J. Algebra Comb. Discrete Appl.
3(2) • 91–96

Received: 18 November 2015
Accepted: 2 Feburary 2016

Journal of Algebra Combinatorics Discrete Structures and Applications

Matrix rings over a principal ideal domain in which elements are nil-clean

Research Article

Somayeh Hadjirezaei, Somayeh Karimzadeh

Abstract: An element of a ring R is called nil-clean if it is the sum of an idempotent and a nilpotent element. A ring is called nil-clean if each of its elements is nil-clean. S. Breaz et al. in [1] proved their main result that the matrix ring $\mathbb{M}_n(F)$ over a field F is nil-clean if and only if $F \cong \mathbb{F}_2$, where $\mathbb{F}_2$ is the field of two elements. M. T. Koşan et al. generalized this result to a division ring. In this paper, we show that the $n \times n$ matrix ring over a principal ideal domain R is a nil-clean ring if and only if R is isomorphic to $\mathbb{F}_2$. Also, we show that the same result is true for the 2×2 matrix ring over an integral domain R. As a consequence, we show that for a commutative ring R, if $\mathbb{M}_2(R)$ is a nil-clean ring, then $\dim R = 0$ and $\operatorname{char} R/J(R) = 2$.

2010 MSC: 15A23, 15B33, 16S50

Keywords: Nil-clean matrix, Idempotent matrix, Nilpotent matrix, Principal ideal domain

1. Introduction

Throughout this paper, all rings are associative with identity. An element in a ring R is said to be (strongly) clean if it is the sum of an idempotent and a unit element(and these commute). A (strongly) clean ring is one in which every element is (strongly) clean. Local rings are obviously strongly clean. Strongly clean rings were introduced by Nicholson [8]. An element in a ring R is said to be (strongly) nil-clean if it is the sum of an idempotent and a nilpotent element(and these commute). A (strongly) nil-clean ring is one in which every element is (strongly) nil-clean. It is easy to see that every strongly nil-clean element is strongly clean and that every nil-clean ring is clean ([3, Proposition 3.1.3]). Nil-clean rings were extensively investigated by Diesl in [3] and [4]. S. Breaz et al. in [1] proved their main result that the matrix ring $\mathbb{M}_n(F)$ over a field F is nil-clean if and only if $F \cong \mathbb{F}_2$, where $\mathbb{F}_2$ is the field of two elements. M. T. Koşan et al. in [6], generalized this result to a division ring. That is, the matrix ring $\mathbb{M}_n(D)$ over a division ring D is nil-clean if and only if $D \cong \mathbb{F}_2$. We show that this is true for a principal ideal domain (PID).

Somayeh Hadjirezaei (Corresponding Author), Somayeh Karimzadeh; Vali-e-Asr University of Rafsanjan (email: s.hajirezaei@vru.ac.ir, karimzadeh@vru.ac.ir).

http://dx.doi.org/10.13069/jacodesmath.82415 ISSN 2148-838X

Throughout this paper an integral domain is a commutative ring without zero divisors and the Jacobson radical of a ring is denoted by $J(R)$. We write $\mathbb{M}_n(R)$ for the $n \times n$ matrix ring over R, I_n for the $n \times n$ identity matrix.

2. Main results

First, we recall from [5, Proposition VII.2.11], the following Proposition.

Proposition 2.1. *If A is an $n \times m$ matrix of rank $r > 0$ over a principal ideal domain R, then A is equivalent to a matrix of the form $\begin{pmatrix} L_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, where L_r is an $r \times r$ diagonal matrix with nonzero diagonal entries $d_1, ..., d_r$ such that $d_1 \mid ... \mid d_r$. The ideals $(d_1), ..., (d_r)$ in R are uniquely determined by the equivalence class of A.*

Further, we use the following lemmas.

Lemma 2.2. *(See [4, Proposition 3.14]) Let R be a nil-clean ring. Then the element 2 is (central) nilpotent and, as such, is always contained in $J(R)$.*

Lemma 2.3. *(See [9, Corollary 5]) Let A be an $n \times n$ idempotent matrix over a ring R. If A is equivalent to a diagonal matrix, then A is similar to a diagonal matrix.*

Next Lemmas are the main results of [1] and [6].

Lemma 2.4. *(See [1, Theorem 3]) Let F be a field and let $n \geq 1$. Then $\mathbb{M}_n(F)$ is a nil-clean ring if and only if $F \cong \mathbb{F}_2$.*

Lemma 2.5. *(See [6, Theorem 3]) Let D be a division ring and let $n \geq 1$. Then $\mathbb{M}_n(D)$ is a nil-clean ring if and only if $D \cong \mathbb{F}_2$.*

Theorem 2.6. *Let R be a principal ideal domain and let $n \geq 1$. Then $\mathbb{M}_n(R)$ is a nil-clean ring if and only if $R \cong \mathbb{F}_2$.*

Proof. If $R \cong \mathbb{F}_2$, then by Lemma 2.4, $\mathbb{M}_n(R)$ is a nil-clean ring.
Now, assume that $\mathbb{M}_n(R)$ is a nil-clean ring. By Lemma 2.2, $2I_n$ is a nilpotent element. Thus $2 = 0$ in R, because R is an integral domain. Proof in the case $n = 1$ is obvious, so assume that $n > 1$. Take $a \in R \setminus \{0, 1\}$ and put

$$A = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = E + N,$$

where E is an idempotent element and N is a nilpotent element of $\mathbb{M}_n(R)$. By Proposition 2.1, E is equivalent to a diagonal matrix. Thus by Lemma 2.3, E is similar to a diagonal matrix where it's entries are 0 and 1. Hence $U^{-1}EU = \begin{pmatrix} I_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, for some invertible matrix $U = (u_{ij}) \in \mathbb{M}_n(R)$. Therefore

$$U^{-1}AU = \begin{pmatrix} I_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + N', \tag{1}$$

where $N' = U^{-1}NU$ is a nilpotent element. Since a is not nilpotent, hence $U^{-1}AU$ is not nilpotent, so $k \geq 1$. If $k = n$, then $A = I_n + N$ is invertible, a contradiction because $\det A = 0$. Thus $1 \leq k < n$. Since $I_n + N'$ is invertible, $U(I_n + N')$ is invertible. We have

$$U(I_n + N') = U \begin{pmatrix} I_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + UN' + U \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{n-k} \end{pmatrix} = AU + U \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{n-k} \end{pmatrix}$$

$$= \begin{pmatrix} au_{11} & \dots & au_{1n} \\ 0 & \dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots & 0 \end{pmatrix} + \begin{pmatrix} 0 & \dots & 0 & u_{1(k+1)} & \dots & u_{1n} \\ 0 & \dots & 0 & u_{2(k+1)} & \dots & u_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & \dots & 0 & u_{n(k+1)} & \dots & u_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} au_{11} & \dots & au_{1k} & (1+a)u_{1(k+1)} & \dots & (1+a)u_{1n} \\ 0 & \dots & 0 & u_{2(k+1)} & \dots & u_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & u_{n(k+1)} & \dots & u_{nn} \end{pmatrix}.$$

We imply that $k = 1$ and $u_{11} \neq 0$. Thus

$$U(I_n + N') = \begin{pmatrix} au_{11} & (1+a)u_{12} & \dots & (1+a)u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & u_{n2} & \dots & u_{nn} \end{pmatrix}.$$

Put

$$U_1 := \begin{pmatrix} u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \vdots \\ u_{n2} & \dots & u_{nn} \end{pmatrix}.$$

Since $\det(U(I_n + N')) = au_{11} \det U_1$, U_1 is invertible in $\mathbb{M}_{n-1}(R)$ and u_{11} is invertible in R, hence (1) implies that

$$\begin{pmatrix} a & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} U = U \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + UN'.$$

This implies that

$$\begin{pmatrix} u_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & U_1^{-1} \end{pmatrix} \begin{pmatrix} a & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} u_{11} & \mathbf{0} \\ \mathbf{0} & U_1 \end{pmatrix} \begin{pmatrix} u_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & U_1^{-1} \end{pmatrix} U$$

$$= \begin{pmatrix} u_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & U_1^{-1} \end{pmatrix} U \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} u_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & U_1^{-1} \end{pmatrix} UN',$$

i.e.,

$$\begin{pmatrix} a & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} V = V \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + VN', \tag{2}$$

where $V = \begin{pmatrix} u_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & U_1^{-1} \end{pmatrix} U = \begin{pmatrix} 1 & X \\ Y & I_{n-1} \end{pmatrix}$. Let $V^{-1} = \begin{pmatrix} c & X' \\ Y' & C_1 \end{pmatrix}$. From $VV^{-1} = V^{-1}V = I_n$, it follows that

$$1 = c + XY' = c + X'Y$$

$$I_{n-1} = YX' + C_1 = Y'X + C_1$$

$$0 = X' + XC_1 = cX + X'$$

$$0 = cY + Y' = Y' + C_1 Y.$$

Since $2 = 0$ in R (by Lemma 2.2, and since R is an integral domain) hence, we have $1 = -1$ in R, so $c = -c$. Therefore $X' = -cX = cX, Y' = cY$ and $C_1 = I_{n-1} + YX' = I_{n-1} + cYX$. Also, $1 = c + XY' = c + cXY = c(1 + XY)$, so c is a unit element of R and

$$XY = 1 + c^{-1}. \tag{3}$$

Hence $V^{-1} = \begin{pmatrix} c & cX \\ cY & I_{n-1} + cYX \end{pmatrix}$. If $XY = 0$, then $c = 1$ and $V^{-1} = \begin{pmatrix} 1 & X \\ Y & I_{n-1} + YX \end{pmatrix}$. Then by (2),

$$N'' := VNV^{-1} = \begin{pmatrix} a & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + V \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} V^{-1} = \begin{pmatrix} 1+a & X \\ Y & YX \end{pmatrix},$$

and, for $k \geq 1$,
$N''^{k+1} = \begin{pmatrix} (1+a)^{k+1} & (1+a)^{k+1}X \\ (1+a)^{k+1}Y & (1+a)^{k+1}YX \end{pmatrix} \neq 0$ (as $(1+a) \neq 0$). This is a contradiction because N'' is a nilpotent matrix. Therefore $XY \neq 0$. From (2) it follows that

$$\begin{pmatrix} 1 & X \\ \mathbf{0} & I_{n-1} \end{pmatrix} \begin{pmatrix} a & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} 1 & X \\ \mathbf{0} & I_{n-1} \end{pmatrix} \begin{pmatrix} 1 & X \\ \mathbf{0} & I_{n-1} \end{pmatrix} V$$

$$= \begin{pmatrix} 1 & X \\ \mathbf{0} & I_{n-1} \end{pmatrix} V \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} 1 & X \\ \mathbf{0} & I_{n-1} \end{pmatrix} VN',$$

i.e.,

$$\begin{pmatrix} a & X \\ \mathbf{0} & \mathbf{0} \end{pmatrix} P = P \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + PN', \tag{4}$$

where

$$P = \begin{pmatrix} 1 & X \\ \mathbf{0} & I_{n-1} \end{pmatrix} V = \begin{pmatrix} 1 & X \\ \mathbf{0} & I_{n-1} \end{pmatrix} \begin{pmatrix} 1 & X \\ Y & I_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 + XY & X + X \\ Y & I_{n-1} \end{pmatrix}.$$

Since $2 = 0$ in R, hence $X + X = 2X = 0$. Also by (3), we have $XY - 1 = XY + 1 = c^{-1}$. Hence $P = \begin{pmatrix} c^{-1} & \mathbf{0} \\ Y & I_{n-1} \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} c & \mathbf{0} \\ cY & I_{n-1} \end{pmatrix}$. It follows from (4) that

$$\triangle := PNP^{-1} = \begin{pmatrix} a & aX \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + P \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} P^{-1} = \begin{pmatrix} 1+a & aX \\ cY & \mathbf{0} \end{pmatrix}.$$

If Q is an $n \times n$ matrix, then we will write Q in block form $Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$, where $Q_{11}, Q_{12}, Q_{21}, Q_{22}$ have size 1×1, $1 \times (n-1)$, $(n-1) \times 1$ and $(n-1) \times (n-1)$, respectively. For $k \geq 1$ we have
$\triangle^{k+1} = \triangle^k \triangle = \begin{pmatrix} (\triangle^k)_{11} & (\triangle)^k_{12} \\ (\triangle^k)_{21} & (\triangle^k)_{22} \end{pmatrix} \begin{pmatrix} 1+a & aX \\ cY & \mathbf{0} \end{pmatrix}$

$$= \begin{pmatrix} (\triangle^k)_{11}(1+a) + (\triangle)^k_{12} cY & a(\triangle^k)_{11} X \\ (\triangle^k)_{21}(1+a) + (\triangle^k)_{22} cY & a(\triangle^k)_{21} X \end{pmatrix}. \tag{5}$$

An easy induction shows that there exist $a_k, b_k, c_k \in R$ such that for $k \geq 1$ we have

$$(\triangle^k)_{12} = b_k X, (\triangle^k)_{21} = c_k Y, (\triangle^k)_{22} = a_k YX. \tag{6}$$

Since $\triangle$ is a nilpotent matrix and $\triangle_{21} = cY \neq 0$, there exists a positive integer s such that $(\triangle^{s+1})_{21} = 0$ but $(\triangle^s)_{21} \neq 0$. Then by (5) and (6),

$$\triangle^{s+1} = \begin{pmatrix} (\triangle^{s+1})_{11} & (\triangle^{s+1})_{12} \\ \mathbf{0} & c_s aYX \end{pmatrix},$$

where $c_s a \neq 0$. For $r \in R$, it is easily seen that $rYX = 0$ if and only if $r = 0$. We have $(\triangle^{s+1})^k_{22} = (c_s a)^k (XY)^{k-1} YX$. Since $c_s a \neq 0$ and $XY \neq 0$, hence $(\triangle^{s+1})^k_{22} \neq 0$, for $k \geq 2$. It is a contradiction because $\triangle$ is nilpotent. □

Theorem 2.7. *Let R be an integral domain . If $M_n(R)$ is a nil-clean ring, then R is a field.*

Proof. Let Q be the field of fractions of R and $0 \neq a \in R$. We know that aI_n is nil-clean. So, $aI_n = E + N$ with E idempotent and N nilpotent. We have $I_n = a^{-1}E + a^{-1}N$, in $M_n(Q)$. Thus $a^{-1}E$ (and consequently E) is invertible in $M_n(Q)$. Since E is idempotent, so $E = I_n$. Therefore aI_n is invertible, hence R is a field. □

Lemma 2.8. *Let R be an integral domain and $0, I_2 \neq A \in \mathbb{M}_2(R)$. Then A is idempotent if and only if* $\operatorname{rank}(A) = 1$ *and* $\operatorname{tr}(A) = 1$.

Proof. By [2, Lemma 1.5]. □

Lemma 2.9. *Let R be an integral domain. If $A \in \mathbb{M}_n(R)$ be a nilpotent matrix, then* $\det(A) = 0$.

Proof. Let A be a nonzero nilpotent matrix. Thus there exists some $k \in \mathbb{N}$ such that $A^k = 0$. Thus $\operatorname{adj}(A)A^k = 0$. Hence $\det(A)A^{k-1} = 0$. So $\det(A)\operatorname{adj}(A)A^{k-1} = 0$. Therefore $(\det(A))^2 A^{k-2} = 0$. Continuing this process we have $(\det(A))^{k-1}A = 0$. Since R is an integral domain and $A \neq 0$, hence $\det(A) = 0$ □

Theorem 2.10. *Let R be an integral domain. Then $\mathbb{M}_2(R)$ is a nil-clean ring if and only if $R \cong \mathbb{F}_2$.*

Proof. $\Longleftarrow$)This is by Theorem 2.6.
$\Longrightarrow$) Assume that R is not isomorphic to $\mathbb{F}_2$. So, there exists $a \in R \setminus \{0, 1\}$. Put $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = E + N$, where E is idempotent and N is a nilpotent matrix. If $E = I_2$, then A is invertible, a contradiction. If $E = 0$, then A is nilpotent. Hence $a = 0$, a contradiction. So by Lemma 2.8, $E = \begin{pmatrix} e & b \\ c & 1-e \end{pmatrix}$, where $e, b, c \in R$ and $e(1-e) = bc$. Hence $N = \begin{pmatrix} n & -b \\ -c & -(1-e) \end{pmatrix}$, for some $n \in R$. By Lemma 2.9, $-n(1-e) = bc$. Therefore $e(1-e) = -n(1-e)$. If $e \neq 1$, then $e = -n$. So $N = -E$, a contradiction. Thus $e = 1$ and $bc = 0$. Hence $b = 0$ or $c = 0$. We consider two cases.
Case 1) Let $b = 0$. So $N = \begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix}$. Since N is nilpotent, hence there exists a positive integer k such that $n^k = 0$. So $n = 0$. Therefore $a = 1$.
Case 2) Let $c = 0$. Thus $N = \begin{pmatrix} n & -b \\ 0 & 0 \end{pmatrix}$. Since N is nilpotent, hence there exists a positive integer k such that $n^k = 0$. So $n = 0$. Therefore $a = 1$. □

Let R be a commutative ring with identity. By a chain of prime ideals of R we mean a finite strictly increasing sequence of prime ideals of R of the type $P_o \subsetneqq P_1 \subsetneqq P_2 \subsetneqq ... \subsetneqq P_n$. The integer n is called the length of the chain.

Definition 2.11. *The Krull dimension of R is the supremum of all lengths of chains of prime ideals of R. Krull dimension of R is denoted by $dimR$.*

Corollary 2.12. *Let R be a commutative ring. If $\mathbb{M}_2(R)$ is a nil-clean ring, then $dimR = 0$ and $charR/J(R) = 2$.*

Proof. Let P be a prime ideal of R. We have $\mathbb{M}_2(R/P) = \mathbb{M}_2(R)/\mathbb{M}_2(P)$ is nil-clean. Hence by Theorem 2.10, $R/P \cong \mathbb{F}_2$. So P is a maximal ideal of R and $2 \in J(R)$. Therefore $\text{char}R/J(R) = 2$. □

Remark 2.13. *Note that all of these results can also be obtained as some consequences of [7, Theorem 6.1].*

Acknowledgment: The authors are grateful to the referees' invaluable comments, which helped to improve our study.

References

[1] S. Breaz, G. Călugăreanu, P. Danchev, T. Micu, Nil-clean matrix rings, Linear Algebra Appl. 439(10) (2013) 3115-3119.
[2] J. Chen, X. Yang, Y. Zhou, On strongly clean matrix and triangular matrix rings, Comm. Algebra. 34(10) (2006) 3659–3674.
[3] A. J. Diesl, Classes of strongly clean rings, Ph. D. thesis, University of California, Berkeley, 2006.
[4] A. J. Diesl, Nil clean rings, J. Algebra. 383 (2013) 197–211.
[5] T. W. Hungerford, Algebra, Springer-Verlag, 1980.
[6] M.T. Koşan, T. K. Lee, Y. Zhou, When is every matrix over a division ring a sum of an idempotent and a nilpotent?, Linear Algebra Appl. 450 (2014) 7–12.
[7] T. Koşan, Z. Wang, Y. Zhou, Nil-clean and strongly nil-clean rings, J. Pure Appl. Algebra. 220(2) (2016) 633–646.
[8] W. K. Nicholson, Strongly clean rings and Fitting's lemma, Comm. Algebra. 27(8) (1999) 3583–3592.
[9] G. Song, X. Guo, Diagonability of idempotent matrices over noncommutative rings, Linear Algebra Appl. 297(1-3) (1999) 1–7.