The Laplace Transform For The Ergodic Distribution Of A Semi-Markovian Random Walk Process With Reflecting And Delaying Barriers

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Abstract

In this paper, a semi-Markovian random walk process with reflecting barrier on the zero-level and delaying barrier on the $\beta(\beta > 0)$-level is constructed and the the Laplace transform for the ergodic distribution of this process is expressed by means of the probability characteristics of random walk $\{Y_n: n \geq 1\}$ and renewal process $\{T_n: n \geq 1\}$.

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1. INTRODUCTION

It is known that the most of the problems of stock control theory is often given by means of random walks or random walks with delaying barriers(see References Afanas'eva & Bulinska (1983), Khaniev & Ünver (1997), etc.). But, for the problem considered in this study, one of the barriers is reflecting and the other one is delaying, and the process representing the quantity of the stock has been given by using a random walk and a renewal process. Such models were rarely considered in literature. The practical state of the problem mentioned above is as follows.

Suppose that some quantity of a stock in a certain warehouse is increasing or decreasing in random discrete portions depending to the demands at discrete times. Then, it is possible to characterize the level of stock by a process called the semi-Markovian random walk process. The processes of this type have been widely studied in literature (see,for example References

Afanas'eva & Bulinska (1983), Gihman & Skorohod (1975) and Lotov (1991), etc.). But sometimes some problems occur in stock control theory such that in order to get an adequate solution we have to consider some processes which are more complex than semi-Markovian random walk processes. For example, if the borrowed quantity is demanded to be added to the warehouse immediately when the qunatity of demanded stock is more than the total quantity of stock in the warehouse then, it is possible to characterize the level of stock in the warehouse by a stochastic process called as semi-Markovian random walk processes with reflecting barrier. But for the model studied in this study an additional condition has been considered. Since the volume of warehouse is finite in real cases, the supply coming to the warehouse is stopped until the next demand when the warehouse becomes full. In order to characterize the quantity of stock in the warehouse under these conditions it is necessary to use a stochastic process called as semi-Markovian random walk process with two barriers in which one of them is reflecting and the other one is delaying. Note that semi-Markovian random walk processes with two barriers, namely reflecting and delaying, have not been considered enough in literature (see, for example References Lotov (1998); Maden (2001)).

This type problems may ocur, for example, in the control of military stocks, refinery stocks, reserve of oil wells, and etc.

The Model. Assume that we observe random motion of a particle, initially at the position $X_0 \in [0,\beta]$, $\beta > 0$, in a stripe bounded by two barriers; the one lying on the zero-level as reflecting and the other lying on β -level as delaying. Furthermore, assume that this motion proceeds according to the following rules: After staying at the position X_0 for as much as random duration ξ_1, the particle wants to reach the position $X_0 + \eta_1$. If $X_0 + \eta_1 > \beta$ then the particle will be kept at the position $X_1 = \beta$ since there is delaying barrier at β -level. If $X_0 + \eta_1 \in [0,\beta]$, then the particle will be at the position $X_1 = X_0 + \eta_1$. Since there is a reflecting barrier at zero-level, when $X_0 + \eta_1 < 0$ the particle will reflect from this barrier as long as $|X_0 + \eta_1|$. In this case, if $|X_0 + \eta_1| \leq \beta$ then the particle will be kept at the position $X_1 = |X_0 + \eta_1|$ and if $|X_0 + \eta_1| > \beta$ then the particle will be at the position β, so the position of the particle will be $X_1 = min\{\beta, |X_0 + \eta_1|\}$.

After staying at the position X_1 for as much as random duration ξ_2 again it will jump to the position $X_2 = min\{\beta, |X_1 + \eta_2|\}$ according to the above mentioned rules. Thus at the end of n-th jump, the particle will be at the position $X_n = min\{\beta, |X_{n-1} + \eta_n|\}$, $n \geq 1$.

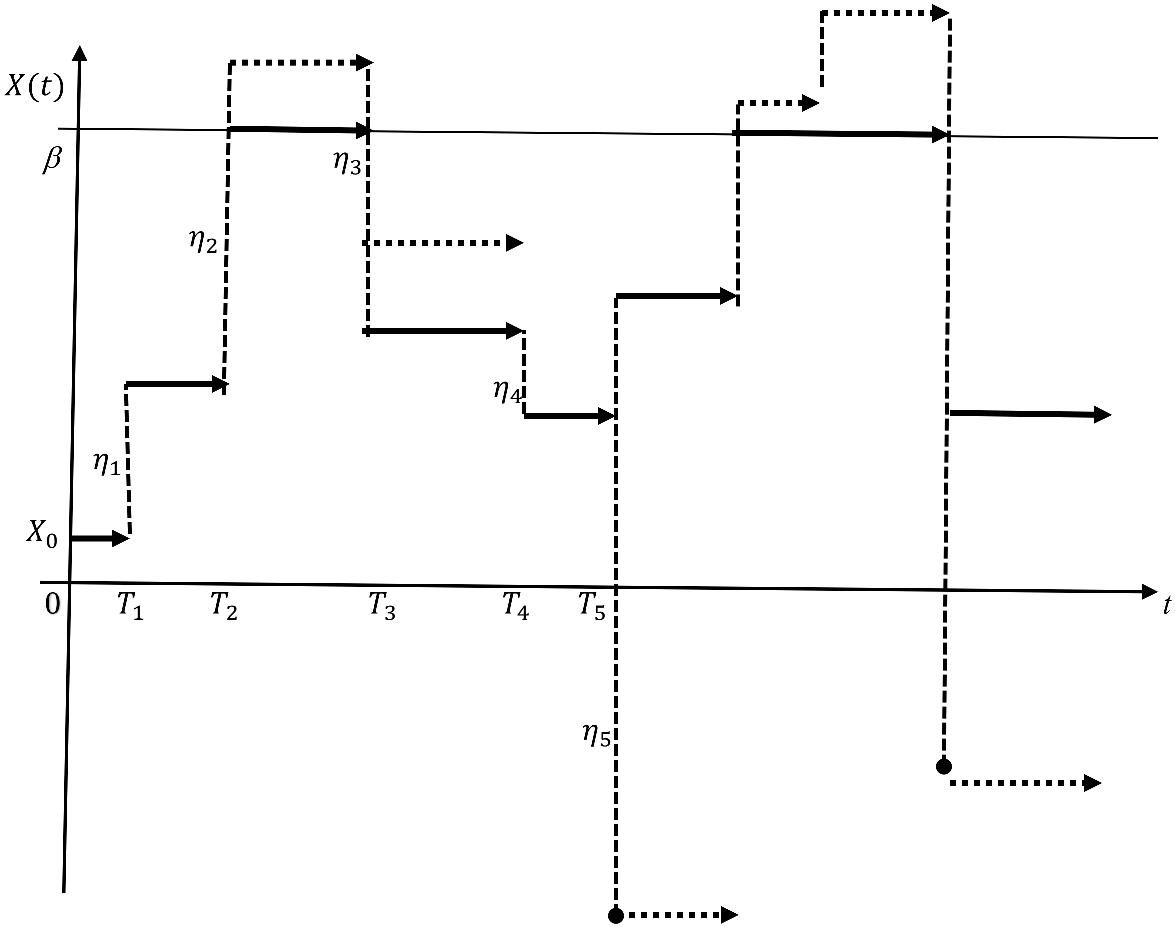

Fig. 1. A View of Semi-Markovian Random Walk Process with Reflecting Barrier on the Zero-level and Delaying Barrier on the $\beta(\beta > 0)$ -level

2. CONSTRUCTION OF THE PROCESS

Suppose $\{(\xi_i, \eta_i)\}$, $i = 1,2,3, \ldots$ is a sequence of identically and independently distributed pairs of random variables, defined in any probability space $(\Omega, \mathcal{F}, P)$ such that ξ_i's are positive valued, i.e., $P\{\xi_i > 0\} = 1, i = 1,2,3, \ldots$. Also let us denote the distribution function of ξ_1 and η_1

$$\Phi(t) = P\{\xi_1 < t\}, \mathrm{F}(x) = P\{\eta_1 < x\}, t \in \mathbb{R}^+, x \in \mathbb{R},$$

respectively. Before stating the corresponding process, let us construct the following sequences of random variables:

$$T_n = \sum_{i=1}^{n} \xi_i, Y_n = \sum_{i=1}^{n} \eta_i, n \geq 1, T_0 = Y_0 = 0.$$

Then the processes $\{T_n: n \geq 1\}$ and $\{Y_n: n \geq 1\}$ forms a renewal process and a random walk respectively. By using the random pairs (ξ_i, η_i) we can construct the random walk process with two barriers in which the reflecting barrier is on the zero-level and the delaying barrier is on $\beta > 0$ -level as follows:

$$X_n = min\{\beta, |X_{n-1} + \eta_n|\},\ n \geq 1,\ z = X_0 \in [0, \beta], \tag{1}$$

where z is the initial position of the process. Now, let us construct the stochastic process $X(t)$ which has the reflecting barrier from below and the delaying barrier from above and which represents the level of stock at the moment t:

$$X(t) = X_n, \text{ if } t \in [T_n, T_{n+1}). \tag{2}$$

This process is called the semi-Markovian random walk with the reflecting barrier on the zero-level and the delaying barrier on β- level.

3. THE LAPLACE TRANSFORM FOR THE ERGODIC DISTRIBUTION OF THE PROCESS $X(t)$

In order to formulate the main results of this paper, let us state the following probability characteristics of random walk $\{Y_n: n \geq 1\}$ and renewal process $\{T_n: n \geq 1\}$:

$$a_n(z, x) = P\{z + Y_i \in [0, \beta]; 1 \leq i \leq n; z + Y_n \in [0, x]\}, n \geq 1,$$

$$c_n(z, v) = P\{z + Y_i \in [0, \beta]; 1 \leq i \leq n - 1; z + Y_n < v\}, v < 0, n \geq 1,$$

$$A(z, x) = \sum_{n=0}^{\infty} a_n(z, x), \quad C(z, dv) = \sum_{n=0}^{\infty} c_n(z, dv),$$

$$A(t, z, x) = \sum_{n=0}^{\infty} a_n(z, x)\, \Delta\Phi_n(t), \quad C(s, z, v) = \sum_{n=0}^{\infty} c_n(z, v)\, \Delta\Phi_n(s),$$

$$C(ds, z, dv) = \sum_{n=0}^{\infty} c_n(z, dv)\, d\Phi_n(s),\ c_n(z, dv) = d_v c_n(z, v),$$

$$\Phi_n(t) = P\{T_n < t\},\ \Delta\Phi_n(t) = \Phi_n(t) - \Phi_{n+1}(t),$$

where $z = X_0 \in [0, \beta]$, $v < 0$, $n \geq 1$ and $a_n(z, x) = 1,\ b_n(z, v) = 0.$

For any function $M(t, z, x)$, let us denote the Laplace transform and Laplace-Stieltjes transform of $M(t, z, x)$

$$\widetilde{M}(\lambda, z, x) = \int_0^{\infty} e^{-\lambda t}\, M(t, z, x) dt \text{ and } M^*(\lambda, z, x) = \int_0^{\infty} e^{-\lambda t}\, d_t M(t, z, x)$$

with respect to t, respectively. Moreover, for any functions $M_1(t,z,x)$ and $M_2(t,z,x)$, the convolution product of $M_1(t,z,x)$ and $M_2(t,z,x)$ as follows:

$$M_1(t,z,x) * M_2(t,z,x) = \int_0^t M_2(t-s,z,x)\, d_s M_1(s,z,x)$$

and k-times convolution product of $M_1(t,z,x)$ with itself

$$[M_1(t,z,x)]_*^k = M_1(t,z,x) * [M_1(t,z,x)]_*^{k-1}.$$

Also let us denote

$$\bar{M}_f(v,*) = \int_0^\beta f(x)\, d_x M(v,x) \text{ and } \bar{\bar{M}}_f(*,*) = \int_0^\beta f(x)\, d_x \bar{M}(*,x).$$

Now let us give the following lemma.

Theorem 3.1. Let ξ_1 and η_1 are independent random variables in the initial squence of random pairs mentioned above. Then the process $X(t)$ is ergodic if the following conditions hold:

i) $E[\xi_1] < \infty$,

ii) $P[\eta_1 > 0] > 0$, $P[\eta_1 < 0] > 0$

iii) η_1 has a non-aritmetic distribution.

Proof: In order to prove that the process $X(t)$ is ergodic, it is sufficient to prove that the conditions of "The ergodic theorem for processes with a discrete chance interference" (see Gihman and Skorohod (1973), p. 244) are satisfied. In accordance with this, firstly it is to necessary to construct a Markov chain and to prove that this chain is ergodic under the conditions of this theorem. For this aim, let us define the natural number valued randon variables ϑ_n, $n \geq 1$, by the following recurrence formula

$$\vartheta_1 = min\{k \geq 1 : X_{k-1} + \eta_k \geq \beta\}, \tag{3}$$

and

$$\vartheta_n = min\{k \geq \vartheta_{n-1} : X_{k-1} + \eta_k \geq \beta\}, \quad n \geq 2. \tag{4}$$

Now, by using these random variables, we can define the random variables $\gamma_{n\beta}$ and $\tilde{\gamma}_{n\beta}$, as follows:

$$\gamma_{n\beta} = \sum_{i=1}^{\vartheta_1} \xi_i \,, \qquad \tilde{\gamma}_{n\beta} = \sum_{i=1}^{\vartheta_n+1} \xi_i \ , \ \ n \geq 1, \tag{5}$$

where $\gamma_{n\beta}$ and $\tilde{\gamma}_{n\beta}, n \geq 1$, denotes the n-th falling moment into yhe delaying barrier and the n-th going out moment from the delaying barrier, respectively. Let us define χ_n as

$$\chi_n = X(\tilde{\gamma}_{n\beta} + 0), n \geq 1. \tag{6}$$

This chain is desired Markov chain. Then we can write the random variables $\chi_n, n \geq 1$, as folllows:

$$\chi_n = min\{\beta, |\beta + \eta_{\vartheta_n+1}|\}, \ n \geq 1.$$

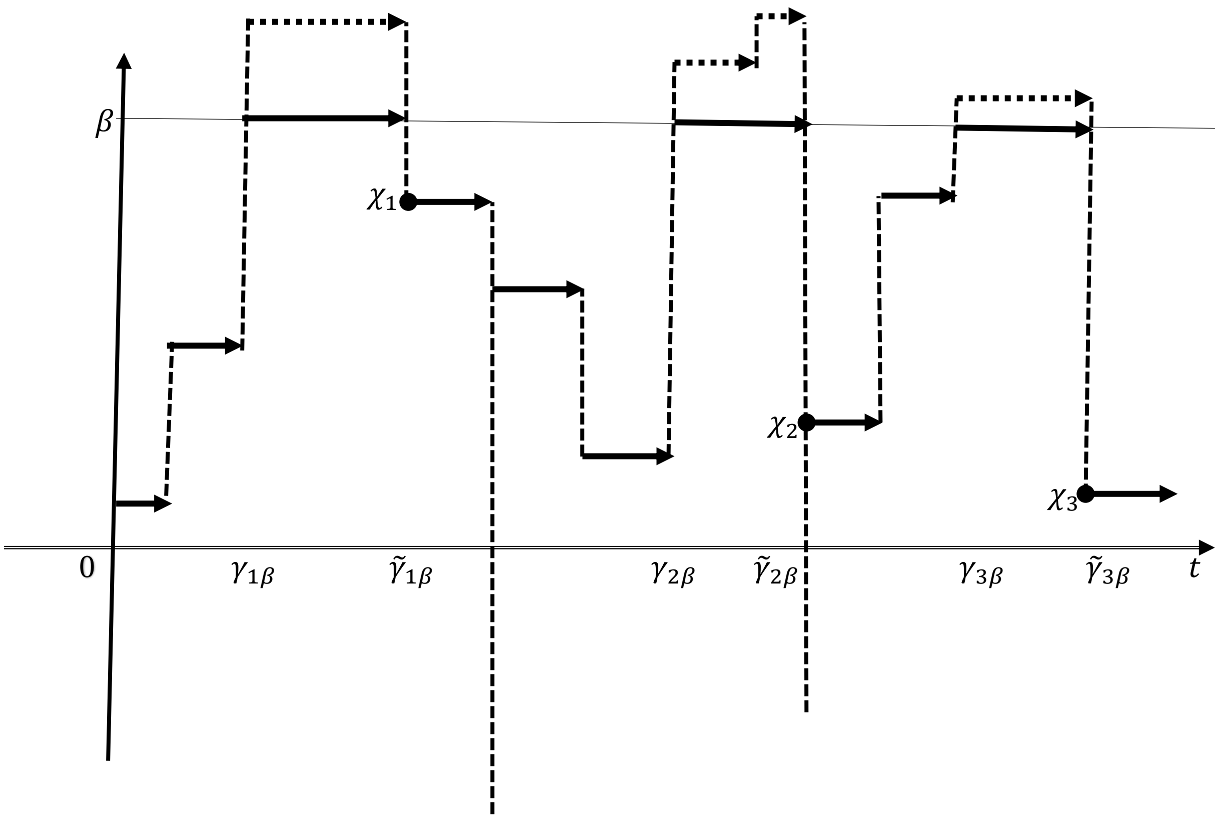

Fig. 2. A View of the Markov Chain χ_n

This Markov chain is ergodic and has a stationary distribution function because of there are reflecting barrier on the zero-level and the delaying barrier on β- level and the random variable η_1 which takes both negative and positive values is non-aritmetic (see Bonokov

(1972)). Let us denote by $\pi(x)$ the stationary distribution function. Then it is obvious that $\pi(x) = 0$ if $x \leq 0$ and $\pi(x) = 1$ if $x > \beta$. Therefore, we consider the case when $x \in (0, \beta]$.

$$\begin{aligned}
\pi(x) = P\{\chi_1 < x\} &= P\{min\{\beta, |\beta + \eta_1|\} < x\} \\
&= 1 - P\{min\{\beta, |\beta + \eta_1|\} \geq x\} \\
&= 1 - P\{\beta \geq x;\ |\beta + \eta_1| \geq x\} \\
&= 1 - P\{\beta \geq x\}P\{\ |\beta + \eta_1| \geq x\} \\
&= 1 - \bar{\varepsilon}(x - \beta)P\{\ |\beta + \eta_1| \geq x\} \\
&= 1 - \bar{\varepsilon}(x - \beta)[1 - P\{\ |\beta + \eta_1| < x\}] \\
&= 1 - \bar{\varepsilon}(x - \beta) + \bar{\varepsilon}(x - \beta)P\{\ |\beta + \eta_1| < x\} \\
&= \varepsilon(x - \beta) + \bar{\varepsilon}(x - \beta)\varepsilon(x)P\{-x < \beta + \eta_1 < x\} \\
&= \varepsilon(x - \beta) + \bar{\varepsilon}(x - \beta)\varepsilon(x)[F(-\beta + x) - F(-\beta - x)], \qquad (7)
\end{aligned}$$

where

$$\varepsilon(u) = \begin{cases} 1, & u \geq 0 \\ 0, & u < 0 \end{cases} \quad \text{and} \quad \bar{\varepsilon}(u) = 1 - \varepsilon(u). \qquad (8)$$

Consequently, under the assumptions of the theorem, the first condition of of "The ergodic theorem for processes with a discrete chance interference" satisfies. Now, let us prove that the second condition of it satisfies. Note that tis is equivalent to show that the expected value of random variable $\tilde{\gamma}_{1\beta}$ is finite, that is, $E[\tilde{\gamma}_{1\beta}] < \infty$. For this aim let us give the following lemma without proof.

Lemma 3.1[7]. If $E[\xi_1] < \infty$, $P[\eta_1 > 0] > 0$ and $P[\eta_1 < 0] > 0$, then there exists numbers $\alpha \in (0,1]$ and $T_\alpha < \infty$ such that

$$\operatorname*{Sup}_{0 \leq z \leq \beta} P_z\left\{\gamma_{1\beta} \geq T_\alpha\right\} \leq \alpha < 1. \qquad (9)$$

Now, we can show that the expected value of random variable $\gamma_{1\beta}$ is finite, that is $E[\gamma_{1\beta}] < \infty$.

Lemma 3.2. If the conditions of lemma 3.1. hold, then $E[\gamma_{1\beta}] < \infty$.

Proof: Under the conditions of lemma 3.1., we can write

$$P_z\{\gamma_{1\beta} \geq 2.T_\alpha\} = \int_0^\beta P_z\{\gamma_{1\beta} \geq T_\alpha; X(T_\alpha) \in dv\}$$

$$= \int_0^\beta P_z\{\gamma_{1\beta} \geq T_\alpha; X(T_\alpha) \in dv\} P_v\{\gamma_{1\beta} \geq T_\alpha\}$$

$$\leq \alpha \int_0^\beta P_z\{\gamma_{1\beta} \geq T_\alpha; X(T_\alpha) \in dv\}$$

$$= \alpha P_v\{\gamma_{1\beta} \geq T_\alpha\}$$

$$\leq \alpha^2. \tag{10}$$

In order to show that $P_z\{\gamma_{1\beta} \geq n.T_\alpha\} \leq \alpha^n$ for every $n \geq 1$ and $z \in [0,\beta]$, we can use the method of mathematical induction. Now, assume that it is true for $n = k$, that is, $P_z\{\gamma_{1\beta} \geq k.T_\alpha\} \leq \alpha^k$ and show that it is true for $n = k + 1$. In this case, we have

$$P_z\{\gamma_{1\beta} \geq (k+1)T_\alpha\} = P_z\{\gamma_{1\beta} \geq k.T_\alpha + T_\alpha\}$$

$$= \int_0^\beta P_z\{\gamma_{1\beta} \geq k.T_\alpha + T_\alpha; X(T_\alpha) \in dv\}$$

$$= \int_0^\beta P_z\{\gamma_{1\beta} \geq k.T_\alpha; Xk.(T_\alpha) \in dv\} P_v\{\gamma_{1\beta} \geq T_\alpha\}$$

$$\leq \alpha \int_0^\beta P_z\{\gamma_{1\beta} \geq k.T_\alpha; X(k.T_\alpha) \in dv\}$$

$$= \alpha P_z\{\gamma_{1\beta} \geq k.T_\alpha\}$$

$$\leq \alpha^{k+1}. \tag{11}$$

Thus, we get $P_z\{\gamma_{1\beta} \geq n.T_\alpha\} \leq \alpha^n$ for every $n \geq 1$ and $z \in [0,\beta]$. On the other hand, we can write

$$E_z[\gamma_{1\beta}] = \int_0^\infty P_z\{\gamma_{1\beta} \geq t\}dt = \sum_{n=1}^\infty \int_{(n-1)T_\alpha}^{nT_\alpha} P_z\{\gamma_{1\beta} \geq t\}dt$$

$$\leq \sum_{n=1}^\infty \int_{(n-1)T_\alpha}^{nT_\alpha} P_z\{\gamma_{1\beta} \geq (n-1)T_\alpha\}dt$$

$$= \sum_{n=1}^\infty T_\alpha . P_z\{\gamma_{1\beta} \geq (n-1)T_\alpha\}$$

$$= T_\alpha \sum_{k=0}^\infty P_z\{\gamma_{1\beta} \geq kT_\alpha\}$$

$$\leq T_\alpha \sum_{k=0}^\infty \alpha^k$$

$$= \frac{1}{1-\alpha} T_\alpha. \tag{12}$$

Thus we get $E_z[\gamma_{1\beta}] < \infty$ for every $z \in [0,\beta]$ because of $\alpha \in (0,1]$ and $T_\alpha < \infty$. Also, we can write

$$E[\gamma_{1\beta}] = \int_0^\beta E_z[\gamma_{1\beta}]\, d\pi(z).$$

In this case, we have

$$E[\gamma_{1\beta}] \le \frac{1}{1-\alpha} T_\alpha \int_0^\beta d\pi(z) = \frac{1}{1-\alpha} T_\alpha < \infty,$$

which completes the proof.

Now, we can show that the expected value of random variable $\tilde{\gamma}_{1\beta}$ is finite, that is $E[\tilde{\gamma}_{1\beta}] < \infty$. By the definitions of random variables $\gamma_{1\beta}$ and $\tilde{\gamma}_{1\beta}$, we can write

$$\tilde{\gamma}_{1\beta} = \sum_{i=1}^{\vartheta_1+1} \xi_i = \sum_{i=1}^{\vartheta_1} \xi_i + \xi_{\vartheta_1+1} = \gamma_{1\beta} + \xi_{\vartheta_1+1}.$$

Thus, we have

$$E[\tilde{\gamma}_{1\beta}] = E[\gamma_{1\beta}] + E[\xi_{\vartheta_1+1}] = E[\gamma_{1\beta}] + E[\xi_1] < \infty.$$

So, under the assumptions of the theorem, the both of conditions of "The ergodic theorem for processes with a discrete chance interference" satisfy. That is, the process $X(t)$ is ergodic.

Theorem 3.2. Under the conditions of Lemma 3.1., the following statement is true for the ergodic distribution of the process $X(t)$:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(X(t))\, dt = \frac{\bar{\bar{R}}_f(*,*)}{\bar{R}_f(*,\infty)} \tag{13}$$

where

$$\bar{\bar{R}}_f(*,*) = \bar{\bar{A}}_f(*,*) + \int_{-\beta}^0 \bar{C}(*, dv_1) \bar{A}_f(|v_1|,*)$$
$$+ \sum_{n=2}^\infty \int_{-\beta}^0 \dots (n) \dots \int_{-\beta}^0 \bar{C}(*, dv_1) \prod_{i=2}^n C(|v_{i-1}|, dv_i) \bar{A}_f(|v_n|,*), \tag{14}$$

$$\bar{R}_f(*,\infty) = \bar{\bar{A}}_f(*,\infty) + \int_{-\beta}^0 \bar{C}(*, dv_1) \bar{A}_f(|v_1|,\infty)$$
$$+ \sum_{n=2}^\infty \int_{-\beta}^0 \dots (n) \dots \int_{-\beta}^0 \bar{C}(*, dv_1) \prod_{i=2}^n C(|v_{i-1}|, dv_i) \bar{A}_f(|v_n|,\infty). \tag{15}$$

Proof: Since the process of $X(t)$ is ergodic the following expression is true for any measurable and bounded function $f(x)$ by "The ergodic theorem for processes with a discrete chance interference" (see Gihman and Skorohod [4], p. 244) are satisfied:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f\big(X(t)\big)\,dt = \left[\int_{z=0}^{\beta}\int_{t=0}^{t\infty}\int_{x=0}^{\beta} f(x)\,P_z\{\tilde{\gamma}_{1\beta}\geq t; X(t)\in dx\}\,dt\pi(dz)\right]$$

$$\times\left[\int_{z=0}^{\beta}\int_{t=0}^{t\infty} P_z\{\tilde{\gamma}_{1\beta}\geq t\}\,dt\pi(dz)\right]^{-1}. \tag{16}$$

Our aim in this section is to express the nominator and denominator of (16) by means of the probability characteristics of random walk $\{Y_n: n\geq 1\}$ and renewal process $\{T_n: n\geq 1\}$. Let us denote

$$R(t,z,x) = P_z\{\tilde{\gamma}_{1\beta}\geq t: X(t) < x\}$$

and

$$r_n(t,z,x) = P_z\{\tilde{\gamma}_{1\beta}\geq t: \upsilon_0(t) = n; X(t) < x\}, n\geq 0.$$

Let us denote by $\upsilon_0(t)$ the number of reflection moments of the process $X(t)$ into the interval $[0,t]$. According to the total probability formula we have

$$R(t,z,x) = P_z\{\tilde{\gamma}_{1\beta}\geq t: X(t) < x\}$$

$$= \sum_{n=0}^{\infty} P_z\{\tilde{\gamma}_{1\beta}\geq t;\ \upsilon_0(t) = n;\ X(t) < x\} = \sum_{n=0}^{\infty} r_n(t,z,x) \tag{17}$$

Now, we expres each $r_n(t,z,x)$ given in (17) by the probability characteristics of both $\{Y_n\}$ and $\{T_n\}$, seperately. For $n=0$, we have

$$r_0(t,z,x) = P_z\{\tilde{\gamma}_{1\beta}\geq t: \upsilon_0(t) = 0; X(t) < x\}$$

$$= \sum_{n=0}^{\infty} P_z\{\tilde{\gamma}_{1\beta}\geq t: \upsilon_0(t) = 0; X(t) < x; T_n\leq t < T_{n+1}\}$$

$$= \sum_{n=0}^{\infty} P\{z+Y_i\in[0,\beta]; 1\leq i\leq n; z+Y_n\in[0,x]\}\,P\{T_n\leq t < T_{n+1}\}$$

$$= \sum_{n=0}^{\infty} a_n(z,x)\,\Delta\Phi_n(t)$$

$$= A(t,z,x)$$

From this, we can write

$$\widetilde{r_0}(\lambda,z,x) = \tilde{A}(\lambda,z,x). \tag{18}$$

Now, we also calculate the conditional distribution $r_1(t,z,x)$ in order to give a general formula for $r_n(t,z,x),\ n\geq 0,$

$$r_1(t,z,x) = P_z\{\tilde{\gamma}_{1\beta} \geq t: v_0(t) = 1; X(t) < x\}$$

$$= \sum_{n=0}^{\infty} P_z\{\tilde{\gamma}_{1\beta} \geq t: v_0(t) = 1; X(t) < x; T_n \leq t < T_{n+1}\}$$

$$= \sum_{n=0}^{\infty} \sum_{i=1}^{n} \int_0^t P_z\{\gamma \geq t: v_0(t) = 1; X(t) < x; T_n \leq t < T_{n+1}; T_i \in ds\}$$

$$= \sum_{n=0}^{\infty} \sum_{i=1}^{n} \int_{-\beta}^{0} \int_0^t P\{z + Y_j \in [0,\beta]; 1 \leq j \leq i-1; z + Y_i \in dv; T_i \in ds\}$$

$$x\ P\{|v| + Y_j \in [0,\beta]; 1 \leq j \leq n-i; |v| + Y_{n-i} < x\}\Delta\Phi_{n-i}(t-s)$$

$$= \sum_{n=0}^{\infty} \sum_{i=1}^{n} \int_{-\beta}^{0} \int_0^t c_i(z,dv) d\Phi_i(s) a_{n-i}(|v|,x)\Delta\Phi_{n-i}(t-s)$$

$$= \int_{-\beta}^{0} \int_0^t \sum_{i=1}^{\infty} c_i(z,dv) d\Phi_i(s) \sum_{n=i}^{\infty} a_{n-i}(|v|,x)\Delta\Phi_{n-i}(t-s)$$

$$= \int_{-\beta}^{0} \int_0^t C(ds,z,dv)\, A(t-s,|v|,x)$$

$$= \int_{-\beta}^{0} A(t,|v|,x) * C(t,z,dv)$$

As such,

$$\tilde{r_1}(t,z,x) = \int_{-\beta}^{0} \tilde{A}(\lambda,|v|,x) C^*(\lambda,z,dv) \qquad (19)$$

is obtained. Analogously, it is possible to prove that

$$\tilde{r_n}(t,z,x) = \int_{-\beta}^{0} \dots (n) \dots \int_{-\beta}^{0} \prod_{i=1}^{n} C^*(\lambda,|v_{i-1}|,dv_i)\tilde{A}(\lambda,|v_n|,x), \qquad (20)$$

for $n \geq 1; |v_0| = z \in [0,\beta]$.

Substituting all of these expressions in the formula for $\tilde{R}(\lambda,z,x)$ given above, we have

$$\tilde{R}(\lambda,z,x) = \tilde{A}(\lambda,z,x) + \sum_{n=1}^{\infty} \int_{-\beta}^{0} \dots \int_{-\beta}^{0} \prod_{i=1}^{n} C^*(\lambda,|v_{i-1}|,x)\tilde{A}(\lambda,|v_n|,x) \qquad (21)$$

as asserted. Thus, we expressed $\tilde{R}(\lambda,z,x)$, the Laplace transform of $R(t,z,x)$, by the probability characteristics of random walk $\{Y_n: n \geq 1\}$ and renewal process $\{T_n: n \geq 1\}$.

This result is very important in theory, but it is not so important in practice because of these formulas are very difficult and complex. Specially, it is very difficult to calculate the inverse of the Laplace transform. In this case, we must notice that when λ tends towards zero ($\lambda \to 0$) with helping $\tilde{R}(\lambda,z,x)$ we can calculate the limit value of $R(t,z,x)$ when $t \to \infty$. For this, it is enough to consider the Tauber's Theorem, see (Feller 1968). In this case, we can write

$$\int_0^\infty P_z\{\tilde{\gamma}_{1\beta} \geq t: X(t) < x\}\, dt = \tilde{R}(\lambda, z, x)\big|_{\lambda=0} = \tilde{R}(0, z, x)$$

$$= \tilde{A}(0, z, x) + \sum_{n=1}^{\infty} \int_{-\beta}^{0} \dots (n) \dots \int_{-\beta}^{0} \prod_{i=1}^{n} C^*(0, |v_{i-1}|, x) \tilde{A}(0, |v_n|, x).$$

Now, we calculate the values of $\tilde{A}(0, z, x)$ and $C^*(0, z, dv)$:

$$\tilde{A}(0, z, x) = \lim_{\lambda \to 0} \int_0^\infty A(t, z, x)\, e^{-\lambda t}\, dt$$

$$= \lim_{\lambda \to 0} \int_0^\infty \sum_{n=0}^{\infty} a_n(z, x)\, \Delta\Phi_n(t)\ e^{-\lambda t} \mathrm{dt}$$

$$= \sum_{n=0}^{\infty} a_n(z, x)\, E[\xi_1]$$

$$= A(z, x) E[\xi_1]$$

and

$$C^*(0, z, dv) = \lim_{\lambda \to 0} C^*(\lambda, z, dv)$$

$$= \lim_{\lambda \to 0} \int_0^\infty\ e^{-\lambda t} \sum_{n=1}^{\infty} P\{z + Y_i \in [0, \beta]; 1 \leq i \leq n-1; z + Y_n \in dv\}\, d\Phi_n(t)$$

$$= \sum_{n=1}^{\infty} P\{z + Y_i \in [0, \beta]; 1 \leq i \leq n-1; z + Y_n \in dv\} \lim_{\lambda \to 0} \int_0^\infty\ e^{-\lambda t}\, d\Phi_n(t)$$

$$= \sum_{n=0}^{\infty} P\{z + Y_i \in [0, \beta]; 1 \leq i \leq n-1; z + Y_n \in dv\}$$

$$= C(z, dv).$$

Thus, we can write

$$\int_0^\infty P_z\{\tilde{\gamma}_{1\beta} \geq t: X(t) < x\}\, dt = A(z, x) E[\xi_1]$$

$$+ E[\xi_1] \sum_{n=1}^{\infty} \int_{-\beta}^{0} \dots (n) \dots \int_{-\beta}^{0} \prod_{i=1}^{n} C(|v_{i-1}|, dv_i) A(|v_n|, x).$$

By limiting as $x \to \infty$, we have

$$\int_0^\infty P_z\{\tilde{\gamma}_{1\beta} \geq t\}\, dt = A(z, \infty) E[\xi_1]$$

$$+ E[\xi_1] \sum_{n=1}^{\infty} \int_{-\beta}^{0} \dots (n) \dots \int_{-\beta}^{0} \prod_{i=1}^{n} C(|v_{i-1}|, dv_i) A(|v_n|, \infty),$$

where

$$A(z; \infty) = A(z) = \sum_{n=1}^{\infty} P\{z + Y_i \in [0, \beta]; 1 \leq i \leq n\}.$$

On the other hand, we can write

$$\int_0^{\infty}\left(\int_0^{\infty} P_z\{\tilde{\gamma}_{1\beta} \geq t\}\, dt\right) d\pi(z) = E[\xi_1]\bar{A}(*,\infty) + E[\xi_1]\int_{-\beta}^{0} \bar{C}(*,dv_1)A(|v_1|,\infty)$$

$$+E[\xi_1]\sum_{n=1}^{\infty}\int_{-\beta}^{0} \dots (n) \dots \int_{-\beta}^{0} \bar{C}(*,dv_1)\prod_{i=2}^{n} C(|v_{i-1}|,dv_i)A(|v_n|,\infty)$$

$$= E[\xi_1]\bar{R}(*,\infty).$$

Analogously, we have

$$\int_0^{\infty}\left(\int_0^{\infty} P_z\{\tilde{\gamma}_{1\beta} \geq t; X(t) < x\}\, dt\right) d\pi(z) = E[\xi_1]\bar{A}(*,x)$$

$$+ E[\xi_1]\int_{-\beta}^{0} \bar{C}(*,dv_1)A(|v_1|,x)$$

$$+E[\xi_1]\sum_{n=1}^{\infty}\int_{-\beta}^{0} \dots (n) \dots \int_{-\beta}^{0} \bar{C}(*,dv_1)\prod_{i=2}^{n} C(|v_{i-1}|,dv_i)A(|v_n|,x).$$

From this, we get

$$\int_0^{\beta}\int_0^{\infty}\left(\int_0^{\infty} P_z\{\tilde{\gamma}_{1\beta} \geq t; X(t) \in dx\}\, dt\right) d\pi(z) = E[\xi_1]\bar{\bar{A}}_f(*,*)$$

$$+ E[\xi_1]\int_{-\beta}^{0} \bar{C}(*,dv_1)\bar{A}_f(|v_1|,*)$$

$$+E[\xi_1]\sum_{n=1}^{\infty}\int_{-\beta}^{0} \dots (n) \dots \int_{-\beta}^{0} \bar{C}(*,dv_1)\prod_{i=2}^{n} C(|v_{i-1}|,dv_i)\bar{A}_f(|v_n|,*)$$

$$= E[\xi_1]\bar{\bar{R}}_f(*,*).$$

Thus, by considering that $E[\xi_1] < \infty$, the proof is completed.

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