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Research Article

Fuzzifying pseudo-quasi-metric topologies on the fuzzy real line

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Abstract

In this paper, three natural fuzzifying topologies are presented on the fuzzy real line. Then the notion of fuzzifying pseudo-quasi-metrics is introduced. It is proved that the three fuzzifying topologies can be induced respectively by three fuzzifying pseudo-quasi-metrics. Our definition of fuzzifying pseudo-metric is slightly different from that of KM-fuzzy metric. A fuzzifying pseudo-metrics can be regarded as a weak form of a KM fuzzy metric.

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1. Introduction

The fuzzy unit interval was first presented by Hutton [7] and the L-fuzzy real line was introduced by Höhle [5] and Gantner et al.[2]. Then Goetschel and Voxman [3] presented the other definition of fuzzy numbers. A GV-fuzzy number can be decomposed into two Hutton's fuzzy numbers. In fact, a Hutton's fuzzy number can be regarded as the complement of a distribution function. However Hutton's fuzzy numbers and GV-fuzzy numbers are very important in the theory of fuzzy sets. They are important not only in L-topology, but also in other fuzzy mathematical fields.

To reflect the characteristics of pointwise L-topology, i.e., the relation between a fuzzy point and its Q-neighborhoods (or R-neighborhoods) [9], a theory of pointwise uniformities and a theory of pointwise metrics were introduced on completely distributive lattices and in L-fuzzy set theory(see [11,12]). Many ideal results in general topology were generalized to L-topology [14,15]. It was proved that the L-fuzzy real line is pointwise pseudo-metrizable and the pointwise pseudo-metric function on the L-fuzzy real line was given [13]. Moreover a natural L-topology is constructed on the set of L-fuzzy numbers in the sense of [3, 6] and it can be induced by a pointwise pseudo-metric [19].

A probabilistic metric space (or a statistical metric space) is classically defined relative to a so-called t-norm [10]. In fact, it is also called a fuzzy metric space in the sense of [8]. In [8], I. Kramosil and J. Michalek presented a definition of fuzzy metric. M. Grabiec revised it as follows:

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Definition 1.1 ([4,8]). A KM fuzzy metric is a fuzzy set $F : X^2 \times [0, \infty) \to [0, 1]$ satisfies the following conditions: $\forall x, y, z \in X$ and $\forall s, t > 0$,

(1) $F(x, y, 0) = 0$;
(2) $F(x, y, t) = 1$ if and only if $x = y$;
(3) $F(x, y, s) * F(y, z, t) \leq F(x, z, s + t)$;
(4) $F(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous;
(5) $F(x, y, t) = F(y, x, t)$;
(6) $\lim_{t\to\infty} F(x, y, t) = 1$.

When $* = \min$ and $L = \mathbf{2}$, a KM fuzzy metric can be regarded as an (L, M)-fuzzy metric [16].

As we know, a KM fuzzy metric can induce a fuzzifying topology [16]. Now we naturally have to ask the following question:

Whether there is a topology on the set of all distribution functions such that the topology can be induced by a probabilistic metric, or analogously whether there is a (fuzzy) topology on the fuzzy real line such that the (fuzzy) topology can be induced by a KM-fuzzy metric?

In fact, there have been some researches about metrics and topologies on the fuzzy real line [1, 21]. In [1], the author only considers the crisp metric and does not consider topology. In addition, Zhang only considers crisp topology but does not consider metric.

In this paper, we shall discuss the above problem. First, three natural fuzzifying topologies are presented on the fuzzy real line. Then the notion of fuzzifying pseudo-quasi-metrics is introduced. It is proved that the three fuzzifying topologies can be induced respectively by three fuzzifying pseudo-quasi-metrics. It is worth noting that our definition of fuzzifying pseudo-metric is slightly different from that of KM-fuzzy metric.

2. Preliminaries

Throughout this paper, M always denotes a completely distributive lattice with an order-reversing involution " $'$ ". The smallest element and the largest element in M are denoted by 0 and 1 respectively. We say that a is wedge-below b, denoted by $a \prec b$, if for every subset $D \subseteq M$, the relation $b \leq \bigvee D$ always implies the existence of a $d \in D$ with $a \leq d$. A complete lattice M is completely distributive if and only if $b = \bigvee\{a \in M \mid a \prec b\}$ for each $b \in M$ [9].

Definition 2.1 ([2,5,7]). An M-fuzzy real number is an equivalence class $[\lambda]$ of antitone maps $\lambda : \mathbb{R} \to M$ satisfying

$$\lambda(-\infty) = \bigvee_{t\in\mathbb{R}} \lambda(t) = 1 \quad \text{and} \quad \lambda(+\infty) = \bigwedge_{t\in\mathbb{R}} \lambda(t) = 0,$$

where the equivalence identifies two such maps λ, μ if and only if $\forall t > 0$, $\lambda(t-) = \mu(t-)$.

The set of all M-fuzzy real numbers is denoted by $\mathbb{R}(M)$.

We shall not distinguish an M-fuzzy real number $[\lambda]$ from its representative function λ being left continuous.

Definition 2.2 ([20]). A map $\mathcal{T} : 2^X \to M$ is called an M-fuzzifying topology if it satisfies the following conditions:

(FYT1) $\mathcal{T}(X) = \mathcal{T}(\emptyset) = 1$;
(FYT2) $\forall A, B \in 2^X, \mathcal{T}(A \cap B) \geq \mathcal{T}(A) \wedge \mathcal{T}(B)$;
(FYT3) $\forall \{A_i \mid i \in \Delta\} \subseteq 2^X, \mathcal{T}\left(\bigcup_{i\in\Delta} A_i\right) \geq \bigwedge_{i\in\Delta} \mathcal{T}(A_i)$.

3. M-fuzzifying topologies on the M-fuzzy real line

In this section, we shall present three M-fuzzifying topologies on $\mathbb{R}(M)$. They are natural extensions of three topologies on $\mathbb{R}$.

Theorem 3.1. *Let $\mathbb{R}(M)$ be the set of M-fuzzy real numbers. Define three mappings $\mathcal{R}, \mathcal{L}, \mathcal{T} : 2^{\mathbb{R}(M)} \to M$ such that for all $A \subseteq \mathbb{R}(M)$,*

(1) $\mathcal{R}(A) = \bigwedge_{\lambda\in A} \bigvee_{s\in\mathbb{R}} \left(\lambda(s+) \wedge \bigwedge_{\mu\notin A} \mu(s+)'\right)$;

(2) $\mathcal{L}(A) = \bigwedge_{\lambda\in A} \bigvee_{r\in\mathbb{R}} \left(\lambda(r-)' \wedge \bigwedge_{\mu\notin A} \mu(r-)\right)$;

(3) $\mathcal{T}(A) = \bigwedge_{\lambda\in A} \bigvee_{r,s\in\mathbb{R}} \left(\lambda(s+) \wedge \lambda(r-)' \wedge \bigwedge_{\mu\notin A} (\mu(s+)' \vee \mu(r-))\right)$.

Then $\mathcal{R}, \mathcal{L}$ and $\mathcal{T}$ are M-fuzzifying topologies on $\mathbb{R}(M)$.

Proof. We only prove that $\mathcal{T}$ is an M-fuzzifying topology on $\mathbb{R}(M)$. The other proofs are analogous. It is obvious that $\mathcal{T}(\emptyset) = \mathcal{T}(\mathbb{R}(M)) = 1$. Now we prove that for all $A, B \in 2^{\mathbb{R}(M)}$,

$$\mathcal{T}(A \cap B) \geq \mathcal{T}(A) \wedge \mathcal{T}(B).$$

Suppose that $a \in M$ and $a \prec \mathcal{T}(A) \wedge \mathcal{T}(B)$. Then $a \prec \mathcal{T}(A)$ and $a \prec \mathcal{T}(B)$. Further we have that

$$a \prec \mathcal{T}(A) = \bigwedge_{\lambda\in A} \bigvee_{s,r\in\mathbb{R}} \left(\lambda(s+) \wedge \lambda(r-)' \wedge \bigwedge_{\mu\notin A} (\mu(s+)' \vee \mu(r-))\right)$$

and

$$a \prec \mathcal{T}(B) = \bigwedge_{\lambda\in B} \bigvee_{s,r\in\mathbb{R}} \left(\lambda(s+) \wedge \lambda(r-)' \wedge \bigwedge_{\mu\notin B} (\mu(s+)' \vee \mu(r-))\right).$$

Hence for all $\lambda \in A$, there exists $s, r \in \mathbb{R}$ such that

$$a \leq \lambda(s+) \wedge \lambda(r-)' \wedge \bigwedge_{\mu\notin A} (\mu(s+)' \vee \mu(r-))$$

and for all $\lambda \in B$, there exists $u, v \in \mathbb{R}$ such that

$$a \leq \lambda(v+) \wedge \lambda(u-)' \wedge \bigwedge_{\mu\notin B} (\mu(v+)' \vee \mu(u-)).$$

This implies for all $\lambda \in A \cap B$ and for all $\mu \notin A \cap B$, it follows

$$a \leq \lambda(s+) \wedge \lambda(v+) \wedge \lambda(r-)' \wedge \lambda(u-)' \wedge \bigwedge_{\mu\notin A} (\mu(s+)' \vee \mu(r-))$$

or

$$a \leq \lambda(s+) \wedge \lambda(v+) \wedge \lambda(r-)' \wedge \lambda(u-)' \wedge \bigwedge_{\mu\notin B} (\mu(v+)' \vee \mu(u-)).$$

Thus we have

$$\begin{aligned} a &\leq \lambda(s+) \wedge \lambda(v+) \wedge \lambda(r-)' \wedge \lambda(u-)' \wedge \bigwedge_{\mu\notin A\cap B} (\mu(s+)' \vee \mu(r-) \vee \mu(v+)' \vee \mu(u-)) \\ &\leq \lambda((s \vee v)+) \wedge \lambda((r \wedge u)-)' \wedge \bigwedge_{\mu\notin A\cap B} \mu((s \vee v)+)' \vee \mu((r \wedge u)-). \end{aligned}$$

This shows

$$\begin{aligned}&\mathcal{T}(A\cap B)\\ =\ &\bigwedge_{\lambda\in A\cap B}\bigvee_{r,s,u,v\in\mathbb{R}}\left(\lambda((s\vee v)+)\wedge\lambda((r\wedge u)-)'\wedge\bigwedge_{\mu\notin A\cap B}\mu((s\vee v)+)'\vee\mu((r\wedge u)-)\right)\\ \geq\ & a.\end{aligned}$$

Therefore we have $\mathcal{T}(A\cap B)\geq\mathcal{T}(A)\wedge\mathcal{T}(B)$.

Next we prove that for any family of sets $\{A_i\mid i\in\Omega\}\subseteq 2^{\mathbb{R}(M)}$,

$$\mathcal{T}\left(\bigcup_{i\in\Omega}A_i\right)\geq\bigwedge_{i\in\Omega}\mathcal{T}(A_i).$$

Suppose that $a\in M$ and $a\prec\bigwedge_{i\in\Omega}\mathcal{T}(A_i)$. Then for all $i\in\Omega$, it follows that

$$a\prec\mathcal{T}(A_i)=\bigwedge_{\lambda\in A_i}\bigvee_{s,r\in\mathbb{R}}\left(\lambda(s+)\wedge\lambda(r-)'\wedge\bigwedge_{\mu\notin A_i}(\mu(s+)'\vee\mu(r-))\right).$$

This implies that for all $i\in\Omega$ and for all $\lambda\in A_i$,

$$a\prec\bigvee_{s,r\in\mathbb{R}}\left(\lambda(s+)\wedge\lambda(r-)'\wedge\bigwedge_{\mu\notin A_i}(\mu(s+)'\vee\mu(r-))\right).$$

Hence for all $\lambda\in\bigcup_{i\in\Omega}A_i$, there exists $k\in\Omega$ and $r_k,s_k\in\mathbb{R}$ such that when $\lambda\in A_k$, for all $\mu\notin A_k$, it follows

$$a\leq\lambda(s_k+)\wedge\lambda(r_k-)'\wedge\left(\mu(s_k+)'\vee\mu(r_k-)\right).$$

In particular, for all $\mu\notin\bigcup_{i\in\Omega}A_i$, we have

$$a\leq\lambda(s_k+)\wedge\lambda(r_k-)'\wedge\left(\mu(s_k+)'\vee\mu(r_k-)\right).$$

This implies

$$\begin{aligned}a\ &\leq\lambda(s_k+)\wedge\lambda(r_k-)'\wedge\bigwedge_{\mu\notin\bigcup_{i\in\Omega}A_i}(\mu(s_k+)'\vee\mu(r_k-))\\ &\leq\bigvee_{u,v\in\mathbb{R}}\left(\lambda(v+)\wedge\lambda(u-)'\wedge\bigwedge_{\mu\notin\bigcup_{i\in\Omega}A_i}(\mu(v+)'\vee\mu(u-))\right).\end{aligned}$$

Therefore we can obtain

$$a\ \leq\bigwedge_{\lambda\in\bigcup_{i\in\Omega}A_i}\bigvee_{u,v\in\mathbb{R}}\left(\lambda(v+)\wedge\lambda(u-)'\wedge\bigwedge_{\mu\notin\bigcup_{i\in\Omega}A_i}(\mu(v+)'\vee\mu(u-))\right)=\mathcal{T}\left(\bigcup_{i\in\Omega}A_i\right).$$

It is proved that $\mathcal{T}\left(\bigcup_{i\in\Omega}A_i\right)\geq\bigwedge_{i\in\Omega}\mathcal{T}(A_i)$ is true. We complete the proof that $\mathcal{T}$ is an M-fuzzifying topology on $\mathbb{R}(M)$. □

Example 3.2. For a real number $\mu\in\mathbb{R}$, define $\underline{\mu}\in\mathbb{R}(M)$ such that

$$\underline{\mu}(t)=\begin{cases}1, & t\leq\mu;\\ 0, & t>\mu.\end{cases}$$

Then $\underline{\mu}$ is an M-fuzzy number. In this case, $\forall s, t \in \mathbb{R}$ with $s < t$, we have

$$\begin{aligned}
\mathcal{L}((-\infty,t)) &= \bigwedge_{\lambda\in(-\infty,t)} \bigvee_{u\in\mathbb{R}} \left(\underline{\lambda}(u-)' \wedge \bigwedge_{\mu\geq t} \underline{\mu}(u-) \right) \\
&\geq \bigwedge_{\lambda\in(-\infty,t)} \left(\underline{\lambda}(t)' \wedge \bigwedge_{\mu\geq t} \underline{\mu}(t-) \right) = 1; \\
\mathcal{R}((s,+\infty)) &= \bigwedge_{\lambda\in(s,+\infty)} \bigvee_{u\in\mathbb{R}} \left(\underline{\lambda}(u+) \wedge \bigwedge_{\mu\leq s} \underline{\mu}(u+)' \right) \\
&\geq \bigwedge_{\lambda\in(s,+\infty)} \left(\underline{\lambda}(s+) \wedge \bigwedge_{\mu\leq s} \underline{\mu}(s+)' \right) = 1; \\
\mathcal{T}((s,t)) &= \bigwedge_{\lambda\in(s,t)} \bigvee_{u,v\in\mathbb{R}} \left(\lambda(v+) \wedge \lambda(u-)' \wedge \bigwedge_{\mu\notin(s,t)} (\mu(v+)' \wedge \mu(u-)) \right) \\
&\geq \bigwedge_{\lambda\in(s,t)} \left(\lambda(s+) \wedge \lambda(t-)' \wedge \bigwedge_{\mu\notin(s,t)} (\mu(s+)' \wedge \mu(t-)) \right) = 1.
\end{aligned}$$

Let $\underline{\mathbb{R}} = \{\underline{u} \mid u \in \mathbb{R}\}$. Then $\underline{\mathbb{R}}$ (or $\mathbb{R}$) can be regarded as a subset of $\mathbb{R}(M)$. The restrictions of $\mathcal{L}$ and $\mathcal{R}$ to $\underline{\mathbb{R}}$ can be regarded as $\{(-\infty,t) \mid t \in \mathbb{R}\}$ and $\{(s,+\infty) \mid s \in \mathbb{R}\}$. The restriction of $\mathcal{T}$ to $\underline{\mathbb{R}}$ can be regarded as the usual topology on $\mathbb{R}$.

4. Fuzzifying pseudo-quasi-metrics on the $[0,1]$-fuzzy real line

In this section, we take $M = I = [0,1]$ and present the definition of $[0,1]$-fuzzifying pseudo-quasi-metrics (fuzzifying pseudo-quasi-metrics for short). Moreover we also present the expression forms of fuzzifying pseudo-quasi-metrics on the $[0,1]$-fuzzy real line (fuzzy real line for short).

Lemma 4.1. *Let $\mathbb{R}(I)$ be the fuzzifying real line. Define two maps $\varepsilon, \sigma : (0,1] \times \mathbb{R}(I) \to \mathbb{R}$ such that for all $a \in (0,1]$ and for all $x \in \mathbb{R}(I)$,*

$$\varepsilon(a,x) = \sup\{t \mid a \leq x(t-)\}, \quad \sigma(a,x) = \inf\{t \mid a \leq x(t+)'\}.$$

Then we have the following results.

(1) $\varepsilon(a,x) = \max\{t \mid a \leq x(t-)\}, \quad \sigma(a,x) = \min\{t \mid a \leq x(t+)'\}$.

(2) *For all $a \in (0,1]$ and for all $x \in \mathbb{R}(I)$,*

$$\varepsilon(a,x) = \bigwedge_{b<a} \varepsilon(b,x), \quad \sigma(a,x) = \bigvee_{b<a} \sigma(b,x).$$

(3) *For all $a, b \in (0,1]$ and for all $x, y \in \mathbb{R}(I)$,*

$$\varepsilon(b,y) < \bigvee_{c>a'} \varepsilon(c,x) + r \text{ if and only if } \sigma(a,x) > \bigwedge_{e>b'} \sigma(e,y) - r.$$

Proof. (1) is obvious. It is easy to see that

$$\varepsilon(a,x) \leq \bigwedge_{b<a} \varepsilon(b,x), \quad \sigma(a,x) \geq \bigvee_{b<a} \sigma(b,x).$$

Thus in order to prove (2) we need only to prove that

$$\varepsilon(a,x) \geq \bigwedge_{b<a} \varepsilon(b,x), \quad \sigma(a,x) \leq \bigvee_{b<a} \sigma(b,x).$$

Suppose that $\varepsilon(a,x) < \bigwedge_{b<a} \varepsilon(b,x)$. Then there exists $s \in \mathbb{R}$ such that

$$\varepsilon(a,x) = \max\{t \mid a \leq x(t)\} < s < \bigwedge_{b<a} \varepsilon(b,x).$$

This implies that $a \not\leq x(s)$. Further there exists $c < a$ such that $c \not\leq x(s)$. Thus we have that $\varepsilon(c,x) < s$. By $s < \bigwedge_{b<a} \varepsilon(b,x)$ we obtain a contradiction. Therefore $\varepsilon(a,x) \geq \bigwedge_{b<a} \varepsilon(b,x)$. Similarly we can prove the other inequalities. (2) is shown.

In order to prove (3), for all $a, b \in (0,1]$ and for all $x, y \in \mathbb{R}(I)$, suppose

$$\varepsilon(b,y) < \bigvee_{c>a'} \varepsilon(c,x) + r.$$

Then there exists $c > a'$ such that $\varepsilon(b,y) < \varepsilon(c,x) + r$. Then there is $t > 0$ such that $\varepsilon(b,y) < \varepsilon(c,x) + r - t$. This implies that

$$b \not\leq y(\varepsilon(c,x) + r - t) \quad \text{or} \quad y(\varepsilon(c,x) + r - t)' \not\leq b'$$

So there exists $e \leq y(\varepsilon(c,x) + r - t)'$ such that $e \not\leq b'$ (i.e., $e > b'$). We can obtain

$$\sigma(e,y) \leq \varepsilon(c,x) + r - t \quad \text{or} \quad \sigma(e,y) - r + t \leq \varepsilon(c,x)$$

since $y(\varepsilon(c,x) + r - t)' \leq y(\varepsilon(c,x) + r - t+)'$. By $c \leq x(\varepsilon(c,x))$ again we know that

$$a > c' \geq x(\varepsilon(c,x))' \geq x(\sigma(e,y) - r + t)' \geq x(\sigma(e,y) - r+)'.$$

Therefore $\sigma(a,x) > \sigma(e,y) - r$. Further we have that $\sigma(a,x) > \bigwedge_{e>b'} \sigma(e,y) - r$.

The inverse of the above proof is obvious. □

Now we present a new kind of fuzzy metric.

Definition 4.2. A fuzzifying pseudo-quasi-metric on a set X is a fuzzy set $D : X^2 \times [0,\infty) \to [0,1]$ which satisfies the following (FM1)–(FM5): $\forall x,y,z \in X$ and $\forall r,s \in (0,+\infty)$,

(FM1) $D(x,y)(0) = 1$;
(FM2) $D(x,x)(r) \leq 0.5$;
(FM3) $\bigwedge_{r>0} D(x,y)(r) = 0$;
(FM4) $D(x,y)(r) = \bigwedge_{s<r} D(x,y)(s)$, i.e., $D(x,y) : [0,\infty) \to [0,1]$ is left continuous;
(FM5) $D(x,z)(s+r) \leq D(x,y)(s) + D(y,z)(r)$.

A fuzzifying pseudo-quasi-metric D is said to be a fuzzifying pseudo-metric if it satisfies the following (FM6):

(FM6) $D(x,y)(r) = D(y,x)(r)$.

Remark 4.3. Suppose that F is a KM fuzzy metric on X and $*$ is a Lukasiewicz T-norm. Let $D(x,y)(r) = F(x,y,r)'$. Then D is a fuzzifying pseudo-metric. But it is easy to check that the inverse is not true.

Theorem 4.4. *Let $\mathbb{R}(I)$ be the fuzzy real line. For all $x, y \in \mathbb{R}(I)$ and for all $r \in \mathbb{R}$, define*

$$D_1(x,y)(r) = \bigvee \left\{ a \in (0,1] \;\middle|\; \max\left\{ \varepsilon(a,y) - \bigvee_{c>a'} \varepsilon(c,x), 0 \right\} \geq r \right\},$$
$$D_2(y,x)(r) = \bigvee \left\{ a \in (0,1] \;\middle|\; \max\left\{ \bigwedge_{c>a'} \sigma(c,y) - \sigma(a,x), 0 \right\} \geq r \right\},$$
$$D(x,y)(r) = D_1(x,y)(r) \vee D_2(x,y)(r).$$

Then

(1) $\forall b \in (0,1]$, $b < D_1(x,y)(r) \Rightarrow \max\left\{ \varepsilon(b,y) - \bigvee_{c>b'} \varepsilon(c,x), 0 \right\} \geq r.$

(2) $\forall b \in (0,1]$, $b < D_2(y,x)(r) \Rightarrow \max\left\{ \bigwedge_{c>b'} \sigma(c,y) - \sigma(b,x), 0 \right\} \geq r.$

(3) D_1, D_2 *are fuzzifying pseudo-quasi-metrics.*

(4) D *is a fuzzifying pseudo-metric.*

Proof. (1) Suppose $b \in (0,1]$ and $b < D_1(x,y)(r)$. By the definition of $D_1(x,y)(r)$ we know that there exists $a \in (0,1]$ such that $b < a$ and

$$\max\left\{\varepsilon(a,y) - \bigvee_{c>a'} \varepsilon(c,x), 0\right\} \geq r.$$

Hence by Lemma 4.1 we have

$$\begin{aligned}\max\left\{\varepsilon(b,y) - \bigvee_{c>b'} \varepsilon(c,x), 0\right\} &\geq \max\left\{\varepsilon(a,y) - \bigvee_{c>b'} \varepsilon(c,x), 0\right\}\\ &\geq \max\left\{\varepsilon(a,y) - \bigvee_{c>a'} \varepsilon(c,x), 0\right\} \geq r.\end{aligned}$$

(2) Suppose $b \in (0,1]$ and $b < D_2(y,x)(r)$. By the definition of $D_2(y,x)(r)$ we know that there exists $a \in (0,1]$ such that $b < a$ and

$$\max\left\{\bigwedge_{c>a'} \sigma(c,y) - \sigma(a,x), 0\right\} \geq r.$$

Hence we have

$$\begin{aligned}\max\left\{\bigwedge_{c>b'} \sigma(c,y) - \sigma(b,x), 0\right\} &\geq \max\left\{\bigwedge_{c>a'} \sigma(c,y) - \sigma(b,x), 0\right\}\\ &\geq \max\left\{\bigwedge_{c>a'} \sigma(c,y) - \sigma(a,x), 0\right\} \geq r.\end{aligned}$$

(3) (i) We first prove that D_1 is a fuzzifying pseudo-quasi-metric.

(FM1) Obviously for all x, y, we have $D_1(x,y)(0) = 1$.

(FM2) For all $a \in (0.5, 1]$, it holds $a > a'$. Thus $\forall x \in \mathbb{R}(I)$ and for all $r \in (0,+\infty)$, by $\max\left\{\varepsilon(a,x) - \bigvee_{c>a'} \varepsilon(c,x), 0\right\} = 0$, we know

$$D_1(x,x)(r) = \bigvee\left\{a \in (0,1] \;\middle|\; \max\left\{\varepsilon(a,x) - \bigvee_{c>a'} \varepsilon(c,x), 0\right\} \geq r\right\} \leq 0.5.$$

(FM3) It is obvious that $\bigwedge_{r>0} D_1(x,y)(r) = 0$.

(FM4) Obviously $D_1(x,y)$ is an antitone map from $\mathbb{R} \to [0,1]$. Now we prove that $D_1(x,y)$ is left continuous, i.e.,

$$D_1(x,y)(r) = \bigwedge_{s<r} D_1(x,y)(s), \quad \forall r \in [0,+\infty).$$

Obviously $\forall r \in [0,+\infty)$, it follows

$$D_1(x,y)(r) \leq \bigwedge_{s<r} D_1(x,y)(s).$$

We need only to prove that

$$D_1(x,y)(r) \geq \bigwedge_{s<r} D_1(x,y)(s).$$

Let $b < \bigwedge_{s<r} D_1(x,y)(s)$. Then $\forall s < r$, we have

$$b < D_1(x,y)(s), \quad \text{i.e.,} \quad \max\left\{\varepsilon(b,y) - \bigvee_{c>b'} \varepsilon(c,x), 0\right\} \geq s.$$

Hence $\max\left\{\varepsilon(b,y) - \bigvee_{c>b'} \varepsilon(c,x), 0\right\} \geq \sup\{s \mid s < r\} = r$. This implies that $b \leq D_1(x,y)(r)$. Therefore

$$D_1(x,y)(r) \geq \bigwedge_{s<r} D_1(x,y)(s).$$

As a result, $D_1(x,y)$ is a fuzzy number.

(FM5) Now we prove $D_1(x,z)(r+s) \leq D_1(x,y)(r) + D_1(y,z)(s)$, $\forall x, y \in \mathbb{R}(I), \forall r, s \in [0,+\infty)$.

If $r = 0$ or $s = 0$, then the above inequality is obvious. If $D_1(x,y)(r) + D_1(y,z)(s) \geq 1$, then the above inequality is also obvious.

Now we let $r, s \in (0,+\infty)$ and suppose $D_1(x,y)(r) + D_1(y,z)(s) < 1$. Take $b, b_1, b_2 \in (0,1]$ such that $1 > b = b_1 + b_2$, $b_1 > D_1(x,y)(r)$ and $b_2 > D_1(y,z)(s)$. Then by (1) we know that

$$\max\left\{\left(\varepsilon(b_1,y) - \bigvee_{c>b_1'} \varepsilon(c,x)\right), 0\right\} < r \quad \text{and} \quad \max\left\{\left(\varepsilon(b_2,z) - \bigvee_{c>b_2'} \varepsilon(c,y)\right), 0\right\} < s.$$

This implies that

$$\begin{aligned}
& \max\left\{\varepsilon(b,z) - \bigvee_{c>b'} \varepsilon(c,x), 0\right\} \\
\leq\ & \max\left\{\varepsilon(b,z) - \bigvee_{c>b_1'} \varepsilon(c,x), 0\right\} \\
\leq\ & \max\left\{\varepsilon(b_2,z) - \bigvee_{c>b_1'} \varepsilon(c,x), 0\right\} \\
\leq\ & \max\left\{\left(\varepsilon(b_2,z) - \bigvee_{c>b_2'} \varepsilon(c,y)\right) + \left(\bigvee_{c>b_2'} \varepsilon(c,y) - \bigvee_{c>b_1'} \varepsilon(c,x)\right), 0\right\} \\
\leq\ & \max\left\{\varepsilon(b_2,z) - \bigvee_{c>b_2'} \varepsilon(c,y), 0\right\} + \max\left\{\varepsilon(b_2',y) - \bigvee_{c>b_1'} \varepsilon(c,x), 0\right\} \\
\leq\ & \max\left\{\varepsilon(b_2,z) - \bigvee_{c>b_2'} \varepsilon(c,y), 0\right\} + \max\left\{\left(\varepsilon(b_1,y) - \bigvee_{c>b_1'} \varepsilon(c,x)\right), 0\right\} \\
<\ & r + s.
\end{aligned}$$

Thus we have $b \geq D_1(x,z)(r+s)$, (FM5) is proved.

So D_1 is a fuzzifying pseudo-quasi-metric.

(ii) Now we prove that D_2 is a fuzzifying pseudo-quasi-metric.

(FM1) Obviously for all x, y, it holds $D_2(y,x)(0) = 1$.

(FM2) For all $a \in (0.5, 1]$, it holds $a > a'$. Thus $\forall x \in \mathbb{R}(I)$ and for all $r \in (0, +\infty)$, by $\max\left\{\bigwedge_{c>a'} \sigma(c,x) - \sigma(a,x), 0\right\} = 0$, we know

$$D_2(x,x)(r) = \bigvee \left\{ a \in (0,1] \;\middle|\; \max\left\{\bigwedge_{c>a'} \sigma(c,x) - \sigma(a,x), 0\right\} \geq r \right\} \leq 0.5.$$

(FM3) $\bigwedge_{r>0} D_2(y,x)(r) = 0$ is obvious.

(FM4) It is clear that $D_2(y,x)$ is an antitone map from $[0,+\infty) \to [0,1]$. Now we prove that $D_2(y,x)$ is left continuous, i.e.,

$$D_2(y,x)(r) = \bigwedge_{s<r} D_2(y,x)(s), \quad \forall r \in [0,+\infty).$$

It is obvious that $\forall r \in [0,+\infty)$,

$$D_2(y,x)(r) \leq \bigwedge_{s<r} D_2(y,x)(s).$$

We need only to prove that

$$D_2(y,x)(r) \geq \bigwedge_{s<r} D_2(y,x)(s).$$

Let $b < \bigwedge_{s<r} D_2(y,x)(s)$. Then $\forall s < r$, we have

$$b < D_2(y,x)(s), \quad \text{i.e.,} \quad \max\left\{\bigwedge_{c>b'} \sigma(c,y) - \sigma(b,x), 0\right\} \geq s.$$

Hence $\max\left\{\bigwedge_{c>b'} \sigma(c,y) - \sigma(b,x), 0\right\} \geq \sup\{s \mid s < r\} = r$. This implies that $b \leq D_2(y,x)(r)$. Therefore

$$D_2(y,x)(r) \geq \bigwedge_{s<t} D_2(y,x)(s).$$

As a result, $D_2(y,x)$ is a fuzzy number.

(FM5) Now we prove $D_2(y,x)(r+s) \leq D_2(y,z)(r) + D_2(z,x)(s)$, $\forall x, y \in \mathbb{R}(I), \forall r, s \in [0,+\infty)$.

If $r = 0$ or $s = 0$, then the above inequality is obvious. If $D_2(y,z)(r) + D_2(z,x)(s) \geq 1$, then the above inequality is also obvious.

Now we let $r, s \in (0,+\infty)$ and suppose $D_2(y,z)(r) + D_2(z,x)(s) < 1$. Take $b, b_1, b_2 \in (0,1]$ such that $1 > b = b_1 + b_2$, $b_1 > D_2(y,z)(r)$ and $b_2 > D_2(z,x)(s)$. Then by (2) we know that

$$\max\left\{\bigwedge_{c>b_1'} \sigma(c,y) - \sigma(b_1,z), 0\right\} < r \quad \text{and} \quad \max\left\{\bigwedge_{c>b_2'} \sigma(c,z) - \sigma(b_2,x), 0\right\} < s.$$

This implies that

$$\begin{aligned}
& \max\left\{\bigwedge_{c>b'}\sigma(c,y)-\sigma(b,x),0\right\}\\
\leq\ & \max\left\{\bigwedge_{c>b'_1}\sigma(c,y)-\sigma(b,x),0\right\}\\
\leq\ & \max\left\{\bigwedge_{c>b'_1}\sigma(c,y)-\sigma(b_2,x),0\right\}\\
\leq\ & \max\left\{\left(\bigwedge_{c>b'_1}\sigma(c,y)-\sigma(b_1,z)\right)+\left(\sigma(b_1,z)-\sigma(b_2,x)\right),0\right\}\\
\leq\ & \max\left\{\bigwedge_{c>b'_1}\sigma(c,y)-\sigma(b_1,z),0\right\}+\max\left\{\sigma(b_1,z)-\sigma(b_2,x),0\right\}\\
\leq\ & \max\left\{\bigwedge_{c>b'_1}\sigma(c,y)-\sigma(b_1,z),0\right\}+\max\left\{\sigma(b'_2,z)-\sigma(b_2,x),0\right\}\\
\leq\ & \max\left\{\bigwedge_{c>b'_1}\sigma(c,y)-\sigma(b_1,z),0\right\}+\max\left\{\bigwedge_{c>b'_2}\sigma(c,z)-\sigma(b_2,x),0\right\}\\
<\ & r+s.
\end{aligned}$$

Thus we have $b \geq D_2(y,x)(r+s)$, (FM5) is proved.

So D_2 is a fuzzifying pseudo-quasi-metric.

(4) From the above (3) we can easily check that D is a fuzzifying pseudo-quasi-metric. In order to prove that D is a fuzzifying pseudo-metric, we need only to check $D(x,y)(r) = D(y,x)(r)$ for all $x, y \in \mathbb{R}(I)$ and $r \in (0,+\infty)$.

Suppose $a > D(x,y)(r) = D_1(x,y)(r) \vee D_2(x,y)(r)$. Then $a > D_1(x,y)(r)$ and $a > D_2(x,y)(r)$. Thus we have

$$\max\left\{\varepsilon(a,y)-\bigvee_{c>a'}\varepsilon(c,x),0\right\}<r \quad\text{and}\quad \max\left\{\bigwedge_{c>a'}\sigma(c,x)-\sigma(a,y),0\right\}<r.$$

From Lemma 4.1(3) we obtain

$$\max\left\{\bigwedge_{c>a'}\sigma(c,y)-\sigma(a,x),0\right\}<r \quad\text{and}\quad \max\left\{\varepsilon(a,x)-\bigvee_{c>a'}\varepsilon(c,y),0\right\}<r.$$

By (1) and (2) this implies $a \geq D_1(y,x)(r) \vee D_2(y,x)(r) = D(y,x)(r)$. This shows $D(x,y)(r) \geq D(y,x)(r)$. Analogously we can prove $D(x,y)(r) \leq D(y,x)(r)$.

The proof is completed. □

Remark 4.5. In Theorem 4.4, $D(x,x)(r) \leq 0.5$ can not be replaced by $D(x,x)(r) = 0$. In fact, we take $x \in \mathbb{R}(I)$ which is defined as

$$x(r)=\begin{cases} 1, & r\leq 3,\\ 0.5, & 3<r\leq 7,\\ 0, & 7<r.\end{cases}$$

Then we can check the following results. $\varepsilon(0.5,x) = 7, \sigma(0.5,x) = 3$. When $c > 0.5$, we can check $\varepsilon(c,x) = 3, \sigma(c,x) = 7$. Thus it follows $D_1(x,x)(4) = D_2(x,x)(4) = D(x,x)(4) = 0.5 \neq 0$.

5. The fuzzifying topologies induced by the fuzzifying pseudo-quasi-metrics

In this section, we can check that when $M = I = [0,1]$, three fuzzifying topologies $\mathcal{L}$,$\mathcal{R}$ and $\mathcal{T}$ in Theorem 3.1 can be respectively induced by three fuzzifying pseudo-quasi-metrics D_1, D_2 and D.

Theorem 5.1. *Let D_1, D_2 be respectively the above fuzzifying pseudo-quasi-metrics on $\mathbb{R}(I)$ and let D be the above fuzzifying pseudo-metric on $\mathbb{R}(I)$. Then $\forall A \subseteq \mathbb{R}(I)$,*

(1) $\mathcal{L}(A) = \bigwedge_{x\in A} \bigvee_{s>0} \bigwedge_{y\notin A} D_1(x,y)(s)$;
(2) $\mathcal{R}(A) = \bigwedge_{x\in A} \bigvee_{s>0} \bigwedge_{y\notin A} D_2(x,y)(s)$;
(3) $\mathcal{T}(A) = \bigwedge_{x\in A} \bigvee_{s>0} \bigwedge_{y\notin A} D(x,y)(s)$.

Proof. (1) In order to prove $\forall A \subseteq \mathbb{R}(I)$,

$$\mathcal{L}(A) = \bigwedge_{x\in A} \bigvee_{s>0} \bigwedge_{y\notin A} D_1(x,y)(s),$$

we need to prove

$$\bigwedge_{x\in A} \bigvee_{r\in\mathbb{R}} \left(x(r-)' \wedge \bigwedge_{y\notin A} y(r-) \right) = \bigwedge_{x\in A} \bigvee_{s>0} \bigwedge_{y\notin A} D_1(x,y)(s).$$

Suppose $a < \mathcal{L}(A) = \bigwedge_{x\in A} \bigvee_{r\in\mathbb{R}} \left(x(r-)' \wedge \bigwedge_{y\notin A} y(r-) \right)$. Then for all $x \in A$, there exists $r \in \mathbb{R}$ such that

$$a < x(r-)' = \bigvee_{s<r} x(s)' \text{ and } a < y(r-), \forall y \notin A.$$

This implies that there exists $t < r$ such that $x(t) = x(t-) < a'$ and

$$\varepsilon(a,y) = \sup\{t \in \mathbb{R} \mid a \leq y(t-)\} \geq r, \ \forall y \notin A.$$

From $x(t-) < a'$ we know that $\forall c > a'$, it follows $\varepsilon(c,x) < t < r$ which implies $\bigvee_{c>a'} \varepsilon(c,x) \leq t < r$. Let $s = r - \bigvee_{c>a'} \varepsilon(c,x) > 0$. Then

$$\varepsilon(a,y) - \bigvee_{c>a'} \varepsilon(c,x) \geq r - \bigvee_{c>a'} \varepsilon(c,x) = s > 0.$$

This shows

$$D_1(x,y)(s) = \bigvee \left\{ a \in (0,1] \,\middle|\, \left\{ \varepsilon(a,y) - \bigvee_{c>a'} \varepsilon(c,x), 0 \right\} \geq s \right\} \geq a.$$

Hence we obtain $a \leq \bigwedge_{x\in A} \bigvee_{s>0} \bigwedge_{y\notin A} D_1(x,y)(s)$. Therefore it follows

$$\mathcal{L}(A) = \bigwedge_{x\in A} \bigvee_{r\in\mathbb{R}} \left\{ x(r-)' \wedge \bigwedge_{y\notin A} y(r-) \right\} \leq \bigwedge_{x\in A} \bigvee_{s>0} \bigwedge_{y\notin A} D_1(x,y)(s).$$

Conversely suppose $a < \bigwedge_{x\in A} \bigvee_{s>0} \bigwedge_{y\notin A} D_1(x,y)(s)$. Then for all $x \in A$, there exists $s \in \mathbb{R}$ with $s > 0$ such that $a < D_1(x,y)(s)$ holds for all $y \notin A$. This implies

$$\varepsilon(a,y) - \bigvee_{c>a'} \varepsilon(c,x) = \bigvee\{t \in \mathbb{R} \mid a \leq y(t-)\} - \bigvee_{c>a'} \bigvee\{t \in \mathbb{R} \mid c \leq x(t-)\} \geq s.$$

Thus we have $\varepsilon(a,y) \geq \bigvee_{c>a'} \varepsilon(c,x) + s$. Let $r = \bigvee_{c>a'} \varepsilon(c,x) + \frac{s}{2}$. Then for all $c > a'$, it holds the inequality $\varepsilon(c,x) < r < \varepsilon(a,y)$. This implies that for all $y \notin A$, $c > x(r-)$ and $a \leq y(r-)$. Hence we obtain

$$a \leq x(r-)' \text{ and } a \leq y(r-), \forall y \notin A.$$

Therefore it holds

$$a \leq \bigvee \left\{ x(r-)' \wedge \bigwedge_{y \notin A} y(r-) \mid r \in \mathbb{R} \right\}.$$

Further it follows

$$a \leq \bigwedge_{x \in A} \bigvee_{r \in \mathbb{R}} \left(x(r-)' \wedge \bigwedge_{y \notin A} y(r-) \right) = \mathcal{L}(A).$$

This shows

$$\mathcal{L}(A) = \bigwedge_{x \in A} \bigvee_{r \in \mathbb{R}} \left(x(r-)' \wedge \bigwedge_{y \notin A} y(r-) \right) \geq \bigwedge_{x \in A} \bigvee_{s>0} \bigwedge_{y \notin A} D_1(x,y)(s).$$

Thus we complete the proof of the following formula

$$\mathcal{L}(A) = \bigwedge_{x \in A} \bigvee_{r \in \mathbb{R}} \left(x(r-)' \wedge \bigwedge_{y \notin A} y(r-) \right) = \bigwedge_{x \in A} \bigvee_{s>0} \bigwedge_{y \notin A} D_1(x,y)(s).$$

(2) In order to prove $\forall A \subseteq \mathbb{R}(I)$,

$$\mathcal{R}(A) = \bigwedge_{x \in A} \bigvee_{s>0} \bigwedge_{y \notin A} D_2(x,y)(s),$$

we need to prove

$$\bigwedge_{x \in A} \bigvee_{r \in \mathbb{R}} \left(x(r+) \wedge \bigwedge_{y \notin A} y(r+)' \right) = \bigwedge_{x \in A} \bigvee_{s>0} \bigwedge_{y \notin A} D_2(x,y)(s).$$

Suppose

$$a < \mathcal{R}(A) = \bigwedge_{x \in A} \bigvee_{r \in \mathbb{R}} \left(x(r+) \wedge \bigwedge_{y \notin A} y(r+)' \right).$$

Then for all $x \in A$, there exists $r \in \mathbb{R}$ such that

$$a < x(r+) = \bigvee_{t>r} x(t+) \text{ and } a < y(r+)', \forall y \notin A.$$

This implies there exists $t > r$ such that $a < x(t+)$ and

$$\sigma(a,y) = \inf\{t \in \mathbb{R} \mid a \leq y(t+)'\} \leq r.$$

From $a < x(t+)$ we know that for all $c > a'$, it holds $c > x(t+)'$, which implies $\bigwedge_{c>a'} \sigma(c,x) \geq t > r$. Let $s = \bigwedge_{c>a'} \sigma(c,x) - r > 0$. Then

$$\bigwedge_{c>a'} \sigma(c,x) - \sigma(a,y) \geq \bigwedge_{c>a'} \sigma(c,x) - r = s > 0.$$

This shows

$$D_2(x,y)(s) = \bigvee \left\{ a \in (0,1] \,\middle|\, \max\left\{ \bigwedge_{c>a'} \sigma(c,x) - \sigma(a,y), 0 \right\} \geq s \right\} \geq a.$$

Hence we obtain $a \le \bigwedge_{x\in A}\bigvee_{s>0}\bigwedge_{y\notin A} D_2(x,y)(s)$. Therefore it follows

$$\mathcal{R}(A) = \bigwedge_{x\in A}\bigvee_{r\in\mathbb{R}}\left\{x(r+)\wedge\bigwedge_{y\notin A} y(r+)'\right\} \le \bigwedge_{x\in A}\bigvee_{s>0}\bigwedge_{y\notin A} D_2(x,y)(s).$$

Conversely suppose $a < \bigwedge_{x\in A}\bigvee_{s>0}\bigwedge_{y\notin A} D_2(x,y)(s)$. Then for all $x \in A$, there exists $s \in \mathbb{R}$ with $s > 0$ such that $a < D_2(x,y)(s)$ holds for all $y \notin A$. This implies

$$\bigwedge_{c>a'}\sigma(c,x) - \sigma(a,y) = \bigwedge_{c>a'}\bigwedge\{t\in\mathbb{R} \mid c \le x(t+)'\} - \bigwedge\{t\in\mathbb{R} \mid a \le y(t+)'\} \ge s.$$

Thus we have $\bigwedge_{c>a'}\sigma(c,x) - s \ge \sigma(a,y)$. Let $r = \bigwedge_{c>a'}\sigma(c,x) - \frac{s}{2}$. Then

$$\bigwedge_{c>a'}\sigma(c,x) > r > \sigma(a,y).$$

This implies that for all $c > a'$, it follows $c > x(r+)'$ and $a \le y(r+)'$, that is,

$$a \le x(r+) \quad \text{and} \quad a \le y(r+)', \forall y \notin A.$$

Therefore it holds

$$a \le \bigvee_{r\in\mathbb{R}}\left(x(r+)\wedge\bigwedge_{y\notin A} y(r+)'\right).$$

Further it follows

$$a \le \bigwedge_{x\in A}\bigvee_{r\in\mathbb{R}}\left(x(t+)\wedge\bigwedge_{y\notin A} y(t+)'\right) = \mathcal{R}(A).$$

This shows

$$\mathcal{R}(A) = \bigwedge_{x\in A}\bigvee_{r\in\mathbb{R}}\left(x(t+)\wedge\bigwedge_{y\notin A} y(t+)'\right) \ge \bigwedge_{x\in A}\bigvee_{s>0}\bigwedge_{y\notin A} D_2(x,y)(s).$$

Thus we complete the proof of the following formula

$$\mathcal{R}(A) = \bigwedge_{x\in A}\bigvee_{r\in\mathbb{R}}\left(x(t+)\wedge\bigwedge_{y\notin A} y(t+)'\right) = \bigwedge_{x\in A}\bigvee_{s>0}\bigwedge_{y\notin A} D_2(x,y)(s).$$

(3) The proof can be obtained from the results of (1) and (2). □

6. Conclusions

A statistical metric space can be regarded as a KM fuzzy metric space and the set of all distribution function can be regarded as a special M-fuzzy real line. In this paper we first presented three fuzzifying topologies on the M-fuzzy real line. Then we constructed three fuzzifying pseudo-quasi-metrics on the fuzzy real line. Finally, we proved that the three fuzzifying topologies can exactly be induced by the three fuzzifying pseudo-quasi-metrics. A fuzzifying pseudo-metric can be regarded as a weak form of a KM fuzzy metric.

In future work, we will consider the fuzzifying convex structure (see [17,18]), the fuzzifying convergence, the fuzzifying completeness, and others on the M-fuzzy real line.

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