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On (k, μ)-Paracontact Manifold Satisfying Some Curvature Conditions

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Abstract. In this work, we studied the curvature tensors of (k, μ) satisfying the conditions $\widetilde{Z}(\xi, \alpha_3) \cdot P = 0$, $\widetilde{Z}(\xi, \alpha_3) \cdot S = 0$, $R(\xi, \alpha_3) \cdot P = 0$, $R(\xi, \alpha_3) \cdot S = 0$ and $P \cdot C = 0$. Besides this, we classify (k, μ)-paracontact manifolds. Also we researched conformally flat and ϕ-conformally flat a $(k, \mu)-$paracontact metric manifolds.

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1. Introduction

Kaneyuki and Williams presented almost paracontact and parahodge structures on manifolds in 1985 [9]. Zamkovoy conducted a comprehensive analysis of paracontact metric manifolds and their subclasses in 2009 [23]. Following that, B. Cappeletti-Montano, I. Kupeli Erken, and C. Murathan proposed a new form of paracontact geometry known as paracontact metric $(k, \mu)-$spaces, where k and μ are real constants. These kinds of manifolds are referred to as $(k, \mu)-$paracontact metric manifolds. Para-Sasakian manifolds are included in the class of $(k, \mu)-$paracontact metric manifolds [6].

Many writers have investigated semi-symmetric spaces as a generalization of locally symmetric spaces. A semi-Riemannian manifold (M^{2n+1}, g), $n \geq 1$, is said to be semi-symmetric if its curvature tensor R satisfies $R(\alpha_2, \alpha_3) \cdot R = 0$, where $R(\alpha_2, \alpha_3)$ is a derivation of the tensor algebra at each point of the manifold for the tangent vectors α_2, α_3 [10, 17]. If $R(\alpha_2, \alpha_3) \cdot S = 0$, a manifold is said to be Ricci semi-symmetric, with S signifying the Ricci tensor of type (0,2). Mirzoyan created a comprehensive taxonomy of these manifolds [14].

Takahashi proposed the concept of locally $\phi-$symmetric manifolds as a weaker variant of locally symmetric manifolds [18]. In Sasakian geometry. U. C. De. and A. Sarkar investigated the features of the projective curvature tensor in terms of generalized Sasakian space form [8]. M. Atçeken researched generalized Sasakian space forms that satisfy specific concircular curvature tensor requirements [2]. C. Özgür and U. C. De investigated certain curvature requirements met by the quasi-conformal curvature tensor in Kenmotsu manifolds [16]. The studies on contact manifold curvature tensor [1] were created by K. Arslan, C. Murathan, and C. Özgür. Since then several geometers studied curvature conditions and obtain various important properties [2–5, 7, 11–13, 21, 24].

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Inspired by the previous authors research, we classify $(k,\mu)-$paracontact manifolds that fulfill the curvature restrictions in this work. $\widetilde{Z}(\xi,\alpha_3)\cdot P=0$, $\widetilde{Z}(\xi,\alpha_3)\cdot S=0$, $R(\xi,\alpha_3)\cdot P=0$, $R(\xi,\alpha_3)\cdot S=0$ and $P\cdot C=0$ where P is the Weyl projective curvature tensor, $\widetilde{Z}$ is the concircular curvature tensor, S is the Ricci tensor, C is the conformal curvature tensor. In addition, conformally flat and ϕ-conformally flat a $(k,\mu)-$paracontact metric manifolds are considered.

2. Preliminaries

A contact manifold is a $C^\infty-(2n+1)$ manifold M^{2n+1} with a global 1-form η such that $\eta\wedge(d\eta)^n\neq 0$ everywhere on M^{2n+1}. Given such a form η, it is widely known that there exists a single vector field ξ, known as the typical vector field, for which $\eta(\xi)=1$ and $d\eta(\alpha_4,\xi)=0$ for any vector field α_4 on M^{2n+1}. A Riemannian metric g is said to be related if a tensor field ϕ of type $(1,1)$ exists such that

$$\phi^2\alpha_1=\alpha_1-\eta(\alpha_1)\xi,\qquad \eta(\xi)=1,\qquad \eta\circ\phi=0,\qquad \phi\xi=0, \tag{2.1}$$

$$g(\phi\alpha_1,\phi\alpha_3)=-g(\alpha_1,\alpha_3)+\eta(\alpha_1)\eta(\alpha_3),\quad g(\alpha_1,\xi)=\eta(\alpha_1) \tag{2.2}$$

for all α_1,α_3 vector fields on M. The structure (ϕ,ξ,η,g) on M is called to as a paracontact metric structure and the manifold equipped with such a structure is referred to as a paracontact metric manifold [22].

If we construct a $(1,1)$ tensor field h as $h=\frac{1}{2}L_\xi\phi$, where L is the Lie derivative, then h is symmetric and satisfies [22].

$$h\phi=-\phi h,\qquad h\xi=0,\qquad Tr.h=Tr.\phi h=0. \tag{2.3}$$

A normal paracontact metric structure (ϕ,ξ,η,g) satisfies $[\phi,\phi]+2d\eta\otimes\xi=0$, which is equivalent to

$$(\nabla_{\alpha_1}\phi)\alpha_3=-g(\alpha_1,\alpha_3)\xi+\eta(\alpha_3)\alpha_1$$

for any $\alpha_1,\alpha_3\in\chi(M)$ [22]. Any para-Sasakian manifold is K-paracontact and the opposite is true for $n=1$, i.e. for 3-dimensional spaces.

If a paracontact manifold M has a Ricci tensor S of type $(0,2)$, it is said to be η-Einstein.

$$S(\alpha_1,\alpha_3)=ag(\alpha_1,\alpha_3)+b\eta(\alpha_1)\eta(\alpha_3), \tag{2.4}$$

where a,b are smooth functions on M. If $b=0$, then the manifold is known as Einstein [19].

A $(k,\mu)-$paracontact manifold is one that fulfills the curvature tensor R.

$$\widetilde{R}(\alpha_1,\alpha_3)\xi=k\left[\eta(\alpha_3)\alpha_1-\eta(\alpha_1)\alpha_3\right]+\mu\left[\eta(\alpha_3)h\alpha_1-\eta(\alpha_1)h\alpha_3\right] \tag{2.5}$$

for all $\alpha_1,\alpha_3\in\chi(M)$ and k,μ are real constants [6].

This class includes the para-Sasakian manifolds as well as the paracontact metric manifolds that fulfill $R(\alpha_1,\alpha_3)\xi=0$ [23].

If $\mu=0$, the paracontact metric $(k,\mu)-$manifold is referred to as the paracontact metric $N(k)$-manifold. Thus, given a $N(k)$-manifold with a paracontact metric, the curvature tensor fulfills the following relationship.

$$R(\alpha_1,\alpha_3)\xi=k(\eta(\alpha_3)\alpha_1-\eta(\alpha_1)\alpha_3)$$

for all $\alpha_1,\alpha_3\in\chi(M)$. Although the geometric behavior of paracontact metric $(k,\mu)-$spaces varies depending on whether $k-1$, or $k>-1$ is used, there are some similar conclusions for both $k-1$ and $k>-1$ [6].

Lemma 2.1. *There is no such thing as a paracontact $(k,\mu)-$manifold of dimension larger than 3 with $k>-1$, but such manifolds exist for $k-1$ [6].*

The following relationship holds in a $(k,\mu)-$paracontact metric manifold $(M^{2n+1}\phi,\xi,\eta,g)$, $n>1$,

$$h^2=(k+1)\phi^2 \tag{2.6}$$

$$(\widetilde{\nabla}_{\alpha_1}\phi)\alpha_3=-g(\alpha_1-h\alpha_1,\alpha_3)\xi+\eta(\alpha_3)(\alpha_1-h\alpha_1),\ \text{for } k\neq-1, \tag{2.7}$$

$$\begin{aligned}S(\alpha_1,\alpha_3) &= [2(1-n)+n\mu]g(\alpha_1,\alpha_3)+[2(n-1)+\mu]g(h\alpha_1,\alpha_3)\\ &\quad+[2(n-1)+n(2k-\mu)]\eta(\alpha_1)\eta(\alpha_3),\end{aligned} \tag{2.8}$$

$$S(\alpha_1,\xi)=2nk\eta(\alpha_1), \tag{2.9}$$

$$Q\alpha_3 = [2(1-n)+n\mu]\alpha_3 + [2(n-1)+\mu]h\alpha_3 + [2(n-1)+n(2k-\mu)]\eta(\alpha_3)\xi, \tag{2.10}$$

$$Q\xi = 2nk\xi, \tag{2.11}$$

$$Q\phi - \phi Q = 2[2(n-1)+\mu]h\phi \tag{2.12}$$

for any vector fields α_1, α_3 on M^{2n+1} , where Q and S denotes the Ricci operator and Ricci tensor of (M^{2n+1}, g), respectively [6].

K. Yano and S. Sawaki [20] developed the notion of quasi-conformal curvature tensor. A $(2n+1)$-dimensional Riemannian manifolds quasi-conformal curvature tensor is denoted as

$$\begin{aligned}\widetilde{C}(\alpha_1,\alpha_5)\alpha_3 &= aR(\alpha_1,\alpha_5)\alpha_3 + b\{S(\alpha_5,\alpha_3)\alpha_1 - S(\alpha_1,\alpha_3)\alpha_5 \\ &\quad + g(\alpha_5,\alpha_3)Q\alpha_1 - g(\alpha_1,\alpha_3)Q\alpha_5\} \\ &- \frac{\tau}{2n+1}\{\frac{a}{2n} + 2b\}\{g(\alpha_5,\alpha_3)\alpha_1 - g(\alpha_1,\alpha_3)\alpha_5\},\end{aligned} \tag{2.13}$$

where a and b are two scalars and r is the manifolds scalar curvature. In (2.13) if $a = 1$ and $b = \frac{-1}{2n-1}$, the quasi conformal curvature tensor is defined as

$$\begin{aligned}C(\alpha_1,\alpha_5)\alpha_3 &= R(\alpha_1,\alpha_5)\alpha_3 - \frac{1}{2n-1}\{S(\alpha_5,\alpha_3)\alpha_1 - S(\alpha_1,\alpha_3)\alpha_5 + g(\alpha_5,\alpha_3)Q\alpha_1 \\ &- g(\alpha_1,\alpha_3)Q\alpha_5\} + \frac{\tau}{2n(2n-1)}\{g(\alpha_5,\alpha_3)\alpha_1 - g(\alpha_1,\alpha_3)\alpha_5\}.\end{aligned} \tag{2.14}$$

Let (M, g) be a Riemannian manifold of $(2n+1)$ dimensions. The concircular curvature tensor $\widetilde{Z}$ is therefore defined as

$$\widetilde{Z}(\alpha_1,\alpha_5)\alpha_3 = R(\alpha_1,\alpha_5)\alpha_3 - \frac{\tau}{2n(2n+1)}\{g(\alpha_5,\alpha_3)\alpha_1 - g(\alpha_1,\alpha_3)\alpha_5\}, \tag{2.15}$$

for all $\alpha_1, \alpha_5, \alpha_3 \in \chi(M)$, where r is the scalar curvature of M [19].

Let (M, g) be an $(2n+1)$-dimensional Riemannian manifold. The Weyl projective curvature tensor field is defined by

$$P(\alpha_2,\alpha_5)\alpha_1 = R(\alpha_2,\alpha_5)\alpha_1 - \frac{1}{2n}[S(\alpha_5,\alpha_1)\alpha_2 - S(\alpha_2,\alpha_1)\alpha_5] \tag{2.16}$$

for any $\alpha_2, \alpha_5, \alpha_1 \in \chi(M)$, where R denotes the Riemannian curvature tensor of M and Q is the Ricci operator given by $g(Q\alpha_2, \alpha_5) = S(\alpha_2, \alpha_5)$ [19].

3. A (k, μ)−Paracontact Manifold Satisfying Some Curvature Conditions

In this part, we shall provide the significant themes of this work.

Let M be $(2n+1)-$dimensional $(k, \mu)-$paracontact metric manifold and the Riemannian curvature tensor be denoted as (2.5)

$$R(\xi,\alpha_2)\alpha_6 = k(g(\alpha_2,\alpha_6)\xi - \eta(\alpha_6)\alpha_2) + \mu(g(h\alpha_2,\alpha_6)\xi - \eta(\alpha_6)h\alpha_2). \tag{3.1}$$

Assuming $\alpha_1 = \xi$ in (2.5), we obtain

$$R(\xi,\alpha_5)\xi = k(\eta(\alpha_5)\xi - \alpha_5) - \mu h\alpha_5. \tag{3.2}$$

In (3.1), we acquire

$$\eta(R(\xi,\alpha_2)\alpha_6) = k(g(\alpha_2,\alpha_6) - \eta(\alpha_2)\eta(\alpha_6)) + \mu g(h\alpha_2,\alpha_6). \tag{3.3}$$

In the same way choosing $\alpha_1 = \xi$ in (2.14), we have

$$\begin{aligned}C(\xi,\alpha_5)\alpha_3 &= (k - \frac{2nk}{2n+1} + \frac{r}{2n(2n+1)})(g(\alpha_5,\alpha_3)\xi - \eta(\alpha_3)\alpha_5) \\ &\quad + \mu(g(h\alpha_5,\alpha_3)\xi - \eta(\alpha_3)h\alpha_5) - \frac{1}{2n-1}(S(\alpha_5,\alpha_3)\xi \\ &\quad - \eta(\alpha_3)Q\alpha_5).\end{aligned} \tag{3.4}$$

Setting $\alpha_3 = \xi$ in (3.4), we get

$$C(\xi,\alpha_5)\xi = (k - \frac{2nk}{2n+1} + \frac{r}{2n(2n+1)})(\eta(\alpha_5)\xi - \alpha_5) - \mu hY - \frac{1}{2n-1}(2nk\eta(\alpha_5))\xi - QY). \tag{3.5}$$

Also from (2.15), we have

$$\widetilde{Z}(\xi,\alpha_5)\alpha_3 = (k - \frac{r}{2n(2n+1)})(g(\alpha_5,\alpha_3)\xi - \eta(\alpha_3)\alpha_5) + \mu(g(hY,\alpha_3)\xi - \eta(\alpha_3)hY) \tag{3.6}$$

and

$$\widetilde{Z}(\xi,\alpha_5)\xi = (k - \frac{r}{2n(2n+1)})(\eta(\alpha_5)\xi - \alpha_5) - \mu h\alpha_5. \tag{3.7}$$

In (2.16), it follows

$$P(\xi,\alpha_5)\alpha_1 = kg(\alpha_5,\alpha_1)\xi + \mu g(h\alpha_5,\alpha_1)\xi - \mu\eta(\alpha_1)h\alpha_5 - \frac{1}{2n}S(\alpha_5,\alpha_1)\xi. \tag{3.8}$$

Theorem 3.1. *Let M be $(2n+1)-$dimensional a (k,μ)-paracontact manifold. Then, $R(\xi,\alpha_2)\cdot P = 0$ if and only if M is an Einstein manifold.*

Proof. Suppose that $R(\xi,\alpha_2)\cdot P = 0$. This implies that,

$$(R(\xi,\alpha_2)P)(\alpha_1,\alpha_3)\alpha_4 = R(\xi,\alpha_2)P(\alpha_1,\alpha_3)\alpha_4 - P(R(\xi,\alpha_2)\alpha_1,\alpha_3)\alpha_4 - P(\alpha_1,R(\xi,\alpha_2)\alpha_3)\alpha_4 - P(\alpha_1,\alpha_3)R(\xi,\alpha_2)\alpha_4 = 0, \tag{3.9}$$

for any $\alpha_1,\alpha_2,\alpha_3,\alpha_4 \in \chi(M)$. Making use of (3.1), (3.2) and (3.3) in (3.9), we obtain

$$\begin{aligned}(R(\xi,\alpha_2)P)(\alpha_1,\alpha_3)\alpha_4 = {} & k(g(\alpha_2,P(\alpha_1,\alpha_3)\alpha_4)\xi - \eta(P(\alpha_1,\alpha_3)\alpha_4)\alpha_2)) \\ & +\mu(g(h\alpha_2,P(\alpha_1,\alpha_3)\alpha_4)\xi - \eta(R(\alpha_1,\alpha_3)\alpha_4)h\alpha_2) \\ & -P(k(g(\alpha_2,\alpha_1)\xi - \eta(\alpha_1)\alpha_2) + \mu(g(h\alpha_2,\alpha_1)\xi \\ & -\eta(\alpha_1)h\alpha_2),\alpha_3)\alpha_4 - P(\alpha_1,k(g(\alpha_2,\alpha_3)\xi \\ & -\eta(\alpha_3)\alpha_2 + \mu(g(h\alpha_2,\alpha_3)\xi - \eta(\alpha_3)h\alpha_2))\alpha_4 \\ & -P(\alpha_1,\alpha_3)(k(g(\alpha_2,\alpha_4)\xi - \eta(\alpha_4)\alpha_2) \\ & +\mu(g(h\alpha_2,\alpha_4)\xi - \eta(\alpha_4)h\alpha_2)) = 0.\end{aligned} \tag{3.10}$$

Taking $\alpha_1 = \alpha_4 = \xi$, by using (2.5), (3.8) and inner product both sides of (3.10) by $\xi \in \chi(M)$, we arrive

$$\frac{\mu}{2n}S(\alpha_2,h\alpha_3) = k^2g(\alpha_2,\alpha_3) + k\mu g(\alpha_2,h\alpha_3) - \frac{k}{2n}S(\alpha_2,\alpha_3). \tag{3.11}$$

In (3.11), using the equations (2.1), (2.6) and choosing $\alpha_3 \to h\alpha_3$, we have

$$\frac{k}{2n}S(\alpha_2,h\alpha_3) = -\frac{\mu}{2n}(k+1)S(\alpha_2,\alpha_3) + k^2g(\alpha_2,h\alpha_3) + k\mu(1+k)g(\alpha_2,\alpha_3). \tag{3.12}$$

Also from (3.11), (3.12) and (2.4), we conclude

$$S(\alpha_2,\alpha_3) = 2n\frac{\mu k - k^3 + \mu k^2}{\mu^2k + \mu^2 - k^2}g(\alpha_2,\alpha_3).$$

So, M is an Einstein manifold. Conversely, let M be an Einstein manifold, i.e., $S(\alpha_2,\alpha_3) = 2n\frac{\mu k-k^3+\mu k^2}{\mu^2k+\mu^2-k^2}g(\alpha_2,\alpha_3)$, then from (3.9)-(3.12), we have $R(\xi,\alpha_2)\cdot P = 0$. □

Theorem 3.2. *Let M be $(2n+1)-$dimensional a (k,μ)-paracontact manifold. Then, $\widetilde{Z}(\xi,\alpha_4)\cdot P = 0$ if and only if M is an Einstein manifold.*

Proof. Suppose that $\widetilde{Z}(\xi,\alpha_4)\cdot P = 0$. Then, we have

$$(\widetilde{Z}(\xi,\alpha_4)P)(\alpha_1,\alpha_3)\alpha_6 = \widetilde{Z}(\xi,\alpha_4)P(\alpha_1,\alpha_3)\alpha_6 - P(\widetilde{Z}(\xi,\alpha_4)\alpha_1,\alpha_3)\alpha_6 - P(\alpha_1,\widetilde{Z}(\xi,\alpha_4)\alpha_3)\alpha_6 - P(\alpha_1,\alpha_3)\widetilde{Z}(\xi,\alpha_4)\alpha_6 = 0 \tag{3.13}$$

for all $\alpha_1, \alpha_3, \alpha_4, \alpha_6 \in \chi(M)$. In (3.13), using (3.6), (3.7), (2.3) for $\alpha = k - \frac{r}{2n(2n+1)}$, we get

$$\begin{aligned}(\widetilde{Z}(\xi, \alpha_4)P)(\alpha_1, \alpha_3)\alpha_6 &= a(g(\alpha_4, P(\alpha_1, \alpha_3)\alpha_6)\xi - \eta(P(\alpha_1, \alpha_3)\alpha_6)\alpha_4) \\ &+\mu(g(hX, P(\alpha_1, \alpha_3)\alpha_6)\xi - \eta(P(\alpha_1, \alpha_3)\alpha_6)h\alpha_4) \\ &-P(a(g(\alpha_4, \alpha_1)\xi - \eta(\alpha_1)\alpha_4) + \mu(g(h\alpha_4, \alpha_1)\xi \\ &-\eta(\alpha_1)h\alpha_4), \alpha_3)\alpha_6 - P(\alpha_1, a(g(\alpha_4, \alpha_3)\xi - \eta(\alpha_3)\alpha_4) \\ &+\mu(g(h\alpha_4, \alpha_3)\xi - \eta(\alpha_3)h\alpha_4)\alpha_6 - P(\alpha_1, \alpha_3)(a(g(\alpha_4, \alpha_6)\xi \\ &-\eta(\alpha_6)\alpha_4) + \mu(g(h\alpha_4, \alpha_6)\xi - \eta(\alpha_6)h\alpha_4) = 0. \end{aligned} \tag{3.14}$$

Taking $\alpha_1 = \alpha_6 = \xi$, using (3.8) and inner product both sides of (3.14) by $\xi \in \chi(M)$ we obtain,

$$\frac{\mu}{2n} S(\alpha_4, h\alpha_3) = \alpha k g(\alpha_4, \alpha_3) + \mu k g(\alpha_4, h\alpha_3) - \frac{a}{2n} S(\alpha_4, \alpha_3). \tag{3.15}$$

In (3.15), substituting $h\alpha_3$ into α_3 and using the equations (2.1), (2.6), we arrive

$$\frac{a}{2n} S(\alpha_4, h\alpha_3) = -\frac{\mu}{2n}(k+1)S(\alpha_4, \alpha_3) + \alpha k g(\alpha_4, h\alpha_3) + \mu k(k+1)g(\alpha_4, \alpha_3). \tag{3.16}$$

Also from (3.15) and (3.16), we have

$$S(\alpha_4, \alpha_3) = 2nkg(\alpha_4, \alpha_3).$$

Thus, M is an Einstein manifold. Conversely, let M be an Einstein manifold i.e. $S(\alpha_4, \alpha_3) = 2nkg(\alpha_4, \alpha_3)$, then from (3.13)-(3.16), we have $\widetilde{Z}(\xi, \alpha_4) \cdot P = 0$. □

Theorem 3.3. *Let M be $(2n+1)-$dimensional a (k, μ)-paracontact manifold. Then, $R(\xi, \alpha_6) \cdot S = 0$ if and only if M is an Einstein manifold.*

Proof. Suppose that $R(\xi, \alpha_6) \cdot S = 0$. This yields to

$$S(R(\xi, \alpha_6)\alpha_1, \alpha_3) + S(\alpha_1, R(\xi, \alpha_6)\alpha_3) = 0, \tag{3.17}$$

for all $\alpha_1, \alpha_3, \alpha_6 \in \chi(M)$. Taking into account that (3.1) and (3.17), we obtain

$$\begin{aligned} &S(k(g(\alpha_6, \alpha_1)\xi - \eta(\alpha_1)\alpha_6) + \mu(g(h\alpha_6, \alpha_1)\xi - \eta(\alpha_1)h\alpha_6), \alpha_3) \\ &+S(\alpha_1, k(g(\alpha_6, \alpha_3)\xi - \eta(\alpha_3)\alpha_6) + \mu(g(h\alpha_6, \alpha_3)\xi - \eta(\alpha_3)h\alpha_6) = 0. \end{aligned} \tag{3.18}$$

Using (2.9) and choosing $\alpha_1 = \xi$ and inner product both sides of (3.18) by $\xi \in \chi(M)$, we have

$$\mu S(h\alpha_6, \alpha_3) = 2nk^2 g(\alpha_6, \alpha_3) + 2nk\mu g(h\alpha_6, \alpha_3) - kS(\alpha_6, \alpha_3). \tag{3.19}$$

Substituting $h\alpha_6$ into α_6 and putting (2.1), (2.6) in (3.19), we arrive

$$\begin{aligned} kS(h\alpha_6, \alpha_3) &= -\mu(1+k)S(\alpha_6, \alpha_3) + 2nk^2 g(h\alpha_6, \alpha_3) \\ &+2nk\mu(1+k)g(\alpha_6, \alpha_3). \end{aligned} \tag{3.20}$$

Also, from (3.19) and (3.20), we reach a conclusion

$$S(\alpha_6, \alpha_3) = 2nkg(\alpha_6, \alpha_3).$$

As a result, M is an η-Einstein manifold. Contrary, let M be an η-Einstein manifold i.e. $S(\alpha_6, \alpha_3) = 2nkg(\alpha_6, \alpha_3)$, hence from (3.17)-(3.20), we acquire $R(\xi, \alpha_6) \cdot S = 0$. □

Theorem 3.4. *Let M be $(2n+1)-$dimensional a (k, μ)-paracontact manifold. Then, $\widetilde{Z}(\xi, \alpha_4) \cdot S = 0$ if and only if M is an Einstein manifold.*

Proof. Assume that $\widetilde{Z}(\xi, \alpha_4) \cdot S = 0$. This implies that

$$S(\widetilde{Z}(\xi, \alpha_4)\alpha_1, \alpha_6) + S(\alpha_1, \widetilde{Z}(\xi, \alpha_4)\alpha_6) = 0, \tag{3.21}$$

for all $\alpha_1, \alpha_4, \alpha_6 \in \chi(M)$. In (3.21), using (3.6) for $\alpha = k - \frac{r}{2n(2n+1)}$, we obtain

$$\begin{aligned} &S(a(g(\alpha_4, \alpha_1)\xi - \eta(\alpha_1)\alpha_4) + \mu(g(h\alpha_4, \alpha_1)\xi - \eta(\alpha_1)h\alpha_4), \alpha_6) \\ &+S(\alpha_1, a(g(\alpha_4, \alpha_6)\xi - \eta(\alpha_6)\alpha_4) + \mu(g(h\alpha_4, \alpha_6)\xi - \eta(\alpha_6)h\alpha_4) = 0. \end{aligned} \tag{3.22}$$

Using (2.9) and choosing $\alpha_1 = \xi$ and inner product both sides of (3.22) by $\xi \in \chi(M)$, we have

$$\mu S(h\alpha_4, \alpha_6) = 2nkag(\alpha_4, \alpha_6) + 2nk\mu g(h\alpha_4, \alpha_6) - aS(\alpha_4, \alpha_6). \tag{3.23}$$

In (3.23), choosing $\alpha_4 \to h\alpha_4$ and putting (2.1) and (2.6), we acquire

$$\begin{aligned} aS(h\alpha_4, \alpha_6) &= -\mu(1+k)S(\alpha_4, \alpha_6) + 2nkag(h\alpha_4, \alpha_6) \\ &\quad +2nk\mu(1+k)g(\alpha_4, \alpha_6). \end{aligned} \tag{3.24}$$

Also from (3.23) and (3.24), we conclude

$$S(\alpha_4, \alpha_6) = 2nkg(\alpha_4, \alpha_6).$$

Hence, M is an Einstein manifold. Conversely, let M be an Einstein manifold i.e. $S(\alpha_4, \alpha_6) = 2nkg(\alpha_4, \alpha_6)$, then from (3.21)-(3.24), we obtain $\widetilde{Z}(\xi, \alpha_4) \cdot S = 0$. □

Theorem 3.5. *Let M be $(2n+1)-$dimensional a (k, μ)-paracontact manifold. If $P \cdot C = 0$ then, $\mu = 2k + 2 - \frac{2}{n}$.*

Proof. Suppose that $P \cdot C = 0$. Then, we have,

$$\begin{aligned} (P(\alpha_4, \alpha_2)C)(\alpha_5, \alpha_3)\alpha_6 &= P(\alpha_4, \alpha_2)C(\alpha_5, \alpha_3)\alpha_6 - C(P(\alpha_4, \alpha_2)\alpha_5, \alpha_3)\alpha_6 \\ &\quad -C(\alpha_5, P(\alpha_4, \alpha_2)\alpha_3)\alpha_6 \\ &\quad -C(\alpha_5, \alpha_3)P(\alpha_4, \alpha_2)\alpha_6 = 0, \end{aligned} \tag{3.25}$$

where $\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in \chi(M)$. Taking $\alpha_4 = \alpha_6 = \xi$ in (3.25) and using (3.4), for $a = k - \frac{2nk}{2n-1} + \frac{r}{2n(2n+1)}$ and $b = \frac{-1}{2n-1}$, we get

$$\begin{aligned} & P(\xi, \alpha_2)(a(\eta(\alpha_3)\alpha_5 - \eta(\alpha_5)\alpha_3) + \mu(\eta(\alpha_3)h\alpha_5 - \eta(\alpha_5)h\alpha_3) \\ & +b(\eta(\alpha_3)Q\alpha_1 - \eta(\alpha_5)Q\alpha_3) - C(kg(\alpha_2, \alpha_5)\xi + \mu(g(h\alpha_2, \alpha_5)\xi \\ & -\eta(\alpha_5)h\alpha_2) - \frac{1}{2n}S(\alpha_2, \alpha_5)\xi, \alpha_3)\xi - C(\alpha_5, kg(\alpha_2, \alpha_3)\xi \\ & +\mu(g(h\alpha_2, \alpha_3)\xi - \eta(\alpha_3)h\alpha_2) - \frac{1}{2n}S(\alpha_2, \alpha_3)\xi)\xi \\ & +C(\alpha_5, \alpha_3)(\mu h\alpha_2) = 0. \end{aligned} \tag{3.26}$$

In (3.26), putting $\alpha_5 = \xi$ and setting (3.5), (3.8), we arrive

$$\begin{aligned} & \frac{a}{2n}S(\alpha_2, \alpha_3)\xi - \mu kg(\alpha_2, h\alpha_3)\xi + \frac{\mu}{2n}S(\alpha_2, h\alpha_3)\xi \\ & -bkS(\alpha_2, \alpha_3)\xi + \frac{b}{2n}S(\alpha_2, Q\alpha_3)\xi - akg(\alpha_2, \alpha_3)\xi = 0. \end{aligned} \tag{3.27}$$

Using $\alpha_3 = \xi$ and making use of (2.10) and inner product both sides of (3.27) by $\xi \in \chi(M)$, we have

$$(-2nkb + 2b - 2nb + n\mu b)\eta(\alpha_2)\xi = 0. \tag{3.28}$$

In (3.28), we conclude

$$\mu = 2k + 2 - \frac{2}{n}.$$

The converse is obvious. □

4. Conformally Flat A $(k, \mu)-$Paracontact Manifold

In this chapter, we investigate at a $(k, \mu)-$paracontact manifold that is conformally flat.

Theorem 4.1. *Let M be (2n+1)-dimensional $\phi-$conformally flat a $(k, \mu)-$paracontact manifold. Then, M is an $\eta-$Einstein manifold.*

Proof. A $(k,\mu)-$paracontact manifold is said to be conformally flat if

$$g(C(\alpha_1,\alpha_5)\alpha_6,\alpha_3)=0, \tag{4.1}$$

where $\alpha_1,\alpha_3,\alpha_5,\alpha_6\in\chi(M)$.

Let a (2n+1)-dimensional $(k,\mu)-$paracontact manifold M be conformally flat. Then using (4.1) in (2.14), we have

$$\begin{aligned} R(\alpha_1,\alpha_5)\alpha_6 &= \frac{1}{2n-1}[S(\alpha_5,\alpha_6)\alpha_1-S(\alpha_1,\alpha_6)\alpha_5+g(\alpha_5,\alpha_6)Q\alpha_1\\ &\quad -g(\alpha_1,\alpha_6)Q\alpha_5]-\frac{r}{2n(2n-1)}[g(\alpha_5,\alpha_6)\alpha_1-g(\alpha_1,\alpha_6)\alpha_5]. \end{aligned}$$

Taking $\alpha_6=\xi$ and using (2.5) and (2.9) we get

$$\begin{aligned} &k[\eta(\alpha_5)\alpha_1-\eta(\alpha_1)\alpha_5]+\mu[\eta(\alpha_5)h\alpha_1-\eta(\alpha_1)h\alpha_5]=\frac{1}{2n-1}[2nk(\eta(\alpha_5)\alpha_1-\eta(\alpha_1)\alpha_5)\\ &+\eta(\alpha_5)Q\alpha_1-\eta(\alpha_1)Q\alpha_5]-\frac{r}{2n(2n-1)}[\eta(\alpha_5)\alpha_1-\eta(\alpha_1)\alpha_5]. \end{aligned} \tag{4.2}$$

Again putting $\alpha_5=\xi$ and inner product both sides of (4.2) by $\alpha_2\in\chi(M)$, we arrive

$$\begin{aligned} S(\alpha_1,\alpha_2) &= \left[\frac{r-2nk}{2n}\right]g(\alpha_1,\alpha_2)-\left[\frac{r+2nk(2n-1)}{2n}\right]\eta(\alpha_1)\eta(\alpha_2)\\ &\quad +\mu g(h\alpha_1,\alpha_2). \end{aligned} \tag{4.3}$$

From (4.3) and also using (2.8), for the sake of brevity, we put

$$\begin{aligned} p_1 &= \frac{[2(n-1)+\mu]-\mu(2n-1)}{(2n-1)[2(n-1)+\mu]},\\ p_2 &= \frac{r-2nk}{2n(2n-1)}+\frac{\mu[2(1-n)+\mu n]}{[2(n-1)+\mu]},\\ p_3 &= -\left[\frac{r+2nk(2n-1)}{2n(2n-1)}\right]+\frac{2[2(n-1)+n(2k-\mu)]}{[2(n-1)+\mu]} \end{aligned}$$

and we conclude

$$S(\alpha_1,\alpha_2)=\frac{p_2}{p_1}g(\alpha_1,\alpha_2)+\frac{p_3}{p_1}\eta(\alpha_1)\eta(\alpha_2).$$

Thus, M is an $\eta-$Einstein manifold. Conversely, let M be an $\eta-$Einstein manifold i.e. $S(\alpha_1,\alpha_2)=\frac{p_2}{p_1}g(\alpha_1,\alpha_2)+\frac{p_3}{p_1}\eta(\alpha_1)\eta(\alpha_2)$, $(p_1\neq 0)$ then from (4.1)-(4.3), we have $g(C(\alpha_1,\alpha_5)\alpha_6,\alpha_3)=0$. □

5. $\phi-$Conformally Flat A $(k,\mu)-$Paracontact Manifold

In this part, we look at $\phi-$conformally flat a $(k,\mu)-$paracontact manifold.

Theorem 5.1. *Let M be (2n+1)-dimensional $\phi-$conformally flat a $(k,\mu)-$paracontact manifold. Then, M is an $\eta-$Einstein manifold.*

Proof. A $(k,\mu)-$paracontact manifold is said to be $\phi-$conformally flat if

$$g(C(\phi\alpha_2,\phi\alpha_4)\phi\alpha_6,\phi\alpha_3)=0, \tag{5.1}$$

for all $\alpha_2,\alpha_3,\alpha_4,\alpha_6\in\chi(M)$. So by the use of (2.7), (2.11), (2.12) and (2.14) $\phi-$conformally flat means

$$\begin{aligned} R(\phi\alpha_2,\phi\alpha_4,\phi\alpha_6,\phi\alpha_3) &= +\frac{1}{2n-1}[S(\phi\alpha_4,\phi\alpha_6)g(\phi\alpha_2,\phi\alpha_3)-S(\phi\alpha_2,\phi\alpha_6)g(\phi\alpha_4,\phi\alpha_3)\\ &\quad +S(\phi\alpha_2,\phi\alpha_3)g(\phi\alpha_4,\phi\alpha_6)-S(\phi\alpha_4,\phi\alpha_3)g(\phi\alpha_2,\phi\alpha_6)]\\ &\quad -\frac{r}{2n(2n-1)}[g(\phi\alpha_4,\phi\alpha_6)g(\phi\alpha_2,\phi\alpha_3)\\ &\quad -g(\phi\alpha_2,\phi\alpha_6)g(\phi\alpha_4,\phi\alpha_3)]. \end{aligned} \tag{5.2}$$

Let $\{e_{i,}..., e_{2n}, \xi\}$ be a local orthonormal basis of vector fields in M. Putting $\alpha_2 = \alpha_3 = e_i$ in (5.2) and summing up from 1 to $2n$, we have

$$\begin{aligned}\sum_{i=1}^{2n} R(\phi e_i, \phi\alpha_4, \phi\alpha_6, \phi e_i) \quad = \quad & \frac{1}{2n-1}\sum_{i=1}^{2n}[S(\phi\alpha_4, \phi\alpha_6)g(\phi e_i, \phi e_i) - S(\phi e_i, \phi\alpha_6)g(\phi\alpha_4, \phi e_i) \\ & + S(\phi e_i, \phi e_i)g(\phi\alpha_4, \phi\alpha_6) - S(\phi\alpha_4, \phi e_i)g(\phi e_i, \phi\alpha_6)] \\ & - \frac{r}{2n(2n-1)}\sum_{i=1}^{2n}[g(\phi\alpha_4, \phi\alpha_6)g(\phi e_i, \phi e_i) \\ & - g(\phi e_i, \phi\alpha_6)g(\phi\alpha_4, \phi e_i)]. \end{aligned} \tag{5.3}$$

It is simple to demonstrate that

$$\sum_{i=1}^{2n} g(R(\phi e_i, \phi\alpha_4)\phi\alpha_6, \phi e_i) = S(\phi\alpha_4, \phi\alpha_6) + g(\phi\alpha_4, \phi\alpha_6), \tag{5.4}$$

$$\sum_{i=1}^{2n} S(\phi e_i, \phi e_i) = r + 2n, \tag{5.5}$$

$$\sum_{i=1}^{2n} S(\phi\alpha_4, \phi e_i)g(\phi e_i, \phi\alpha_6) = S(\phi\alpha_4, \phi\alpha_6), \tag{5.6}$$

$$\sum_{i=1}^{2n} g(\phi e_i, \phi e_i) = 0 \tag{5.7}$$

and

$$\sum_{i=1}^{2n} g(\phi\alpha_4, \phi e_i)g(\phi e_i, \phi\alpha_6) = g(\phi\alpha_4, \phi\alpha_6). \tag{5.8}$$

As a result of (5.4)-(5.8), the equation (5.3) can be written as

$$S(\phi\alpha_4, \phi\alpha_6) = \frac{8n^2 + 2nr - 2n + r}{2n-1} g(\phi\alpha_4, \phi\alpha_6). \tag{5.9}$$

Then, by making use of (2.1) and (2.2), the equation (5.9) takes the form

$$S(\alpha_4, \alpha_6) = -\frac{8n^2 + 2nr - 2n + r}{2n-1} g(\alpha_4, \alpha_6) + (2nk + \frac{8n^2 + 2nr - 2n + r}{2n-1})\eta(\alpha_4)\eta(\alpha_6),$$

which implies M is an $\eta-$Einstein manifold. Conversely, let M be an $\eta-$Einstein manifold i.e. $S(\alpha_4, \alpha_6) = -\frac{8n^2+2nr-2n+r}{2n-1} g(\alpha_4, \alpha_6) + (2nk + \frac{8n^2+2nr-2n+r}{2n-1})\eta(\alpha_4)\eta(\alpha_6)$ then from (5.1)-(5.9), we have $g(C(\phi\alpha_2, \phi\alpha_4)\phi\alpha_6, \phi\alpha_3) = 0$. □

6. Conclusion

In this paper, we studied the curvature tensors of (k, μ) satisfying the conditions $\widetilde{Z}(\xi, \alpha_3) \cdot P = 0$, $\widetilde{Z}(\xi, \alpha_3) \cdot S = 0$, $R(\xi, \alpha_3) \cdot P = 0$, $R(\xi, \alpha_3) \cdot S = 0$ and $P \cdot C = 0$. Besides this, we classify (k, μ)-paracontact manifolds. Also we researched conformally flat and ϕ-conformally flat a $(k, \mu)-$paracontact metric manifolds.

7. Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

8. Authors Contribution Statement

All authors have contributed sufficiently in the planning, execution or analysis of this study to be included as authors. All authors jointly worked on the results and they have read and agreed to the published version of the manuscript.

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