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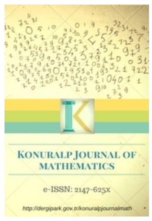


# Hyper-Fibonacci and Hyper-Lucas Hybrinomials

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### Abstract

The hybrid numbers which are accepted as a generalization of complex, hyperbolic and dual numbers, have attracted the attention of many researchers recently. In this paper, hyper-Fibonacci and hyper-Lucas hybrinomials are defined. The recurrence relations, generation functions, as well as some algebraic and combinatoric properties are examined for the newly defined hybrinomials.

***Keywords:*** *Hybrinomials; Hyper-Fibonacci numbers; Hyper-Lucas numbers; Polynomials*
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## 1. Introduction

The Fibonacci numbers, Lucas numbers and their generalizations have wide application area in mathematics and other sciences. The Fibonacci and Lucas numbers are generated by the recurrence relations ($n \geq 1$)

$$F_{n+1} = F_n + F_{n-1} \quad \text{with} \quad F_0 = 0, \quad F_1 = 1 \tag{1.1}$$

and

$$L_{n+1} = L_n + L_{n-1} \quad \text{with} \quad L_0 = 2, \quad L_1 = 1, \tag{1.2}$$

respectively [1]. There are many generalizations for the Fibonacci and Lucas numbers [2, 3, 4, 5, 6, 7, 8, 9]. The hyper generalizations defined by Dil and Mező [10] as follows:

$$F_n^{(r)} = \sum_{k=0}^{n} F_k^{(r-1)} \quad \text{with} \quad F_n^{(0)} = F_n, \quad F_0^{(r)} = 0, \quad F_1^{(r)} = 1 \tag{1.3}$$

and

$$L_n^{(r)} = \sum_{k=0}^{n} L_k^{(r-1)} \quad \text{with} \quad L_n^{(0)} = L_n, \quad L_0^{(r)} = 2, \quad L_1^{(r)} = 2r+1, \tag{1.4}$$

where $r$ is a positive number. $F_n^{(r)}$ and $L_n^{(r)}$ are called hyper-Fibonacci number and hyper-Lucas number, respectively [10]. Hyper-Fibonacci and hyper-Lucas numbers have the recurrence relations $F_n^{(r)} = F_{n-1}^{(r)} + F_n^{(r-1)}$ and $L_n^{(r)} = L_{n-1}^{(r)} + L_n^{(r-1)}$, respectively [10]. Also, hyper-Fibonacci and hyper-Lucas numbers have the properties for $n \geq 1$ and $r \geq 1$ [11].

$$F_n^{(r)} = \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} F_s, \qquad L_n^{(r)} = \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} L_s, \tag{1.5}$$

$$\sum_{s=0}^{r} F_n^{(s)} = F_{n+1}^{(r)} - F_{n-1} \quad \text{and} \quad \sum_{s=0}^{r} L_n^{(s)} = L_{n+1}^{(r)} - L_{n-1}. \tag{1.6}$$

Catalan and Bicknell introduced polynomial generalizations named Fibonacci polynomial and Lucas polynomial by the recurrence relations

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x) \quad \text{with} \quad F_0(x) = 0, \quad F_1(x) = 1 \tag{1.7}$$

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and

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x) \quad \text{with} \quad L_0(x) = 2, \quad L_1(x) = x, \tag{1.8}$$

where $x$ is any variable quantity and $n \geq 2$. In recent years, Fibonacci hybrid numbers and Fibonacci hybrinomials have been the subject of research [12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. Özdemir [22] introduced hybrid numbers, as the generalization of complex, hyperbolic and dual numbers, sets by

$$\mathbb{K} = \{a + bi + c\varepsilon + dh : a, b, c, d \in \mathbb{R}, i^2 = -1, \varepsilon^2 = 0, h^2 = 1, ih = hi = \varepsilon + i\}. \tag{1.9}$$

For the detailed information we refer to [22]. Kızılateş and Kone [23] introduced Fibonacci divisor hybrid numbers by using the Fibonacci divisor numbers and investigated some of their algebraic properties. Szynal-Liana [24] defined Horadam hybrid numbers and examined some of their properties such as Binet formula, character and generating function. Kilic [25] introduced $k$-Horadam hybrid numbers and investigated some of their applications related to $k$-Horadam hybrid numbers in matrices. For $n \geq 2$, Szynal-Liana and Wloch [17] defined $n$-th Fibonacci hybrid number $FH_n = F_n + iF_{n+1} + \varepsilon F_{n+2} + hF_{n+3}$ and $n$-th Lucas hybrid number $LH_n = L_n + iL_{n+1} + \varepsilon L_{n+2} + hL_{n+3}$. Kızılateş [18] introduced $q$-Fibonacci hybrid numbers and $q$-Lucas hybrid numbers as a generalization of Fibonacci and Lucas hybrid numbers. The author obtained exponential generating functions, summation formulas, Binet-like formulas, Catalan, Cassini and d'Ocagne like identities for $q$-Fibonacci and $q$-Lucas hybrid numbers. Asci and Aydinyuz [19] described generalized k-order Fibonacci and Lucas hybrid numbers and gave some of their algebraic properties. Kızılateş [26] defined Horadam hybrid polynomials by using the Horadam polynomials. The author obtained some special cases and properties of the Horadam hybrinomials. Szynal-Liana and Wloch [20] introduced Fibonacci and Lucas hybrinomials as follows:

$$FH_n(x) = F_n(x) + iF_{n+1}(x) + \varepsilon F_{n+2}(x) + hF_{n+3}(x) \tag{1.10}$$

and

$$LH_n(x) = L_n(x) + iL_{n+1}(x) + \varepsilon L_{n+2}(x) + hL_{n+3}(x), \tag{1.11}$$

where $F_n(x)$ and $L_n(x)$ are $n$-th Fibonacci and Lucas polynomials, respectively. The generating functions and recurrence relations for the Fibonacci and Lucas hybrinomials are [20]:

$$g(t) = \frac{FH_0(x) + (FH_1(x) - FH_0(x)x)t}{1 - xt - t^2}, \text{(for Fibonacci hybrinomials)}, \tag{1.12}$$

$$G(t) = \frac{LH_0(x) + (LH_1(x) - LH_0(x)x)t}{1 - xt - t^2}, \text{(for Lucas hybrinomials)}, \tag{1.13}$$

$$FH_n(x) = xFH_{n-1}(x) + FH_{n-2}(x), \text{(for Fibonacci hybrinomials)}, \tag{1.14}$$

and

$$LH_n(x) = xLH_{n-1}(x) + LH_{n-2}(x), \text{(for Lucas hybrinomials)}, \tag{1.15}$$

respectively. The generalized Lucas hybrinomials with two variables are described by Sevgi [27]. The author obtained the matrix representation and some properties for these hybrinomials. Szynal-Liana and Wloch [21] defined a wide generalization of the Fibonacci hybrinomials which is called $(k, \alpha, q)$-Fibonacci-Pell hybrinomials. They also investigated generating function, Binet formula, Catalan, Cassini, d'Ocagne identities and some other algebraic properties for $(k, \alpha, q)$-Fibonacci-Pell hybrinomials.
The aim of this paper is to define hyper-Fibonacci and hyper-Lucas hybrinomials as a generalization of Fibonacci and Lucas hybrinomials, and to examine some properties of the newly defined hybrinomials such as recurrence relations, summation formulas and generating functions. Another aim is to introduce hyper-Fibonacci and hyper-Lucas hybrid numbers, using hyper-Fibonacci and hyper-Lucas hybrinomials, respectively.

## 2. Main Results

**Definition 2.1.** *Hyper-Fibonacci and hyper-Lucas hybrinomials are defined as*

$$HF_n^{(r)}(x) = \sum_{s=0}^{n} HF_s^{(r-1)}(x) \quad \textit{with} \quad HF_n^{(0)}(x) = HF_n(x), \quad HF_0^{(r)}(x) = HF_0(x) \tag{2.1}$$

*and*

$$HL_n^{(r)}(x) = \sum_{s=0}^{n} HL_s^{(r-1)}(x) \quad \textit{with} \quad HL_n^{(0)}(x) = HL_n(x), \quad HL_0^{(r)}(x) = HL_0(x), \tag{2.2}$$

*where $r$ is a positive integer, $HF_n(x)$ and $HL_n(x)$ are the ordinary Fibonacci and Lucas hybrinomials.*

It is clear that $HF_n^{(r)}(x)$ and $HL_n^{(r)}(x)$ have the recurrence relations for $n \geq 1$ and $r \geq 1$:

$$HF_n^{(r)}(x) = HF_{n-1}^{(r)}(x) + HF_n^{(r-1)}(x) \tag{2.3}$$

and

$$HL_n^{(r)}(x) = HL_{n-1}^{(r)}(x) + HL_n^{(r-1)}(x). \tag{2.4}$$

The first few of hyper-Fibonacci hybrinomials and hyper-Lucas hybrinomials are:

$$\begin{aligned}
HF_0^{(1)}(x) &= i + \varepsilon x + h\left(x^2+1\right), \\
HF_1^{(1)}(x) &= 1 + i(x+1) + \varepsilon\left(x^2+x+1\right) + h\left(x^3+x^2+2x+1\right), \\
HF_2^{(1)}(x) &= (x+1) + i\left(x^2+x+2\right) + \varepsilon\left(x^3+x^2+3x+1\right) + h\left(x^4+x^3+4x^2+2x+2\right),
\end{aligned}$$

$$\begin{aligned}
HF_0^{(2)}(x) &= i + \varepsilon x + h\left(x^2+1\right), \\
HF_1^{(2)}(x) &= 1 + i(x+2) + \varepsilon\left(x^2+2x+1\right) + h\left(x^3+2x^2+2x+2\right), \\
HF_2^{(2)}(x) &= (x+2) + i\left(x^2+2x+4\right) + \varepsilon\left(x^3+2x^2+5x+2\right) + h\left(x^4+2x^3+6x^2+4x+4\right),
\end{aligned}$$

$$\begin{aligned}
HL_0^{(1)}(x) &= 2 + ix + \varepsilon\left(x^2+2\right) + h\left(x^3+3x\right), \\
HL_1^{(1)}(x) &= (x+2) + i\left(x^2+x+2\right) + \varepsilon\left(x^3+x^2+3x+2\right) + h\left(x^4+x^3+4x^2+3x+2\right), \\
HL_2^{(1)}(x) &= \left(x^2+x+4\right) + i\left(x^3+x^2+4x+2\right) + \varepsilon\left(x^4+x^3+5x^2+3x+4\right) + h\left(x^5+x^4+6x^3+4x^2+8x+2\right),
\end{aligned}$$

and

$$\begin{aligned}
HL_0^{(2)}(x) &= 2 + ix + \varepsilon\left(x^2+2\right) + h\left(x^3+3x\right), \\
HL_1^{(2)}(x) &= (x+4) + i\left(x^2+2x+2\right) + \varepsilon\left(x^3+2x^2+3x+4\right) + h\left(x^4+2x^3+4x^2+6x+2\right), \\
HL_2^{(2)}(x) &= \left(x^2+2x+8\right) + i\left(x^3+2x^2+6x+4\right) + \varepsilon\left(x^4+2x^3+7x^2+6x+8\right) + h\left(x^5+2x^4+8x^3+8x^2+14x+4\right).
\end{aligned}$$

For $x = 1$, hyper-Fibonacci and hyper-Lucas hybrinomials give the numbers which we will call hyper-Fibonacci and hyper-Lucas hybrid numbers, respectively.

**Definition 2.2.** *Hyper-Fibonacci and hyper-Lucas hybrid numbers are defined as*

$$HF_n^{(r)} = \sum_{s=0}^{n} HF_s^{(r-1)} \quad \text{with} \quad HF_n^{(0)} = HF_n \quad \text{and} \quad HF_0^{(r)} = HF_0 \tag{2.5}$$

*and*

$$HL_n^{(r)} = \sum_{s=0}^{n} HL_s^{(r-1)} \quad \text{with} \quad HL_n^{(0)} = HL_n \quad \text{and} \quad HL_0^{(r)} = HL_0, \tag{2.6}$$

*where $r$ is a positive integer, $HF_n$ and $HL_n$ are the ordinary Fibonacci hybrid and Lucas hybrid numbers, respectively.*

Next two tables contain some values of the hyper-Fibonacci and hyper-Lucas hybrid numbers.

| | $r=0$ | $r=1$ | $r=2$ | $r=3$ | $r=4$ |
|---|---|---|---|---|---|
| n=0 | $i+\varepsilon+2h$ | $i+\varepsilon+2h$ | $i+\varepsilon+2h$ | $i+\varepsilon+2h$ | $i+\varepsilon+2h$ |
| n=1 | $1+i+2\varepsilon+3h$ | $1+2i+3\varepsilon+5h$ | $1+3i+4\varepsilon+7h$ | $1+4i+5\varepsilon+9h$ | $1+5i+6\varepsilon+11h$ |
| n=2 | $1+2i+3\varepsilon+5h$ | $2+4i+6\varepsilon+10h$ | $3+7i+10\varepsilon+17h$ | $4+11i+15\varepsilon+26h$ | $5+16i+21\varepsilon+37h$ |
| n=3 | $2+3i+5\varepsilon+8h$ | $4+7i+11\varepsilon+18h$ | $7+14i+21\varepsilon+35h$ | $11+25i+36\varepsilon+61h$ | $16+41i+57\varepsilon+98h$ |
| n=4 | $3+5i+8\varepsilon+13h$ | $7+12i+19\varepsilon+31h$ | $14+26i+40\varepsilon+66h$ | $25+51i+76\varepsilon+127h$ | $41+92i+133\varepsilon+225h$ |

**Table 1:** The values of the hyper-Fibonacci hybrid numbers $HF_n^{(r)}$ for $n, r = 0, 1, 2, 3, 4$.

| | $r=0$ | $r=1$ | $r=2$ | $r=3$ | $r=4$ |
|---|---|---|---|---|---|
| n=0 | $2+i+3\varepsilon+4h$ | $2+i+3\varepsilon+4h$ | $2+i+3\varepsilon+4h$ | $2+i+3\varepsilon+4h$ | $2+i+3\varepsilon+4h$ |
| n=1 | $1+3i+4\varepsilon+7h$ | $3+4i+7\varepsilon+11h$ | $5+5i+10\varepsilon+15h$ | $7+6i+13\varepsilon+19h$ | $9+7i+16\varepsilon+23h$ |
| n=2 | $3+4i+7\varepsilon+11h$ | $6+8i+14\varepsilon+22h$ | $11+13i+24\varepsilon+37h$ | $18+19i+37\varepsilon+56h$ | $27+26i+53\varepsilon+79h$ |
| n=3 | $4+7i+11\varepsilon+18h$ | $10+15i+25\varepsilon+40h$ | $21+28i+49\varepsilon+77h$ | $39+47i+86\varepsilon+133h$ | $66+73i+139\varepsilon+212h$ |
| n=4 | $7+11i+18\varepsilon+29h$ | $17+26i+43\varepsilon+69h$ | $38+54i+92\varepsilon+146h$ | $77+101i+178\varepsilon+279h$ | $143+174i+317\varepsilon+491h$ |

**Table 2:** The values of the hyper-Lucas hybrid numbers $HL_n^{(r)}$ for $n, r = 0, 1, 2, 3, 4$.

Hyper-Fibonacci and hyper-Lucas hybrid numbers have also the recurrence relations for $n \geq 1$ and $r \geq 1$:

$$HF_n^{(r)} = HF_{n-1}^{(r)} + HF_n^{(r-1)} \tag{2.7}$$

and

$$HL_n^{(r)} = HL_{n-1}^{(r)} + HL_n^{(r-1)}, \tag{2.8}$$

respectively. Now, we shall give our main results.

**Theorem 2.3.** *The generating function for the hyper-Fibonacci hybrinomials is:*

$$g(r) = \sum_{n=0}^{\infty} HF_n^{(r)}(x)\,t^n = \frac{HF_0(x) + (HF_1(x) - HF_0(x)x)\,t}{(1 - xt - t^2)(1-t)^r}, \tag{2.9}$$

*where $HF_n(x)$ is the ordinary Fibonacci hybrinomial.*

*Proof.* We use the mathematical induction on $r$. Since

$$g(0) = \sum_{n=0}^{\infty} HF_n^{(0)}(x)\,t^n = \sum_{n=0}^{\infty} HF_n(x)\,t^n = \frac{HF_0(x) + (HF_1(x) - HF_0(x)x)\,t}{1 - xt - t^2}, \tag{2.10}$$

the result is true for $r = 0$. Assume that the result is true for $r = k$. Then, we have

$$G(k) = \sum_{n=0}^{\infty} HF_n^{(k)}(x)\,t^n = HF_0^{(k)}(x) + HF_1^{(k)}(x)\,t + HF_2^{(k)}(x)\,t^2 + HF_3^{(k)}(x)\,t^3 + \ldots. \tag{2.11}$$

We must show that the result is true for $r = k+1$.

$$\begin{aligned} G(k+1) &= \sum_{n=0}^{\infty} HF_n^{(k+1)}(x)\,t^n = HF_0^{(k+1)}(x) + HF_1^{(k+1)}(x)\,t + HF_2^{(k+1)}(x)\,t^2 + HF_3^{(k+1)}(x)\,t^3 + \ldots \\ tG(k+1) &= HF_0^{(k+1)}(x)\,t + HF_1^{(k+1)}(x)\,t^2 + HF_2^{(k+1)}(x)\,t^3 + \ldots. \end{aligned}$$

Subtracting the above equalities, then considering the recurrence relation in Equation (2.3), we have

$$\begin{aligned} (1-t)\,G(k+1) &= HF_0^{(k+1)}(x) + \left(HF_1^{(k+1)}(x) - HF_0^{(k+1)}(x)\right)t + \left(HF_2^{(k+1)}(x) - HF_1^{(k+1)}(x)\right)t^2 \\ &\qquad + \left(HF_3^{(k+1)}(x) - HF_2^{(k+1)}(x)\right)t^3 \ldots \\ &= HF_0^{(k)}(x) + HF_1^{(k)}(x)\,t + HF_2^{(k)}(x)\,t^2 + HF_3^{(k)}(x)\,t^3 + \ldots \\ &= G(k). \end{aligned}$$

So, the proof is completed. □

**Corollary 2.4.** *The generating function for the hyper-Fibonacci hybrid numbers is:*

$$g(r) = \sum_{n=0}^{\infty} HF_n^{(r)} t^n = \frac{HF_0 + (HF_1 - HF_0)\,t}{(1 - t - t^2)(1-t)^r}, \tag{2.12}$$

*where $HF_n$ is the ordinary Fibonacci hybrid number.*

**Theorem 2.5.** *The generating function for the hyper-Lucas hybrinomials is*

$$G(r) = \sum_{n=0}^{\infty} HL_n^{(r)}(x)\,t^n = \frac{HL_0(x) + (HL_1(x) - HL_0(x)x)\,t}{(1 - xt - t^2)(1-t)^r}, \tag{2.13}$$

*where $HL_n(x)$ is the ordinary Lucas hybrinomial.*

*Proof.* We use the induction method on $r$. Since

$$G(0) = \sum_{n=0}^{\infty} HL_n^{(0)}(x)\,t^n = \sum_{n=0}^{\infty} HL_n(x)\,t^n = \frac{HL_0(x) + (HL_1(x) - HL_0(x)x)\,t}{1 - xt - t^2}, \tag{2.14}$$

the result is true for $r = 0$. Suppose that the result is true for $r$. Then, we have

$$G(r) = \sum_{n=0}^{\infty} HL_n^{(r)}(x)\,t^n = \frac{HL_0(x) + (HL_1(x) - HL_0(x)x)\,t}{(1 - xt - t^2)(1-t)^r}. \tag{2.15}$$

For $r+1$, considering the Cauchy product, we have

$$\begin{aligned} G(r+1) &= \sum_{n=0}^{\infty} HL_n^{(r+1)}(x)\,t^n \\ &= \sum_{n=0}^{\infty} \left( \sum_{s=0}^{n} HL_n^{(r)}(x) \right) t^n \\ &= \left( \sum_{i=0}^{\infty} HL_i^{(r)}(x)\,t^i \right) \left( \sum_{j=0}^{\infty} t^j \right) \\ &= \frac{HL_0(x) + (HL_1(x) - HL_0(x))\,t}{(1 - xt - t^2)(1-t)^{r+1}}. \end{aligned}$$

□

**Corollary 2.6.** *The generating function for the hyper-Lucas hybrid numbers is*

$$G(r) = \sum_{n=0}^{\infty} HL_n^{(r)} t^n = \frac{HL_0 + (HL_1 - HL_0)t}{(1-t-t^2)(1-t)^r}, \tag{2.16}$$

*where $HL_n$ is the ordinary Lucas hybrid number.*

The following theorem gives the relation between the hyper-Fibonacci hybrinomials and Fibonacci hybrinomials, similarly the relation between the hyper-Lucas hybrinomials and Lucas hybrinomials, respectively.

**Theorem 2.7.** *If $n \geq 1$ and $r \geq 1$, then*

*(i)* $HF_n^{(r)}(x) = \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} HF_s(x)$,

*(ii)* $HL_n^{(r)}(x) = \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} HL_s(x)$

*are hold.*

*Proof.* (i) For two real initial sequences $(a_n)$ and $(a^n)$, the symmetric infinite matrix with entries $a_n^r$ has the following recurrence relation [28]:

$$a_n^0 = a_n, \quad a_0^n = a^n \quad (n \geq 0),$$

$$a_n^r = a_n^{r-1} + a_{n-1}^r \quad (n \geq 1, r \geq 1).$$

Also the entries $a_n^r$ have the following symmetric relation [10]:

$$a_n^r = \sum_{i=1}^{r} \binom{n+r-i-1}{n-1} a_0^i + \sum_{s=1}^{n} \binom{n+r-s-1}{r-1} a_s^0. \tag{2.17}$$

For the case $a_n^r = HF_n^{(r)}$, Equation (2.17) is of the form:

$$HF_n^{(r)}(x) = \sum_{i=1}^{r} \binom{n+r-i-1}{n-1} HF_0^{(i)}(x) + \sum_{s=1}^{n} \binom{n+r-s-1}{r-1} HF_s^{(0)}(x). \tag{2.18}$$

By considering the initial conditions in Definition 2.1, we get

$$\begin{aligned} HF_n^{(r)}(x) &= \sum_{i=1}^{r} \binom{n+r-i-1}{n-1} HF_0(x) + \sum_{s=1}^{n} \binom{n+r-s-1}{r-1} HF_s(x) \\ &= \sum_{i=0}^{r-1} \binom{n+r-i-2}{n-1} HF_0(x) + \sum_{s=0}^{n-1} \binom{n+r-s-2}{r-1} HF_{s+1}(x) \\ &= HF_0(x) \sum_{k=0}^{r-1} \binom{n+k-1}{n-1} + \sum_{b=0}^{n-1} \binom{r+b-1}{r-1} HF_{n-b}(x), \end{aligned}$$

where $k = r-i-1$ and $b = n-s-1$. By means of [29], we have

$$\sum_{t=a}^{c} \binom{t}{a} = \binom{c+1}{a+1}. \tag{2.19}$$

Thus,

$$\begin{aligned} HF_n^{(r)}(x) &= HF_0(x) \binom{n+r-1}{n} + \sum_{b=0}^{n-1} \binom{r+b-1}{r-1} HF_{n-b}(x) \\ &= \sum_{b=0}^{n} \binom{r+b-1}{r-1} HF_{n-b}(x), \\ &= \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} HF_s(x). \end{aligned}$$

(ii) The proof is similar to the proof of (i).

□

**Corollary 2.8.** *If $n \geq 1$ and $r \geq 1$, then there are the relation between the hyper-Fibonacci hybrid numbers and Fibonacci hybrid numbers, similarly the relation between the hyper-Lucas hybrid numbers and Lucas hybrid numbers, respectively:*

*(i)* $HF_n^{(r)} = \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} HF_s$,

*(ii)* $HL_n^{(r)} = \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} HL_s$.

**Theorem 2.9.** *If $n\geq 1$ and $r\geq 1$, then there are the summation formulas for the hyper-Fibonacci and hyper-Lucas hybrinomials,*

*(i)* $\sum_{s=0}^{r} HF_n^{(s)}(x) = HF_{n+1}^{(r)}(x) - HF_{n-1}(x)$,

*(ii)* $\sum_{s=0}^{r} HL_n^{(s)}(x) = HL_{n+1}^{(r)}(x) - HL_{n-1}(x)$,

*where $HF_n(x)$ and $HL_n(x)$ are the ordinary Fibonacci and Lucas hybrinomials, respectively.*

*Proof.* (i) The proof is similar to the proof of (ii).

(ii) Considering Theorem 2.7, we have

$$\begin{aligned}
\sum_{s=1}^{r} HL_n^{(s)}(x) &= \sum_{s=1}^{r}\left(\sum_{t=0}^{n}\binom{n+s-t-1}{s-1}HL_t(x)\right)\\
&= \sum_{t=0}^{n}\left(HL_t(x)\sum_{s=1}^{r}\binom{n+s-t-1}{s-1}\right)\\
&= \sum_{t=0}^{n}\binom{n+r-t}{r-1}HL_t(x)\\
&= \sum_{t=0}^{n+1}\binom{n+r-t}{r-1}HL_t(x) - HL_{n+1}(x).
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{s=0}^{r} HL_n^{(s)}(x) &= HL_{n+1}^{(r)}(x) - HL_{n+1}(x) + HL_n^{(0)}(x)\\
&= HL_{n+1}^{(r)}(x) - (x-1)HL_n(x) - HL_{n-1}(x).
\end{aligned}$$

□

**Corollary 2.10.** *If $n\geq 1$ and $r\geq 1$, then the following identities hold:*

*(i)* $\sum_{s=0}^{r} HF_n^{(s)} = HF_{n+1}^{(r)} - HF_{n-1}$,

*(ii)* $\sum_{s=0}^{r} HL_n^{(s)} = HL_{n+1}^{(r)} - HL_{n-1}$,

*where $HF_n$ and $HL_n$ are the ordinary Fibonacci and Lucas hybrid numbers, respectively.*

**Theorem 2.11.** *For $n\geq 2$ and $r\geq 1$, there are the following recurrence relations for the hyper-Fibonacci and hyper-Lucas hybrinomials, respectively:*

*(i)* $HF_n^{(r)}(x) = xHF_{n-1}^{(r)}(x) + HF_{n-2}^{(r)}(x) + \binom{n+r-1}{r-1}\left(i+\varepsilon x + h\left(x^2+1\right)\right) + \binom{n+r-2}{r-1}(1+\varepsilon+hx)$,

*(ii)* $HL_n^{(r)}(x) = xHL_{n-1}^{(r)}(x) + HL_{n-2}^{(r)}(x) + \binom{n+r-1}{r-1}\left(2+ix+\varepsilon\left(x^2+2\right)+h\left(x^3+3x\right)\right) + \binom{n+r-2}{r-1}\left(-x+2i+\varepsilon x+h\left(x^2+2\right)\right)$.

*Proof.* (i) Considering Theorem 2.5 and Equation (1.14),

$$\begin{aligned}
HF_n^{(r)}(x) &= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}HF_s(x)\\
&= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}\left(xHF_{s-1}(x)+HF_{s-2}(x)\right)\\
&= x\sum_{s=0}^{n}\binom{n+r-s-1}{r-1}HF_{s-1}(x) + \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}HF_{s-2}(x)\\
&= x\sum_{s=-1}^{n-1}\binom{(n-1)+r-s-1}{r-1}HF_s(x) + \sum_{s=-2}^{n-2}\binom{(n-2)+r-s-1}{r-1}HF_s(x)\\
&= x\sum_{s=0}^{n-1}\binom{(n-1)+r-s-1}{r-1}HF_s(x) + x\binom{n+r-1}{r-1}HF_{-1}(x) + \sum_{s=0}^{n-2}\binom{(n-2)+r-s-1}{r-1}HF_s(x)\\
&\qquad + \binom{n+r-2}{r-1}HF_{-1}(x) + \binom{n+r-1}{r-1}HF_{-2}(x).\\
&= xHF_{n-1}^{(r)}(x) + x\binom{n+r-1}{r-1}(1+\varepsilon+hx) + HF_{n-2}^{(r)}(x) + \binom{n+r-2}{r-1}(1+\varepsilon+hx) + \binom{n+r-1}{r-1}(-x+i+h)\\
&= xHF_{n-1}^{(r)}(x) + HF_{n-2}^{(r)}(x) + \binom{n+r-1}{r-1}\left(i+\varepsilon x+h\left(x^2+1\right)\right) + \binom{n+r-2}{r-1}(1+\varepsilon+hx).
\end{aligned}$$

(ii) The proof is similar to the proof of (i).

□

**Corollary 2.12.** *If $n \geq 2$ and $r \geq 1$, then the recurrence relations for the hyper-Fibonacci and hyper-Lucas hybrid numbers are as follows:*

*(i)* $HF_n^{(r)} = HF_{n-1}^{(r)} + HF_{n-2}^{(r)} + \binom{n+r-1}{r-1}(i+\varepsilon+2h) + \binom{n+r-2}{r-1}(1+\varepsilon+h)$,

*(ii)* $HL_n^{(r)} = HL_{n-1}^{(r)} + HL_{n-2}^{(r)} + \binom{n+r-1}{r-1}(2+i+3\varepsilon+4h) + \binom{n+r-2}{r-1}(-1+2i+\varepsilon+3h)$.

**Theorem 2.13.** *If $n \geq 1$ and $r \geq 1$, then there are the following relations:*

*(i)* $HF_n^{(r)}(x) - \left(F_n^{(r)}(x) + iF_{n+1}^{(r)}(x) + \varepsilon F_{n+2}^{(r)}(x) + hF_{n+3}^{(r)}(x)\right) = -\binom{n+r}{r-1}(\varepsilon+hx) - \binom{n+r+1}{r-1}h$,

*where $F_n^{(r)}(x) = \binom{n+r-s-1}{r-1}F_s(x)$ and $F_s(x)$ is the ordinary Fibonacci polynomial,*

*(ii)* $HL_n^{(r)}(x) - \left(L_n^{(r)}(x) + iL_{n+1}^{(r)}(x) + \varepsilon L_{n+2}^{(r)}(x) + hL_{n+3}^{(r)}(x)\right) = -\binom{n+r}{r-1}\left(2i+\varepsilon x+h\left(x^2+2\right)\right) - \binom{n+r+1}{r-1}(2\varepsilon+hx)$

$$-\binom{n+r+2}{r-1}2h,$$

*where $L_n^{(r)}(x) = \binom{n+r-s-1}{r-1}L_s(x)$ and $L_s(x)$ is the ordinary Lucas polynomial.*

*Proof.* (i) The proof is similar to the proof of (ii).

(ii) By using Theorem 2.7, we have

$$\begin{aligned}
&\sum_{s=0}^{n}\binom{n+r-s-1}{r-1}HL_s(x) - \left(L_n^{(r)}(x) + iL_{n+1}^{(r)}(x) + \varepsilon L_{n+2}^{(r)}(x) + hL_{n+3}^{(r)}(x)\right)\\
&= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}\left(L_s(x) + iL_{s+1}(x) + \varepsilon L_{s+2}(x) + hL_{s+3}(x)\right) - \left(\sum_{s=0}^{n}\binom{n+r-s-1}{r-1}L_s(x) + i\sum_{s=0}^{n+1}\binom{(n+1)+r-s-1}{r-1}L_s(x)\right.\\
&\qquad\left. + \varepsilon\sum_{s=0}^{n+2}\binom{(n+2)+r-s-1}{r-1}L_s(x) + h\sum_{s=0}^{n+3}\binom{(n+3)+r-s-1}{r-1}L_s(x)\right)\\
&= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}L_s(x) + i\sum_{s=1}^{n+1}\binom{(n+1)+r-s-1}{r-1}L_s(x) + \varepsilon\sum_{s=2}^{n+2}\binom{(n+2)+r-s-1}{r-1}L_s(x) + h\sum_{s=3}^{n+3}\binom{(n+3)+r-s-1}{r-1}L_s(x)\\
&\quad - \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}L_s(x) - i\sum_{s=0}^{n+1}\binom{(n+1)+r-s-1}{r-1}L_s(x) - \varepsilon\sum_{s=0}^{n+2}\binom{(n+2)+r-s-1}{r-1}L_s(x) - h\sum_{s=0}^{n+3}\binom{(n+3)+r-s-1}{r-1}L_s(x)\\
&= -i\binom{n+r}{r-1}L_0(x) - \varepsilon\left(\binom{n+r}{r-1}L_1(x) + \binom{n+r+1}{r-1}L_0(x)\right) - h\left(\binom{n+r}{r-1}L_2(x) + \binom{n+r+1}{r-1}L_1(x) + \binom{n+r+2}{r-1}L_0(x)\right)\\
&= -i\binom{n+r}{r-1}2 - \varepsilon\left(\binom{n+r}{r-1}x + \binom{n+r+1}{r-1}2\right) - h\left(\binom{n+r}{r-1}\left(x^2+2\right) + \binom{n+r+1}{r-1}x + \binom{n+r+2}{r-1}2\right)\\
&= -\binom{n+r}{r-1}\left(2i+\varepsilon x+h\left(x^2+2\right)\right) - \binom{n+r+1}{r-1}(2\varepsilon+hx) - \binom{n+r+2}{r-1}2h.
\end{aligned}$$

□

**Corollary 2.14.** *If $n \geq 1$ and $r \geq 1$, then the relations are hold:*

*(i)* $HF_n^{(r)} - \left(F_n^{(r)} + iF_{n+1}^{(r)} + \varepsilon F_{n+2}^{(r)} + hF_{n+3}^{(r)}\right) = -\binom{n+r}{r-1}(\varepsilon+h) - \binom{n+r+1}{r-1}h$,

*(ii)* $HL_n^{(r)} - \left(L_n^{(r)} + iL_{n+1}^{(r)} + \varepsilon L_{n+2}^{(r)} + hL_{n+3}^{(r)}\right) = -\binom{n+r}{r-1}(2i+\varepsilon+3h) - \binom{n+r+1}{r-1}(2\varepsilon+h) - \binom{n+r+2}{r-1}2h$,

*where $F_n^{(r)}$ and $L_n^{(r)}$ are the ordinary hyper-Fibonacci and hyper-Lucas numbers, respectively.*

## 3. Conclusion

In this paper, hyper-Fibonacci and hyper-Lucas hybrinomials are defined as a generalization of the Fibonacci and Lucas hybrinomials. For the value x=1, hyper-Fibonacci and hyper-Lucas hybrinomials gave the hybrid numbers, which we called hyper-Fibonacci and hyper-Lucas hybrid numbers. The generating functions, summation formulas and recurrence relations are investigated for the newly defined hybrinomials and hybrid numbers. Hyper-Horadam hybrinomials may be explored in future papers.


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The author contributed to the writing of this paper. The author read and approved the final manuscript.

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