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Analysis of Nonlinear Mathematical Model of COVID-19 via Fractional-Order Piecewise Derivative

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ABSTRACT Short memory and long memory terms are excellently explained using the concept of piecewise fractional order derivatives. In this research work, we investigate dynamical systems addressing COVID-19 under piecewise equations with fractional order derivative (FOD). Here, we study the sensitivity of the proposed model by using some tools from the nonlinear analysis. Additionally, we develop a numerical scheme to simulate the model against various fractional orders by using Matlab 2016. All the results are presented graphically.

KEYWORDS
Nonlinear dynamical system
Crossover behavior
Mathematical biology
Sensitivity analysis

INTRODUCTION

Fractional calculus has been recognized as a powerful tool to investigate various dynamical problems with more detail and a realistic approach. The foundation of this branch was laid by Newton and some known mathematicians of that time. Later on Reimann, Liouville, Hadamard, Hilfer and other researchers developed this branch further by introducing various differential and integral operators (Machado *et al.* 2011). The great advantage of using fractional calculus instead of classical in the description of real-world problems is its global nature. By fractional derivatives, we can describe global dynamics for various evolutionary processes in a more realistic way. Also, the mentioned operators are keeping a greater degree of freedom as compared to ordinary operators of derivatives which are local in nature, (see some detail in (Hilfer *et al.* 2008) and (Agarwal *et al.* 2010)).

Keeping the mentioned characteristics in mind researchers have increasingly used the concept of fractional calculus in the mathematical modeling of various phenomena and processes. In this regard, we can find literature full of such types of articles, books, and monographs addressing the applications of fractional calculus. Here we remark that fractional derivative has not a unique definition. There have been introduced various definitions by researchers including singular and non-singular operators (Rahman *et al.* 2021). Recently in this connection, see more work as (Ahmad *et al.* 2021c; Alqahtani *et al.* 2021; Ojo and Goufo 2022, 2023). Both forms have been used extensively in various research problems. Both operators have merits and sometimes some de-merits which have been discussed by researchers. For instance, authors have investigated fractal fractional chaotic attractor behavior in (Saifullah *et al.* 2021), a physical model in (Ahmad *et al.* 2021b), and using the Caputo-Fabrizio derivative in (Ahmad *et al.* 2021c).

On the other hand, for epidemiological purposes, the said concept has been used very well. Large numbers of models have been investigated under the concept of fractional order derivatives and integrals. As we know that infectious diseases have greatly affected our society from ancient times. Due to this disease, millions of people have lost their lives in the past as well as in the recent two-three years. Currently, the outbreak of COVID-19 has greatly destroyed the world and more than fifty million people have died within two years all over the globe. The said infection has also affected the economic situation of various countries around the globe. Further, to control the disease researchers, physicians and authorities are working day and night to overcome or control this disease from further spreading.

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In this regards various procedures have been introduced in the last two years to overcome the infection. Some work done on mathematical models of COVID can be seen as (Atangana and İğret Araz 2020), (Arfan *et al.* 2021), and (Abdo *et al.* 2020). Among one which is very important of vaccine which has been prepared and is now available in the market. Further, to aware people of the individual measures to save their lives and their family. Various measures for safety have been implemented by various countries including keeping social distance, regularly washing mouth, hands, etc, and wearing a face mask in gatherings, avoiding joining the huge crowd.

One important tool from a research perspective to investigate the transmission dynamics of the disease in the community through a scientific approach is devoted to mathematical modeling. In this regards various models have been introduced to study the mentioned process, for instance, authors investigated the time fractal-Klein-Gordon equation in (Saifullah *et al.* 2022), the complex behavior of multi-structure dynamical system (Ahmad *et al.* 2021a), Zika virus model in (Zhou *et al.* 2017) and some heat problems in(Doungmo Goufo 2016). For this purpose, various differential operators have been used properly. Along the same line fractional calculus has been used extensively. In the same fashion authors (Doungmo Goufo 2016) have discussed the dynamics of the KDV-Berger equation. Also in (Doungmo Goufo 2015), the authors have applied the concept of fractal-fractional to investigate the cellulose degradation model.

Applications of the newly introduced ABC derivative have been discussed in (Atangana 2020). The existence and uniqueness of the epidemiological model has been studied in (Shah *et al.* 2023). Some authors investigated different TB models under the concept of the fractional derivative with simulation in (Shatanawi *et al.* 2021). Authors (Nawaz *et al.* 2022) established some computational and theoretical analysis for TB model by using ABC derivative of fractional order.

We should keep in mind that many evolutionary processes often suffer from abrupt changes in their dynamics, which can be determined by ordinary derivatives and even fractional derivatives also. For such a situation, we need to use a fractional type derivative with piecewise nature which has the ability to clarify the crossover behavior of the dynamics more properly. In this regard recently some authors have introduced the concept of piecewise derivative to detect the said behavior in the dynamical problems (Atangana and Araz 2021). For further details on piecewise derivatives, recent contributions can be seen as (Shah *et al.* 2022a,b,c).

Motivated by the said analysis, literature, and features of fractional calculus, we will investigate the following models of COVID-19 under the global piecewise derivative of fractional order. Our concerned model is given by

$$\begin{aligned}
{}^{\mathbf{pABC}}_{0}\mathcal{D}^{\chi}_{t}\mathscr{S}(t) &= \beta-\xi\mathscr{S}(t)\mathscr{I}(t)-(\tau+\theta)\,\mathscr{S}(t)+\eta\mathscr{R}(t),\\
{}^{\mathbf{pABC}}_{0}\mathcal{D}^{\chi}_{t}\mathscr{E}(t) &= \xi\mathscr{S}(t)\mathscr{I}(t)-(\delta+\tau+\theta)\,\mathscr{E}(t),\\
{}^{\mathbf{pABC}}_{0}\mathcal{D}^{\chi}_{t}\mathscr{I}(t) &= \delta\mathscr{E}(t)-(\theta+\tau+\Delta+\omega)\,\mathscr{I}(t),\\
{}^{\mathbf{pABC}}_{0}\mathcal{D}^{\chi}_{t}\mathscr{V}(t) &= \theta\mathscr{I}(t)-(\tau+\kappa)\,\mathscr{V}(t)+\theta\mathscr{E}(t)+\theta\mathscr{S}(t),\\
{}^{\mathbf{pABC}}_{0}\mathcal{D}^{\chi}_{t}\mathscr{R}(t) &= \Delta\mathscr{I}(t)+\kappa\mathscr{V}(t)-(\tau+\eta)\,\mathscr{R}(t).
\end{aligned}\tag{1}$$

Here we remark in determinacy form the model (8) is given as

$$\begin{aligned}
\frac{d\mathscr{S}(t)}{dt} &= \beta-\xi\mathscr{S}(t)\mathscr{I}(t)-(\tau+\theta)\,\mathscr{S}(t)+\eta\mathscr{R}(t),\\
\frac{d\mathscr{E}(t)}{dt} &= \xi\mathscr{S}(t)\mathscr{I}(t)-(\delta+\tau+\theta)\,\mathscr{E}(t),\\
\frac{d\mathscr{I}(t)}{dt} &= \delta\mathscr{E}(t)-(\theta+\tau+\Delta+\omega)\,\mathscr{I}(t),\\
\frac{d\mathscr{V}(t)}{dt} &= \theta\mathscr{I}(t)-(\tau+\kappa)\,\mathscr{V}(t)+\theta\mathscr{E}(t)+\theta\mathscr{S}(t),\\
\frac{d\mathscr{R}(t)}{dt} &= \Delta\mathscr{I}(t)+\kappa\mathscr{V}(t)-(\tau+\eta)\,\mathscr{R}(t).
\end{aligned}\tag{2}$$

The complete detailed description and explanations of compartments and parameters are given in Tables 2 and 3 respectively. We obtained the basic reproduction number (R_0) using the next-generation matrix on the disease-free equilibrium point and investigated the global sensitivity analysis of the basic reproduction number (R_0). Then, we focused on some numerical techniques based on the Euler method to simulate the given model under the concept of piecewise fractional order derivatives. We use some real values of parameters to present results graphically.

PRELIMINARIES

Here we recall some definitions results, lemmas from (Doungmo Goufo 2015).

Definition 0.1. If $f(t)\in\mathscr{H}^1(0,T)$ and $\chi\in(0,1]$, then the **ABC** derivative is defined as

$${}^{\mathbf{ABC}}_{0}D^{\chi}_{t}\mathbf{u}(t)=\frac{\mathbf{ABC}(\chi)}{1-\chi}\int_0^t E_{\chi}\left[\frac{-\chi}{1-\chi}(t-\tau)^{\chi}\right]\frac{d}{d\tau}\mathbf{u}(\tau)d\tau,\varepsilon \tag{3}$$

Definition 0.2. Let $\mathbf{u}(t)\in L[0,T]$, then the fractional integral in **ABC** sense as:

$${}^{\mathbf{ABC}}_{0}I^{\chi}_{t}\mathbf{u}(t)=\frac{1-\chi}{\mathbf{ABC}(\chi)}\mathbf{u}(t)+\frac{\chi}{\mathbf{ABC}(\chi)\Gamma(\chi)}\int_0^t(t-\zeta)^{\chi-1}\mathbf{u}(\zeta)d\zeta. \tag{4}$$

Definition 0.3. Let, $\mathbf{u}(t)$ is a differentiable function at interval $[0,t_1]$ and $[t_1,t]$, then the piecewise derivative is defined as:

$${}^{\mathbf{pABC}}_{0}D\mathbf{u}(t)=\begin{cases}\frac{d\mathbf{u}}{dt}, & 0<t<t_1\\ {}^{\mathbf{pABC}}_{0}D^{\chi}_{t}\mathbf{u}. & t_1<t<t_2\end{cases}=\begin{cases}g(t,\mathbf{u}(t)),\\ t\in[0,t_2]\end{cases} \tag{5}$$

Definition 0.4. Suppose, we consider the generic piecewise fractional order differential equation with fractional order χ, such that

$${}^{\mathbf{pABC}}_{0}\mathcal{D}^{\chi}_{t}\mathbf{u}(t)=\rho(t,\mathbf{u}(t)),\ \text{with}\ \mathbf{u}(0)=\mathbf{u}_0. \tag{6}$$

For the differential equation (6) we propose a numerical Euler's scheme that is

$$\mathbf{u}(t_{n+1})=\begin{cases} y_n+hf(t_{n-1},\mathbf{u}(t_{n-1})), & 0<t<t_1\\ \mathbf{u}(t_1)+\frac{(1-\chi)}{\mathbf{ABC}(\chi)}f(t_n,\mathbf{u}_n)+\frac{h\chi(1-\chi)}{\mathbf{ABC}(\varnothing)}f(t_n,\mathbf{u}_n),\ t_1<t<t_2, & 0<\chi<1.\end{cases} \tag{7}$$

MATHEMATICAL MODEL OF COVID-19

We investigate the mathematical model given in (2) by using the Caputo and Atangana-Baleanu piecewise differential operators. We formulated the proposed model in the aforementioned operators form with $0 < \chi \leq 1, t \in [0, T], 0 \leq t \leq T, T < \infty$ as

$$\left.\begin{aligned} {}_0^{\mathbf{pABC}}\mathcal{D}_t^{\chi}\mathscr{S}(t) &= \beta - \xi\mathscr{S}(t)\mathscr{I}(t) - (\tau+\theta)\,\mathscr{S}(t) + \eta\mathscr{R}(t), \\ {}_0^{\mathbf{pABC}}\mathcal{D}_t^{\chi}\mathscr{E}(t) &= \xi\mathscr{S}(t)\mathscr{I}(t) - (\delta+\tau+\theta)\,\mathscr{E}(t), \\ {}_0^{\mathbf{pABC}}\mathcal{D}_t^{\chi}\mathscr{I}(t) &= \delta\mathscr{E}(t) - (\theta+\tau+\Delta+\omega)\,\mathscr{I}(t), \\ {}_0^{\mathbf{pABC}}\mathcal{D}_t^{\chi}\mathscr{V}(t) &= \theta\mathscr{I}(t) - (\tau+\kappa)\,\mathscr{V}(t) + \theta\mathscr{E}(t) + \theta\mathscr{S}(t), \\ {}_0^{\mathbf{pABC}}\mathcal{D}_t^{\chi}\mathscr{R}(t) &= \Delta\mathscr{I}(t) + \kappa\mathscr{V}(t) - (\tau+\eta)\,\mathscr{R}(t). \end{aligned}\right\} \tag{8}$$

In more explicit form the model (8) can also be write as

$$\left.\begin{aligned} &{}_0^{\mathbf{pABC}}\mathcal{D}_t^{\chi}(\mathscr{S}(t)) \\ &= \begin{cases} \frac{d\mathscr{S}(t)}{dt} = \mathscr{H}_1(\mathscr{S},\mathscr{E},\mathscr{I},\mathscr{V},\mathscr{R},t), & 0 < t \leq t_1, \\ {}_0^{\mathbf{ABC}}\mathcal{D}_t^{\chi}(\mathscr{S}(t)) = \mathscr{H}_1(\mathscr{S},\mathscr{E},\mathscr{I},\mathscr{V},\mathscr{R},t), t_1 < t \leq T. \end{cases} \\ &{}_0^{\mathbf{pABC}}\mathcal{D}_t^{\chi}(\mathscr{E}(t)) \\ &= \begin{cases} \frac{d\mathscr{E}(t)}{dt} = \mathscr{H}_2(\mathscr{S},\mathscr{E},\mathscr{I},\mathscr{V},\mathscr{R},t), & 0 < t \leq t_1, \\ {}_0^{\mathbf{ABC}}\mathcal{D}_t^{\chi}(\mathscr{E}(t)) = \mathscr{H}_2(\mathscr{S},\mathscr{E},\mathscr{I},\mathscr{V},\mathscr{R},t), t_1 < t \leq T. \end{cases} \\ &{}_0^{\mathbf{pABC}}\mathcal{D}_t^{\chi}(\mathscr{I}(t)) \\ &= \begin{cases} \frac{d\mathscr{I}(t)}{dt} = \mathscr{H}_3(\mathscr{S},\mathscr{E},\mathscr{I},\mathscr{V},\mathscr{R},t), & 0 < t \leq t_1, \\ {}_0^{\mathbf{ABC}}\mathcal{D}_t^{\chi}(\mathscr{I}(t)) = \mathscr{H}_3(\mathscr{S},\mathscr{E},\mathscr{I},\mathscr{V},\mathscr{R},t), t_1 < t \leq T. \end{cases} \\ &{}_0^{\mathbf{pABC}}\mathcal{D}_t^{\chi}(\mathscr{V}(t)) \\ &= \begin{cases} \frac{d\mathscr{V}(t)}{dt} = \mathscr{H}_4(\mathscr{S},\mathscr{E},\mathscr{I},\mathscr{V},\mathscr{R},t), & 0 < t \leq t_1, \\ {}_0^{\mathbf{ABC}}\mathcal{D}_t^{\chi}(\mathscr{V}(t)) = \mathscr{H}_4(\mathscr{S},\mathscr{E},\mathscr{I},\mathscr{V},\mathscr{R},t), t_1 < t \leq T. \end{cases} \\ &{}_0^{\mathbf{pABC}}\mathcal{D}_t^{\chi}(\mathscr{R}(t)) \\ &= \begin{cases} \frac{d\mathscr{R}(t)}{dt} = \mathscr{H}_5(\mathscr{S},\mathscr{E},\mathscr{I},\mathscr{V},\mathscr{R},t), & 0 < t \leq t_1, \\ {}_0^{\mathbf{ABC}}\mathcal{D}_t^{\chi}(\mathscr{R}(t)) = \mathscr{H}_5(\mathscr{S},\mathscr{E},\mathscr{I},\mathscr{V},\mathscr{R},t), t_1 < t \leq T. \end{cases} \end{aligned}\right\} \tag{9}$$

EQUILIBRIUM POINT AND BASIC REPRODUCTION NUMBER (R_0)

The Disease–Free equilibrium point is computed as:

$$E^0 = \left(\mathscr{S}^0, 0, 0, \mathscr{V}^0, \mathscr{R}^0\right). \tag{10}$$

Where,

$$\begin{aligned} \mathscr{S}^0 &= \frac{\beta(\eta\,\tau+\eta\,\kappa+\tau\kappa+\tau^2)}{\eta\,\tau^2+\tau^2\xi+\tau^2\kappa+\tau^3-\theta\eta\,\kappa+\eta\,\tau\xi+\eta\,\tau\kappa+\eta\,\xi\kappa+\tau\xi\kappa}, \\ \mathscr{V}^0 &= \frac{\theta\,\beta(\eta+\tau)}{\eta\,\tau^2+\tau^2\xi+\tau^2\kappa+\tau^3-\theta\eta\,\kappa+\eta\,\tau\xi+\eta\,\tau\kappa+\eta\,\xi\kappa+\tau\xi\kappa}, \\ \mathscr{R}^0 &= \frac{\theta\,\beta\kappa}{\eta\,\tau^2+\tau^2\xi+\tau^2\kappa+\tau^3-\theta\eta\,\kappa+\eta\,\tau\xi+\eta\,\tau\kappa+\eta\,\xi\kappa+\tau\xi\kappa}. \end{aligned} \tag{11}$$

The basic reproduction number at disease-free equilibrium point for the model (8) is computed such that considering the equation:

$$\left.\frac{dZ}{dt}\right|_{E^0} = \mathbf{f} - \mathbf{v}. \tag{12}$$

The non–linear and linear terms from the infected classes in matrix f and v, respectively:

$$f = \begin{pmatrix} \xi\mathscr{S}\mathscr{I} \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} (\delta+\tau+\theta)\,\mathscr{E}(t) \\ (\theta-\tau-\Delta-\omega)\,\mathscr{I}(t) - \delta\mathscr{E}(t) \end{pmatrix}. \tag{13}$$

Now, the jacobian matrix of f and v is given by:

$$\mathscr{F} = \begin{pmatrix} 0 & \xi\mathscr{S}^0 \\ 0 & 0 \end{pmatrix}, \quad \mathscr{V} = \begin{pmatrix} \theta+\delta+\tau & 0 \\ -\delta & \theta+\tau+\Delta+\omega \end{pmatrix}. \tag{14}$$

Calculating the inverse of matrix $\mathscr{V}$ and the next generation matrix G, such that:

$$\mathscr{V}^{-1} = \begin{pmatrix} \frac{1}{\theta+\delta+\tau} & 0 \\ \frac{\delta}{(\theta+\delta+\tau)\,(\theta+\tau+\Delta+\omega)} & \frac{1}{\theta+\tau+\Delta+\omega} \end{pmatrix}. \tag{15}$$

Thus, the non-zero and largest eigenvalue is the basic reproduction number R_0 is:

$$R_0 = \frac{\delta\,\xi\,\mathscr{S}^0}{(\theta+\delta+\tau)\,(\theta+\tau+\Delta+\omega)}. \tag{16}$$

Where,

$$\mathscr{S}^0 = \frac{\beta(\eta\,\tau+\eta\,\kappa+\tau\,\kappa+\tau^2)}{\eta\,\tau^2+\tau^2\xi+\tau^2\kappa+\tau^3-\theta\eta\,\kappa+\eta\,\tau\xi+\eta\,\tau\,\kappa+\eta\,\xi\kappa+\tau\xi\kappa}.$$

SENSITIVITY ANALYSIS

It is vital to understand the relative relevance of the many elements involved in COVID-19 transmissions and prevalence in order to determine how best to decrease human mortality and morbidity as a result of the virus. The endemic equilibrium point is directly connected to R_0, and the initial illness transmission is directly related to R_0. The infectious human percentage, $\mathscr{I}(t)$, is particularly noteworthy since it reflects persons who may get clinically sick and is proportional to the overall number of COVID-19 fatalities. The reproductive number, R_0, and sensitivity indices to the model parameters are calculated. These indices indicate the importance of each parameter in disease transmission and prevalence. To assess the resilience of model predictions to parameter values, sensitivity analysis is widely performed (since there are usually errors in data collection and presumed parameter values). Using the explicit formula for R_0, we derive an analytical expression for the sensitivity of R_0

$$\mathbf{s}_{(\mathbf{p})}^{R_0} = \frac{\mathbf{p}}{R_0}\left[\frac{\partial R_0}{\partial \mathbf{p}}\right]. \tag{17}$$

Now, according to the above relation, we have

$$\mathbf{s}_{\beta}^{R_0} = \frac{\beta}{R_0}\left[\frac{\delta\,\xi(\eta+\tau)(\tau+\kappa)}{(\theta+\delta+\tau)(\theta+\tau+\Delta+\omega)\phi_1}\right], \tag{18}$$

$$\mathbf{s}_{\tau}^{R_0} = \frac{\tau}{R_0}\left[\frac{\delta\,\beta\,\xi\,(\eta+2\,\tau+\kappa)}{(\theta+\delta+\tau)\,(\theta+\tau+\Delta+\omega)\,\phi_1} - \frac{\delta\,\beta\,\xi\,\phi_2}{(\theta+\delta+\tau)\,(\theta+\tau+\Delta+\omega)^2\,\phi_1} - \frac{\delta\,\beta\,\xi\,\phi_2}{(\theta+\delta+\tau)^2\,(\theta+\tau+\Delta+\omega)\,\phi_1} - \frac{\delta\,\beta\,\xi\,\phi_2\,(2\,\eta\,\tau+\eta\,\xi+2\,\tau\,\xi+\eta\,\kappa+2\,\tau\,\kappa+\xi\,\kappa+3\,\tau^2)}{(\theta+\delta+\tau)\,(\theta+\tau+\Delta+\omega)\,{\phi_1}^2}\right],$$

where

$$\phi_1 = \eta\,\tau^2+\tau^2\,\xi+\tau^2\,\kappa+\tau^3-\theta\,\eta\,\kappa+\eta\,\tau\,\xi+\eta\,\tau\,\kappa+\eta\,\xi\,\kappa+\tau\,\xi\,\kappa,$$
$$\phi_2 = \eta\,\tau+\eta\,\kappa+\tau\,\kappa+\tau^2.$$

$$\mathbf{s}_{\eta}^{R_0} = \frac{\eta}{R_0}\left[\frac{\theta\,\delta\,\beta\,\tau\,\xi\,\kappa\,(\tau+\kappa)}{(\theta+\delta+\tau)\,(\theta+\tau+\Delta+\omega)\,{\Phi_1}^2}\right],$$

$$\mathbf{s}_{\kappa}^{R_0} = \frac{\kappa}{R_0}\left[\frac{\theta\,\delta\,\beta\,\tau\,\xi\,\kappa\,(\tau+\kappa)}{(\theta+\delta+\tau)\,(\theta+\tau+\Delta+\omega)\,{\Phi_1}^2}\right],$$

$$\mathbf{s}_{\theta}^{R_0} = \frac{\theta}{R_0}\left[\frac{\delta\,\beta\,\eta\,\xi\,\kappa\,(\eta+\tau)\,(\tau+\kappa)}{(\theta+\delta+\tau)\,(\theta+\tau+\Delta+\omega)\,{\Phi_1}^2} - \frac{\Phi_2}{(\theta+\delta+\tau)^2\,(\theta+\tau+\Delta+\omega)\,\Phi_1} - \frac{\Phi_2}{(\theta+\delta+\tau)\,(\theta+\tau+\Delta+\omega)^2\,\Phi_1}\right],$$

where

$$\Phi_1 = \eta\,\tau^2+\tau^2\,\xi+\tau^2\,\kappa+\tau^3-\theta\,\eta\,\kappa+\eta\,\tau\,\xi+\eta\,\tau\,\kappa+\eta\,\xi\,\kappa+\tau\,\xi\,\kappa,$$
$$\Phi_2 = \delta\,\beta\,\xi\,(\eta+\tau)\,(\tau+\kappa).$$

$$\mathbf{s}_{\xi}^{R_0} = \frac{\xi}{R_0}\left[\frac{\delta\,\beta\,(\eta+\tau)\,(\tau+\kappa)\,(\eta\,\tau^2+\tau^2\,\kappa+\tau^3-\theta\,\eta\,\kappa+\eta\,\tau\,\kappa)}{(\theta+\delta+\tau)\,(\theta+\tau+\Delta+\omega)\,{\Phi_1}^2}\right],$$

$$\mathbf{s}_{\delta}^{R_0} = \frac{\delta}{R_0}\left[\frac{\beta\,\xi\,(\theta+\tau)\,(\eta+\tau)\,(\tau+\kappa)}{(\theta+\delta+\tau)^2\,(\theta+\tau+\Delta+z)\,\Phi_1}\right],$$

$$\mathbf{s}_{\Delta}^{R_0} = -\frac{\Delta}{R_0}\left[\frac{\delta\,\beta\,\xi\,(\eta+\tau)\,(\tau+\kappa)}{(\theta+\delta+\tau)\,(\theta+\tau+\Delta+\omega)^2\,\Phi_1}\right],$$

$$\mathbf{s}_{\omega}^{R_0} = -\frac{\omega}{R_0}\left[\frac{\delta\,\beta\,\xi\,(\eta+\tau)\,(\tau+\kappa)}{(\theta+\delta+\tau)\,(\theta+\tau+\Delta+\omega)^2\,\Phi_1}\right].$$

Table 1 Sensitivity of the R_0 versus proposed parameters

Parameter	Sensitivity Index	Value	Sign
β	$\mathbf{s}_{(\beta)}^{R_0}$	1.0000	+ve
η	$\mathbf{s}_{(\eta)}^{\mathcal{R}_0}$	-0.0006	-ve
θ	$\mathbf{s}_{(\theta)}^{\mathcal{R}_0}$	-3.4078	-ve
δ	$\mathbf{s}_{(\delta)}^{\mathcal{R}_0}$	0.9434	+ve
ω	$\mathbf{s}_{(\omega)}^{\mathcal{R}_0}$	-0.0001	-ve
τ	$\mathbf{s}_{(\tau)}^{\mathcal{R}_0}$	0.0010	+ve
κ	$\mathbf{s}_{(\kappa)}^{\mathcal{R}_0}$	-0.0004	-ve
ξ	$\mathbf{s}_{(\xi)}^{\mathcal{R}_0}$	1.5554	+ve
Δ	$\mathbf{s}_{(\Delta)}^{\mathcal{R}_0}$	-0.0909	-ve

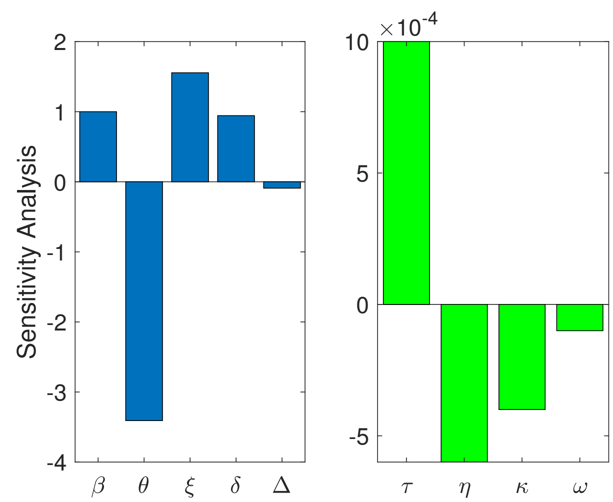

Figure 1 Plot of Sensitivity Analysis with a graphical representation of sensitivity indices $\mathbf{s}_{(\mathbf{p})}^{R_0}$ bases on the expression (17).

In Table (1), the sensitivity indices are provided for each parameter associated with basic reproduction number (R_0) computed based on the expression (17). There is a positive and negative effect of each parameter in the basic reproduction number (R_0) and thus the parameters with positive signs increase the basic reproduction number (R_0) and negative decreases, respectively. Considering the Table (1) and Figure (1), we observed that with the increase in the value parameters β, ξ, δ, and τ cause growth in basic reproduction number (R_0) while decay by parameters θ, Δ, η, κ, and ω. Thus, having negative indices must be minimized in the environment.

NUMERICAL SCHEME

Consider the model (8), we use the proposed Euler's scheme from the Definition (7) and implement on the given problem, such that

$$\mathscr{S}(t_{n+1}) = \begin{cases} \mathscr{S}_n + hf(t_{n-1}, \mathscr{S}(t_{n-1})), & 0 < t < t_1 \\ z_1, & t_1 < t < t_2, 0 < \chi < 1. \end{cases} \tag{19}$$

where, $z_1 = \mathscr{S}(t_1) + \frac{(1-\chi)}{\mathbf{ABC}(\chi)} f(t_n, \mathscr{S}_n) + \frac{h^{\chi}(1-\chi)}{\mathbf{ABC}(\chi)} f(t_n, \mathscr{S}_n)$.

$$\mathscr{E}(t_{n+1}) = \begin{cases} \mathscr{E}_n + hf(t_{n-1}, \mathscr{E}(t_{n-1})), & 0 < t < t_1 \\ z_2, & t_1 < t < t_2, 0 < \chi < 1. \end{cases} \tag{20}$$

where, $z_2 = \mathscr{E}(t_1) + \frac{(1-\chi)}{\mathbf{ABC}(\chi)} f(t_n, \mathscr{E}_n) + \frac{h^{\chi}(1-\chi)}{\mathbf{ABC}(\chi)} f(t_n, \mathscr{E}_n)$.

$$\mathscr{I}(t_{n+1}) = \begin{cases} \mathscr{I}_n + hf(t_{n-1}, \mathscr{I}(t_{n-1})), & 0 < t < t_1 \\ z_3, & t_1 < t < t_2, 0 < \chi < 1. \end{cases} \tag{21}$$

where, $z_3 = \mathscr{I}(t_1) + \frac{(1-\chi)}{\mathbf{ABC}(\chi)} f(t_n, \mathscr{I}_n) + \frac{h^{\chi}(1-\chi)}{\mathbf{ABC}(\chi)} f(t_n, \mathscr{I}_n)$.

$$\mathscr{V}(t_{n+1}) = \begin{cases} \mathscr{V}_n + hf(t_{n-1}, \mathbf{u}(t_{n-1})), & 0 < t < t_1 \\ z_4, & t_1 < t < t_2, 0 < \chi < 1. \end{cases} \tag{22}$$

where, $z_4 = \mathscr{V}(t_1) + \frac{(1-\chi)}{\mathbf{ABC}(\chi)} f(t_n, \mathscr{V}_n) + \frac{h^{\chi}(1-\chi)}{\mathbf{ABC}(\chi)} f(t_n, \mathscr{V}_n)$.

$$\mathscr{R}(t_{n+1}) = \begin{cases} \mathscr{R}_n + hf(t_{n-1}, \mathscr{R}(t_{n-1})), & 0 < t < t_1 \\ z_5, & t_1 < t < t_2, 0 < \chi < 1. \end{cases} \tag{23}$$

where, $z_5 = \mathscr{R}(t_1) + \frac{(1-\chi)}{\mathbf{ABC}(\chi)} f(t_n, \mathscr{R}_n) + \frac{h^{\chi}(1-\chi)}{\mathbf{ABC}(\chi)} f(t_n, \mathscr{R}_n)$.

NUMERICAL INTERPRETATION AND DISCUSSION

Here we apply the aforesaid scheme to simulate the results for different fractional order under piecewise derivative to see the crossover behavior in the transmission dynamics of the disease and the effect of vaccination.

In Figures 2-6, we have presented the approximate solutions corresponding to piecewise derivatives using various fractional orders. We have taken here $t_1 = 5$ and $T = 120$. The crossover effect is clearly observed near the point $t_1 = 5$, and the dynamics after that point shows variation in behavior. This multi-behavior of the dynamics is known as crossover. This effect cannot be determined by using a usual derivative of fractional order. As the vaccination procedure increases more people are giving vaccines, and the security from the infection is also increasing, and hence recovered class is growing up.

Table 2 Table of description and Initial Condition of Compartment of Population.

Symbols	Description of Compartment	Initial Condition
$\mathscr{S}(t)$	Susceptible Human Population	$\mathscr{N} - (\mathscr{E} + \mathscr{I} + \mathscr{V} + \mathscr{R})$
$\mathscr{E}(t)$	Exposed Human Population	10
$\mathscr{I}(t)$	Infected Human Population	20
$\mathscr{V}(t)$	Vaccinated Human Population	30
$\mathscr{R}(t)$	Recovered Human Population	50
$\mathscr{N}$	Total Population	200

Table 3 Table of description and values of Parameters.

Symbol	Description of Parameter	Value
τ	Natural Death Rate	$\frac{1}{67.7 \times 365}$
β	Recruitment Rate	$\tau \times N$
ξ	Transmission rate	0.1784
θ	Vaccination Rate	0.5
η	Lose of Immunity in Recovered Population	0.1
δ	Rate of Infection of Exposed Population	0.03
Δ	Recovery Rate of Infected Population	0.05
κ	Recovery Rate of Vaccinated Population.	0.15
ω	Death Rate of Infected Population due to COVID–19 Infection	0.32

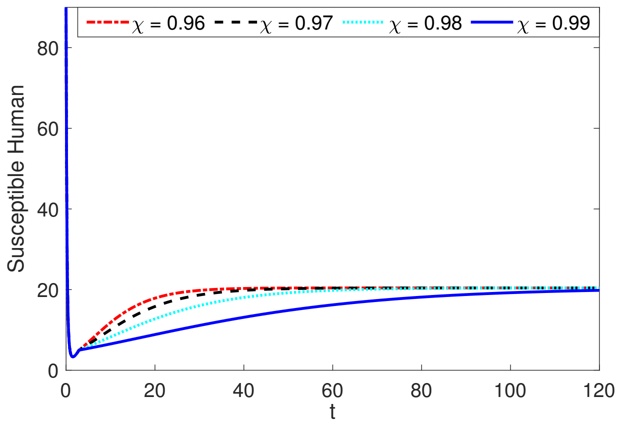

Figure 2 Plot of susceptible class at various fractional order derivatives.

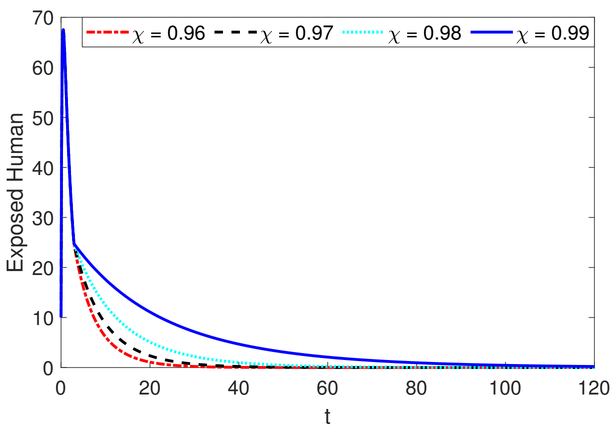

Figure 3 Plot of exposed class at various fractional order derivatives.

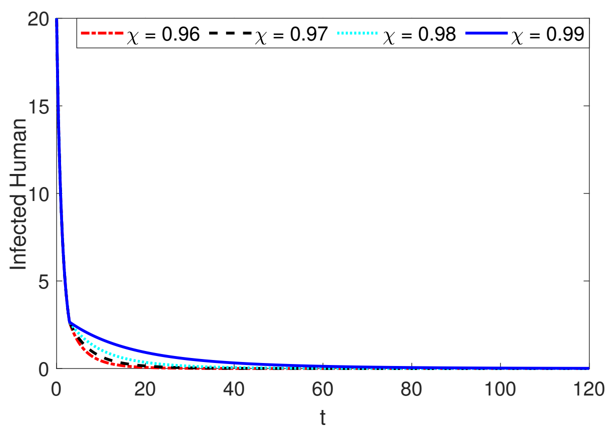

Figure 4 Plot of infected class at various fractional order derivatives.

CONCLUSION

We have extended the concept of piecewise ABC fractional order derivative concept to a dynamical system of COVID-19 with a vaccinated class. We investigated global sensitivity analysis of parameters associated with the basic reproduction number (R_0) of the

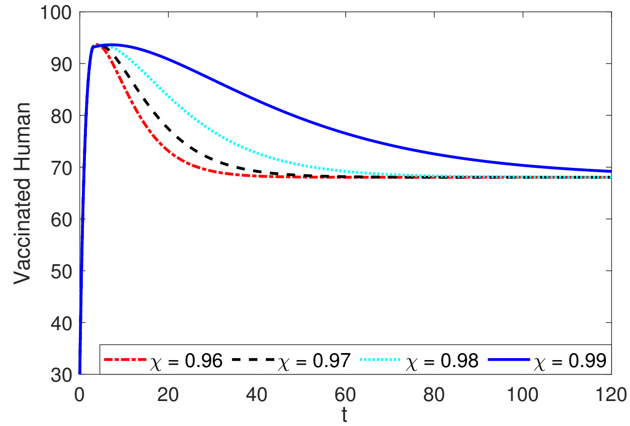

Figure 5 Plot of recovered class at various fractional order derivatives.

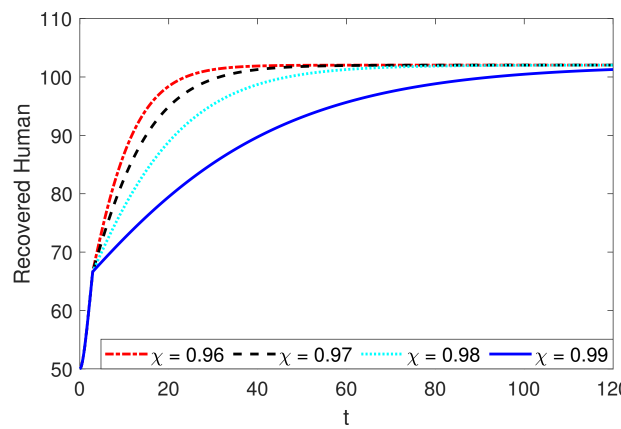

Figure 6 Plot of vaccinated class at various fractional order derivatives.

given model and as a result, we have some potential parameters on which the basic reproduction number (R_0) depends. Due to both increase and decrease, there is an associated increase and decrease in (R_0). We present the sensitivity indices graphically using a bar chart for justification. We have also simulated the results by using some real values for the parameters and initial data. We see that at point $t_1 = 5$, the behavior of the dynamics has shown variation. This is due to the piecewise derivative. Such effect is called crossover and can be well explained by using piecewise derivative as compared to ordinary or usual fractional order. Hence we concluded that piecewise derivative can be used as a powerful tool to investigate the transmission dynamics of infectious diseases that suffer from abrupt changes in their dynamical evolution.

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Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Availability of data and material

Not applicable.

LITERATURE CITED

Abdo, M. S., K. Shah, H. A. Wahash, and S. K. Panchal, 2020 On a comprehensive model of the novel coronavirus (covid-19) under mittag-leffler derivative. Chaos, Solitons & Fractals **135**: 109867.

Agarwal, R. P., V. Lakshmikantham, and J. J. Nieto, 2010 On the concept of solution for fractional differential equations with uncertainty. Nonlinear Analysis: Theory, Methods & Applications **72**: 2859–2862.

Ahmad, S., A. Ullah, and A. Akgül, 2021a Investigating the complex behaviour of multi-scroll chaotic system with caputo fractal-fractional operator. Chaos, Solitons & Fractals **146**: 110900.

Ahmad, S., A. Ullah, A. Akgül, and M. De la Sen, 2021b A study of fractional order ambartsumian equation involving exponential decay kernel. AIMS Math **6**: 9981–9997.

Ahmad, S., A. Ullah, M. Partohaghighi, S. Saifullah, A. Akgül, *et al.*, 2021c Oscillatory and complex behaviour of caputo-fabrizio fractional order hiv-1 infection model. Aims Math **7**: 4778–4792.

Alqahtani, R. T., S. Ahmad, and A. Akgül, 2021 Dynamical analysis of bio-ethanol production model under generalized nonlocal operator in caputo sense. Mathematics **9**: 2370.

Arfan, M., H. Alrabaiah, M. U. Rahman, Y.-L. Sun, A. S. Hashim, *et al.*, 2021 Investigation of fractal-fractional order model of covid-19 in pakistan under atangana-baleanu caputo (abc) derivative. Results in Physics **24**: 104046.

Atangana, A., 2020 Extension of rate of change concept: from local to nonlocal operators with applications. Results in Physics **19**: 103515.

Atangana, A. and S. İ. Araz, 2021 New concept in calculus: Piecewise differential and integral operators. Chaos, Solitons & Fractals **145**: 110638.

Atangana, A. and S. İğret Araz, 2020 Mathematical model of covid-19 spread in turkey and south africa: theory, methods, and applications. Advances in Difference Equations **2020**: 1–89.

Chitnis, N., J. M. Hyman, and J. M. Cushing, 2008 Determining important parameters in the spread of malaria through the sensitivity analysis of a mathematical model. Bulletin of mathematical biology **70**: 1272–1296.

Doungmo Goufo, E. F., 2015 A biomathematical view on the fractional dynamics of cellulose degradation. Fractional Calculus and Applied Analysis **18**: 554–564.

Doungmo Goufo, E. F., 2016 Application of the caputo-fabrizio fractional derivative without singular kernel to korteweg-de vries-burgers equation. Mathematical Modelling and Analysis **21**: 188–198.

Grace, S., R. Agarwal, P. Wong, and A. Zafer, 2012 On the oscillation of fractional differential equations. Fractional Calculus and Applied Analysis **15**: 222–231.

Hajiseyedazizi, S. N., M. E. Samei, J. Alzabut, and Y. ming Chu, 2021 On multi-step methods for singular fractional q-integro-differential equations. Open Mathematics **19**: 1378–1405.

Hilfer, R. *et al.*, 2008 Threefold introduction to fractional derivatives. Anomalous transport: Foundations and applications pp. 17–73.

Machado, J. T., V. Kiryakova, and F. Mainardi, 2011 Recent history of fractional calculus. Communications in nonlinear science and numerical simulation **16**: 1140–1153.

Nawaz, Y., M. S. Arif, and W. Shatanawi, 2022 A new numerical scheme for time fractional diffusive seair model with non-linear incidence rate: An application to computational biology. Fractal and Fractional **6**: 78.

Ojo, M. M. and E. F. D. Goufo, 2022 Modeling, analyzing and simulating the dynamics of lassa fever in nigeria. Journal of the Egyptian Mathematical Society **30**: 1.

Ojo, M. M. and E. F. D. Goufo, 2023 The impact of covid-19 on a malaria dominated region: A mathematical analysis and simulations. Alexandria Engineering Journal **65**: 23–39.

Rahman, F., A. Ali, and S. Saifullah, 2021 Analysis of time-fractional ϕ 4-equation with singular and non-singular kernels. International Journal of Applied and Computational Mathematics **7**: 192.

Saifullah, S., A. Ali, and E. F. D. Goufo, 2021 Investigation of complex behaviour of fractal fractional chaotic attractor with mittag-leffler kernel. Chaos, Solitons & Fractals **152**: 111332.

Saifullah, S., A. Ali, and Z. A. Khan, 2022 Analysis of nonlinear time-fractional klein-gordon equation with power law kernel. AIMS Math **7**: 5275–5290.

Shah, K., B. Abdalla, T. Abdeljawad, and R. Gul, 2023 Analysis of multipoint impulsive problem of fractional-order differential equations. Boundary Value Problems **2023**: 1–17.

Shah, K., T. Abdeljawad, B. Abdalla, and M. S. Abualrub, 2022a Utilizing fixed point approach to investigate piecewise equations with non-singular type derivative. AIMS Math **7**: 14614–14630.

Shah, K., T. Abdeljawad, and A. Ali, 2022b Mathematical analysis of the cauchy type dynamical system under piecewise equations with caputo fractional derivative. Chaos, Solitons & Fractals **161**: 112356.

Shah, K., T. Abdeljawad, and H. Alrabaiah, 2022c On coupled system of drug therapy via piecewise equations. Fractals **30**: 2240206.

Shatanawi, W., M. S. Abdo, M. A. Abdulwasaa, K. Shah, S. K. Panchal, *et al.*, 2021 A fractional dynamics of tuberculosis (tb) model in the frame of generalized atangana–baleanu derivative. Results in Physics **29**: 104739.

Zhou, H., J. Alzabut, and L. Yang, 2017 On fractional langevin differential equations with anti-periodic boundary conditions. The European Physical Journal Special Topics **226**: 3577–3590.

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