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Constacyclic and Negacyclic Codes over $\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2$ and their Equivalents over $\mathbb{F}_2$

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Abstract

In this work, we consider the finite ring $\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2$, $u^2=1, v^2=0$, $u\cdot v=v\cdot u=0$ which is not Frobenius and chain ring. We studied constacyclic and negacyclic codes in $\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2$ with odd length. These codes are compared with codes that had priorly been obtained on the finite field $\mathbb{F}_2$. Moreover, we indicate that the Gray image of a constacyclic and negacyclic code over $\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2$ with odd length n is a quasicyclic code of index 4 with length $4n$ in $\mathbb{F}_2$. In particular, the Gray images are applied to two different rings $S_1=\mathbb{F}_2+v\mathbb{F}_2$, $v^2=0$ and $S_2=\mathbb{F}_2+u\mathbb{F}_2$, $u^2=1$ and negacyclic and constacyclic images of these rings are also discussed.

1. Introduction

The fundamental problem in coding theory, such as distance, polynomial representation over codes, weight, etc. were examined in [1]. The Gray images of cyclic and negacyclic codes defined on $\mathbb{Z}_4$ were studied, and their relationships on $\mathbb{Z}_2$ were researched in [2]. In [3], differently in the previously studied the ring $\mathbb{Z}_4$, the images of the $(1+u)-constacyclic$ codes on the finite chain ring $\mathbb{F}_2+u\mathbb{F}_2$ were studied in the case $u^2=0$, and the relationship of cyclic codes between this ring and field $\mathbb{F}_2$ has been mentioned. Moreover, in [4] gray images of $(1+u^2)-constacyclic$ codes on $\mathbb{F}_2+u\mathbb{F}_2+u^2\mathbb{F}_2$ with 8 elements were given on the field $\mathbb{F}_2$ by the same authors in [3]. X. Xiaofang [5] investigated $(1+v)-constacyclic$ codes over $\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2, u^2=v^2=0, v\cdot u=u\cdot v=0$, and $(1+v)-constacyclic$ codes in $\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2$ of odd length were described through cyclic codes over $\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2$.

In this study, unlike in [6], we take the properties of the variables in the ring structure differently. Therefore, a different ring structure emerged. In the next section, we give the primary form of the ring and define the Gray transformations. In the third section, we show that the images of the codes on this ring correspond to codes in the finite rings. Finally, in the last part, we also match the codes found to codes on $\mathbb{F}_2$.

2. Preliminaries

We denote $R=\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2$ as a ring with characteristic 2, where $u^2=1, v^2=0, u\cdot v=v\cdot u=0$. It is clearly see that $\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2\cong\mathbb{F}_2[u,v]/\langle u^2=1, v^2=0, u\cdot v=v\cdot u=0\rangle$. Consider $R=\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2=\{0,1,u,1+u,v,1+v,u+v,1+u+v\}$. Thus R is a ring under "$+$" and "." operations. Also, 1 and $1+v$ are units in R, and all the ideals of R can be given by $\{0\}=I_0$, I_u, I_v, I_{u+v}, $I_{1+u}=I_{1+u+v}$, $I_{1+v}=R$. We consider R as a natural extension of $S_1=\mathbb{F}_2+v\mathbb{F}_2$, $v^2=0$. Thus, $S_1\cong\mathbb{F}_2[v]/\langle v^2\rangle$. Then, the elements of S_1 are $0,1,v,1+v$ where the units in S_1 are 1 and $1+v$. We consider R as a natural

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extension of $S_2 = \mathbb{F}_2 + u\mathbb{F}_2$, $u^2 = 1$. Therefore, $S_2 \cong \mathbb{F}_2[u]/\langle u^2 \rangle$. The elements of S_2 are $0, 1, u$ and $1+u$. Then, the only units in S_2 are 1 and u. Let us take C as a linear code with length n over R (S_1^n or S_2^n). Thus C is a R (S_1^n or S_2^n) submodule of R^n (S_1^n or S_2^n). If D is a linear code with length n in $\mathbb{F}_2$, in this case, D is a $\mathbb{F}_2$ subvector space $\mathbb{F}_2^n$. An element of C and D is called a codeword.

Let Γ_1 denote the Gray map on R (see [6]).

$$\Gamma_1 : R \mapsto S_1^2$$

$$a+ub+vc \mapsto \Gamma_1(a+ub+vc) = \Gamma_1(r+uq) = (v\cdot r, q)$$

where $r = a+vc$ and $q = b+vc$. It can be extended to R^n as shown below:
$\Gamma_1(c_0, c_1, \ldots, c_{n-1}) = (v\cdot r_0, v\cdot r_1, \ldots, v\cdot r_{n-1}, q_0, q_1, \ldots, q_{n-1})$ where $c_i = r_i + u\cdot q_i$ for all $0 \le i \le n-1$. Let the Gray map Ψ_1 on R be defined as indicated below:

$$\Psi_1 : R \mapsto S_2^2$$

$$a+ub+vc \mapsto \Psi_1(a+ub+vc) = \Psi_1(r+vq) = (u\cdot r, q) \tag{2.1}$$

such that $r = a+ub$ and $q = c+ub$. We will extend Ψ_1 to R^n, that is, $\Psi_1(c_0, c_1, \ldots, c_{n-1}) = (u\cdot r_0, u\cdot r_1, \ldots, u\cdot r_{n-1}, q_0, q_1, \ldots, q_{n-1})$ where $c_i = r_i + v\cdot q_i$ for all $0 \le i \le n-1$.

Let us define the Gray map Γ_2 on S_1 as the following:

$$\Gamma_2 : S_1 \mapsto \mathbb{F}_2^2$$

$$s+vt \mapsto (s, s+t) \tag{2.2}$$

where $s, t \in \mathbb{F}_2$. The extension of Γ_2 to S_1^n is given by

$$\Gamma_2 : S_1^n \mapsto \mathbb{F}_2^{2n}$$

$$(c_0, c_1, \ldots, c_{n-1}) \mapsto (s_0, \ldots, s_{n-1}, s_0+t_0, \ldots, s_{n-1}+t_{n-1})$$

where $c_i = s_i + v\cdot t_i$, $s_i, t_i \in \mathbb{F}_2$ for all $0 \le i \le n-1$. The Gray map Ψ_2 on S_2 is given by

$$\Psi_2 : S_2 \mapsto \mathbb{F}_2^2$$

$$s+ut \mapsto (s, s+t) \tag{2.3}$$

where $s, t \in \mathbb{F}_2$. The extension of Ψ_2 to S_2^n is given by

$$\Psi_2 : S_2^n \mapsto \mathbb{F}_2^{2n}$$

$$(c_0, c_1, \ldots, c_{n-1}) \mapsto (s_0, \ldots, s_{n-1}, s_0+t_0, \ldots, s_{n-1}+t_{n-1})$$

where $c_i = s_i + u\cdot t_i$, $s_i, t_i \in \mathbb{F}_2$ for all $0 \le i \le n-1$. For $r \in R$, we define the weight function $w_1(r)$ by

$$w_1(r) = \begin{cases} 0\,;\ r = 0 \\ 1\,;\ r = 1 \\ 2\,;\ r = v, 1+u, 1+u+v \\ 3\,;\ r = u, u+v, 1+v \end{cases}$$

For $r \in R$, we define the weight function $w_2(r)$ by

$$w_2(r) = \begin{cases} 0\,;\ r = 0 \\ 1\,;\ r = 1, u, u+v \\ 2\,;\ r = u, 1+v, 1+u+v \\ 3\,;\ r = 1+u \end{cases}$$

Then $w_1(r)$ and $w_2(r)$ extend to a weight function in R^n. If $r = (r_0, r_1, \ldots, r_{n-1}) \in R^n$, then we write $w_1(r) = \sum_{i=0}^{n-1} w_1(r_i)$ and $w_2(r) = \sum_{i=0}^{n-1} w_2(r_i)$. Let $x,\ y \in R^n$ be any distinct vectors. The distance $d_1(x,y)$ and $d_2(x,y)$ can be defined to be $w_1(x-y)$ and $w_2(x-y)$. The d_1 and d_2 minimum distance of C can be given by $d_1(C) = min\{d_1(x,y)\}$ and $d_2(C) = min\{d_2(x,y)\}$ for any $x, y \in C$ such that $x \neq y$. The weights $w_3(t)$ of $t \in S_1$ and $w_4(t)$ of $t \in S_2$ can be given by

$$w_3(t)=\begin{cases} 0\,;\, t=0 \\ 1\,;\, t=v,1+v \\ 2\,;\, t=1 \end{cases}$$

$$w_4(t)=\begin{cases} 0\,;\, t=0 \\ 1\,;\, t=u,1+u \\ 2\,;\, t=1 \end{cases}$$

These extend to w_3 and w_4 weight functions in S_1^n and S_2^n. If $t=(t_0,t_1,\dots,t_{n-1})\in S_1^n,\ S_2^n$, then we have $w_3(t)=\sum_{i=0}^{n-1} w_3(t_i)$ and $w_4(t)=\sum_{i=0}^{n-1} w_4(t_i)$. Let $x,y\in S_1^n,S_2^n$ be any distinct vectors. The distance $d_3(x,y)$ and $d_4(x,y)$ between x,y can be given by $w_{S_1}(x-y)$ and $w_{S_2}(x-y)$, respectively. Also, the d_3 and d_4 minimum distance of C is defined as $d_3(C)=min\ \{\ d_3\ (x,y)\ \}$ and $d_4(C)=min\{d_4(x,y)\}$ for any $x,y\in C$ such that $x\neq y$. Let D as a code with length n over $\mathbb{F}_2$ and $c=(c_0,c_1,\dots,c_{n-1})$ be a codeword of D. The Hamming weight of D is defined as $w_H(c)=\sum_{i=0}^{n-1} w_H(c_i)$ where $w_H(c_i)=1$ if $c_i=1$ and $w_H(c_i)=0$ if $c_i=0$. In addition, we can define the minimum Hamming distance of D such as $d_H=min\ \{d_H(c,\tilde{c})\}$ for any $c,\tilde{c}\in D$, $c\neq\tilde{c}$. The elements of R as $a+ub+vc=r+vq$ where $r=a+ub$ and $q=c+ub$ are in S_2, we have

$$w_1(a+ub+vc)=w_1(r+vq)=w_4(ur,q)=w_4(b+ua,c+ub)=w_H(b,c,b+a,c+b)$$

Similarly, the elements of R as $a+ub+vc=r+vq$ where $r=a+vc$ and $q=b+vc$ are in S_1 and so we obtain the following

$$w_2(a+ub+vc)=w_2(r+uq)=w_3(vr,q)=w_3(av,b+vc)=w_H(0,b,a,b+c)$$

Definition 2.1. [1] Let C be a linear code over R with length n. A cyclic shift on R^n is a permutation σ such that $\sigma\ (c_0,c_1,\dots,c_{n-1})=(c_{n-1},c_0,\dots,c_{n-1})$. If $\sigma(C)=C$, the code C is said to be cyclic code. A $(1+v)-constacylic$ shift μ act on R^n as $\mu(c_0,c_1,\dots,c_{n-1})=((1+v)c_{n-1},c_0,\dots,c_{n-2})$. The code C is called $(1+v)-constacyclic$ code if $\mu(C)=C$. A *negacylic* shift δ act on R^n as $\delta(c_0,c_1,\dots,c_{n-1})=(-c_{n-1},c_0,\dots,c_{n-2})$. If $\delta(C)=C$, C is said to be *negacyclic* code.

Let $P(C)=\left\{\sum_{i=0}^{n-1} r_ix^i:(r_0,r_1,\dots,r_{n-1})\in C\right\}$. $P(C)$ is a polynomial representation of code C with length n over R. Note that C is cyclic if and only if $P(C)$ is an ideal of $R[x]/\langle x^n-1\rangle$ and C is $(1+v)-constacyclic$ if and only if $P(C)$ is an ideal of $R[x]/\langle x^n-(1+v)\rangle$.

Definition 2.2. [1] Let $a\in S_1^{2n}$ with $a=(a_0,a_1,\dots,a_{2n-1})=(a^{(0)}|a^{(1)})$, $a^{(i)}\in S_1^n$ for all $i=0,1$ and σ be the usual cyclic shift.

$$\sigma_1^{*2}:S_1^{2n}\mapsto S_1^{2n}$$
$$a\mapsto\sigma_1^{*2}(a)=(\sigma(a^{(0)})|\sigma(a^{(1)}))$$

A code $\hat{C}$ of length $2n$ in S_1 is called *quasicyclic* code with index 2 if $\sigma_1^{*2}(\hat{C})=\hat{C}$. Let $a\in S_2^{2n}$ with $a=(a_0,a_1,\dots,a_{2n-1})=(a^{(0)}|a^{(1)})$, $a^{(i)}\in S_2^n$ for all $i=0,1$ and σ be the usual cyclic shift.

$$\sigma_2^{*2}:S_2^{2n}\mapsto S_2^{2n}$$
$$a\mapsto\sigma_2^{*2}(a)=(\sigma(a^{(0)})|\sigma(a^{(1)}))$$

A code $\hat{C}$ with length $2n$ in S_2 is called *quasicyclic* code with index 2 if $\sigma_2^{*2}(\hat{C})=\hat{C}$. Take $a\in\mathbb{F}_2^{4n}$ with $a=(a_0,a_1,\dots,a_{4n-1})=(a^{(0)}|a^{(1)}|a^{(2)}|a^{(3)})$, $a^{(i)}\in\mathbb{F}_2^n$ for all $i=0,1,2,3$ and let σ be the usual cyclic shift.

$$\sigma^{*4}:\mathbb{F}_2^{4n}\mapsto\mathbb{F}_2^{4n}$$
$$a\mapsto\sigma^{*4}(a)=(\sigma(a^{(0)})|\sigma(a^{(1)})|\sigma(a^{(2)})|\sigma(a^{(3)}))$$

A code $\hat{D}$ of length $4n$ over $\mathbb{F}_2$ is called *quasicyclic* code with index 4 if $\sigma^{*4}(\hat{D})=\hat{D}$.

3. Negacyclic codes and their gray images

We get *quasicyclic* code of index 2 with even length in S_2 as the Gray image Ψ_1 of *negacyclic* code over R. Therefore, we construct the Gray image Ψ_2 of *quasicyclic* code of index 2 in S_2 with even length.

Proposition 3.1. $\sigma_2^{*2}\Psi_1=\Psi_1\delta$

Proof. Ψ_1, σ_2^{*2} and δ are defined in (2.1) and in [1], respectively. Let $c=(c_0,c_1,\dots,c_{n-1})\in R^n$ such that $c_i=r_i+v\cdot q_i$ for $i=0,1,\dots,n-1$.

$$\Psi_1(c_0,c_1,\dots,c_{n-1})=\Psi_1(r_0+v\cdot q_0,r_1+v\cdot q_1,\dots,r_{n-1}+v\cdot q_{n-1})=(u\cdot r_0,u\cdot r_1,\dots,u\cdot r_{n-1},q_0,q_1,\dots,q_{n-1})$$

By applying σ_2^{*2}, we have

$$\Psi_1(c_0,c_1,\ldots,c_{n-1})=\sigma_2^{*2}(u\cdot r_0,u\cdot r_1,\ldots,u\cdot r_{n-1},q_0,q_1,\ldots,q_{n-1})=(u\cdot r_{n-1},u\cdot r_0,\ldots,u\cdot r_{n-2},q_{n-1},q_0,q_1,\ldots,q_{n-2})$$

Conversely, $\delta(c_0,\ldots,c_{n-1})=(-c_{n-1},c_0,\ldots,c_{n-2})$ where $-c_{n-1}=r_{n-1}+v\cdot q_{n-1}$. Therefore,

$$\Psi_1(\delta(c))=\Psi_1(r_{n-1}+v\cdot q_{n-1},r_0+v\cdot q_0,\ldots,r_{n-2}+v\cdot q_{n-2})=(u\cdot r_{n-1},u\cdot r_0,\ldots,u\cdot r_{n-2},q_{n-1},q_0,\ldots,q_{n-2})$$

Equality is obtained by using the above equations. □

Theorem 3.1 A code C_1 of length n over R is a *negacyclic* code if and only if $\Psi_1(C_1)$ is a *quasicyclic* code of index 2 and length $2n$ over S_2.

Proof. Assume that C_1 is a *negacyclic* code. Then we write $\delta(C_1)=C_1$. By applying Ψ_1, we have $\Psi_1(\delta(C_1))=\Psi_1(C_1)$. By using Proposition 3.1, we have $\sigma_2^{*2}(\Psi_1(C_1))=\Psi_1(\delta(C_1))=\Psi_1(C_1)$. Therefore $\Psi_1(C_1)$ is a quasicyclic code with index 2. On the contrary, if $\Psi_1(C_1)$ is a quasicyclic code with index 2, then $\sigma_2^{*2}(\Psi_1(C_1))=\Psi_1(C_1)$. Again by Proposition 3.1, we have $\sigma_2^{*2}(\Psi_1(C_1))=\Psi_1(\delta(C_1))=\Psi_1(C_1)$. Since $\delta(C_1)=C_1$, C_1 is a *negacyclic* code. □

Proposition 3.2. $\sigma^{*4}\Psi_2=\Psi_2\sigma_2^{*2}$

Proof. Ψ_2, σ_2^{*2} and σ^{*4} are given in (2.3) and in [1], respectively.

$$\begin{aligned}\sigma_2^{*2}(a)&=\sigma_2^{*2}(a_0,a_1,\ldots,a_{2n-1})=(\sigma(a^{(0)})|\sigma(a^{(1)}))\\&=(\sigma(a_0,a_1,\ldots,a_{n-1})|\sigma(a_n,\ldots,a_{2n-1}))\\&=(a_{n-1},a_0,\ldots,a_{n-2},a_{2n-1},a_n,\ldots,a_{2n-2})\end{aligned}$$

where $a_{n-1}=s_{n-1}+u\cdot t_{n-1},a_0=s_0+u\cdot t_0,\ldots,a_{2n-2}=s_{2n-2}+u\cdot t_{2n-2}$. By applying Ψ_2, we have

$$\begin{aligned}&\Psi_2(\sigma_2^{*2}(a))=\Psi_2(a_{n-1},a_0,\ldots,a_{n-2},a_{2n-1},a_n,\ldots,a_{2n-2})\\&=\Psi_2(s_{n-1}+u\cdot t_{n-1},s_0+u\cdot t_0,\ldots,s_{n-2}+u\cdot t_{n-2},s_{2n-1}+u\cdot t_{2n-1},\ldots,s_{2n-2}+u\cdot t_{2n-2})\\&=(s_{n-1},s_0,\ldots,s_{n-2},s_{2n-1},\ldots,s_{2n-2},s_{n-1}+t_{n-1},s_0+t_0,\ldots,s_{n-2}+t_{n-2},s_{2n-1}+t_{2n-1},\ldots,s_{2n-2}+t_{2n-2})\end{aligned}$$

Conversely, $\Psi_2(a)=\Psi_2(a_0,a_1,\ldots,a_{2n-1})=(s_0,s_1,\ldots,s_{2n-1},s_0+t_0,s_1+t_1,\ldots,s_{2n-1}+t_{2n-1})$ where $a_0=s_0+ut_0$,

$a_1=s_1+ut_1$ $,\ldots,a_{2n-1}=s_{2n-1}+ut_{2n-1}$. By applying σ^{*4}, we have

$$\begin{aligned}&\sigma^{*4}(\Psi_2(a))=\sigma^{*4}(\Psi_2(a_0,a_1,\ldots,a_{2n-1}))=\sigma^{*4}(s_0,s_1,\ldots,s_{2n-1},s_0+t_0,s_1+t_1,\ldots,s_{2n-1}+t_{2n-1})\\&=(\sigma(s_0,s_1,\ldots,s_{n-1})|\sigma(s_n,\sigma(s_{n+1},\ldots,s_{2n-1})|\sigma(s_0+t_0,s_1+t_1,\ldots,s_{n-1}+t_{n-1})|\sigma(s_n+t_n,s_{n+1}+t_{n+1},\ldots,s_{2n-1}+t_{2n-1}))\\&=(s_{n-1},s_0,\ldots,s_{n-2},s_{2n-1},s_{n+1},\ldots,s_{2n-2},s_{n-1}+t_{n-1},s_0+t_0,\ldots,s_{n-2}+t_{n-2},s_{2n-1}+t_{2n-1},\ldots,s_{2n-2}+t_{2n-2})\end{aligned}$$

Equality is obtained by using the above equations. □

Theorem 3.2 A code C_2 with length $2n$ over S_2 is a quasicyclic code of index 2 if and only if $\Psi_2(C_2)$ is a *quasicyclic* code with length $4n$ over $\mathbb{F}_2$ and has index 4.

Proof. Assume C_2 is a *quasicyclic* code with index 2. Then $\sigma_2^{*2}(C_2)=C_2$. By applying Ψ_2, we get $\Psi_2(\sigma_2^{*2}(C_2))=\Psi_2(C_2)$. Using Proposition 3.2, we can write $\sigma^{*4}(\Psi_2(C_2))=\Psi_2(\sigma_2^{*2}(C_2))=\Psi_2(C_2)$. So $\Psi_2(C_2)$ is a quasicyclic code with index 4. Conversely, if $\Psi_2(C_2)$ is a quasicyclic code of index 4, then we say that $\sigma^{*4}(\Psi_2(C_2))=\Psi_2(C_2)$. From Proposition 3.2, we have $\sigma^{*4}(\Psi_2(C_2))=\Psi_2(\sigma_2^{*2}(C_2))=\Psi_2(C_2)$. Since Ψ_2 is injective, it follows that $\sigma_2^{*2}(C_2)=C_2$. □

4. Constacyclic codes and their gray images

In this part, we present even length quasicyclic code of index 2 over S_1 as the Gray image Γ_1 of *constacyclic* code over R and we also give the Gray image Γ_2 of *constacyclic* code with index 2 over S_1 with even length.

Proposition 4.1. $\sigma_1^{*2}\Gamma_1=\Gamma_1\delta$

Proof. The proof is given in [5]. □

Theorem 4.1 A code C_3 of length n in R is a *constacyclic* code if and only if $\Gamma_1(C_3)$ is a *quasicyclic* code with length $2n$ over S_1 and has index 2.

Proof. The proof is given in [5]. □

Proposition 4.2. $\sigma^{*4}\Gamma_2=\Gamma_2\sigma_1^{*2}$

Proof. Γ_2, σ_1^{*2} and σ^{*4} are given in (2.2) and in [1], respectively.

$$\begin{aligned}\sigma_1^{*2}(a) &= \sigma_1^{*2}(a_0,a_1,\ldots,a_{2n-1}) = (\sigma(a^{(0)})|\sigma(a^{(1)}))\\ &= (\sigma(a_0,a_1,\ldots,a_{n-1})|\sigma(a_n,\ldots,a_{2n-1}))\\ &= (a_{n-1},a_0,\ldots,a_{n-2},a_{2n-1},a_n,\ldots,a_{2n-2})\end{aligned}$$

where $a_{n-1} = s_{n-1}+v\cdot t_{n-1}, a_0 = s_0+v\cdot t_0,\ldots,a_{2n-2} = s_{2n-2}+v\cdot t_{2n-2}$. By applying Γ_2, we have

$$\begin{aligned}&\Gamma_2(\sigma_1^{*2}(a)) = \Psi_2(a_{n-1},a_0,\ldots,a_{n-2},a_{2n-1},a_n,\ldots,a_{2n-2})\\ &= \Psi_2(s_{n-1}+v\cdot t_{n-1},s_0+v\cdot t_0,\ldots,s_{n-2}+v\cdot t_{n-2},s_{2n-1}+v\cdot t_{2n-1},\ldots,s_{2n-2}+v\cdot t_{2n-2})\\ &= (s_{n-1},s_0,\ldots,s_{n-2},s_{2n-1},\ldots,s_{2n-2},s_{n-1}+t_{n-1},s_0+t_0,\ldots,s_{n-2}+t_{n-2},s_{2n-1}+t_{2n-1},\ldots,s_{2n-2}+t_{2n-2})\end{aligned}$$

Conversely, $\Gamma_2(a) = \Gamma_2(a_0,a_1,\ldots,a_{2n-1}) = (s_0,s_1,\ldots,s_{2n-1},s_0+t_0,s_1+t_1,\ldots,s_{2n-1}+t_{2n-1})$ where $a_0 = s_0+v\cdot t_0$,

$a_1 = s_1+v\cdot t_1,\ldots,a_{2n-1} = s_{2n-1}+v\cdot t_{2n-1}$. By applying σ^{*4}, we have

$$\begin{aligned}&\sigma^{*4}(\Gamma_2(a)) = \sigma^{*4}(\Gamma_2(a_0,a_1,\ldots,a_{2n-1})) = \sigma^{*4}(s_0,s_1,\ldots,s_{2n-1},s_0+t_0,s_1+t_1,\ldots,s_{2n-1}+t_{2n-1})\\ &= (\sigma(s_0,s_1,\ldots,s_{n-1})|\sigma(s_n,\sigma(s_{n+1},\ldots,s_{2n-1})|\sigma(s_0+t_0,s_1+t_1,\ldots,s_{n-1}+t_{n-1})|\sigma(s_n+t_n,s_{n+1}+t_{n+1},\ldots,s_{2n-1}+t_{2n-1}))\\ &= (s_{n-1},s_0,\ldots,s_{n-2},s_{2n-1},s_{n+1},\ldots,s_{2n-2},s_{n-1}+t_{n-1},s_0+t_0,\ldots,s_{n-2}+t_{n-2},s_{2n-1}+t_{2n-1},\ldots,s_{2n-2}+t_{2n-2})\end{aligned}$$

Equality is obtained by using the above equations.

□

Theorem 4.2 A code C_4 with length $2n$ over S_1 is a quasicyclic code of index 2 if and only if $\Gamma_2(C_4)$ is a *quasicyclic* code of index 4 over $\mathbb{F}_2$ with length $4n$.

Proof. Assume that C_4 is a *quasicyclic* code with index 2. So $\sigma_1^{*2}(C_4) = C_4$. By applying Γ_2, we have $\Gamma_2(\sigma_1^{*2}(C_4)) = \Gamma_2(C_4)$. From Proposition 4.2, it follows that $\sigma^{*4}(\Gamma_2(C_4)) = \Gamma_2(\sigma_1^{*2}(C_4)) = \Gamma_2(C_4)$. Hence $\Gamma_2(C_4)$ is a quasicyclic code with index 4. Conversely, if $\Gamma_2(C_4)$ is a quasicyclic code of index 4, then $\sigma^{*4}(\Gamma_2(C_4)) = \Gamma_2(C_4)$. By Proposition 4.2, it can be written as $\sigma^{*4}(\Gamma_2(C_4)) = \Gamma_2(\sigma_1^{*2}(C_4)) = \Gamma_2(C_4)$. Since Γ_2 is injective, it follows that $\sigma_1^{*2}(C_4) = C_4$. □

5. Conclusion

We examined the constacyclic and negacyclic codes over $R = \mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2$, $u^2 = 1$, $v^2 = 0$, $u\cdot v = v\cdot u = 0$ which is not Frobenius and chain ring. We compare these codes with the codes over finite field $\mathbb{F}_2$. Besides, we mention the Gray image of constacyclic and negacyclic codes over R with odd length n, and it is a quasicyclic code of index 4 with length $4n$ in $\mathbb{F}_2$.

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