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ON SOME NEW HADAMARD TYPE INEQUALITIES FOR (s,r)-PREINVEX FUNCTIONS IN THE SECOND SENSE

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Abstract. In this paper the authors introduce a new class of preinvexity called (s,r)- preinvex functions in the second sense and establish some new Hadamard-type inequalities.

1. Introduction

One of the most well-known inequalities in mathematics for convex functions is so called Hermite-Hadamard integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_{a}^{b} f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where f is a real continuous convex function on the finite interval $[a,b]$. If the function f is concave, then (1.1) holds in the reverse direction (see [26]).

The Hermite-Hadamard inequality play an important role in nonlinear analysis and optimization. The above double inequality has attracted many researchers, various generalizations, refinements, extensions and variants of (1.1) have appeared in the literature, we can mention the works [5, 7, 9, 10, 13, 17, 18, 21, 24, 25, 28, 29, 30, 34] and the references cited therein.

In recent years, lot of efforts have been made by many mathematicians to generalize the classical convexity. Hanson [11], introduced a new class of generalized convex functions, called invex functions. In [7], the authors gave the concept of preinvex function which is special case of invexity. Pini [27], Noor [19, 20] and Weir [33], have studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems.

In [18], N.P.G. Ngoc et al. proved the following theorems for r-convex functions

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Theorem 1.1. [18, Theorem 2.1] *Let $f:[a,b]\to(0,\infty)$ be r-convex function on $[a,b]$ with $a<b$. Then the following inequality holds for $0<r\leq 1$:*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \left(\frac{r}{r+1}\right)\left\{f^r(a)+f^r(b)\right\}^{\frac{1}{r}}.$$

Theorem 1.2. [18, Theorem 2.6] *Let $f,g:[a,b]\to(0,\infty)$ be r-convex and s-convex functions respectively on $[a,b]$ with $a<b$. Then the following inequality holds for $r>1$ and $\frac{1}{r}+\frac{1}{s}=1$*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \left\{\frac{f^r(a)+f^r(b)}{2}\right\}^{\frac{1}{r}}\left\{\frac{g^s(a)+g^s(b)}{2}\right\}^{\frac{1}{s}}.$$

In [34], G. Zabandan et al. established the following theorems

Theorem 1.3. [34, Theorem 2.1] *Let $f:[a,b]\to(0,\infty)$ be r-convex and $r\geq 1$. Then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \left\{\frac{f^r(a)+f^r(b)}{2}\right\}^{\frac{1}{r}}.$$

Theorem 1.4. [34, Theorem 2.8] *Let $f,g:[a,b]\to(0,\infty)$ be r-convex and s-convex functions respectively on $[a,b]$ and $r,s>0$ with $f(b)\neq f(a)$ and $g(b)\neq g(a)$. Then for the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{2}\left(\frac{r}{r+2}\right)\frac{f^{r+2}(b)-f^{r+2}(a)}{f^r(b)-f^r(a)}+\frac{1}{2}\left(\frac{s}{s+2}\right)\frac{g^{s+2}(b)-g^{s+2}(a)}{g^s(b)-g^s(a)}.$$

In [30], W. Ul-Haq and J. Iqbal proved the following Hadamard's inequality for r-preinvex functions

Theorem 1.5. [30, Theorem 4] *Let $f:K=[a,a+\eta(b,a)]\to(0,\infty)$ be an r-preinvex function on the interval of real numbers K° (interior of K) and $a,b\in K^\circ$ with $a<a+\eta(b,a)$. Then the following inequality holds:*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \leq \left[\frac{f^r(a)+f^r(b)}{2}\right]^{\frac{1}{r}},\quad r\geq 1.$$

Theorem 1.6. [30, Theorem 11] *Let $f,g:K=[a,a+\eta(b,a)]\to(0,\infty)$ be an r-preinvex and s-preinvex functions respectively with $r,s>0$ on the interval of real numbers K° (interior of K) and $a,b\in K^\circ$ with $a<a+\eta(b,a)$ with $f(b)\neq f(a)$ and $g(b)\neq g(a)$. Then the following inequality holds:*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx \leq \frac{1}{2}\frac{r}{r+2}\left[\frac{f^{r+2}(a)-f^{r+2}(b)}{f^r(a)-f^r(b)}\right]+\frac{1}{2}\frac{s}{s+2}\left[\frac{g^{s+2}(a)-g^{s+2}(b)}{g^s(a)-g^s(b)}\right].$$

Theorem 1.7. [30, Theorem 12] *Let $f,g:K=[a,a+\eta(b,a)]\to(0,\infty)$ be an r-preinvex $(r>0)$ and 0-preinvex functions respectively on the interval of real numbers K° (interior of K) and $a,b\in K^\circ$ with $a<a+\eta(b,a)$ with $f(b)\neq f(a)$. Then*

the following inequality holds:

$$\frac{1}{\eta(b,a)}\int\limits_{a}^{a+\eta(b,a)} f(x)g(x)dx \leq \frac{1}{2}\frac{r}{r+2}\left[\frac{f^{r+2}(a)-f^{r+2}(b)}{f^{r}(a)-f^{r}(b)}\right]+\frac{1}{4}\left[\frac{g^{2}(a)-g^{2}(b)}{\ln g(a)-\ln g(b)}\right].$$

In [21, 22], Noor proved the following Hadamard's inequality for log-preinvex functions and product of two log-preinvex functions

Theorem 1.8. [21, 22, Theorem 2.8] *Let f be a log-preinvex function on the interval $[a, a+\eta(b,a)]$ with $f(b) \neq f(a)$. Then*

$$\frac{1}{\eta(b,a)}\int\limits_{a}^{a+\eta(b,a)} f(x)dx \leq \frac{f(a)-f(b)}{\ln f(a)-\ln f(b)}.$$

Theorem 1.9. [21, Theorem 3.1] *Let $f, g : K = [a, a+\eta(b,a)] \to (0,\infty)$ be preinvex functions on the interval of real numbers K° (the interior of K) and $a, b \in K^{\circ}$ with $a < a+\eta(b,a)$ with $f(b) \neq f(a)$ and $g(b) \neq g(a)$. Then the following inequality holds.*

$$\frac{1}{\eta(b,a)}\int\limits_{a}^{a+\eta(b,a)} f(x)g(x)dx \leq \frac{1}{4}\left(\frac{\left[f^{2}(b)-f^{2}(a)\right]}{\ln f(b)-\ln f(a)}+\frac{\left[g^{2}(b)-g^{2}(a)\right]}{\ln g(b)-\ln g(a)}\right).$$

In [9], Dragomir et. al. established the following Hadamaed's inequality for s-convex function in the second sense

Theorem 1.10. *Let $f : [0,\infty) \to [0,\infty)$ be s-convex function in the second sense where $s \in (0,1)$ and let $a, b \in [0,\infty)$, $a < b$. If $f \in L^{1}([a,b])$, then the following inequality holds:*

$$2^{s-1}f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int\limits_{a}^{b} f(x)dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1.2}$$

Motivated by the above results, in this paper we first introduce a new kind of preinvexity by combining Definition 2.9 with Definition 2.11, then we establish some new Hadamard-type inequalities for this novel class.

2. Preliminaries

In this section we recall some concepts of convexity and preinvexity which are well known in the literature. Throughout this section I is an interval of $\mathbb{R}$.

Definition 2.1. [26] A function $f : I \to \mathbb{R}$ is said to be convex, if

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y),$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

Definition 2.2. [26] A positive function $f : I \to \mathbb{R}$ is said to logarithmically convex, if

$$f(tx+(1-t)y) \leq [f(x)]^{t}[f(y)]^{(1-t)},$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

Definition 2.3. [8] A nonnegative function $f : I \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in (0,1]$, if

$$f(tx+(1-t)y) \ \leq t^s f(x)+(1-t)^s f(y),$$

holds for all $x,y \in I$ and $t \in [0,1]$.

Definition 2.4. [1] A positive function $f : I \subset [0,\infty) \to \mathbb{R}$ is said to be s-logarithmically convex in the second sense on I,for some $s \in (0,1]$, if

$$f(tx+(1-t)y) \ \leq [f(x)]^{t^s} [f(y)]^{(1-t)^s},$$

holds for all $x,y \in I$ and $t \in [0,1]$.

Definition 2.5. [25] A positive function $f : I \to \mathbb{R}$ is said to be r-convex on I, where $r \geq 0$, if

$$f(tx+(1-t)y) \leq \begin{cases} [tf^r(x)+(1-t)f^r(y)]^{\frac{1}{r}} , & r \neq 0 \\ [f(x)]^{1-t}[f(y)]^t , & r=0, \end{cases}$$

holds for all $x,y \in I$ and $t \in [0,1]$.

Let K be a subset in $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}^n$ be continuous functions.

Definition 2.6. [33] A set K is said to be invex at x with respect to η, if $x + t\eta(y,x) \in K$, for all $x,y \in K$ and $t \in [0,1]$.

K is said to be an invex set with respect to η if K is invex at each $x \in K$.

Definition 2.7. [33] A function f on the invex set K is said to be preinvex with respect to η, if

$$f(x+t\eta(y,x)) \leq (1-t)f(x)+tf(y),$$

for all $x,y \in K$ and $t \in [0,1]$.

Definition 2.8. [19] A positive function f on the invex set K is said to be log-preinvex with respect to η, if

$$f(x+t\eta(y,x)) \leq [f(x)]^{(1-t)}[f(y)]^t,$$

holds for all $x,y \in K$ and $t \in [0,1]$.

Definition 2.9. [31] A nonnegative function f on the invex set $K \subseteq [0,\infty)$ is said to be s-preinvex in the second sense with respect to η for some fixed $s \in (0,1]$, if

$$f(x+t\eta(y,x)) \leq (1-t)^s f(x)+t^s f(y),$$

holds for all $x,y \in K$ and $t \in [0,1]$.

Definition 2.10. [32] A positive function f on the invex set $K \subseteq [0,\infty)$ is said to be s-log-preinvex in the second sense with respect to η for some fixed $s \in (0,1]$, if

$$f(x+t\eta(y,x)) \leq [f(x)]^{(1-t)^s}[f(y)]^{t^s},$$

holds for all $x,y \in K$ and $t \in [0,1]$.

Definition 2.11. [2] A positive function f on the invex set K is said to be r-preinvex with respect to η for $r \geq 0$, if

$$f(x+t\eta(y,x)) \leq \begin{cases} [(1-t)f^r(x)+tf^r(y)]^{\frac{1}{r}}, & r \neq 0 \\ [f(x)]^{1-t}[f(y)]^t, & r=0, \end{cases}$$

for all $x, y \in K$ and $t \in [0,1]$.

Lemma 2.1. [4] *Let $0 < \phi \leq 1 \leq \psi$ and $t, s \in (0,1]$. Then*

$$\begin{aligned} \phi^{t^s} &\leq \phi^{st}, \\ \psi^{t^s} &\leq \psi^{st+1-s}. \end{aligned}$$

Lemma 2.2. [15] *For $a \geq 0$ and $b \geq 0$, the following algebraic inequalities are true*

$$\begin{aligned} (a+b)^\lambda &\leq 2^{\lambda-1}\left(a^\lambda+b^\lambda\right), \qquad \text{for } \lambda \geq 1 \\ (a+b)^\lambda &\leq a^\lambda+b^\lambda, \qquad \text{for } 0 \leq \lambda \leq 1. \end{aligned}$$

3. Main results

At first, we introduce the class of (s,r)- preinvexity in the second sense.

Definition 3.1. A positive function f on the invex set K, is said to be (s,r)-preinvex in the second sense, if

$$f(x+t\eta(y,x)) \leq \begin{cases} [(1-t)^s f^r(x)+t^s f^r(y)]^{\frac{1}{r}}, & r \neq 0 \\ [f(x)]^{(1-t)^s}[f(y)]^{t^s}, & r=0, \end{cases}$$

holds, for some fixed $s \in (0,1]$ and all $x, y \in K$ and $t \in [0,1]$.

Remark 3.1. Obviously a Definition 3.1 recapture all definitions cited above, for well-chosen values of $\eta(.,.)$, s and r. With the exception of Definition 2.6.

Now, we are in position to stating our results.

Theorem 3.1. *Let $f : K = [a, a+\eta(b,a)] \to \mathbb{R}^+$ be (s,r)- preinvex function in the second sense with respect to η on K°(K° interior of K) for some fixed $s \in (0,1], a, b \in K^\circ$ with $a < a+\eta(b,a)$ and $r \geq 1$, then the following inequality holds*

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \leq \left[\frac{1}{s+1}\left[f^r(a)+f^r(b)\right]\right]^{\frac{1}{r}}. \tag{3.1}$$

Proof. For $x = a+t\eta(b,a)$, we have

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx = \int_0^1 f(a+t\eta(b,a))dt. \tag{3.2}$$

Since $r \geq 1$, using Jensen's inequality and (s,r)- preinvexity in the second sense of f, we get

$$\begin{aligned}
\left[\int_0^1 f(a+t\eta(b,a))dt\right]^r &\leq \int_0^1 (f(a+t\eta(b,a)))^r\,dt \\
&= \int_0^1 \left[(1-t)^s f^r(a) + t^s f^r(b)\right] dt \\
&= f^r(a)\int_0^1 (1-t)^s\,dt + f^r(b)\int_0^1 t^s dt \\
&= \frac{1}{s+1}\left[f^r(a)+f^r(b)\right]. \qquad (3.3)
\end{aligned}$$

Thus the inequality (3.3) gives the desired inequality in (3.1). Which completes the proof. □

Remark 3.2. If we put $s=1$ in Theorem 3.1, we obtain Theorem 4 from [30]. Moreover if we choose $\eta(b,a)=b-a$, we get Theorem 2.1 from [34].

Theorem 3.2. *Let $f : K = [a, a+\eta(b,a)] \to \mathbb{R}^+$ be (s,r)-preinvex function in the second sense with respect to η on K° (K° interior of K) with $s, r \in (0,1]$, $a, b \in K^\circ$ with $a < a+\eta(b,a)$, then the following inequality holds*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \leq \left(\frac{r}{s+r}\right)\left[f^r(a)+f^r(b)\right]^{\frac{1}{r}}. \qquad (3.4)$$

Proof. Since f is (s,r)- preinvex function in the second sense, we have

$$\begin{aligned}
\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx &= \int_0^1 f(a+t\eta(b,a))dt \\
&\leq \int_0^1 \left[(1-t)^s f^r(a)+t^s f^r(b)\right]^{\frac{1}{r}} dt. \qquad (3.5)
\end{aligned}$$

Since $0 < r \leq 1$, Minkowski's inequality get

$$\begin{aligned}
\int_0^1 \left[(1-t)^s f^r(a)+t^s f^r(b)\right]^{\frac{1}{r}} dt &\leq \left[\left(\int_0^1 (1-t)^{\frac{s}{r}} f(a)\,dt\right)^r + \left(\int_0^1 t^{\frac{s}{r}} f(b)dt\right)^r\right]^{\frac{1}{r}} \\
&= \left[f^r(a)\left(\int_0^1 (1-t)^{\frac{s}{r}}\,dt\right)^r + f^r(b)\left(\int_0^1 t^{\frac{s}{r}}dt\right)^r\right]^{\frac{1}{r}} \\
&= \left[\left(\frac{r}{s+r}\right)^r \left[f^r(a)+f^r(b)\right]\right]^{\frac{1}{r}} \\
&= \left(\frac{r}{s+r}\right)\left[f^r(a)+f^r(b)\right]^{\frac{1}{r}}.
\end{aligned}$$

The proof is completed. ☐

Remark 3.3. In Theorem 3.2, if we take $s = 1$ and $\eta(b,a) = b - a$, we obtain Theorem 2.1 from [18]. Also we obtain the right hand side of inequality (1.2), if we choose $\eta(b,a) = b - a$ and $r = 1$.

Theorem 3.3. *Let $f : K = [a, a+\eta(b,a)] \to \mathbb{R}^+$ be (s,r)- preinvex function in the second sense with respect to η on K° (K° interior of K) for some fixed $s \in (0,1]$, $a, b \in K^\circ$ with $a < a+\eta(b,a)$ and $r > 0$, then the following inequality holds*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \leq \begin{cases} 2^{\frac{1-r}{r}}\frac{r}{s+r}\left[f(a)+f(b)\right] & \text{if } 0<r\leq 1 \\ \frac{r}{s+r}\left[f(a)+f(b)\right] & \text{if } r\geq 1. \end{cases} \tag{3.6}$$

Proof. Since f is (s,r)-preinvex function in the second sense, we have

$$\begin{aligned} \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx &= \int_0^1 f(a+t\eta(b,a))dt \\ &\leq \int_0^1 \left[(1-t)^s f^r(a) + t^s f^r(b)\right]^{\frac{1}{r}} dt. \end{aligned} \tag{3.7}$$

Using Lemma 2.2, we obtain

$$\left[(1-t^s) f^r(a) + t^s f^r(b)\right]^{\frac{1}{r}} \leq \begin{cases} 2^{\frac{1-r}{r}}\left[(1-t)^{\frac{s}{r}} f(a) + t^{\frac{s}{r}} f(b)\right] & \text{if } 0<r\leq 1 \\ (1-t)^{\frac{s}{r}} f(a) + t^{\frac{s}{r}} f(b) & \text{if } r\geq 1. \end{cases} \tag{3.8}$$

Integrating a both sides of inequality (3.8) with respect to t on $[0,1]$, we get

$$\begin{aligned} \int_0^1 \left[(1-t^s) f^r(a) + t^s f^r(b)\right]^{\frac{1}{r}} dt &\leq \begin{cases} 2^{\frac{1-r}{r}} f(a)\int_0^1 (1-t)^{\frac{s}{r}} dt + 2^{\frac{1-r}{r}} f(b)\int_0^1 t^{\frac{s}{r}} dt & \text{if } 0<r\leq 1 \\ f(a)\int_0^1 (1-t)^{\frac{s}{r}} dt + f(b)\int_0^1 t^{\frac{s}{r}} dt & \text{if } r\geq 1 \end{cases} \\ &= \begin{cases} 2^{\frac{1-r}{r}}\frac{r}{s+r}\left[f(a)+f(b)\right] & \text{if } 0<r\leq 1 \\ \frac{r}{s+r}\left[f(a)+f(b)\right] & \text{if } r\geq 1. \end{cases} \end{aligned} \tag{3.9}$$

Which is the desired inequality. The proof is achieved. ☐

Remark 3.4. In Theorem 3.3 if we take $\eta(b,a) = b - a$ and $r = 1$, then (3.6) will be reduced to the right hand side of inequality (1.2).

Theorem 3.4. *Suppose that all the assumptions of Theorem 3.3 are satisfied. Then the following inequality holds*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \leq \frac{2r}{s+r}\left[f^r(a)+f^r(b)\right]^{\frac{1}{r}}\left[1-\left(\frac{1}{2}\right)^{\frac{s+r}{r}}\right]. \tag{3.10}$$

Proof. Since f is (s,r)- preinvex function in the second sense, we have

$$\begin{aligned}\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx &= \int_0^1 f(a+t\eta(b,a))dt \\ &\leq \int_0^1 \left[(1-t)^s f^r(a)+t^s f^r(b)\right]^{\frac{1}{r}} dt\end{aligned}$$

$$\begin{aligned}&= \int_0^{\frac{1}{2}} \left[(1-t)^s f^r(a)+t^s f^r(b)\right]^{\frac{1}{r}} dt+\int_{\frac{1}{2}}^1 \left[(1-t)^s f^r(a)+t^s f^r(b)\right]^{\frac{1}{r}} dt \\ &\leq \left[f^r(a)+f^r(b)\right]^{\frac{1}{r}}\int_0^{\frac{1}{2}} (1-t)^{\frac{s}{r}} dt \ +\left[f^r(a)+f^r(b)\right]^{\frac{1}{r}}\int_{\frac{1}{2}}^1 t^{\frac{s}{r}} dt \\ &= \frac{2r}{s+r}\left[f^r(a)+f^r(b)\right]^{\frac{1}{r}}\left[1-\left(\frac{1}{2}\right)^{\frac{s+r}{r}}\right].\end{aligned}$$

The proof is completed. □

Theorem 3.5. *Let $f:K=[a,a+\eta(b,a)]\to\mathbb{R}^+$ be $(s,0)$- preinvex function in the second sense with respect to η on K° (K° interior of K) for some fixed $s\in(0,1]$, $a,b\in K^\circ$, then the following inequality holds*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \leq E(f(a),f(b),s)\times N(f(a),f(b),s), \tag{3.11}$$

where

$$E(f(a),f(b),s)=\begin{cases} 1 \ \textit{if} \ \ f(b)=f(a) \\ \frac{(f(b))^s-(f(a))^s}{s(f(a))^s(\ln f(b)-\ln f(a))} \ \textit{if } f(b)\neq f(a), \end{cases} \tag{3.12}$$

and

$$N(f(a),f(b),s)=\begin{cases} (f(a))^s \ \ \textit{if } f(b)\neq f(a) \textit{with } f(b),f(a)\leq 1 \\ (f(a))^s \ \ \textit{if } \ f(b)=f(a) \ \ \textit{with } f(b),f(a)\leq 1 \\ f(a) \ \ \textit{if } \ f(b)\leq 1\leq f(a) \\ (f(a))^s(f(b))^{1-s} \ \ \textit{if } \ f(a)\leq 1\leq f(b) \\ (f(a))^s(f(b))^{1-s} \textit{if } f(b)\neq f(a) \ \textit{with } f(b),f(a)\geq 1 \\ (f(a))^s(f(b))^{1-s} \ \ \textit{if } \ f(b)=f(a) \ \textit{with } f(b),f(a)\geq 1. \end{cases} \tag{3.13}$$

Proof. Since f is $(s,0)$- preinvex in the second sense, we have

$$\begin{aligned}\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx &= \int_0^1 f(a+t\eta(b,a))dt \\ &\le \int_0^1 (f(a))^{(1-t)^s}(f(b))^{t^s}dt. \quad (3.14)\end{aligned}$$

From Lemma 2.1, (3.14) gives

$$\int_0^1 (f(a))^{(1-t)^s}(f(b))^{t^s}dt$$
$$\le \begin{cases} (f(a))^s \int_0^1 \left(\frac{f(b)}{f(a)}\right)^{st} dt \text{ if } f(b)\ne f(a) \text{with } f(b),f(a)\le 1 \\ (f(a))^s \text{ if } f(b)=f(a) \text{ with } f(b),f(a)\le 1 \\ f(a)\int_0^1 \left(\frac{f(b)}{f(a)}\right)^{st} dt \text{ if } f(b)\le 1\le f(a) \\ (f(a))^s (f(b))^{1-s}\int_0^1 \left(\frac{f(b)}{f(a)}\right)^{st} dt \text{ if } f(a)\le 1\le f(b) \\ (f(a))^s (f(b))^{1-s}\int_0^1 \left(\frac{f(b)}{f(a)}\right)^{st} dt \text{ if if } f(b)\ne f(a) \text{ with } f(b),f(a)\ge 1 \\ (f(a))^s (f(b))^{1-s} \text{ if } f(b)=f(a) \text{ with } f(b),f(a)\ge 1. \end{cases} \quad (3.15)$$

Note that

$$\int_0^1 \left(\frac{f(b)}{f(a)}\right)^{st} dt = \frac{(f(b))^s-(f(a))^s}{s(f(a))^s(\ln f(b)-\ln f(a))}. \quad (3.16)$$

Substituting (3.16) into (3.15), using (3.12) and (3.13), we get the desired result in (3.11). The proof is completed. □

Remark 3.5. In Theorem 3.5 if we put $s=1$, we obtain Theorem 2.8 from [21].

Theorem 3.6. *Let $f,g:K=[a,a+\eta(b,a)]\to\mathbb{R}^+$ be (s_1,r_1) and (s_2,r_2)- preinvex functions in the second sense respectively with respect to η on K° (K° interior of K), $a,b\in K^\circ$ with $a<a+\eta(b,a)$ and $(s_1,r_1),(s_2,r_2)\in(0,1]\times(0,\infty)$, then the following inequality holds for $r_1>1$ and $\frac{1}{r_1}+\frac{1}{r_2}=1$*

$$\begin{aligned}\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx &\le \left[\frac{1}{1+s_1}\left[[f(a)]^{r_1}+[f(a)]^{r_1}\right]\right]^{\frac{1}{r_1}} \\ &\times\left[\frac{1}{1+s_2}\left[[g(a)]^{r_2}+[g(b)]^{r_2}\right]\right]^{\frac{1}{r_2}}. \quad (3.17)\end{aligned}$$

Proof. Since f and g are (s_1,r_1) and (s_2,r_2)- preinvex functions in the second sense respectively, we have

$$\begin{aligned}
\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx &= \int_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))dt \\
&\leq \int_0^1 \left[\left[(1-t)^{s_1}[f(a)]^{r_1}+t^{s_1}[f(b)]^{r_1}\right]^{\frac{1}{r_1}}\right. \\
&\quad \times \left.\left[(1-t)^{s_2}[g(a)]^{r_2}+t^{s_2}[g(b)]^{r_2}\right]^{\frac{1}{r_2}}\right]dt,
\end{aligned} \tag{3.18}$$

using Hölder's inequality, we obtain

$$\begin{aligned}
&\int_0^1 \left[(1-t)^{s_1}[f(a)]^{r_1}+t^{s_1}[f(b)]^{r_1}\right]^{\frac{1}{r_1}}\left[(1-t)^{s_2}[g(a)]^{r_2}+t^{s_2}[g(b)]^{r_2}\right]^{\frac{1}{r_2}}dt \\
\leq &\left[\int_0^1 \left[(1-t)^{s_1}[f(a)]^{r_1}+t^{s_1}[f(b)]^{r_1}\right]dt\right]^{\frac{1}{r_1}} \\
&\times \left[\int_0^1 \left[(1-t)^{s_2}[g(a)]^{r_2}+t^{s_2}[g(b)]^{r_2}\right]dt\right]^{\frac{1}{r_2}} \\
= &\left[[f(a)]^{r_1}\int_0^1 (1-t)^{s_1}dt+[f(a)]^{r_1}\int_0^1 t^{s_1}dt\right]^{\frac{1}{r_1}} \\
&\times \left[[g(a)]^{r_2}\int_0^1 (1-t)^{s_2}dt+[g(b)]^{r_2}\int_0^1 t^{s_2}dt\right]^{\frac{1}{r_2}} \\
= &\left[\frac{1}{1+s_1}\left[[f(a)]^{r_1}+[f(a)]^{r_1}\right]\right]^{\frac{1}{r_1}}\left[\frac{1}{1+s_2}\left[[g(a)]^{r_2}+[g(b)]^{r_2}\right]\right]^{\frac{1}{r_2}}.
\end{aligned}$$

The proof is completed. □

Remark 3.6. In Theorem 3.6 if we choose $s_1=s_2=1$ and $\eta(b,a)=b-a$, we obtain Theorem 2.6 from [18].

Theorem 3.7. *Let $f,g:K=[a,a+\eta(b,a)]\to\mathbb{R}^+$ be (s_1,r_1) and (s_2,r_2)- preinvex functions in the second sense respectively with respect to η on K° $(K^\circ$ interior of $K)$, $a,b\in K^\circ$ with $a<a+\eta(b,a)$ and $(s_1,r_1),(s_2,r_2)\in(0,1]\times(0,\infty)$ and $f(b)\neq f(a)$*

and $g(b) \neq g(a)$, *then the following inequality holds*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx \leq \frac{r_1}{4+2r_1}\frac{\left([s_1 f^{r_1}(b)+(1-s_1)f^{r_1}(a)]^{\frac{2+r_1}{r_1}}-f^{2+r_1}(a)\right)}{s_1[f^{r_1}(b)-f^{r_1}(a)]}$$
$$+\frac{r_2}{4+2r_2}\frac{(s_2 g^{r_2}(b)+(1-s_2)g^{r_2}(a))^{\frac{2+r_2}{r_2}}-g^{2+r_2}(a)}{s_2[g^{r_2}(b)-g^{r_2}(a)]}. \tag{3.19}$$

Proof. Since f and g are (s_1,r_1) and (s_2,r_2)- preinvex functions in the second sense respectively, we have

$$\begin{aligned}\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx &= \int_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))dt\\ &\leq \int_0^1 \Big[[(1-t)^{s_1}f^{r_1}(a)+t^{s_1}f^{r_1}(b)]^{\frac{1}{r_1}}\\ &\qquad\times[(1-t)^{s_2}g^{r_2}(a)+t^{s_2}g^{r_2}(b)]^{\frac{1}{r_2}}\Big]dt.\end{aligned} \tag{3.20}$$

Applying Cauchy's inequality, we get

$$\begin{aligned}&\int_0^1 [(1-t)^{s_1}f^{r_1}(a)+t^{s_1}f^{r_1}(b)]^{\frac{1}{r_1}}[(1-t)^{s_2}g^{r_2}(a)+t^{s_2}g^{r_2}(b)]^{\frac{1}{r_2}}dt\\ \leq\ &\frac{1}{2}\int_0^1 [(1-t)^{s_1}f^{r_1}(a)+t^{s_1}f^{r_1}(b)]^{\frac{2}{r_1}}dt+\frac{1}{2}\int_0^1 [(1-t)^{s_2}g^{r_2}(a)+t^{s_2}g^{r_2}(b)]^{\frac{2}{r_2}}dt.\end{aligned} \tag{3.21}$$

From Lemma 2.1, we have

$$(1-t)^{s_1}f^{r_1}(a)+t^{s_1}f^{r_1}(b) \leq s_1[f^{r_1}(b)-f^{r_1}(a)]t+f^{r_1}(a). \tag{3.22}$$

Thus

$$\begin{aligned}&\int_0^1 [(1-t)^{s_1}f^{r_1}(a)+t^{s_1}f^{r_1}(b)]^{\frac{2}{r_1}}dt\\ \leq\ &\int_0^1 [s_1[f^{r_1}(b)-f^{r_1}(a)]t+f^{r_1}(a)]^{\frac{2}{r_1}}dt\\ =\ &\frac{r_1}{2+r_1}\frac{(s_1 f^{r_1}(b)+(1-s_1)f^{r_1}(a))^{\frac{2+r_1}{r_1}}-f^{2+r_1}(a)}{s_1[f^{r_1}(b)-f^{r_1}(a)]}.\end{aligned} \tag{3.23}$$

Similarly,

$$\int_0^1 \left[(1-t)^{s_2} g^{r_2}(a) + t^{s_2} g^{r_2}(b)\right]^{\frac{2}{r_2}} dt$$

$$\leq \frac{r_2}{2+r_2} \frac{\left(s_2 g^{r_2}(b) + (1-s_2) g^{r_2}(a)\right)^{\frac{2+r_2}{r_2}} - g^{2+r_2}(a)}{s_2 \left[g^{r_2}(b) - g^{r_2}(a)\right]}. \tag{3.24}$$

Substituting (3.24) and (3.23) into (3.21), we obtain the desired inequality in (3.19). The proof is completed. □

Remark 3.7. Theorem 3.7 will be reduced to Theorem 11 from [30] if we choose $s_1 = s_2 = 1$. Moreover if $\eta(b,a) = b - a$ then we obtain Theorem 2.8 from [34].

Theorem 3.8. *Let $f, g : K = [a, a+\eta(b,a)] \to \mathbb{R}^+$ be (s_1, r_1) and (s_2, r_2)- preinvex functions in the second sense respectively with respect to η on K° (K° interior of K), $a, b \in K^\circ$ with $a < a+\eta(b,a)$ and $(s_1, r_1), (s_2, r_2) \in (0,1] \times (0,2]$, then the following inequality holds*

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)dx \leq \frac{r_1}{2s_1 + r_1} 2^{\frac{2}{r_1}-1} \left[f^2(a) + f^2(b)\right] + \frac{r_2}{2s_2 + r_2} 2^{\frac{2}{r_2}-1} \left[g^2(a) + g^2(b)\right]. \tag{3.25}$$

Proof. Since f and g are (s_1, r_1) and (s_2, r_2)- preinvex functions in the second sense respectively, we have

$$\begin{aligned} \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)dx &= \int_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))dt \\ &\leq \int_0^1 \Big[\left[(1-t)^{s_1} f^{r_1}(a) + t^{s_1} f^{r_1}(b)\right]^{\frac{1}{r_1}} \\ &\quad \times \left[(1-t)^{s_2} g^{r_2}(a) + t^{s_2} g^{r_2}(b)\right]^{\frac{1}{r_2}}\Big] dt. \end{aligned} \tag{3.26}$$

Applying Cauchy's inequality, we get

$$\int_0^1 \left[(1-t)^{s_1} f^{r_1}(a) + t^{s_1} f^{r_1}(b)\right]^{\frac{1}{r_1}} \left[(1-t)^{s_2} g^{r_2}(a) + t^{s_2} g^{r_2}(b)\right]^{\frac{1}{r_2}} dt$$

$$\leq \frac{1}{2}\int_0^1 \left[(1-t)^{s_1} f^{r_1}(a) + t^{s_1} f^{r_1}(b)\right]^{\frac{2}{r_1}} dt + \frac{1}{2}\int_0^1 \left[(1-t)^{s_2} g^{r_2}(a) + t^{s_2} g^{r_2}(b)\right]^{\frac{2}{r_2}} dt. \tag{3.27}$$

From Lemma 2.2 and taking into account that $0 < r_1, r_2 \leq 2$, we have

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \left[(1-t)^{s_1} f^{r_1}(a) + t^{s_1} f^{r_1}(b)\right]^{\frac{2}{r_1}} dt \\
&+\frac{1}{2}\int_0^1 \left[(1-t)^{s_2} g^{r_2}(a) + t^{s_2} g^{r_2}(b)\right]^{\frac{2}{r_2}} dt \\
\leq\ & \frac{2^{\frac{2}{r_1}}}{2}\left[\int_0^1 (1-t)^{\frac{2s_1}{r_1}} f^2(a)\, dt + \int_0^1 t^{\frac{2s_1}{r_1}} f^2(b) dt\right] \\
&+\frac{2^{\frac{2}{r_2}}}{2}\left[\int_0^1 (1-t)^{\frac{2s_2}{r_2}} g^2(a)\, dt + \int_0^1 t^{\frac{2s_2}{r_2}} g^2(b) dt\right] \\
=\ & \frac{r_1}{2s_1+r_1} 2^{\frac{2}{r_1}-1}\left[f^2(a) + f^2(b)\right] + \frac{r_2}{2s_2+r_2} 2^{\frac{2}{r_2}-1}\left[g^2(a) + g^2(b)\right]. \qquad (3.28)
\end{aligned}$$

The proof is completed. □

Theorem 3.9. *Let $f, g : K = [a, a+\eta(b,a)] \to \mathbb{R}^+$ be (s_1, r_1) and (s_2, r_2)- preinvex functions in the second sense respectively with respect to η on K° (K° interior of K), $a, b \in K^\circ$ with $a < a+\eta(b,a)$ and $(s_1, r_1), (s_2, r_2) \in (0,1] \times [2, \infty)$ then the following inequality holds*

$$\begin{aligned}
\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)dx &\leq \frac{r_1}{4s_1+2r_1}\left[f^2(a) + f^2(b)\right] \\
&+\frac{r_2}{4s_2+2r_2}\left[g^2(a) + g^2(b)\right]. \qquad (3.29)
\end{aligned}$$

Proof. Since f and g are (s_1, r_1) and (s_2, r_2)- preinvex functions in the second sense respectively, we have

$$\begin{aligned}
\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)dx &= \int_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))dt \\
&\leq \int_0^1 \Big[\left[(1-t)^{s_1} f^{r_1}(a) + t^{s_1} f^{r_1}(b)\right]^{\frac{1}{r_1}} \\
&\quad \times \left[(1-t)^{s_2} g^{r_2}(a) + t^{s_2} g^{r_2}(b)\right]^{\frac{1}{r_2}}\Big] dt. \qquad (3.30)
\end{aligned}$$

Applying Cauchy's inequality, we get

$$\begin{aligned}
&\int_0^1 \left[(1-t)^{s_1} f^{r_1}(a) + t^{s_1} f^{r_1}(b)\right]^{\frac{1}{r_1}} \left[(1-t)^{s_2} g^{r_2}(a) + t^{s_2} g^{r_2}(b)\right]^{\frac{1}{r_2}} dt \\
\leq\ & \frac{1}{2}\int_0^1 \left[(1-t)^{s_1} f^{r_1}(a) + t^{s_1} f^{r_1}(b)\right]^{\frac{2}{r_1}} dt + \frac{1}{2}\int_0^1 \left[(1-t)^{s_2} g^{r_2}(a) + t^{s_2} g^{r_2}(b)\right]^{\frac{2}{r_2}} dt. \qquad (3.31)
\end{aligned}$$

Since $r_1, r_2 \geq 2$, from Lemma 2.2, (3.31) gives

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \left[(1-t)^{s_1} f^{r_1}(a) + t^{s_1} f^{r_1}(b)\right]^{\frac{2}{r_1}} dt + \\
&\frac{1}{2}\int_0^1 \left[(1-t)^{s_2} g^{r_2}(a) + t^{s_2} g^{r_2}(b)\right]^{\frac{2}{r_2}} dt \\
\leq\ &\frac{1}{2}\left[\int_0^1 (1-t)^{\frac{2s_1}{r_1}} f^2(a)\, dt + \int_0^1 t^{\frac{2s_1}{r_1}} f^2(b) dt\right] \\
&+\frac{1}{2}\left[\int_0^1 (1-t)^{\frac{2s_2}{r2}} g^2(a)\, dt + \int_0^1 t^{\frac{2s_2}{r2}} g^2(b) dt\right] \\
=\ &\frac{r_1}{4s_1+2r_1}\left[f^2(a) + f^2(b)\right] + \frac{r_2}{4s_2+2r_2}\left[g^2(a) + g^2(b)\right]. \qquad (3.32)
\end{aligned}$$

The proof is completed. □

Theorem 3.10. *Let $f, g : K = [a, a+\eta(b,a)] \to \mathbb{R}^+$ be (s_1, r_1) and $(s_2, 0)$-preinvex functions in the second sense respectively with respect to η on K° $(K^\circ$ interior of $K)$, $a, b \in K^\circ$ with $a < a+\eta(b,a)$ and $(s_1, r_1) \in (0,1] \times [2,\infty)$, $s_2 \in (0,1]$ satisfying $g(b) > g(a) > 1$, then the following inequality holds*

$$\begin{aligned}
&\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx \leq \frac{r_1}{4s_1+2r_1}\left[[f(a)]^2 + [f(b)]^2\right] \\
&+E_1(g(a), g(b), s_2) \times N_1(g(a), g(b), s_2), \qquad (3.33)
\end{aligned}$$

where

$$E_1(g(a), g(b), s_2) = \begin{cases} 1 \ \textit{if} \ \ g(b) = g(a) \\ \frac{(g(b))^{2s_2} - (g(a))^{2s_2}}{2s_2 (g(a))^{2s_2} (\ln g(b) - \ln g(a))} \textit{if} \ \ g(b) \neq g(a), \end{cases} \qquad (3.34)$$

and
(3.35)

$$N_1(g(a), g(b), s_2) = \begin{cases} \frac{1}{2}(g(a))^{2s_2} \quad \textit{if} \ \ g(b) \neq g(a) \ \ \textit{with} \ g(b), g(a) \leq 1 \\ \frac{1}{2}(g(a))^{2s_2} \ \ \textit{if} \ \ g(b) = g(a) \ \ \textit{with} \ g(b), g(a) \leq 1 \\ \frac{1}{2}(g(a))^2 \ \ \textit{if} \ \ g(b) \leq 1 \leq g(a) \\ \frac{1}{2}(g(a))^{2s_2} (g(b))^{2(1-s_2)} \ \ \textit{if} \ \ g(a) \leq 1 \leq g(b) \\ \frac{1}{2}(g(a))^{2s_2} (g(b))^{2(1-s_2)} \ \ \textit{if} \ \ g(b) \neq g(a) \ \textit{with} \ g(b), g(a) \geq 1 \\ \frac{1}{2}(g(a))^{2s_2} (g(b))^{2(1-s_2)} \ \ \textit{if} \ \ g(b) = g(a) \ \textit{with} \ g(b), g(a) \geq 1. \end{cases}$$

Proof. Since f and g are (s_1, r_1) and$(s_2, 0)$- preinvex functions in the second sense respectively, we have

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)dx = \int_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))dt$$
$$\leq \int_0^1 \left[(1-t)^{s_1} f^{r_1}(a) + t^{s_1} f^{r_1}(b)\right]^{\frac{1}{r_1}} [g(a)]^{(1-t)^{s_2}} [g(b)]^{t^{s_2}} dt. \tag{3.36}$$

Using Cauchy's inequality, we get

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)dx \leq \frac{1}{2}\int_0^1 \left[(1-t)^{s_1} f^{r_1}(a) + t^{s_1} f^{r_1}(b)\right]^{\frac{2}{r_1}} dt$$
$$+\frac{1}{2}\int_0^1 \left[(g(a))^2\right]^{(1-t)^{s_2}} \left[(g(b))^2\right]^{t^{s_2}} dt. \tag{3.37}$$

From Lemma 2.2, we have

$$\frac{1}{2}\int_0^1 \left[(1-t)^{s_1} [f(a)]^{r_1} + t^{s_1} [f(b)]^{r_1}\right]^{\frac{2}{r_1}} dt$$
$$\leq \frac{[f(a)]^2}{2}\int_0^1 (1-t)^{\frac{2s_1}{r_1}} dt + \frac{[f(b)]^2}{2}\int_0^1 t^{\frac{2s_1}{r_1}} dt$$
$$= \frac{r_1}{4s_1+2r_1}\left[[f(a)]^2 + [f(b)]^2\right]. \tag{3.38}$$

Using Lemma 2.1, we get

$$\frac{1}{2}\int_0^1 \left[(g(a))^2\right]^{(1-t)^{s_2}} \left[(g(b))^2\right]^{t^{s_2}} dt$$
$$\leq \begin{cases} \frac{1}{2}(g(a))^{2s_2} \int\limits_0^1 \left(\frac{g(b)}{g(a)}\right)^{2s_2 t} dt \text{ if } g(b) \neq g(a) \text{ with } g(b), g(a) \leq 1 \\ \frac{1}{2}(g(a))^{2s_2} \text{ if } g(b) = g(a) \text{ with } g(b), g(a) \leq 1 \\ \frac{1}{2}(g(a))^{2} \int\limits_0^1 \left(\frac{g(b)}{g(a)}\right)^{2s_2 t} dt \text{ if } g(b) \leq 1 \leq g(a) \\ \frac{1}{2}(g(a))^{2s_2} (g(b))^{2(1-s_2)} \int\limits_0^1 \left(\frac{g(b)}{g(a)}\right)^{2s_2 t} dt \text{ if } g(a) \leq 1 \leq g(b) \\ \frac{1}{2}(g(a))^{2s_2} (g(b))^{2(1-s_2)} \int\limits_0^1 \left(\frac{g(b)}{g(a)}\right)^{2s_2 t} dt \text{ if } g(b) \neq g(a) \text{ with } g(b), g(a) \geq 1 \\ \frac{1}{2}(g(a))^{2s_2} (g(b))^{2(1-s_2)} \text{ if } g(b) = g(a) \text{ with } g(b), g(a) \geq 1. \end{cases} \tag{3.39}$$

A simple computation gives

$$\int_0^1 \left(\frac{g(b)}{g(a)}\right)^{2s_2 t} dt = \frac{(g(b))^{2s_2} - (g(a))^{2s_2}}{2s_2 (g(a))^{2s_2} (\ln g(b) - \ln g(a))}. \tag{3.40}$$

Substituting (3.40) into (3.39) and using (3.34) and (3.35), we obtain

$$\frac{1}{2}\int_0^1 \left[(g(a))^2\right]^{(1-t)^{s_2}} \left[(g(b))^2\right]^{t^{s_2}} dt \leq E_1(g(a), g(b), s_2) \times N_1(g(a), g(b), s_2). \tag{3.41}$$

Substituting (3.38) and (3.41) into (3.37), we get the desired result in (3.33). □

Theorem 3.11. *Suppose that all the assumptions of Theorem 3.10 are satisfied. Then the following inequality holds*

$$\begin{aligned} \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)dx &\leq \frac{r_1}{4+2r_1} \frac{\left([s_1 f^{r_1}(b) + (1-s_1) f^{r_1}(a)]\right)^{\frac{2+r_1}{r_1}} - f^{2+r_1}(a)}{s_1 [f^{r_1}(b) - f^{r_1}(a)]} \\ &\quad + E_1(g(a), g(b), s_2) \times N_1(g(a), g(b), s_2), \end{aligned} \tag{3.42}$$

where $E_1(g(a), g(b), s_2)$ *and* $N_1(g(a), g(b), s_2)$ *are defined as in* (3.34) *and* (3.35) *respectively.*

Proof. Since f is (s_1, r_1)- preinvex function in the second sense and g is $(s_2, 0)$-preinvex function, we have

$$\begin{aligned} \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)dx &= \int_0^1 f(a+t\eta(b,a))g(a+t\eta(b,a))dt \\ &\leq \int_0^1 \left[(1-t)^{s_1} f^{r_1}(a) + t^{s_1} f^{r_1}(b)\right]^{\frac{1}{r_1}} [g(a)]^{(1-t)^{s_2}} [g(b)]^{t^{s_2}} dt. \end{aligned} \tag{3.43}$$

Using Cauchy's inequality, we get

$$\begin{aligned} \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)dx &\leq \frac{1}{2}\int_0^1 \left[(1-t)^{s_1} f^{r_1}(a) + t^{s_1} f^{r_1}(b)\right]^{\frac{2}{r_1}} dt \\ &\quad + \frac{1}{2}\int_0^1 \left[(g(a))^2\right]^{(1-t)^{s_2}} \left[(g(b))^2\right]^{t^{s_2}} dt. \end{aligned} \tag{3.44}$$

Like in Theorem 3.7, inequality (3.44) gives

$$\begin{aligned} &\frac{1}{2}\int_0^1 \left[(1-t)^{s_1} f^{r_1}(a) + t^{s_1} f^{r_1}(b)\right]^{\frac{2}{r_1}} dt \\ &\leq \frac{r_1}{4+2r_1} \frac{\left([s_1 f^{r_1}(b) + (1-s_1) f^{r_1}(a)]\right)^{\frac{2+r_1}{r_1}} - f^{2+r_1}(a)}{s_1 [f^{r_1}(b) - f^{r_1}(a)]}, \end{aligned} \tag{3.45}$$

and

$$\frac{1}{2}\int_0^1 \left[(g(a))^2\right]^{(1-t)^{s_2}} \left[(g(b))^2\right]^{t^{s_2}} dt \leq E_1(g(a), g(b), s_2) \times N_1(g(a), g(b), s_2), \tag{3.46}$$

where $E_1(g(a), g(b), s_2)$ and $N_1(g(a), g(b), s_2)$ are defined as in (3.34) and (3.35).

Using (3.46) and (3.45) into (3.44), we obtain the desired inequality in (3.42). □

Remark 3.8. In Theorem 3.11 if we put $s_1 = s_2 = 1$, we obtain Theorem 12 from [30].

Theorem 3.12. *Let $f, g : K = [a, a + \eta(b, a)] \to \mathbb{R}^+$ be $(s_1, 0)$-preinvex and $(s_2, 0)$- preinvex functions in the second sense respectively with respect to η on K° (K° interior of K), $a, b \in K^\circ$ with $a < a + \eta(b, a)$, then the following inequality holds*

$$\begin{aligned}\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx \quad &\leq \quad E_1(f(a), f(b), s_1) \times N_1(f(a), f(b), s_1) \\ &\quad + E_1(g(a), g(b), s_2) \times N_1(g(a), g(b), s_2),\end{aligned} \tag{3.47}$$

where $E_1(.,.,.)$ and $N_1(.,.,.)$ are defined as in (3.34) *and* (3.35) *respectively.*

Proof. Since f and g are $(s_1, 0)$ and $(s_2, 0)$-preinvex functions in the second sense respectively, we have

$$\begin{aligned}\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx \quad &= \quad \int_0^1 f(a + t\eta(b,a))g(a + t\eta(b,a))dt \\ &\leq \quad \int_0^1 [f(a)]^{(1-t)^{s_1}} [f(b)]^{t^{s_1}} [g(a)]^{(1-t)^{s_2}} [g(b)]^{t^{s_2}} dt.\end{aligned} \tag{3.48}$$

Using Cauchy's inequality, we get

$$\begin{aligned}\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx \quad &\leq \quad \frac{1}{2}\int_0^1 \left[(f(a))^2\right]^{(1-t)^{s_1}} \left[(f(b))^2\right]^{t^{s_1}} dt \\ &\quad + \frac{1}{2}\int_0^1 \left[(g(a))^2\right]^{(1-t)^{s_2}} \left[(g(b))^2\right]^{t^{s_2}} dt.\end{aligned} \tag{3.49}$$

Like Theorem 3.10, we have

$$\frac{1}{2}\int_0^1 \left[(f(a))^2\right]^{(1-t)^{s_1}} \left[(f(b))^2\right]^{t^{s_1}} dt \leq E_1(f(a), f(b), s_1) \times N_1(f(a), f(b), s_1), \tag{3.50}$$

and

$$\frac{1}{2}\int_{0}^{1}\left[(g(a))^2\right]^{(1-t)^{s_2}}\left[(g(b))^2\right]^{t^{s_2}}dt\le E_1(g(a),g(b),s_2)\times N_1(g(a),g(b),s_2).$$
(3.51)

Substituting (3.51) and (3.50) into (3.49), we get the desired result in (3.47). $\square$

Remark 3.9. Theorem 3.12 will be reduced to Theorem 3.1 from [21], if we take $s_1 = s_2 = 1$.

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