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# On Jacobsthal–Narayana and Jacobsthal-Narayana-Lucas Sequences

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**Abstract**

This paper introduces two new integer sequences that are the third-order recurrence relations. These are called Jacobsthal–Narayana and Jacobsthal-Narayana-Lucas sequences. In particular, great attention is focused on the identification of the Binet type representations for our new sequence, including the generating functions, some important identities, and generating matrix. Finally, we consider the circulant matrix whose entries are Jacobsthal–Narayana sequence and present an appropriate formula to find eigenvalues of that matrix.

***Keywords:*** *Binet formula; Circulant matrix; Jacobsthal–Narayana sequence; Third-order sequence, Vajda identity.*
***2010 Mathematics Subject Classification:*** *11B83, 11B37, 05A15.*

## 1. Introduction

Second-order linear recurrence sequences have received substantial attention because they are regularly encountered in different branches of modern sciences. As an example, Fibonacci, Lucas, Pell, and Pell-Lucas sequences have been under dense study by a great number of researchers. Due to the outside of main aim of the paper, no detailed review about the mentioned sequences is given here, but more detailed information is available in the monographs by Vajda [1] and Koshy [2].
Herein, the scope of this paper is related to the following. The usual Jacobsthal sequence is defined as

$$J_0 = 0,\ J_1 = 1, \text{ and } J_{n+1} = J_n + 2J_{n-1} \text{ for } n \geqslant 2 \tag{1.1}$$

and the Jacobsthal-Lucas sequence satisfies the same recurrence relation with the initial terms $j_2 = 0$ and $j_1 = 1$. The Narayana sequence is a third-order one and is defined as

$$N_0 = 0,\ N_1 = N_2 = 1, \text{ and } N_{n+1} = N_n + N_{n-2} \text{ for } n \geqslant 3. \tag{1.2}$$

Besides, the Binet's formulas for the Jacobsthal and Jacobsthal-Lucas sequences are

$$J_n = \frac{2^n - (-1)^n}{3} \text{ and } j_n = 2^n + (-1)^n, \tag{1.3}$$

respectively. Now, herein, we present the Binet's formula for the Narayana sequence that has not been given up to now as follows:

$$N_n = \frac{\mathrm{A}\alpha^n + \mathrm{B}\beta^n + \mathrm{E}\varepsilon^n}{\Delta} \tag{1.4}$$

where

$$\alpha = \frac{1+a+b}{3},\ \beta = \frac{1-\frac{1}{2}(a+b) - i\frac{\sqrt{3}}{2}(a-b)}{3},\ \varepsilon = \frac{1-\frac{1}{2}(a+b) + i\frac{\sqrt{3}}{2}(a-b)}{3},\ a = \sqrt[3]{\frac{29+3\sqrt{93}}{2}},\ b = \sqrt[3]{\frac{29-3\sqrt{93}}{2}}$$
$$\Delta = (\alpha-\beta)(\alpha-\gamma)(\beta-\varepsilon),\ \mathrm{A} = (\beta-\varepsilon)(1-\beta-\varepsilon),\ \mathrm{B} = (\varepsilon-\alpha)(1-\varepsilon-\alpha), \text{ and } \mathrm{E} = (\alpha-\beta)(1-\alpha-\beta).$$

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The equation given in (1.4) has been obtained after very extensive operations. Here, we do not give any proof, but the similar process will be applied in next section.
Each sequence in the above has an emergence process and a quite significant character in the current literature so that many researchers have given a significant attraction on the subject. For example, the solution to the problem about a herd of cows and calves of the great Indian mathematician Narayana Pandita has been led to obtain a third-order recurrence sequence that is named after himself; the Narayana sequence. As summarized, the above-stated sequences have many applications in mathematics and the other fields of modern sciences.
We refer to some papers regarding the usual Jacobsthal and Jacobsthal-Lucas sequences in the current literature herein. Horadam [3] presented a systematic survey for the Jacobsthal and Jacobsthal-Lucas sequences, while Cerin [4] derived some formulas for sums of squares of their terms and products. Atanassov [5] generalized the mentioned sequences. In [6] and [7], Daşdemir developed a matrix approach and some interesting identities for the usual Jacobsthal and Jacobsthal-Lucas sequences. Petroudi and Pirouz [8] investigated the eigenvalues and determinant of special circulant matrix involving $(k,h)$-Jacobsthal and $(k,h)$-Jacobsthal-Lucas sequence. Goy [9] gave a very nice application for the Jacobsthal sequence. In [10], Daşdemir characterized the usual Mersenne, Jacobsthal, and Jacobsthal-Lucas numbers with negative subscripts. For the sake of not further burdening the reader, a similar literature review for the Narayana sequence is not provided here, but a few of these can be found in [11, 12, 13].
Based on the current literature, to our knowledge, new third-order integer sequences presented herein have yet to be studied. We recommend that these are called the Jacobsthal–Narayana and Jacobsthal-Narayana-Lucas sequences due to their special structures. To fill this gap, we will show a mathematical approach in terms of the theory of the recurrence relation in number theory. Within the scope of this mentality, we also display the Binet type formulas for our new sequence and derive some important identities, including the generating functions and generating matrix. In addition to these results, we define a circulant matrix with the Jacobsthal–Narayana and Jacobsthal-Narayana-Lucas sequences and compute its eigenvalues.

## 2. Main definitions and results

Our new definitions are as follows.

**Definition 2.1.** *We define the Jacobsthal-Narayana sequence $JN_r$ by the recursive relation*

$$JN_r = JN_{r-1} + 2JN_{r-3} \tag{2.1}$$

*with the initial values $JN_0 = 0$ and $JN_1 = JN_2 = 1$.*

The first values of Jacobsthal-Narayana sequence are $0,1,1,1,3,5,7,13,23,37,63,109\ldots$.

**Definition 2.2.** *We define the Jacobsthal-Narayana-Lucas sequence $jN_r$ by the recursive relation*

$$jN_r = jN_{r-1} + 2jN_{r-3} \tag{2.2}$$

*with the initial values $jN_0 = 2$ and $jN_1 = jN_2 = 1$.*

The first values of Jacobsthal-Narayana-Lucas sequence are $2,1,1,5,7,9,19,33,51,89,155\ldots\ldots$
According to our definitions, Eqs. (2.1) and (2.2) are a third-order linear homogeneous difference equation, with constant coefficients, in the form of

$$x_n = x_{n-1} + 2x_{n-3} \tag{2.3}$$

We can then explore a solution to Eq. (2.3) as $x_n = \phi^n$, which $\phi$ is a unknown constant. On the substitution of this linear solution into our difference equation, we, therefore, obtain

$$\phi^3 = \phi^2 + 2. \tag{2.4}$$

From the cubic formula for the roots, we find three independent solutions as follows:

$$\alpha = \frac{1+\lambda_1+\lambda_2}{3},\ \beta = \frac{2-\lambda_1-\lambda_2}{6} + i\frac{\lambda_1-\lambda_2}{2\sqrt{3}},\ \text{and}\ \gamma = \frac{2-\lambda_1-\lambda_2}{6} - i\frac{\lambda_1-\lambda_2}{2\sqrt{3}}, \tag{2.5}$$

where $i=\sqrt{-1}$, $\lambda_1 = \sqrt[3]{28+3\sqrt{87}}$ and $\lambda_2 = \sqrt[3]{28-3\sqrt{87}}$. Furthermore, since a linear combination of the solutions in Eq. (2.5) satisfies Eq. (2.4); i.e. $JN_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n$, with the initial terms, we can write

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_1\alpha + c_2\beta + c_3\gamma &= 1 \\ c_1\alpha^2 + c_2\beta^2 + c_3\gamma^2 &= 1 \end{aligned}$$

and obtain the solution

$$c_1 = \frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)},\ c_2 = \frac{\beta}{(\beta-\alpha)(\beta-\gamma)},\ \text{and}\ c_3 = \frac{\gamma}{(\gamma-\beta)(\gamma-\alpha)}, \tag{2.6}$$

respectively. Besides, repeating the same technique for the Jacobsthal-Narayana-Lucas sequence gives the solution

$$\tilde{c}_1 = \frac{\alpha+2\beta\gamma}{(\alpha-\beta)(\alpha-\gamma)},\ \tilde{c}_2 = \frac{\beta+2\alpha\gamma}{(\beta-\alpha)(\beta-\gamma)},\ \text{and}\ \tilde{c}_3 = \frac{\gamma+2\alpha\beta}{(\gamma-\beta)(\gamma-\alpha)}, \tag{2.7}$$

respectively.
As summarized, we can write the following main results of the paper without the proof.

**Theorem 2.3** (Binet's formulas)**.** *Let n be any integer. Binet's formulas for the Jacobsthal-Narayana sequence and the Jacobsthal-Narayana-Lucas sequence are*

$$JN_r=\frac{\alpha^{r+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{r+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{r+1}}{(\gamma-\alpha)(\gamma-\beta)} \tag{2.8}$$

*and*

$$jN_r=\frac{\alpha+2\beta\gamma}{(\alpha-\beta)(\alpha-\gamma)}\alpha^r+\frac{\beta+2\alpha\gamma}{(\beta-\alpha)(\beta-\gamma)}\beta^r+\frac{\gamma+2\alpha\beta}{(\gamma-\alpha)(\gamma-\beta)}\gamma^r, \tag{2.9}$$

*respectively.*

By the way, there are many interesting properties between the roots of the cubic equation $x^3-x^2-2=0$. We can give some of the ones in the following.

**Remark 2.4.** *For $\alpha$, $\beta$, and $\gamma$, we can write the following.*

- $\alpha+\beta+\gamma=1$
- $\alpha\beta\gamma=2$
- $\alpha\beta+\alpha\gamma+\beta\gamma=0$
- $k_1+k_2+k_3=0$
- $k_1+k_2=\frac{-\gamma}{(\alpha-\gamma)(\beta-\gamma)}$, $k_1+k_3=\frac{-\beta}{(\alpha-\beta)(\gamma-\beta)}$, $k_2+k_3=\frac{-\alpha}{(\beta-\alpha)(\gamma-\alpha)}$
- $(k_1+k_3)\beta+(k_2+k_3)\alpha+(k_1+k_2)\gamma=-(\alpha k_1+\beta k_2+\gamma k_3)$
- $\frac{k_1}{\alpha}+\frac{k_2}{\beta}+\frac{k_3}{\gamma}=0$

*where* $k_1=\frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)}$, $k_2=\frac{\beta}{(\beta-\alpha)(\beta-\gamma)}$, *and* $k_3=\frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}$.

**Theorem 2.5.** *The generating functions for the Jacobsthal-Narayana sequence and the Jacobsthal-Narayana-Lucas sequence are*

$$\sum_{r=0}^{\infty}JN_rx^r=\frac{x}{1-x-2x^3} \text{ and } \sum_{r=0}^{\infty}jN_rx^r=\frac{3-2x-x^2}{1-x-2x^3} \tag{2.10}$$

*respectively.*

*Proof.* Define $g(x)=\sum_{n=0}^{\infty}JN_rx^n$. Then, summing the statements $g(x)$, $-xg(x)$, and $2x^3g(x)$, with some mathematical manipulations, the proof can be completed. □

**Theorem 2.6.** *Let $r>0$ be an integer and $k$ be an arbitrary integer. Then,*

- $JN_{r+k}+JN_{r-k}=\left[\frac{\alpha^{2k}+1}{(\alpha-\beta)(\alpha-\gamma)}\right]\alpha^{r-k+1}+\left[\frac{\beta^{2k}+1}{(\beta-\alpha)(\beta-\gamma)}\right]\beta^{r-k+1}+\left[\frac{\gamma^{2k}+1}{(\gamma-\alpha)(\gamma-\beta)}\right]\gamma^{r-k+1}$
- $JN_{r+k}-JN_{r-k}=\left[\frac{\alpha^{2k}-1}{(\alpha-\beta)(\alpha-\gamma)}\right]\alpha^{r-k+1}+\left[\frac{\beta^{2k}-1}{(\beta-\alpha)(\beta-\gamma)}\right]\beta^{r-k+1}+\left[\frac{\gamma^{2k}-1}{(\gamma-\alpha)(\gamma-\beta)}\right]\gamma^{r-k+1}$
- $jN_{r+k}+jN_{r-k}=\left[\frac{(3\alpha+1)(\alpha-1)(\alpha^{2k}+1)}{(\alpha-\beta)(\alpha-\gamma)}\right]\alpha^{r-k}+\left[\frac{(3\beta+1)(\beta-1)(\beta^{2k}+1)}{(\beta-\alpha)(\beta-\gamma)}\right]\beta^{r-k}+\left[\frac{(3\gamma+1)(\gamma-1)(\gamma^{2k}+1)}{(\gamma-\alpha)(\gamma-\beta)}\right]\gamma^{r-k}$
- $jN_{r+k}-jN_{r-k}=\left[\frac{(3\alpha+1)(\alpha-1)(\alpha^{2k}-1)}{(\alpha-\beta)(\alpha-\gamma)}\right]\alpha^{r-k}+\left[\frac{(3\beta+1)(\beta-1)(\beta^{2k}-1)}{(\beta-\alpha)(\beta-\gamma)}\right]\beta^{r-k}+\left[\frac{(3\gamma+1)(\gamma-1)(\gamma^{2k}-1)}{(\gamma-\alpha)(\gamma-\beta)}\right]\gamma^{r-k}$

*Proof.* They can be proved by direct calculations according to Theorem 2.3. □

In particular, for $k=1$ in the last theorem, we get the following cases.

- $JN_{r+1}+JN_{r-1}=\left[\frac{\alpha^2+1}{(\alpha-\beta)(\alpha-\gamma)}\right]\alpha^r+\left[\frac{\beta^2+1}{(\beta-\alpha)(\beta-\gamma)}\right]\beta^r+\left[\frac{\gamma^2+1}{(\gamma-\alpha)(\gamma-\beta)}\right]\gamma^r$
- $JN_{r+1}-JN_{r-1}=\left[\frac{\alpha^2-1}{(\alpha-\beta)(\alpha-\gamma)}\right]\alpha^r+\left[\frac{\beta^2-1}{(\beta-\alpha)(\beta-\gamma)}\right]\beta^r+\left[\frac{\gamma^2-1}{(\gamma-\alpha)(\gamma-\beta)}\right]\gamma^r$
- $jN_{r+1}+jN_{r-1}=\left[\frac{(3\alpha^2-2\alpha-1)(\alpha^2+1)}{(\alpha-\beta)(\alpha-\gamma)}\right]\alpha^{r-1}+\left[\frac{(3\beta^2-2\beta-1)(\beta^2+1)}{(\beta-\alpha)(\beta-\gamma)}\right]\beta^{r-1}+\left[\frac{(3\gamma^2-2\gamma-1)(\gamma^2+1)}{(\gamma-\alpha)(\gamma-\beta)}\right]\gamma^{r-1}$
- $jN_{r+1}-jN_{r-1}=\left[\frac{(3\alpha^2+4\alpha+1)(\alpha-1)^2}{(\alpha-\beta)(\alpha-\gamma)}\right]\alpha^{r-1}+\left[\frac{(3\beta^2+4\beta+1)(\beta-1)^2}{(\beta-\alpha)(\beta-\gamma)}\right]\beta^{r-1}+\left[\frac{(3\gamma^2+4\gamma+1)(\gamma-1)^2}{(\gamma-\alpha)(\gamma-\beta)}\right]\gamma^{r-1}$

**Theorem 2.7** (Vajda identity)**.** *Let n, r, and s be a positive integer. Then, we have*

$$JN_{n+r}JN_{n+s}-JN_nJN_{n+r+s}=\frac{-i}{2\sqrt{29}}\left[\frac{(\alpha^r-\beta^r)(\alpha^s-\beta^s)}{\alpha-\beta}(\alpha\beta)^{n+1}+\frac{(\gamma^r-\alpha^r)(\gamma^s-\alpha^s)}{\gamma-\alpha}(\alpha\gamma)^{n+1}+\frac{(\beta^r-\gamma^r)(\beta^s-\gamma^s)}{\beta-\gamma}(\gamma\beta)^{n+1}\right] \tag{2.11}$$

*Proof.* Since the proof has very extensive and complex operations, we omit the details. □

This theorem can be reduced to the following cases.

- For $r=-s$, the Catalan's identity is obtained:

$$JN_{n-s}JN_{n+s}-JN_n{}^2=\frac{i}{2\sqrt{29}}\left[\frac{(\alpha^s-\beta^s)^2}{\alpha-\beta}(\alpha\beta)^{n-s+1}+\frac{(\gamma^s-\alpha^s)^2}{\gamma-\alpha}(\gamma\alpha)^{n-s+1}+\frac{(\beta^s-\gamma^s)^2}{\beta-\gamma}(\beta\gamma)^{n-s+1}\right] \tag{2.12}$$

- For $s=-r=1$, the Cassini's identity is obtained:

$$JN_{n-1}JN_{n+1}-JN_n{}^2=\frac{i}{2\sqrt{29}}\left[(\alpha-\beta)(\alpha\beta)^n+(\gamma-\alpha)\,(\gamma\alpha)^n+(\beta-\gamma)(\beta\gamma)^n\right] \tag{2.13}$$

- For $s=m-n$ and $r=1$, the d'Ocagne's identity is obtained:

$$JN_{n+1}JN_m-JN_nJN_{m+1}=\frac{-i}{2\sqrt{29}}\left[\left(\alpha^{m-n}-\beta^{m-n}\right)(\alpha\beta)^{n+1}+\left(\gamma^{n-n}-\alpha^{m-n}\right)\,(\gamma\alpha)^{n+1}+\left(\beta^{m-n}-\gamma^{n-n}\right)(\beta\gamma)^{n+1}\right] \tag{2.14}$$

## 3. Matrix approach

In this section, we explore some identities using matrix technique. First, define the matrix

$$\varphi=\left[\begin{array}{ccc}0&1&0\\0&0&1\\2&0&1\end{array}\right]. \tag{3.1}$$

This is a special matrix that satisfies the characteristic equation in (8), i.e. $\varphi^3-\varphi^2-2I=0$, where $I$ is the identity matrix. This result can be seen from the well-known Cayley Hamilton theorem.
We can, thus, start with the following main result.

**Theorem 3.1.** *Let r be a positive integer. Then, we have*

$$\varphi^n=\begin{pmatrix}2JN_{n-2}&2JN_{n-3}&JN_{n-1}\\2JN_{n-1}&2JN_{n-2}&JN_n\\2JN_n&2JN_{n-1}&JN_{n+1}\end{pmatrix}. \tag{3.2}$$

*Proof.* Considering the recurrence relation in Eq. (2.1), the proof is completed easily. □

**Theorem 3.2.** *For an integer n, we have*

$$\det(\varphi^n)=2^n. \tag{3.3}$$

*Proof.* Using the determinant properties of a matrix leads to complete the proof. □

Next theorem presents an interesting result.

**Theorem 3.3.** *For the matrix $\varphi$, we have the matrix-polynomial identity*

$$\varphi^{n+5}=\varphi^{n+4}-\varphi^{n+3}+3\varphi^{n+2}+2\varphi^n \tag{3.4}$$

*Proof.* From the equation $\varphi^3-\varphi^2-2I=0$, we can write

$$I=\frac{1}{2}\left(\varphi^3-\varphi^2\right)=\frac{1}{2}\varphi^2(\varphi-I)=\frac{1}{2}\varphi^2\left(\varphi^3-\varphi^2+\varphi-3I\right)=\frac{1}{2}\left(\varphi^5-\varphi^4+\varphi^3-3\varphi^2\right).$$

Multiplying both sides of the above equality by $\varphi^n$, the proof is completed. □

Theorem 3.2 gives us the following interesting results for the Jacobsthal-Narayana and Jacobsthal-Narayana-Lucas sequences.

**Theorem 3.4.** *Let $r\geqslant 0$ be an integer. Then, we have*

$$JN_{r+5}=2JN_r+3JN_{r+2}-JN_{r+3}+JN_{r+4}$$

*and*

$$jN_{r+5}=2jN_r+3jN_{r+2}-jN_{r+3}+jN_{r+4}.$$

*Proof.* We apply the induction method on $n$. Since $JN_{1+5}=2JN_1+3JN_{1+2}+JN_{1+4}-JN_{1+3}$, the result is true for $n=1$. Now based on the assumption that $JN_{t+5}=2JN_t+3JN_{t+2}+JN_{t+4}-JN_{t+3}a$ is satisfied for all $t<n$. Then, we write

$$\begin{aligned}JN_{n+5}&=JN_{n+4}+2JN_{n+2}\\&=(2JN_{n-1}+3JN_{n+1}+JN_{n+3}-JN_{n+2})+2(2JN_{n-3}+3JN_{n-1}+JN_{n+1}-JN_n)\\&=2(JN_{n-1}+2JN_{n-3})+3(JN_{n+1}+2JN_{n-1})+(JN_{n+3}+2JN_{n+1})-(JN_{n+2}-JN_n)\\&=2JN_n+3JN_{n+2}+JN_{n+4}-JN_{n+3},\end{aligned}$$

which is desired result. □

This theorem shows that the representations of our integer sequences are not unique. Furthermore, using this strategy, we can find another representation of them. For instance, after similar mathematical operations, the following can be obtained:

$$\varphi^{n+r} = \frac{1}{4}\left[\varphi^{n+r+10} - 2\varphi^{n+r+9} + 3\varphi^{n+r+8} - 8\varphi^{n+r+7} + 7\varphi^{n+r+6} - 6\varphi^{n+r+5} + 9\varphi^{n+r+4}\right]$$

$$JN_{n+r} = \frac{1}{4}\left[JN_{n+r+10} - 2JN_{n+r+9} + 3JN_{n+r+8} - 8JN_{n+r+7} + 7JN_{n+r+6} - 6JN_{n+r+5} + 9JN_{n+r+4}\right]$$

$$jN_{n+r} = \frac{1}{4}\left[jN_{n+r+10} - 2jN_{n+r+9} + 3jN_{n+r+8} - 8jN_{n+r+7} + 7jN_{n+r+6} - 6jN_{n+r+5} + 9jN_{n+r+4}\right]$$

It should be noted these recurrence relations are a seventh-order finite difference equations and accordingly an another Binet type formulas can also be given.

## 4. An application of Jacobsthal-Narayana sequence

As known, a Circulant matrix $C = \left[c_{i,j}\right] \in M_{n\times n}$ is denoted briefly by $C = Circ(c_0, c_1, \ldots, c_{n-1})$ and is defined as

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \end{bmatrix}.$$

In addition, we recall the following lemma.

**Lemma 4.1** (Zhang, [14]). *Let $C = Circ(c_0, c_1, \ldots, c_{n-1})$ be a $n\times n$ circulant matrix. Then, we have*

$$\rho_j(C) = \sum_{k=0}^{n-1} c_k w^{-jk}, \tag{4.1}$$

*where $\rho_j$ is the eigenvalue of the circulant matrix $C$ for $j = 0, 1, 2, \cdots, n-1$, $w = e^{\frac{2\pi i}{n}}$, and $i = \sqrt{-1}$.*

In this situation, we can give the following important theorem.

**Theorem 4.2.** *Let $C = Cir(JN_0, JN_1, \cdots, JN_{n-1})$ be a $n\times n$ circulant matrix whose entries are the Jacobsthal-Narayana sequence $(JN_n)$. Then, the eigenvalues of $C$ are*

$$\rho_j(C) = \frac{(2JN_{n-1})w^{-2j} + (2JN_{n-1} - 1)w^{-j} + JN_n}{2w^{-3j} + w^{-2j} - 1},$$

*where $j = 0, 1, 2, \cdots, n-1$, $w = e^{\frac{2\pi i}{n}}$, and $i = \sqrt{-1}$.*

*Proof.* By Lemma 4.1 for the eigenvalues of circulant matrix $C = Cir(JN_0, JN_1, \cdots, JN_{n-1})$, we have

$$\begin{aligned}
\rho_j(C) &= \sum_{k=0}^{n-1} JN_k w^{-jk} = \sum_{k=0}^{n-1}\left[\frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)}\alpha^k + \frac{\beta}{(\beta-\alpha)(\beta-\gamma)}\beta^k + \frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\gamma^k\right]w^{-jk} \\
&= \frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)}\sum_{k=0}^{n-1}\alpha^k w^{-jk} + \frac{\beta}{(\beta-\alpha)(\beta-\gamma)}\sum_{k=0}^{n-1}\beta^k w^{-jk} + \frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\sum_{k=0}^{n-1}\gamma^k w^{-jk} \\
&= k_1\left(\frac{(\alpha w^{-j})^n - 1}{\alpha w^{-j} - 1}\right) + k_2\left(\frac{(\beta w^{-j})^n - 1}{\beta w^{-j} - 1}\right) + k_3\left(\frac{(\gamma w^{-j})^n - 1}{\gamma w^{-j} - 1}\right) \\
&= k_1\left(\frac{\alpha^n - 1}{\alpha w^{-j} - 1}\right) + k_2\left(\frac{\beta^n - 1}{\beta w^{-j} - 1}\right) + k_3\left(\frac{\gamma^n - 1}{\gamma w^{-j} - 1}\right) \\
&= \frac{\begin{array}{c} k_1(\alpha^n - 1)\left(\beta w^{-j} - 1\right)\left(\gamma w^{-j} - 1\right) + k_2(\beta^n - 1)\left(\alpha w^{-j} - 1\right)\left(\gamma w^{-j} - 1\right) \\ + k_3(\gamma^n - 1)\left(\alpha w^{-j} - 1\right)\left(\beta w^{-j} - 1\right) \end{array}}{(\alpha w^{-j} - 1)(\beta w^{-j} - 1)(\gamma w^{-j} - 1)} \\
&= \frac{\begin{array}{c} -(k_1 + k_2 + k_3) + (k_1\alpha^n + k_2\beta^n + k_3\gamma^n) + (k_1\alpha^n\beta\gamma + k_2\beta^n\alpha\,\gamma + k_3\gamma^n\alpha\beta)w^{-2j} - (k_1\alpha^n\beta + k_2\beta^n\alpha + k_3\gamma^n\alpha)w^{-j} \\ -(k_1\alpha^n\gamma + k_2\beta^n\gamma + k_3\gamma^n\beta)w^{-j} - (k_1\beta\gamma + k_2\alpha\gamma + k_3\alpha\beta)w^{-2j} + (k_1\beta + k_2\alpha + k_3\alpha + k_1\gamma + k_2\gamma + k_3\beta)w^{-j} \end{array}}{(\alpha\beta\gamma)w^{-3j} - (\alpha\beta + \alpha\gamma + \beta\gamma)w^{-2j} + (\alpha + \beta + \gamma)w^{-j} - 1}
\end{aligned}$$

and according to Remark 2.4, after some computations we get

$$\begin{aligned}
\rho_j(\mathrm{C}) &= \frac{(2JN_{n-1})\,w^{-2j}+(2JN_{n-1}-JN_1)\,w^{-j}+JN_n}{2w^{-3j}+w^{-2j}-1} \\
&= \frac{(2JN_{n-1})\,w^{-2j}+(2JN_{n-1}-1)\,w^{-j}+JN_n}{2w^{-3j}+w^{-2j}-1} \\
&= \frac{(2JN_{n-1})\,w^{-2j}+(2JN_{n-1}-JN_1)\,w^{-j}+JN_n}{2w^{-3j}+w^{-2j}-1} \\
&= \frac{(2JN_{n-1})\,w^{-2j}+(2JN_{n-1}-1)\,w^{-j}+JN_n}{2w^{-3j}+w^{-2j}-1}.
\end{aligned}$$

Thus, the proof is completed. □

**Example 4.3.** *The following table represents the eigenvalues of circulant matrix $C = Cir(JN_0, JN_1, \cdots, JN_{n-1})$ for some values of $n$.*

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| Eigenvalues | -1 | 2 | 3 | 6 | -3 | 18 |
| | -1 | -1 | -1 | $-2.6180+1.1755i$ | 11 | $0.0490+9.4585i$ |
| | | -1 | -1 | $-2.6180-1.1755i$ | $7.1054\times10^{-15}+5.1962i$ | $0.0490-9.4585i$ |
| | | | -1 | $-0.3819+1.9021i$ | $7.1054\times10^{-15}-5.1962i$ | $-4.6920+2.5504i$ |
| | | | | $-0.3819-1.9021i$ | $-4+1.73205i$ | $-4.6920-2.5504i$ |
| | | | | | $-4-1.73205i$ | $-4.3569+1.4258i$ |
| | | | | | | $-4.3569-1.4258i$ |

## 5. Conclusion

In this paper, we introduced new integer sequences, named the Jacobsthal-Narayana and Jacobsthal-Narayana-Lucas sequences. We obtained the Binet-like formula and generating functions for these sequences. We gave some interesting identities and examples about these sequences. Finally, we represented a formula to find the eigenvalues of the circulant matrix involving the Jacobsthal-Narayana sequence.

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