-}}.$$

Similarly we obtain

$$\left(\xi_i^{p^+}\right)^{\frac{1}{p^-}} = \xi_i^{\frac{p^+}{p^-}} \le \xi_i \le \xi_i^{\frac{p^-}{p^+}} = \left(\xi_i^{p^-}\right)^{\frac{1}{p^+}}. \tag{1.8}$$

Summing (1.8) over i, we have

$$\sum_{i=1}^{N}\xi_i \ge \sum_{i=1}^{N}\left(\xi_i^{p^+}\right)^{\frac{1}{p^-}} \ge \left(\sum_{i=1}^{N}\xi_i^{p^+}\right)^{\frac{1}{p^-}},$$

and

$$\sum_{i=1}^{N}\xi_i \le \sum_{i=1}^{N}\left(\xi_i^{p^-}\right)^{\frac{1}{p^+}} \le N^{\frac{p^+-1}{p^+}}\left(\sum_{i=1}^{N}\xi_i^{p^-}\right)^{\frac{1}{p^+}}.$$

Thus the proof of Lemma 1.4 is complete. □

For weak solution of problem (1.1), we have the following definition.

Definition 1.5. We define a function $u \in L^\infty(0,T;W_0^{1,p(\cdot)}(\Omega)) \cap C([0,T], L^2(\Omega))$ with $u' \in L^2(0,T;L^2(\Omega))$ to be a weak solution of problem (1.1), if it satisfies the initial condition $u(\cdot,0) := u^0 \in W_0^{1,p(\cdot)}(\Omega)$, $\lambda > 0$ and

$$(u', \upsilon) + \sum_{i=1}^{N}\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p(x)-2}\frac{\partial u}{\partial x_i}.\frac{\partial \upsilon}{\partial x_i}dx = \left(\lambda\,|u|^{q(x)-2}\,u, \upsilon\right)$$

for all $\upsilon \in W_0^{1,p(\cdot)}(\Omega)$, and for a.e. $t \in (0,T)$.

2. Main result

We assume that p^-, p^+, q^-, q^+ satisfy

$$2 \le\ p^- \le p(x) \le p^+ < q^- \le q(x) \le q^+ < p^*(x), \forall x \in \overline{\Omega}, \tag{2.1}$$

if $p(x) < N$, for all $x \in \overline{\Omega}$; and that q satisfies $q(x) > 1$ if $p(x) > N$, for all $x \in \overline{\Omega}$, and

$$\left(p^+ - p^-\right)N < p^+p^-. \tag{2.2}$$

By using (2.1), (2.2), Proposition 1.2(ii) and Proposition 1.3, we obtain

$$W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q^+}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega) \hookrightarrow L^{q^-}(\Omega) \hookrightarrow L^2(\Omega). \tag{2.3}$$

In order to simplify the notations, we denote the space $W_0^{1,p(\cdot)}(\Omega)$ by X_0. By (2.3) there exists a constant $K_1 > 0$ such that

$$\|u\|_{q(\cdot)} \le K_1 \|u\|_{X_0},\ \forall u \in X_0. \tag{2.4}$$

We further, set

$$K = \max\{1, K_1\}, \tag{2.5}$$

where K_1 is the embedding constant of the (2.4).

Now, our main results can be stated as follows.

Theorem 2.1. (Global solutions) *Assume that hypotheses* $p \in P_{\log}(\Omega)$, $q \in C_+(\overline{\Omega})$, $0 < \lambda < \lambda^*$, (2.1) and (2.2) *hold. If* $u^0 \in X_0$ *satisfies*

$$\left\|u^0\right\|_{X_0} < \delta_0 \leq 1, \tag{2.6}$$

and

$$\left\|u^0\right\|_{X_0}^2 + \left\|u^0\right\|_{X_0}^{q^-} < \left(\frac{p^-}{N^{q^- -1}p^+} - \frac{2\lambda p^- K^{q^+}}{q^-}\right)\delta_0^{q^-}, \tag{2.7}$$

where $\lambda^* = \frac{q^-}{2N^{q^- -1}K^{q^-}p^+}$ *and* K *is a constant given in* (2.5), *then there exists a function* $u \in L^\infty(0,\infty;X_0)$ *with* $u' \in L^2(0,\infty;L^2(\Omega))$ *that satisfies*

$$u' - \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p(.)-2}\frac{\partial u}{\partial x_i}\right) = \lambda\,|u|^{q(x)-2}\,u \ \text{in}\ L^2_{loc}(0,\infty;X_0^*), \tag{2.8}$$

and $u(0) = u^0$ *in* Ω.

Theorem 2.2. (Decay estimates) *Let* u *be the solution given by Theorem 2.1. Suppose that* $p \in P_{\log}(\Omega)$, $q \in C_+(\overline{\Omega})$, $0 < \lambda < \lambda^{**}$ *and* (2.1), (2.2), (2.6), (2.7) *hold. Then*

$$\|u(t)\|_2 \leq \left[\left\|u^0\right\|_2^{2-q^-} + \frac{q^- - 2}{2}\eta t\right]^{\frac{1}{2-q^-}}, \tag{2.9}$$

where $\eta = \frac{2}{N^{q^- -1}S^{q^-}} - \frac{4\lambda K^{q^-}}{S^{q^-}} > 0$, $\lambda^{**} = \frac{1}{2N^{q^- -1}K^{q^-}}$ *and* S, K *are constants given in* (1.4) *and* (2.5) *respectively.*

The following Theorem gives us exact decay estimates solutions to extinction.

Theorem 2.3. (Extinction of solutions) *Assume that hypotheses* $p \in P_{\log}(\Omega)$, $q \in C_+(\overline{\Omega})$, (2.6) *and the following condition*

$$\frac{2N}{N+2} < p^- \leq p^+ < q^- \leq q^+ < 2$$

hold. If $0 < \lambda < \lambda_0^*$, *then the non-negative weak solution of problem* (1.1) *vanishes in finite time for any initial data* u_0 *where*

$$\lambda_0^* = \frac{1}{2S^{p^+}\left(|\Omega|+1\right)^{(2-q^-)/2}\max\left\{\|u^0\|_2^{q^- -p^+}, \|u^0\|_2^{q^+ -p^+}\right\}}.$$

More precisely speaking, we have the following estimates

$$\begin{cases} \|u(t)\|_2 \leq \left(\|u^0\|_2^{2-p^+} - \frac{\beta_0\left(2-p^+\right)}{2}t\right)^{\frac{1}{2-p^+}}, \ \forall t \in (0,T_0), \\ \|u(t)\|_2 \equiv 0, \ \forall t \in [T_0,+\infty), \end{cases}$$

where

$$\beta_0 = 2S^{-p^+} - 4\lambda\left(|\Omega|+1\right)^{\left(2-q^-\right)/2}\max\left\{\left\|u^0\right\|_2^{q^- -p^+}, \left\|u^0\right\|_2^{q^+ -p^+}\right\},$$

and

$$T_0 = \frac{2\left\|u^0\right\|_2^{2-p^+}}{\left(2-p^+\right)\beta_0},$$

and S *is constant given in* (1.4).

3. Global existence

We employ the Galerkin's method to obtain the global existence of problem (1.1). Consider a Schauder basis $\{\omega_1, \omega_2, ..., \omega_N, ...\}$ of X_0. Let u_m be an approximate solution of problem (1.1) defined by

$$u_m(x,t) = \sum_{j=1}^{m} g_{jm}(t)\,\omega_j(x),\ m = 1, 2, ...,$$

where the coefficients $g_{jm}(t) \in C^1[0,T]$ $(1 \le j \le m)$ and satisfying

$$\begin{aligned}\langle u'_m(t), \upsilon\rangle + \sum_{i=1}^{N}\int_\Omega \left|\frac{\partial u_m(t)}{\partial x_i}\right|^{p(x)-2} \frac{\partial u_m(t)}{\partial x_i}\cdot\frac{\partial \upsilon}{\partial x_i}dx \\ = \lambda\int_\Omega |u_m(t)|^{q(x)-2} u_m(t)\,\upsilon dx \qquad (3.1)\end{aligned}$$

for all $\upsilon \in V_m = span\{\omega_1, \omega_2, ..., \omega_m\}$, $\lambda > 0$. The subspace of dimension m of X_0 generated by $\omega_1, \omega_2, ..., \omega_m$ and $\omega_m(0) := \omega_m^0 \in V_m$,

$$u_m(x,0) = \sum_{j=1}^{m} g_{jm}(0)\,\omega_j(x) \to u^0(x) \text{ strongly in } X_0$$

as $m \to +\infty$.

By (2.6) and (2.7), we obtain

$$\left\|u_m^0\right\|_{X_0} < \delta_0 \le 1,\ \forall m \ge m_0,$$

and

$$\frac{1}{p^-}\left\|u_m^0\right\|_{X_0}^2 + \frac{1}{p^-}\left\|u_m^0\right\|_{X_0}^{p^-} < \left(\frac{1}{N^{q^- -1}p^+} - \frac{2\lambda K^{q^+}}{q^-}\right)\delta_0^{q^-}.$$

Fixing m such that $m \ge m_0$, we have the folowing estimate:

Lemma 3.1. *Assume that hypotheses $p \in P_{\log}(\Omega)$, $q \in C_+(\overline{\Omega})$,* (2.1) *and* (2.2) *hold. Suppose that u^0 and δ_0 satisfy the conditions* (2.6) *and* (2.7) *of Theorem 2.1. Then, we have $\|u_m(t)\|_{X_0} < \delta_0$, $\forall t \in [0, +\infty)$.*

Proof. We argue by contradiction. In fact, suppose there exists $m_0 \in \mathbb{N}$ and that there exists $t_1 \in (0, t_m)$ such that $\|u_m(t_1)\|_{X_0} \ge \delta_0$. Consider the subset σ of $(0, t_m)$ defined by:

$$\Re = \left\{\sigma \in (0, t_m) : \|u_m(\sigma)\|_{X_0} \ge \delta_0\right\}, \qquad (3.2)$$

and $\inf_{\sigma\in\Re}\sigma = t_0$. Then we have $\|u_m(t_0)\|_{X_0} = \delta_0$ and $t_0 > 0$. $\Re$ is not empty, because of (3.2). It is a closed set because the function $\psi(t) := \|u_m(t)\|_{X_0}$ is continuous on $[0, t_m)$. In fact, the function ψ is continuous on $[0, t_m)$ then $\psi(t_0) \ge \delta_0$. If $\psi(t_0) > \delta_0$, the Intermediate Value Theorem and noting that $\psi(0) < \delta_0$, imply that t_0 is not the infimum on $\Re$, which is a contradiction. Thus $\psi(t_0) = \delta_0$. Also $t_0 > 0$ because $\psi(0) < \delta_0$. Note that $\psi(t) < \delta_0$ for all $0 \le t < t_0$.

Consider $0 \le t < t_0$ and $\upsilon = u'_m$ in (3.1), we have

$$\begin{aligned}\|u'_m(\tau)\|_2^2 + \sum_{i=1}^{N}\int_\Omega \left|\frac{\partial u_m(t)}{\partial x_i}\right|^{p(x)-2} \frac{\partial u_m(t)}{\partial x_i}\frac{\partial u'_m(t)}{\partial x_i}dx \\ = \lambda\int_\Omega |u_m(t)|^{q(x)-2} u_m(t)u'_m(t)\,dx.\end{aligned}$$

It follows

$$\|u'_m(\tau)\|_2^2 + \sum_{i=1}^{N}\frac{d}{dt}\int_\Omega \frac{1}{p(x)}\left|\frac{\partial u_m(t)}{\partial x_i}\right|^{p(x)} dx = \lambda\frac{d}{dt}\int_\Omega \frac{|u_m(t)|^{q(x)}}{q(x)}dx,$$

and integrating with respect to t from 0 to t, we obtain

$$\begin{aligned} &\int_0^t \left\|u_m'(\tau)\right\|_2^2 d\tau + \sum_{i=1}^N \int_\Omega \frac{1}{p(x)} \left|\frac{\partial u_m(t)}{\partial x_i}\right|^{p(x)} dx \\ = &\sum_{i=1}^N \int_\Omega \frac{1}{p(x)} \left|\frac{\partial u_m^0}{\partial x_i}\right|^{p(x)} dx + \lambda \int_\Omega \frac{|u_m(t)|^{q(x)}}{q(x)} dx \\ &-\lambda \int_\Omega \frac{1}{q(x)} \left|u_m^0\right|^{q(x)} dx. \end{aligned} \tag{3.3}$$

It follows from (3.3) that

$$\begin{aligned} &\int_0^t \left\|u_m'(\tau)\right\|_2^2 d\tau + \frac{1}{p^+} \sum_{i=1}^N \int_\Omega \left|\frac{\partial u_m(t)}{\partial x_i}\right|^{p(x)} dx \\ \leq &\frac{1}{p^-} \sum_{i=1}^N \int_\Omega \left|\frac{\partial u_m^0}{\partial x_i}\right|^{p(x)} dx + \frac{\lambda}{q^-} \int_\Omega |u_m(t)|^{q(x)} dx. \end{aligned} \tag{3.4}$$

By using (2.4), Proposition 1.1 (iii), since $\|u_m(t)\|_{X_0} < 1$ for $0 \leq t < t_0$, we have

$$\begin{aligned} &\frac{\lambda}{q^-} \int_\Omega |u_m(t)|^{q(x)} dx \\ \leq &\frac{\lambda}{q^-} \|u_m(t)\|_{q(\cdot)}^{q^-} + \frac{\lambda}{q^-} \|u_m(t)\|_{q(\cdot)}^{q^+} \\ \leq &\frac{\lambda K^{q^-}}{q^-} \|u_m(t)\|_{X_0}^{q^-} + \frac{\lambda K^{q^+}}{q^-} \|u_m(t)\|_{X_0}^{q^+} \\ \leq &\frac{2\lambda K^{q^+}}{q^-} \|u_m(t)\|_{X_0}^{q^-}. \end{aligned} \tag{3.5}$$

Therefore, it follows from Proposition 1.1 (ii) that

$$\frac{1}{p^+} \sum_{i=1}^N \int_\Omega \left|\frac{\partial u_m(t)}{\partial x_i}\right|^{p(x)} dx \geq \frac{1}{p^+} \sum_{i=1}^N \left\|\frac{\partial u_m(t)}{\partial x_i}\right\|_{p(\cdot)}^{p^+}.$$

By (1.5), we have

$$\begin{aligned} \frac{1}{p^+ N^{q^- -1}} \|u_m(t)\|_{X_0}^{q^-} &\leq \frac{1}{p^+} \sum_{i=1}^N \left\|\frac{\partial u_m(t)}{\partial x_i}\right\|_{p(\cdot)}^{p^+} \\ &\leq \frac{1}{p^+} \sum_{i=1}^N \int_\Omega \left|\frac{\partial u_m(t)}{\partial x_i}\right|^{p(x)} dx, \end{aligned} \tag{3.6}$$

and from (1.5), (1.6), we have

$$\begin{aligned} &\frac{1}{p^-} \sum_{i=1}^N \int_\Omega \left|\frac{\partial u_m^0}{\partial x_i}\right|^{p(x)} dx \\ \leq &\frac{1}{p^-} \sum_{i=1}^N \left\|\frac{\partial u_m^0}{\partial x_i}\right\|_{p(\cdot)}^{p^-} + \frac{1}{p^-} \sum_{i=1}^N \left\|\frac{\partial u^0}{\partial x_i}\right\|_{p(\cdot)}^{p^+} \\ \leq &\frac{1}{p^-} \left\|u_m^0\right\|_{X_0}^2 + \frac{1}{p^-} \left\|u_m^0\right\|_{X_0}^{p^-}. \end{aligned} \tag{3.7}$$

Plugging (3.5), (3.6) and (3.7) into (3.4), we obtain

$$\begin{aligned}&\int_0^t \|u_m'(\tau)\|_2^2\,d\tau + \left(\frac{1}{N^{q^- -1}p^+} - \frac{2\lambda K^{q^+}}{q^-}\right)\|u_m(t)\|_{X_0}^{q^-}\\ \le\ & \frac{1}{p^-}\left\|u_m^0\right\|_{X_0}^2 + \frac{1}{p^-}\left\|u_m^0\right\|_{X_0}^{p^-}\end{aligned} \tag{3.8}$$

for all $0 \le t < t_0$. By (2.7) and (3.8), we obtain

$$\frac{1}{p^-}\left\|u_m^0\right\|_{X_0}^2 + \frac{1}{p^-}\left\|u_m^0\right\|_{X_0}^{p^-} < \left(\frac{1}{N^{q^- -1}p^+} - \frac{2\lambda K^{q^+}}{q^-}\right)\delta_0^{q^-}.$$

Therefore,

$$\begin{aligned}\left(\frac{1}{N^{q^- -1}p^+} - \frac{2\lambda K^{q^+}}{q^-}\right)\|u_m(t)\|_{X_0}^{q^-} &< \frac{1}{p^-}\left\|u_m^0\right\|_{X_0}^2 + \frac{1}{p^-}\left\|u_m^0\right\|_{X_0}^{p^-}\\ &< r < \left(\frac{1}{N^{q^- -1}p^+} - \frac{2\lambda K^{q^+}}{q^-}\right)\delta_0^{q^-}\end{aligned}$$

for some $r \in \mathbb{R}$. Taking the limit $t \to t_0$, $t < t_0$, in the above inequality, we obtain

$$\left(\frac{1}{N^{q^- -1}p^+} - \frac{2\lambda K^{q^+}}{q^-}\right)\|u_m(t_0)\|_{X_0}^{q^-} \le r < \left(\frac{1}{N^{q^- -1}p^+} - \frac{2\lambda K^{q^+}}{q^-}\right)\delta_0^{q^-}$$

which is a contradiction because $\|u_m(t_0)\|_{X_0} = \delta_0$. Thus the Lemma 3.1 is proved. $\square$

Proof of Theorem 2.1. By Lemma 3.1, (3.8) and properties of operator L, there exist u, χ and a subsequence of $\{u_m\}$ (still denoted by $\{u_m\}$), such that, as $m \to \infty$,

$$u_m \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,\infty;X_0), \tag{3.9}$$

$$u_m' \rightharpoonup u' \text{ in } L^2(0,\infty;L^2(\Omega)), \tag{3.10}$$

and

$$Lu_m \overset{*}{\rightharpoonup} \chi \text{ in } L^\infty(0,\infty;X_0^*).$$

The next step is to prove that $\chi = Lu$ and for that we need to show that

$$\int_0^T\int_\Omega |u_m|^{q(x)}\,dxdt \to \int_0^T\int_\Omega |u|^{q(x)}\,dxdt,\ \forall T>0. \tag{3.11}$$

By compactness $X_0 \hookrightarrow\hookrightarrow L^{q^+}(\Omega)$ and Aubin-Lions-Simon Lemma (see [25], Corollary 6 in [36]) and convergences (3.9) and (3.10), we find

$$u_m \to u \text{ in } C([0,T];L^{q^+}(\Omega)).$$

So,

$$u_m \rightharpoonup u \text{ in } L^{q^+}(Q_T), \tag{3.12}$$

where $\Omega\times(0,T) := Q_T$, and

$$u_m(x,t) \to u(x,t) \text{ a.e. in } Q_T.$$

This implies

$$|u_m|^{q(x)-2}u_m \to |u|^{q(x)-2}\,u \text{ a.e. in } Q_T,\ \forall T>0. \tag{3.13}$$

By (3.12), we have

$$\begin{aligned}
& \int_{Q_T} \left| |u_m(x,t)|^{q(x)-2} u_m(x,t) \right|^{\frac{q^+}{q^+-1}} dxdt \\
\leq & \int_{Q_T} \left(|u_m(x,t)|^{q(x)-1} \right)^{\frac{q^+}{q^+-1}} dxdt \\
\leq & \int_{\{x\in Q_T : |u_m(x,t)|\leq 1\}} \left(|u_m(x,t)|^{q(x)-1} \right)^{\frac{q^+}{q^+-1}} dxdt \\
& + \int_{\{x\in Q_T : |u_m(x,t)| > 1\}} \left(|u_m(x,t)|^{q(x)-1} \right)^{\frac{q^+}{q^+-1}} dxdt \\
\leq & \; T|\Omega| + \int_{Q_T} |u_m(x,t)|^{q^+} dxdt \leq C,
\end{aligned}$$

that is,

$$\int_{Q_T} \left(|u_m(x,t)|^{q(x)-1} \right)^{\frac{q^+}{q^+-1}} dxdt \leq C, \ \forall m \in \mathbb{N}. \tag{3.14}$$

From (3.13), (3.14), Lions' Lemma [25] and by applying the diagonalization process to the sequence of (u_m), it follows that

$$|u_m|^{q(x)-2} u_m \rightharpoonup |u|^{q(x)-2} u \text{ in } L^{\frac{q^+}{q^+-1}}(Q_T), \forall T > 0. \tag{3.15}$$

This result and convergence (3.12) imply convergence (3.11).

By the method of Browder and Minty in the theory of monotone operators $s \longmapsto |s|^{p-2} s$ and (3.11), (3.15), we deduce (see [3, 25])

$$\chi = Lu. \tag{3.16}$$

Convergences (3.9), (3.15) and (3.16) allows us to pass to the limit in the approximate equation (3.1) and so it holds that

$$\begin{aligned}
& \int_0^\infty (u'(t), \upsilon)\, dt + \int_0^\infty \sum_{i=1}^N \int_\Omega \left| \frac{\partial u(t)}{\partial x_i} \right|^{p(x)-2} \frac{\partial u(t)}{\partial x_i} . \frac{\partial \upsilon}{\partial x_i} dxdt \\
= & \; \lambda \int_0^\infty \int_\Omega |u(t)|^{q(x)-2} u(t)\, \upsilon dxdt
\end{aligned}$$

for all $\upsilon \in L^2_{loc}\left(0, \infty; W_0^{1,p(\cdot)}(\cdot)\right)$ and supp υ compact in $(0, \infty)$. Taking $\upsilon \in C_0^\infty(\Omega \times (0, T))$ in the last equality, we find equation (2.8). The initial condition $u(0) = u^0$ in (2.8) follows by convergences (3.9) and (3.10). This concludes the proof of Theorem 2.1. □

4. Decay estimates

We define the energy $E(t)$ by

$$E(t) = \|u(t)\|_2^2, \ \forall t \geq 0. \tag{4.1}$$

By $u \in L^\infty(0, \infty; X_0)$ and $u' \in L^2(0, \infty; L^2(\Omega))$, we have $E \in C([0, \infty); L^2(\Omega))$.

Proof of Theorem 2.2. Multiply both sides of (2.8) by u and integrate on Ω. We obtain

$$\frac{1}{2}\frac{d}{dt} \|u(t)\|_2^2 + \sum_{i=1}^N \int_\Omega \left| \frac{\partial u(t)}{\partial x_i} \right|^{p(x)} dx = \lambda \int_\Omega |u(t)|^{q(x)} dx. \tag{4.2}$$

By Lemma 3.1 we have $\|u(t)\|_{X_0} < 1$. Therefore, from (1.5) (see (3.6)) it follows that

$$\frac{1}{N^{q^- -1}} \|u(t)\|_{X_0}^{q^-} \leq \sum_{i=1}^N \int_\Omega \left| \frac{\partial u(t)}{\partial x_i} \right|^{p(x)} dx, \tag{4.3}$$

and from (3.5), we have

$$\lambda \int_{\Omega} |u(t)|^{q(x)} dx \leq 2\lambda K^{q^-} \|u(t)\|_{X_0}^{q^-}. \tag{4.4}$$

Plugging (4.3) and (4.4) into (4.2), we get

$$\frac{d}{dt}\|u(t)\|_2^2 + \left(\frac{2}{N^{q^--1}} - 4\lambda K^{q^-}\right)\|u(t)\|_{X_0}^{q^-} \leq 0. \tag{4.5}$$

By using (4.1) and (4.5), we obtain

$$E'(t) + \left(\frac{2}{N^{q^--1}S^{q^-}} - \frac{4\lambda K^{q^-}}{S^{q^-}}\right) E(t)^{\frac{q^-}{2}} \leq 0. \tag{4.6}$$

We make the following considerations. If $u^0 = 0$, we take $u^0 \equiv 0$ as the solution of problem (1.1). Assume $u^0 \neq 0$. If there exists $t_1 \in (0, +\infty)$ such that $E(t_1) = 0$, we consider the set $\Im = \{\nu \in (0, +\infty) : E(\nu) = 0\}$ and $\inf_{\nu \in \Im} \nu = t_0$. Then $t_0 > 0$ because $E(0) > 0$. Also $E(t_0) = 0$. As $E'(t) \leq 0$ a.e. in $(0, +\infty)$, then $E(t)$ is decreasing, therefore $E(t) = 0$ for all $t \geq t_0$. Therefore, either $E(t) = 0$, for all $t \geq t_0$ or $E(t) > 0$, for all $t > 0$. We prove inequality (2.9) for the second case, that is, $E(t) > 0$, for all $t \in [0, +\infty)$. The inequality (2.9) for $t \in [0, t_0)$ is derived in a similar way. By (4.6), we obtain

$$E'(t) + \eta E(t)^{\frac{q^-}{2}} \leq 0,$$

where $\eta = \frac{2}{N^{q^--1}S^{q^-}} - \frac{4\lambda K^{q^-}}{S^{q^-}} > 0$. Thus we have

$$\frac{(2 - q^-)E'(t)}{2E^{\frac{q^-}{2}}(t)} \geq \frac{\eta(q^- - 2)}{2}.$$

Therefore,

$$\left(E^{\frac{2-q^-}{2}}(t)\right)' \geq \frac{\eta(q^- - 2)}{2}$$

that is,

$$E^{\frac{2-q^-}{2}}(t) \leq E^{\frac{2-q^-}{2}}(0) + \frac{\eta(q^- - 2)}{2}t.$$

This concludes the proof of Theorem 2.2. □

5. Extinction of weak solutions

It is well known that Eq. (1.1) is degenerate if $p > 2$ or singular if $1 < p < 2$, since the modulus of ellipticity is degenerate ($p > 2$) or blows up ($1 < p < 2$) at points where $\nabla u = 0$, and therefore there is no classical solution in general. Unlike the linear case, for $p \neq 2$ the solutions of the Dirichlet problem for Eq. (1.1) are localized either in space, or in time. More precisely, the following alternative holds: if u is a solution of the Dirichlet problem for Eq. (1.1) with $p \neq 2$, then either

1) $1 < p < 2$ (fast diffusion)$\Longrightarrow \exists T_1 : u \equiv 0$ for all $t \geq T_1$.

The local existence of such a weak solution can be obtained by similar argument to that in [5, 18], by using a priori estimates.

In combustion theory, for instance, the function $u(\cdot, t)$ represents the temperature, the term $\Delta_{p(\cdot)}u$ represents the thermal diffusion, and $u^{q(\cdot)}$ is a source.

2) $p > 2$ (slow diffusion) and $u_0 \equiv 0$ in

$$B_r(x_0) = \left\{x \in \mathbb{R}^N : |x - x_0| < r\right\} \Longrightarrow \exists t_*(x_0) : u(x_0, t) \equiv 0$$

for all $t \in [0, t_*(x_0)]$. These properties complement each other: the former is called extinction in a finite time, the latter is usually referred to as finite speed of propagation of disturbances from the data.

In order to obtain the extinction properties of weak solutions, we introduce an auxiliary lemma on the ordinary differential inequality as follows.

Lemma 5.1. (see [7]) *Assume* $0 < l_1 \leq l_2 < r_1 \leq r_2 \leq 1$ *and* $\alpha \geq 0$, $\beta \geq 0$ *and* φ *is a nonnegative and absolutely continuous function, which satisfies*

$$\varphi'(t) + \alpha \min\left\{\varphi^{l_1}(t), \varphi^{l_2}(t)\right\} \leq \beta \max\left\{\varphi^{r_1}(t), \varphi^{r_2}(t)\right\}, \ t \geq 0,$$
$$\varphi(0) > 0, \ \beta \max\left\{\varphi^{r_1 - l_1}(0), \varphi^{r_2 - l_1}(0)\right\} < \alpha \min\left\{1, \varphi^{l_2 - l_1}(0)\right\},$$

then it holds

$$\begin{cases} \varphi(t) \leq \left[\varphi^{1-l_1}(0) - \alpha_0 (1 - l_1) t\right]^{\frac{1}{1-l_1}}, \ 0 < t < T_0, \\ \varphi(t) \equiv 0, t \geq T_0, \end{cases}$$

where

$$\alpha_0 = \alpha \min\left\{1, \varphi^{l_2 - l_1}(0)\right\} - \beta \max\left\{\varphi^{r_1 - l_1}(0), \varphi^{r_2 - l_1}(0)\right\} > 0,$$

and

$$T_0 = \alpha_0^{-1} (1 - l_1)^{-1} \varphi^{1-l_1}(0) > 0.$$

Proof of Theorem 2.3. By using (4.2), we have

$$\frac{d}{dt} \|u(t)\|_2^2 + 2 \sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u(t)}{\partial x_i}\right|^{p(x)} dx = 2\lambda \int_\Omega |u(t)|^{q(x)} dx. \tag{5.1}$$

Furthermore, by using Proposition 1.1 (ii), (1.4) and (4.1), we obtain

$$2 \sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u(t)}{\partial x_i}\right|^{p(x)} dx \geq 2 \|u\|_{X_0}^{p+} \geq 2S^{-p^+} \|u\|_2^{p^+} = 2S^{-p^+} E^{\frac{p^+}{2}}(t). \tag{5.2}$$

Also by Proposition 1.2 (i), we have

$$\begin{aligned} 2\lambda \int_\Omega |u|^{q(x)} dx &\leq 4\lambda \left\| |u|^{q(\cdot)} \right\|_{\frac{2}{q(\cdot)}} \|1\|_{\frac{2}{2-q(\cdot)}} \\ &\leq 4\lambda (|\Omega| + 1)^{(2-q^-)/2} \max\left\{\|u\|_2^{q^-}, \|u\|_2^{q^+}\right\} \\ &= 4\lambda (|\Omega| + 1)^{(2-q^-)/2} \max\left\{E^{\frac{q^-}{2}}(t), E^{\frac{q^+}{2}}(t)\right\}. \end{aligned} \tag{5.3}$$

By (5.1), (5.2) and (5.3), we arrive at the following relation

$$E'(t) + 2S^{-p^+} E^{\frac{p^+}{2}}(t) \leq 4\lambda (|\Omega| + 1)^{(2-q^-)/2} \max\left\{E^{\frac{q^-}{2}}(t), E^{\frac{q^+}{2}}(t)\right\}.$$

Since $1 < p^- \leq p^+ < q^- \leq q^+ < 2$, we have $\frac{1}{2} < \frac{p^-}{2} \leq \frac{p^+}{2} < \frac{q^-}{2} \leq \frac{q^+}{2} < 1$. By using Lemma 5.1, we obtain

$$E'(t) \leq -\beta_0 E^{\frac{p^+}{2}}(t),$$

where

$$\beta_0 = 2S^{-p^+} - 4\lambda (|\Omega| + 1)^{(2-q^-)/2} \max\left\{E^{\frac{q^- - p^+}{2}}(0), E^{\frac{q^+ - p^+}{2}}(0)\right\} > 0.$$

Thus, from $E(t) > 0$ with $E(0) > 0$, we get

$$E(t) \leq \left(E^{\frac{2-p^+}{2}}(0) - \frac{\beta_0 (2 - p^+)}{2} t\right)^{\frac{2}{2-p^+}},$$

for $t \in (0, T_0)$, and

$$E(t) \equiv 0,$$

for $t \in [T_0, +\infty)$, where

$$T_0 = \frac{2E^{\frac{2-p^+}{2}}(0)}{\beta_0 \left(2 - p^+\right)}.$$

Thus the proof of Theorem 2.3 is complete. □

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