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A Sequential Differential Problem With Caputo and Riemann Liouville Derivatives Involving Convergent Series

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Abstract

In this paper, we study a new nonlinear differential problem with nonlocal integral conditions and convergent series. The problem involves three fractional order operators: Riemann-Liouville integral, Caputo and Riemann-Liouville derivatives. The introduced Caputo derivatives in the problem have neither the commutativity property nor the semi-group one. The considered problem can be seen as a more general case for the problem considered in the recent paper: [Existence and Mittag-Leffler-Ulam-Stability Results for Duffing Type Problem Involving Sequential Fractional Derivatives] that is published in the International Journal of Applied and Computational Mathematics, (2022). We begin by proving a first auxiliary integral result. Then, we demonstrate an existence and uniqueness result by applying Banach contraction principle. Also, we establish a new existence result using Leray-Schauder fixed point theorem. We end our paper by presenting some illustrative examples.

Mathematics Subject Classification (2010): 30C45, 39B72, 39B82. **Key words and phrases:** Caputo derivative fixed point Riemann-Liouville derivative existence of solution.

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1. Introduction

Fractional differential equations theory is attracting more popularity and increasing importance, due to its numerous applications in various areas, such as optics, medicine, statistical physics, electrochemistry, automatics, and control theory, see [5, 6, 7, 9, 14, 15, 17, 20, 22]. To be more specific, many models are discussed; among them: glycolysis [11], viscous nanofluid holding [12, 24, 25], The dynamics of vector–host infectious diseases[13], the magnetohydrodynamic [33], The second grade fluid flow for generalized thermal and molecular diffusion by applying the constant proportional Caputo fractional derivative [10]. Moreover, the nonlinear case is one of the most important mathematical tools used to model real-world problems in many domains of science. The reader is invited to consult the paper [1, 4, 8, 23, 26, 27, 31, 32]. In particular, one of these nonlinear equations called the Duffing equation that has become very important in engineering sciences, see [3, 34]. In this context, many authors have been interested in studying the question of the existence and uniqueness of solutions for certain types of such equations. We refer the interested reader to [2, 16, 29] for more details.
In [2], the authors were concerned with the following sequential Duffing problem:

$$\begin{cases} D^{\alpha}\left(D^{2-\beta}+\lambda D^{\alpha}\right)x(t)+k_1f_1\left(t,x(t),D^{\alpha}x(t)\right)+k_2f_2\left(t,x(t),J^{p}x(t)\right)=h(t), \\ \\ x(1)=0, \quad D^{1-(\alpha-\beta)}D^{\alpha-\beta}x(1)=A^{*}\in\mathbb{R}, \quad x(T)=0, \\ 0\leq\beta<\alpha\leq 1, \quad 0\leq\alpha+\beta<1, \quad 0<p, \quad t\in I, \end{cases}$$

where $D^{\alpha}, D^{2-\beta}$, are the Caputo-Hadamard fractional derivatives, J^{p} is the Hadamard fractional integral $I=[1,T], k_1, k_2$ are real constants, and the functions f_1, f_2 and h are continuous. The authors have investigated the existence, uniqueness and stability of solutions for a new sequential Van der Pol-Duffing (VdPD) jerk fractional differential oscillator with Caputo-Hadamard derivatives. Their arguments are based on Banach contraction principle and Krasnoselskii fixed point theorem. They have also studied Ulam-Hyers stabilities for their proposed problem.

Also in [18], Y. Gouari et al. have studied the following three sequential fractional problem of Duffing type:

$$\begin{cases} D^{\alpha}(D^{\beta}(D^{\delta}y(t)))+f(t,y(t),D^{p}y(t))+g(t,y(t),I^{q}y(t))+h(t,y(t))=l(t), \\ \\ y(0)=\xi\in\mathbb{R}, \\ y(1)=\displaystyle\int_0^{\eta}y(s)ds, \ 0<\eta<1, \\ I^{e}y(\theta)=D^{\delta}y(1), \ 0<u<1, \\ 0<\alpha,\beta,\delta,p\leq 1, \quad q>0, \quad t\in J, \end{cases}$$

where $J:=[0,1]$, $D^{\alpha}, D^{\beta}, D^{\delta}, D^{p}$ are derivatives of Caputo , I^{q} denotes the Riemann-Liouville fractional integral of order q, and $f,g:J\times\mathbb{R}^2\to\mathbb{R}$ are two given functions, also $h:J\times\mathbb{R}\to\mathbb{R}$ is a given function and l is a function which is defined on J. The authors have proved the existence and uniqueness of solutions by application of Banach contraction principle, then, by means of Schaefer fixed point theorem, they have studied the existence of at least one solution for the problem.

In [19], by the applications of singular differential equations in fluid dynamics, the authors have considered the following problem:

$$\begin{cases} D^{\alpha}u(t)+\lambda f(u(t),u^{''}(t))=\delta g(t,u(t),D^{\gamma}u(t))+\sum_{i=1}^{\infty}\nu_i\Phi_i(t)I^{\alpha}h_i(t,u(t)), t\in(0,1],\\ u^{''}(0)+u^{''}(1)=\kappa_1\int_0^{\xi}u(s)ds,\quad 0<\xi<1,\\ u^{'}(0)+u^{'}(1)=\kappa_2\int_0^{\theta}u(s)ds,\quad 0<\theta<1,\\ u(0)+u(1)=\kappa_3\int_0^{\eta}u(s)ds,\quad 0<\eta<1,\\ 2<\alpha\leq 3,\quad 0<\gamma<1,\quad \kappa_1,\kappa_2,\kappa_3,\lambda,\delta,\nu_i\in\mathbb{R}, \end{cases}$$

where we note that $J:=[0,1]$, the functions f, h_i and Φ_i will be specified later, g is singular at $t=0$, ξ,θ,η are constants, the operators D^{α} and D^{γ} are the derivatives in the sense of Caputo.

Very recently, M. Houas et al. [21] have studied the existence, uniqueness and Mittag-Leffler-Ulam-stability of solutions for sequential Caputo-Riemann-Liouville fractional Duffing problem, given by

$$\begin{cases} {}_CD^{\eta}[{}_CD^{\omega}[{}_{RL}D^{\theta}v(s)]]=m(s)-Ap(s,v(s),{}_{RL}D^{\mu}v(s))-q(s,v(s),{}_{RL}I^{\gamma}v(s)),\\ \\ v(0)=0,\\ {}_CD^{\omega}[{}_{RL}D^{\theta}v(1)]]=0,\\ {}_{RL}D^{\theta}v(1)]-{}_{RL}D^{\alpha}v(\beta)]=0, \end{cases}$$

with $s\in\Omega:=[0,1],\ \eta,\omega,\theta,\beta\in(0,1),\ \mu<\theta,\gamma,\alpha\geq 0, A>0,$
and ${}_CD^{\vartheta},\vartheta\in\{\eta,\omega\},{}_{RL}D^{\theta}$ denote the Caputo and Riemann-Liouville fractional derivatives, ${}_{RL}I^{\varsigma},\varsigma\in\{\gamma,\alpha\}$ is the Riemann-Liouville fractional integral of order ς, $p,q:\Omega\times\mathbb{R}^2\to\mathbb{R}$ and $m:\Omega\to\mathbb{R}$ are given continuous functions. A uniqueness result for solutions of the underlying Duffing problem has been presented by the authors with the aid of Banach fixed point theorem, while the existence result has been derived from Leray-Schauder alternative. Also the Mittag-Leffler-Ulam stability has been obtained by using generalized singular Gronwall inequality.

The present paper deals with the existence and uniqueness of solutions to the following sequential fractional problem:

$$\begin{cases} {}_CD^{\alpha_1}[{}_CD^{\alpha_2}[{}_CD^{\alpha_3}...[{}_CD^{\alpha_n}[{}_{RL}D^{\beta}u(s)]]...]]=\Lambda_1 f(t,u(t),{}_{RL}D^{\gamma}u(t))\\ +\Lambda_2 g(t,u(t),I^{\zeta}u(t))+\sum_{j=1}^{\infty}\Phi_j(t)I^{\zeta}[h_j(t,u(t))+l_j(t)],\quad t\in[0,1],\\ u(0)=0,\\ {}_CD^{\alpha_2}[{}_CD^{\alpha_3}[{}_CD^{\alpha_4}...[{}_CD^{\alpha_n}[{}_{RL}D^{\beta}u(0)]]...]]=\theta,\quad\theta>0,\\ {}_CD^{\alpha_3}[{}_CD^{\alpha_4}[{}_CD^{\alpha_5}...[{}_CD^{\alpha_n}[{}_{RL}D^{\beta}u(0)]]...]]=0,\\ {}_CD^{\alpha_4}[{}_CD^{\alpha_5}[{}_CD^{\alpha_6}...[{}_CD^{\alpha_n}[{}_{RL}D^{\beta}u(0)]]...]]=0,\\ \quad .\\ \quad .\\ \quad .\\ {}_CD^{\alpha_{n-1}}[{}_CD^{\alpha_n}[{}_{RL}D^{\beta}u(0)]]=0,\\ {}_CD^{\alpha_n}[{}_{RL}D^{\beta}u(0)]=\int_0^{\eta}u(s)ds,\quad 0<\eta<1,\\ {}_{RL}D^{\beta}u(1)={}_{RL}D^{\varphi}u(\tau),\quad 0<\tau<1. \end{cases}\tag{1}$$

For (1), we take $J:=[0,1],0\leq\beta,\varphi<1,0\leq\alpha_i<1;\gamma<\alpha_i,i=1,2,...,n,$ and $\gamma,\Lambda_1,\Lambda_2>0$, the sequential derivatives are in the sense of Caputo and Riemann-Liouville, I^{ζ} denotes the Riemann-Liouville fractional

integral of order ζ, and $f, g : J \times \mathbb{R}^2 \to \mathbb{R}$ are two given functions, also $h_j : J \times \mathbb{R} \to \mathbb{R}$ is a given functions and l_j, Φ_j are functions which are defined on J, $j \in \mathbb{N}^*$.
We think that our problem is more general than the problem considered in [21], since it includes several parameters of sequential Caputo derivations. The parameters allow us to introduce a new sequential problem with ACSG; absence of commutativity and semi group properties between Caputo derivatives. So, to study the problem, we shall find new arguments to overcome this type of ACSG problems.
In general, the aim is to present a new contribution in this field of interest and try to fill this gap. Especially, we study the question of existence and uniqueness of solutions by using both Banch fixed point theorem and integral inequalities, then we pass to the investigate the existence of solutions by using Leray-Schauder fixed point theorem. Another particularity of the above problem is the introduction of the Riemann-Liouville derivative in both sides of the problem. Also, we introduce the Riemann-Liouville integral in one nonlinearity of the right hand side of the sequential problem.
To the best of our knowledge this is the first time in the literature where problem, involving fractional calculus and convergent series on Riemann-Liouville integrals and other nonlinear terms, is investigated.
The paper is organised as follow: in section 2, we recall some results and definitions that are used for the proof of our main results. In section 3, we prove the main theorems of this paper, and we discuss some illustrative examples.

2. Fractional Calculus

We recall some definitions and lemmas [28].

Definition 2.1. *Let $\alpha > 0$, and $f : [0, 1] \to \mathbb{R}$ be a continuous function. The Riemann-Liouville integral of order $\alpha > 0$ is defined by:*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau,$$

where, $\Gamma(\alpha) := \int_0^{\infty} e^{-u} u^{\alpha-1} du$.

Definition 2.2. *For a function $f \in C^n([0, 1], \mathbb{R})$ and $n - 1 < \alpha \leq n$, the Caputo fractional derivative is defined by:*

$$\begin{aligned} {}_C D^{\alpha} f(t) &= I^{n-\alpha} \frac{d^n}{dt^n}(f(t)) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds. \end{aligned}$$

Definition 2.3. *For a function $f \in C^n([0, 1], \mathbb{R})$, the Riemann-Liouville fractional derivative of f is defined by:*

$${}_{RL}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds,$$

where, $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

To study (1), we need the following lemmas [28]:

Lemma 2.4. *Let $n \in \mathbb{N}^*$, and $n - 1 < \alpha < n$. Then, the solutions of the equation $D^{\alpha} y(t) = 0, t \in [0, 1]$ are:*

$$y(t) = \sum_{i=0}^{n-1} c_i t^i, c_i \in \mathbb{R}, i = 0, 1, 2, .., n-1.$$

Lemma 2.5. *If* $n \in \mathbb{N}^*$, *and* $n-1 < \alpha < n$, *then, we have*

$$I^{\alpha} D^{\alpha} y(t) = y(t) + \sum_{i=0}^{n-1} c_i t^i,$$

such that $c_i \in \mathbb{R}$, $i = 0, 1, 2, .., n-1$.

Lemma 2.6. *Suppose that* $0 < \lambda \leq 1$. *Then for* $y \in C(0,1) \cap L_1(0,1)$ *and* ${}_{RL}D^{\lambda}y \in C(0,1) \cap L_1(0,1)$, *we have*

$$I^{\lambda}[{}_{RL}D^{\lambda}y(t)] = y(t) + e_0 t^{\lambda-1}.$$

Lemma 2.7. *Let* $B : W \to W$ *be a completely continuous operator and*

$$F(B) := \{x \in W : x = \sigma\ Bx, \sigma \in]0,1[\}.$$

Then either the set $F(B)$ *is unbounded, or* B *has at least one fixed point.*

Now, we prove the following auxiliary integral result.

Lemma 2.8. *Let* Υ *in* $C(]0,1])$, $(\Psi_j)_{j=1,...,r}$ *and* $(\Phi_j)_{j=1,...,r}$ *in* $C(J), r \in \mathbb{N}^*$, *such that* $M_\Phi = \sum_{j=1}^{\infty} \|\Phi_j\|_\infty < +\infty$.

Then, the differential problem

$$\begin{cases} {}_{C}D^{\alpha_1}[{}_{C}D^{\alpha_2}[{}_{C}D^{\alpha_3}...[{}_{C}D^{\alpha_n}[{}_{RL}D^{\beta}u(s)]]...]] = \Upsilon(t) + \sum_{j=1}^{\infty} \Phi_j(t) I^{\zeta}\Psi_j(t), \quad t \in [0,1], \\ u(0) = 0, \\ {}_{C}D^{\alpha_2}[{}_{C}D^{\alpha_3}[{}_{C}D^{\alpha_4}...[{}_{C}D^{\alpha_n}[{}_{RL}D^{\beta}u(0)]]...]] = \theta, \quad \theta > 0, \\ {}_{C}D^{\alpha_3}[{}_{C}D^{\alpha_4}[{}_{C}D^{\alpha_5}...[{}_{C}D^{\alpha_n}[{}_{RL}D^{\beta}u(0)]]...]] = 0, \\ {}_{C}D^{\alpha_4}[{}_{C}D^{\alpha_5}[{}_{C}D^{\alpha_6}...[{}_{C}D^{\alpha_n}[{}_{RL}D^{\beta}u(0)]]...]] = 0, \\ \quad . \\ \quad . \\ \quad . \\ {}_{C}D^{\alpha_{n-1}}[{}_{C}D^{\alpha_n}[{}_{RL}D^{\beta}u(0)]] = 0, \\ {}_{C}D^{\alpha_n}[{}_{RL}D^{\beta}u(0)] = \int_0^{\eta} u(s)ds, \quad 0 < \eta < 1, \\ {}_{RL}D^{\beta}u(1) = {}_{RL}D^{\varphi}u(\tau), \quad 0 < \tau < 1, \end{cases} \tag{2}$$

has as an integral solution the following expression:

$$
\begin{aligned}
u(t) = {} & I^{\beta+\sum_{i=1}^{n}\alpha_i}\Upsilon(t) + \sum_{j=1}^{\infty} I^{\beta+\sum_{i=1}^{n}\alpha_i}[\Phi_j(t)I^{\zeta}\Psi_j(t)] + \frac{\theta t^{\beta+\sum_{i=2}^{n}\alpha_i}}{\Gamma(\beta+\sum_{i=2}^{n}\alpha_i+1)} + \Big[I^{\sum_{i=1}^{n}\alpha_i}\Upsilon(1) \\
& + \sum_{j=1}^{\infty} I^{\sum_{i=1}^{n}\alpha_i}[\Phi_j(1)I^{\zeta}\Psi_j(1)] + \chi_1 - I^{\varphi+\beta+\sum_{i=1}^{n}\alpha_i}\Upsilon(\tau) - \sum_{j=1}^{\infty} I^{\varphi+\beta+\sum_{i=1}^{n}\alpha_i}[\Phi_j(\tau)I^{\zeta}\Psi_j(\tau)] \\
& - \chi_2 + Q_1\iota^*\Big(\int_0^{\eta} I^{\beta+\sum_{i=1}^{n}\alpha_i}\Upsilon(s)ds + \int_0^{\eta}\sum_{j=1}^{\infty} I^{\beta+\sum_{i=1}^{n}\alpha_i}[\Phi_j(s)I^{\zeta}\Psi_j(s)]ds + \chi_3\Big)\Big] \\
& \times \frac{t^{\beta+\alpha_n}}{\Pi\Gamma(\beta+\alpha_n+1)} + \Big[\frac{Q_2\iota^*}{\Pi}\Big(I^{\sum_{i=1}^{n}\alpha_i}\Upsilon(1) + \sum_{j=1}^{\infty} I^{\sum_{i=1}^{n}\alpha_i}[\Phi_j(1)I^{\zeta}\Psi_j(1)] + \chi_1 \\
& + I^{\varphi+\beta+\sum_{i=1}^{n}\alpha_i}\Upsilon(\tau) + \sum_{j=1}^{\infty} I^{\varphi+\beta+\sum_{i=1}^{n}\alpha_i}[\Phi_j(\tau)I^{\zeta}\Psi_j(\tau)] + \chi_2 + \big(\frac{Q_2Q_1\iota^{*2}}{\Pi} - \iota^*\big)\Big) \\
& \times \Big(\int_0^{\eta} I^{\beta+\sum_{i=1}^{n}\alpha_i}\Upsilon(s)ds + \int_0^{\eta}\sum_{j=1}^{\infty} I^{\beta+\sum_{i=1}^{n}\alpha_i}[\Phi_j(s)I^{\zeta}\Psi_j(s)]ds + \chi_3\Big)\Big]\frac{t^{\beta}}{\Pi\Gamma(\beta+1)},
\end{aligned}
\tag{3}
$$

where,

$$
\chi_1 = \frac{\theta}{\Gamma(\sum_{i=2}^{n}\alpha_i+1)}, \quad \chi_2 = \frac{\theta\tau^{\varphi+\beta+\sum_{i=2}^{n}\alpha_i}}{\Gamma(\varphi+\beta+\sum_{i=2}^{n}\alpha_i+1)}, \quad \chi_3 = \frac{\theta\eta^{\beta+\sum_{i=2}^{n}\alpha_i+1}}{\Gamma(\beta+\sum_{i=2}^{n}\alpha_i+2)},
$$

$$
Q_1 = \frac{\tau^{\varphi+\beta}}{\Gamma(\varphi+\beta+1)} - 1, \quad Q_2 = 1 - \frac{\eta^{\beta+\alpha_n+1}}{\Gamma(\beta+\alpha_n+2)}, \quad \iota^* = \frac{\Gamma(\beta+2)}{\eta^{\beta+1}},
$$

$$
\Pi = \iota^*Q_1Q_2 + \frac{\tau^{\beta+\alpha_n+1}}{\Gamma(\beta+\alpha_n+2)} - \frac{1}{\Gamma(\alpha_n+1)},
$$

and

$$
\Big(\iota^*Q_1Q_2\Gamma(\alpha_n+1)+1\Big)\Gamma(\beta+\alpha_n+2) \neq \tau^{\beta+\alpha_n+1}\Gamma(\alpha_n+1).
$$

Proof. We apply Lemma 2.5 and Lemma 2.6 to (1). So, we find that

$$
\begin{aligned}
u(t) &= I^{\beta+\sum_{i=1}^{n}\alpha_i}\Upsilon(t)+\sum_{j=1}^{\infty} I^{\beta+\sum_{i=1}^{n}\alpha_i}[\Phi_j(t)_{RL}I^{\zeta}\Psi_j(t)]+\frac{c_0}{\Gamma(\sum_{i=2}^{n}\alpha_i+\beta+1)}t^{\sum_{i=2}^{n}\alpha_i+\beta} \\
&+\frac{c_1}{\Gamma(\sum_{i=3}^{n}\alpha_i+\beta+1)}t^{\sum_{i=3}^{n}\alpha_i+\beta}+\frac{c_2}{\Gamma(\sum_{i=4}^{n}\alpha_i+\beta+1)}t^{\sum_{i=4}^{n}\alpha_i+\beta}+... \\
&+\frac{c_{n-2}}{\Gamma(\alpha_n+\beta+1)}t^{\alpha_n+\beta}+\frac{c_{n-1}}{\Gamma(\beta+1)}t^{\beta}+c_n t^{\beta-1}.
\end{aligned} \tag{4}
$$

The initial conditions allow us to write:

$$
\begin{gathered}
u(0)=0 \Rightarrow c_n=0 \\
{}_CD^{\alpha_2}[{}_CD^{\alpha_3}[{}_CD^{\alpha_4}...[{}_CD^{\alpha_n}[{}_{RL}D^{\beta}u(0)]]...]]=\theta \Rightarrow c_0=\theta, \\
{}_CD^{\alpha_3}[{}_CD^{\alpha_4}[{}_CD^{\alpha_5}...[{}_CD^{\alpha_n}[{}_{RL}D^{\beta}u(0)]]...]]=0 \Rightarrow c_1=0, \\
{}_CD^{\alpha_4}[{}_CD^{\alpha_5}[{}_CD^{\alpha_6}...[{}_CD^{\alpha_n}[{}_{RL}D^{\beta}u(0)]]...]]=0 \Rightarrow c_2=0, \\
. \\
. \\
. \\
{}_CD^{\alpha_{n-1}}[{}_CD^{\alpha_n}[{}_{RL}D^{\beta}u(0)]]=0 \Rightarrow c_{n-3}=0.
\end{gathered} \tag{5}
$$

Thanks to the following conditions

$$
\begin{aligned}
{}_CD^{\alpha_n}[{}_{RL}D^{\beta}u(0)] &= \int_0^{\eta}u(s)ds, \quad 0<\eta<1, \\
{}_{RL}D^{\beta}u(1) &= {}_{RL}D^{\varphi}u(\tau),
\end{aligned}
$$

we end the proof.

□

We shall use fixed point theory to study the above problem. So, we consider the space

$$X := \{x \in C(J,\mathbb{R}), D^{\gamma}x \in C(J,\mathbb{R})\},$$

and the norm:

$$\|x\|_X = Max\{\|x\|_{\infty}\ , \|D^{\gamma}x\|_{\infty}\},$$

where,

$$\|x\|_{\infty}=\sup_{t\in J}|x(t)|\ , \|D^{\gamma}x\|_{\infty}=\sup_{t\in J}|D^{\gamma}x(t)|.$$

Then, we take the nonlinear operator $H_{\beta^*}: X \to X$ that is defined by:

$$
\begin{aligned}
H_{\beta^*}u(t) &= I^{\beta^*+\sum_{i=1}^{n}\alpha_i}\Upsilon_u^*(t) + \sum_{j=1}^{\infty} I^{\beta^*+\sum_{i=1}^{n}\alpha_i}[\Phi_j(t)I^{\zeta}(\Psi_j)_u^*(t)] + \frac{\theta t^{\beta^*+\sum_{i=2}^{n}\alpha_i}}{\Gamma(\beta^*+\sum_{i=2}^{n}\alpha_i+1)} \\
&+\Big[I^{\sum_{i=1}^{n}\alpha_i}\Upsilon_u^*(1) + \sum_{j=1}^{\infty} I^{\sum_{i=1}^{n}\alpha_i}[\Phi_j(1)I^{\zeta}(\Psi_j)_u^*(1)] + \chi_1 - I^{\varphi+\beta+\sum_{i=1}^{n}\alpha_i}\Upsilon_u^*(\tau) \\
&-\sum_{j=1}^{\infty} I^{\varphi+\beta+\sum_{i=1}^{n}\alpha_i}[\Phi_j(\tau)I^{\zeta}(\Psi_j)_u^*(\tau)] - \chi_2 + Q_1\iota^*\Big(\int_0^{\eta} I^{\beta+\sum_{i=1}^{n}\alpha_i}\Upsilon_u^*(s)ds \\
&+\int_0^{\eta}\sum_{j=1}^{\infty} I^{\beta+\sum_{i=1}^{n}\alpha_i}[\Phi_j(s)I^{\zeta}(\Psi_j)_u^*(s)]ds + \chi_3\Big)\Big]\frac{t^{\beta^*+\alpha_n}}{\Pi\Gamma(\beta^*+\alpha_n+1)} + \Big[\frac{Q_2\iota^*}{\Pi} \\
&\times\Big(I^{\sum_{i=1}^{n}\alpha_i}\Upsilon_u^*(1) + \sum_{j=1}^{\infty} I^{\sum_{i=1}^{n}\alpha_i}[\Phi_j(1)I^{\zeta}(\Psi_j)_u^*(1)] + \chi_1 + I^{\varphi+\beta+\sum_{i=1}^{n}\alpha_i}\Upsilon_u^*(\tau) \\
&+\sum_{j=1}^{\infty} I^{\varphi+\beta+\sum_{i=1}^{n}\alpha_i}[\Phi_j(\tau)I^{\zeta}(\Psi_j)_u^*(\tau)] + \chi_2\Big) + \Big(\frac{Q_2Q_1{\iota^*}^2}{\Pi} - \iota^*\Big)\Big(\int_0^{\eta} I^{\beta+\sum_{i=1}^{n}\alpha_i} \\
&\Upsilon_u^*(s)ds + \int_0^{\eta}\sum_{j=1}^{\infty} I^{\beta+\sum_{i=1}^{n}\alpha_i}[\Phi_j(s)I^{\zeta}(\Psi_j)_u^*(s)]ds + \chi_3\Big)\Big]\frac{t^{\beta^*}}{\Pi\Gamma(\beta^*+1)},
\end{aligned}
$$

where

$$\Upsilon_u^*(t) = \Lambda_1 f(t, u(t),_{RL} D^{\gamma}u(t)) + \Lambda_2 g(t, u(t)_{RL}I^{\zeta}u(t)), \quad (\Psi_j)_u^*(t) = h_j(t, u(t)) + l_j(t).$$

Remark 2.9. *We have* $_{RL}D^{\gamma}H_{\beta}u(t) = H_{\beta-\gamma}u(t)$.

Now, we are ready to prove our main results.

3. Main Results

We need the following hypotheses:

$(A1)$: The given functions of (1) are continuous.

$(A2)$: There exist nonnegative real $\varpi_{f1}, \varpi_{f2}, \varpi_{g1}, \varpi_{g2}$ such that, for any t $\in J$, $s_i, s_i^* \in \mathbb{R}$,

$$|f(t, s_1, s_2) - f(t, {s_1}^*, {s_2}^*)| \le \varpi_{f1}|s_1 - {s_1}^*| + \varpi_{f2}|s_2 - {s_2}^*|, \tag{6}$$

$$|g(t, s_1, s_2) - g(t, {s_1}^*, {s_2}^*)| \le \varpi_{g1}|s_1 - {s_1}^*| + \varpi_{g2}|s_2 - {s_2}^*|, \tag{7}$$

and there exists positive numbers K_j such that, for any $t \in J$, $s, s^{'} \in \mathbb{R}$,

$$|h_j(t,s) - h_j(t,s^{'})| \quad \leq \quad K_j|s - s^{'}|, \tag{8}$$

and

$$\sum_{j=1}^{\infty} K_j \quad \leq \quad K. \tag{9}$$

We take:

$$\varpi_1 := Max(\varpi_{f1}, \varpi_{f2}), \;\; \varpi_2 := Max(\varpi_{g1}, \varpi_{g2}). \tag{10}$$

$(A3)$: We take: $\sum\limits_{j=1}^{\infty} \|l_j\|_\infty = O_l$.

$(A4)$: There exist real constants $\epsilon_i, \varepsilon_i \geq 0$ $(i = 0,1,2)$ such that for any $x, y \in \mathbb{R}$, we have
$|f(t,x,y)| \leq \epsilon_0 + \epsilon_1|x| + \epsilon_2|y|$,
$|g(t,x,y)| \leq \varepsilon_0 + \varepsilon_1|x| + \varepsilon_2|y|$,
and, there exist real constants $\rho_j, \rho_j^{'}$, such that for any $x \in \mathbb{R}$, we have
$|h_j(t,x)| \leq \rho_j + \rho_j^{'}|x|$ and $\sum\limits_{j=1}^{\infty} \rho_j \to \rho_0, \sum\limits_{j=1}^{\infty} \rho_j^{'} \to \rho_1$.

Also, we consider the quantity:

$$\begin{aligned} \Theta_{\beta^*} \;\; &= \;\; \Big[2\Lambda_1\varpi_1 + \Lambda_2\big(\varpi_2 + \frac{\varpi_2}{\Gamma(\zeta+1)}\big) + M_\Phi K\Big]\Big[\frac{1}{\Gamma(\sum\limits_{i=1}^{n}\alpha_i + \beta^* + 1)} + \frac{1}{\Gamma(\sum\limits_{i=1}^{n}\alpha_i + 1)} \\ &+ \;\; \frac{1}{\Gamma(\sum\limits_{i=1}^{n}\alpha_i + \beta + \varphi + 1)} + \frac{Q_1\iota^*\eta}{\Gamma(\sum\limits_{i=1}^{n}\alpha_i + 1)}\Big]\frac{1}{|\Pi|\Gamma(\beta^* + \alpha_n + 1)} + \Big[2\Lambda_1\varpi_1 \\ &+ \;\; \Lambda_2\big(\varpi_2 + \frac{\varpi_2}{\Gamma(\zeta+1)}\big) + M_\Phi K\Big]\Big[|Q_2|\iota^*\Big(\frac{1}{\Gamma(\sum\limits_{i=1}^{n}\alpha_i + 1)} + \frac{1}{\Gamma(\sum\limits_{i=1}^{n}\alpha_i + \beta + \varphi + 1)}\Big) \\ &+ \;\; |\frac{Q_1Q_2(\iota^*)^2}{\Pi} - \iota^*|\frac{\eta}{\Gamma(\sum\limits_{i=1}^{n}\alpha_i + \beta + 1)}\Big]\frac{1}{|\Pi|\Gamma(\beta^* + 1)}. \end{aligned} \tag{11}$$

The first main result is given by the following theorem:

Theorem 3.1. *Let the conditions $(Ai)_{i=1,2}$ be satisfied and $\Theta < 1$, where, $\Theta := max\left\{\Theta_\beta, \Theta_{\beta-\gamma}\right\}$. Then, the problem (1) has a unique solution on J.*

Proof. It is sufficient for us to prove that H_β is a contraction mapping.
Let $(x,y) \in X^2$. Then, we can write

$$
\begin{aligned}
\|H_{\beta^*}y-H_{\beta^*}x\|_\infty \quad \leq \quad & \Big[2\Lambda_1\varpi_1+\Lambda_2\big(\varpi_2+\frac{\varpi_2}{\Gamma(\zeta+1)}\big)+M_\Phi K\Big]\Big[\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i+\beta^*+1)} \\
& +\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i+1)}+\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i+\beta+\varphi+1)}+\frac{Q_1\iota^*\eta}{\Gamma(\sum_{i=1}^{n}\alpha_i+1)}\Big] \\
& \times\frac{1}{|\Pi|\Gamma(\beta^*+\alpha_n+1)}\|y-x\|_X \\
& +\Big[2\Lambda_1\varpi_1+\Lambda_2\big(\varpi_2+\frac{\varpi_2}{\Gamma(\zeta+1)}\big)+M_\Phi K\Big]\Big[|Q_2|\iota^*\Big(\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i+1)} \\
& +\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i+\beta+\varphi+1)}\Big)+|\frac{Q_1Q_2(\iota^*)^2}{\Pi}-\iota^*|\frac{\eta}{\Gamma(\sum_{i=1}^{n}\alpha_i+\beta+1)}\Big] \\
& \times\frac{1}{|\Pi|\Gamma(\beta^*+1)}\|y-x\|_X.
\end{aligned}
\tag{12}
$$

We deduce that

$$\|H_{\beta^*}y-H_{\beta^*}x\|_\infty \quad \leq \quad \Theta_{\beta^*}\|x-y\|_X.$$

Thanks to (12), we obtain

$$
\begin{aligned}
\|H_\beta y-H_\beta x\|_\infty \quad &\leq \quad \Theta_\beta\|x-y\|_X, \\
\|H_{\beta-\gamma}y-H_{\beta-\gamma}x\|_\infty \quad &\leq \quad \Theta_{\beta-\gamma}\|x-y\|_X,
\end{aligned}
$$

and

$$\|_{RL}D^\gamma H_\beta y-_{RL}D^\gamma H_\beta x\|_\infty \quad \leq \quad \Theta_{\beta-\gamma}\|x-y\|_X.$$

Thus,

$$\|H_\beta y-H_\beta x\|_X \quad \leq \quad \Theta\|x-y\|_X.$$

The proof is thus achieved.

□

We prove also the following theorem that guarantees the existence of at least one solution.

Theorem 3.2. *Under the hypotheses* $(A1)$, $(A3)$ *and* $(A4)$, *if* $\Lambda_1(\epsilon_1+\epsilon_2)+\Lambda_2(\varepsilon_1+\varepsilon_2)+M_\Phi\rho_1<\Xi_{\beta^*}^{-1}$, *where* Ξ_{β^*} *are given by (14), then problem (1) has at least one solution* $u(t), t\in J$.

Proof. Let us prove the result by considering the following three steps:

Step 1. Firstly, we will prove that the operator $H_\beta : X\to X$ is completely continuous.
Let $\Sigma\subset X$ be bounded. Then there exist positive constants M_f, M_g, M_h, such that, for any $t\in J,\ x\in\mathbb{R}^2, y\in\mathbb{R}$, we take

$$|f(t,x)|\leq O_f,\quad |g(t,x)|\leq O_g,\sum_{j=1}^{\infty}|h_j(t,y)|\leq O_h,$$

Thus, for any $u \in X$, we observe that

$$\begin{aligned} \|H_{\beta^*}u\|_\infty &\leq \Big[\Lambda_1 O_f + \Lambda_2 O_g + M_\Phi(O_h + O_l)\Big]\Xi_{\beta^*} + \frac{\theta}{\Gamma(\beta^* + \sum_{i=2}^{n}\alpha_i + 1)} \\ &\quad + \frac{\chi_{1,2,3}}{\Pi\Gamma(\beta^* + \alpha_n + 1)} + \frac{\chi^*_{1,2,3}}{\Pi\Gamma(\beta^* + 1)} < +\infty, \end{aligned} \tag{13}$$

where,

$$\begin{aligned} \Xi_{\beta^*} &= \Bigg[\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i + \beta^* + 1)} + \frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i + 1)} + \frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i + \beta + \varphi + 1)} + \frac{Q_1\iota^*\eta}{\Gamma(\sum_{i=1}^{n}\alpha_i + 1)}\Bigg] \\ &\quad \times \frac{1}{|\Pi|\Gamma(\beta^* + \alpha_n + 1)} + \Bigg[|Q_2|\iota^*\Bigg(\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i + 1)} + \frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i + \beta + \varphi + 1)}\Bigg) \\ &\quad + |\frac{Q_1Q_2(\iota^*)^2}{\Pi} - \iota^*|\frac{\eta}{\Gamma(\sum_{i=1}^{n}\alpha_i + \beta + 1)}\Bigg]\frac{1}{|\Pi|\Gamma(\beta^* + 1)}. \end{aligned} \tag{14}$$

$$\begin{aligned} \chi_{1,2,3} &= \chi_1 + \chi_2 + Q_1\iota^*\chi_3, \\ \chi^*_{1,2,3} &= |Q_2|\iota^*\big(\chi_1 + \chi_2\big) + |\frac{Q_1Q_2(\iota^*)^2}{\Pi} - \iota^*|\chi_3. \end{aligned}$$

which yields to the following inequalities:

$$\|H_\beta u\|_\infty \leq +\infty,$$

$$\|H_{\beta-\gamma}u\|_\infty \leq +\infty,$$

and

$$\|{}_{RL}D^\gamma H_\beta u\|_\infty \leq +\infty.$$

We conclude that

$$\|H_\beta u\|_X \leq +\infty.$$

Thus, it follows from the above inequalities that the operator H_β is uniformly bounded.

Step 2. Now we show that the operator H_β is equicontinuous. For $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we obtain

$$\begin{aligned}
\|H_{\beta^*}u(t_2) \;\; - \;\; & H_{\beta^*}u(t_1)\|_\infty \\
& \leq \frac{\Lambda_1 O_f + \Lambda_2 O_g + M_\Phi(O_h + O_l)}{\Gamma(\beta^* + \sum_{i=1}^{n}\alpha_i + 1}\Bigg)\Bigg((t_2 - t_1)^{\beta^* + \sum_{i=1}^{n}\alpha_i} + |t_2^{\beta^* + \sum_{i=1}^{n}\alpha_i} \\
& -t_1^{\beta^* + \sum_{i=1}^{n}\alpha_i}|\Bigg) + \frac{\theta|t_2^{\beta^* + \sum_{i=2}^{n}\alpha_i} - t_1^{\beta^* + \sum_{i=2}^{n}\alpha_i}|}{\Gamma(\beta^* + \sum_{i=2}^{n}\alpha_i + 1)} + \Bigg(\Big[\Lambda_1 O_f + \Lambda_2 O_g + M_\Phi \\
& \times(O_h + O_l)\Big]\Bigg[\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i + 1)} + \frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i + \beta + \varphi + 1)} + \frac{Q_1\iota^*\eta}{\Gamma(\sum_{i=1}^{n}\alpha_i + 1)}\Bigg] \\
& +\chi_{1,2,3}\Bigg)\frac{|t_2^{\beta^*+\alpha_n} - t_1^{\beta^*+\alpha_n}|}{|\Pi|\Gamma(\beta^* + \alpha_n + 1)} + \Bigg(\Big[\Lambda_1 O_f + \Lambda_2 O_g + M_\Phi(O_h + O_l)\Big]\Big[|Q_2|\iota^* \\
& \times\Bigg(\frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i + 1)} + \frac{1}{\Gamma(\sum_{i=1}^{n}\alpha_i + \beta + \varphi + 1)}\Bigg) + |\frac{Q_1 Q_2(\iota^*)^2}{\Pi} - \iota^*| \\
& \times\frac{\eta}{\Gamma(\sum_{i=1}^{n}\alpha_i + \beta + 1)}\Big] + \chi^*_{1,2,3}\Bigg)\frac{|t_2^{\beta^*} - t_1^{\beta^*}|}{|\Pi|\Gamma(\beta^* + 1)}.
\end{aligned} \tag{15}$$

Thanks to the above inequality, we can state that $\|H_\beta u(t_2) - H_\beta u(t_1)\|_X \to 0$ as $s_2 - s_1 \to 0$. Therefore, $H_\beta : X \to X$ is completely continuous by application of the Arzelá-Ascoli theorem.

Step 3. We show that $A_\varrho := \{u \in X : u = \varrho\, H_\beta u, \varrho \in]0,1[\}$ is bounded.
Let $u \in A_\varrho$. Then we have $u = \varrho H_\beta u$, for some $0 < \varrho < 1$. Hence, we have

$$\begin{aligned}
\|u\|_\infty \;\; \leq \;\; & \Bigg(\Big[\Lambda_1(\epsilon_1 + \epsilon_2) + \Lambda_2(\varepsilon_1 + \varepsilon_2) + M_\Phi\rho_1\Big]\|u\|_\infty + \epsilon_0 + \varepsilon_0 + M_\Phi\rho_0\Bigg)\Xi_\beta \\
& + \frac{\theta}{\Gamma(\beta + \sum_{i=2}^{n}\alpha_i + 1)} + \frac{\chi_{1,2,3}}{|\Pi|\Gamma(\beta + \alpha_n + 1)} + \frac{\chi^*_{1,2,3}}{|\Pi|\Gamma(\beta + 1)},
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
\|_{RL}D^\gamma u\|_\infty \;\; \leq \;\; & \Bigg(\Big[\Lambda_1(\epsilon_1 + \epsilon_2) + \Lambda_2(\varepsilon_1 + \varepsilon_2) + M_\Phi\rho_1\Big]\|_{RL}D^\gamma u\|_\infty + \epsilon_0 + \varepsilon_0 + M_\Phi\rho_0\Bigg)\Xi_{\beta-\gamma} \\
& + \frac{\theta}{\Gamma(\beta - \gamma + \sum_{i=2}^{n}\alpha_i + 1)} + \frac{\chi_{1,2,3}}{|\Pi|\Gamma(\beta - \gamma + \alpha_n + 1)} + \frac{\chi^*_{1,2,3}}{|\Pi|\Gamma(\beta - \gamma + 1)}.
\end{aligned} \tag{17}$$

Therefore,

$$\|u\|_\infty \leq \frac{\left(\epsilon_0+\varepsilon_0+M_\Phi\rho_0\right)\Xi_\beta+\dfrac{\theta}{\Gamma(\beta+\sum\limits_{i=2}^{n}\alpha_i+1)}+\dfrac{\chi_{1,2,3}}{|\Pi|\Gamma(\beta+\alpha_n+1)}+\dfrac{\chi^*_{1,2,3}}{|\Pi|\Gamma(\beta+1)}}{1-\Big[\Lambda_1(\epsilon_1+\epsilon_2)+\Lambda_2(\varepsilon_1+\varepsilon_2)+M_\Phi\rho_1\Big]\Xi_\beta}, \tag{18}$$

and

$$\|_{RL}D^\gamma u\|_\infty \leq \frac{\left(\epsilon_0+\varepsilon_0+M_\Phi\rho_0\right)\Xi_{\beta-\gamma}+\dfrac{\theta}{\Gamma(\beta-\gamma+\sum\limits_{i=2}^{n}\alpha_i+1)}+\dfrac{\chi_{1,2,3}}{|\Pi|\Gamma(\beta-\gamma+\alpha_n+1)}+\dfrac{\chi^*_{1,2,3}}{|\Pi|\Gamma(\beta-\gamma+1)}}{1-\Big[\Lambda_1(\epsilon_1+\epsilon_2)+\Lambda_2(\varepsilon_1+\varepsilon_2)+M_\Phi\rho_1\Big]\Xi_{\beta-\gamma}}. \tag{19}$$

These show that A_ϱ is bounded. Thus, the operator H_β has at least one fixed point. Hence, problem (1) has at least one solution on J. The proof is complete. $\square$

In what follows, we present two examples to illustrate the validity of the main results.

Example 3.3. *We consider the following problem:*

$$\begin{cases} {}_CD^{0.5}[{}_CD^{0.8}[{}_CD^{0.9}[{}_CD^{0.6}[{}_{RL}D^{0.5}u(s)]]]]=15\Big(\frac{\cos(u(t))}{e^{t+3}}+\frac{|{}_{RL}D^{\frac{1}{2}}u(t)|}{200(1+|{}_{RL}D^{\frac{1}{2}}u(t)|)}+e^t\Big)\\ +\dfrac{20}{e^{t+4}}\left(\dfrac{|u(t)|}{(1+|u(t)|)}+\cos(I^{\frac{1}{10}}u(t))\right)\\ +\sum\limits_{j=1}^{\infty}\dfrac{6e^{-jt^2}}{(j\pi)^4}I^{\frac{1}{10}}\Big(\dfrac{e^{-jt}|u(t)|}{15j^2\big[(t^2+1)+|u(t)|\big]}+\dfrac{e^{-jt^2-2}}{j^2}\Big), \quad t\in[0,1],\\ u(0)=0,\\ {}_CD^{0.8}[{}_CD^{0.9}[{}_CD^{0.6}[{}_{RL}D^{0.5}u(0)]]]=2,\\ {}_CD^{0.9}[{}_CD^{0.6}[{}_{RL}D^{0.5}u(0)]]=0,\\ {}_CD^{0.6}[{}_{RL}D^{0.5}u(0)]=\displaystyle\int_0^{\frac{1}{2}}u(s)ds,\\ {}_{RL}D^{0.5}u(1)={}_{RL}\,D^{\frac{2}{5}}u(\tfrac{1}{10}), \end{cases} \tag{20}$$

where,

$$
\begin{aligned}
&f(t,x_1,x_2) = \tfrac{\cos(x_1)}{e^{t+3}} + \tfrac{|x_2|}{200(1+|x_2|)} + e^t,\\
&g(t,x_1,x_2) = \frac{1}{e^{t+4}}\left(\frac{|x_1|}{(1+|x_1|)} + \cos(x_2)\right),\\
&h_j(t,x) = \tfrac{e^{-jt}|x|}{15j^2\left[(t^2+1)+|x|\right]},\\
&l_j(t) = \frac{e^{-jt^2-2}}{j^2},
\end{aligned}
$$

and

$\alpha_1 = 0.5,\ \alpha_2 = 0.8,\ \alpha_3 = 0.9,\ \alpha_4 = 0.6,\ \beta = 0.5,\ \theta = 2,$

$\zeta = \frac{1}{10},\ \tau = \frac{1}{10},\ \varphi = \frac{2}{5},\ \eta = \frac{1}{2},\ \gamma = \frac{1}{2},$

$\Theta 1 = 0.6487, \quad \Theta 2 = 0.6440,$

$\Theta = max\,\{0.6487, 0.6440\} = 0.6487.$

So, thanks to Theorem 3.1, we confirm that this example has a unique solution on $[0,1]$.

Example 3.4. *As a second illustrative example, we consider the following problem:*

$$
\begin{cases}
{}_CD^{\frac{1}{2}}[{}_CD^{\frac{3}{4}}[{}_CD^{\frac{1}{3}}[{}_CD^{\frac{2}{3}}[{}_CD^{\frac{9}{10}}[{}_CD^{\frac{4}{5}}[{}_CD^{\frac{1}{4}}[{}_CD^{\frac{3}{4}}[{}_CD^{0.9}[{}_{RL}D^{\frac{5}{6}}u(s)]]]]]]]]] = \dfrac{50}{3}\\
\times\left(\frac{\cos(u(t)+D^{\frac{1}{5}}u(t))}{100e^{t^2+1}} + t^2\right) + \dfrac{30}{(t^2+1)}\left(\cos(\pi t) + \dfrac{1}{200}u(t) + \dfrac{1}{300}I^{\frac{1}{2}}u(t)\right)\\
+\displaystyle\sum_{j=1}^{\infty}\frac{12}{(j\pi)^2e^{jt^2}}I^{\frac{1}{2}}\left(\frac{u(t)}{30j^2e^{jt}} + \frac{1}{j(t^2+1)}\right), \quad t\in[0,1],\\
u(0) = 0,\\
{}_CD^{\frac{3}{4}}[{}_CD^{\frac{1}{3}}[{}_CD^{\frac{2}{3}}[{}_CD^{\frac{9}{10}}[{}_CD^{\frac{4}{5}}[{}_CD^{\frac{1}{4}}[{}_CD^{\frac{3}{4}}[{}_CD^{0.9}[{}_{RL}D^{\frac{5}{6}}u(0)]]]]]]]] = \frac{1}{2},\\
{}_CD^{\frac{1}{3}}[{}_CD^{\frac{2}{3}}[{}_CD^{\frac{9}{10}}[{}_CD^{\frac{4}{5}}[{}_CD^{\frac{1}{4}}[{}_CD^{\frac{3}{4}}[{}_CD^{0.9}[{}_{RL}D^{\frac{5}{6}}u(0)]]]]]]] = 0,\\
{}_CD^{\frac{2}{3}}[{}_CD^{\frac{9}{10}}[{}_CD^{\frac{4}{5}}[{}_CD^{\frac{1}{4}}[{}_CD^{\frac{3}{4}}[{}_CD^{0.9}[{}_{RL}D^{\frac{5}{6}}u(0)]]]]]] = 0,\\
{}_CD^{\frac{9}{10}}[{}_CD^{\frac{4}{5}}[{}_CD^{\frac{1}{4}}[{}_CD^{\frac{3}{4}}[{}_CD^{0.9}[{}_{RL}D^{\frac{5}{6}}u(0)]]]]] = 0,\\
{}_CD^{\frac{4}{5}}[{}_CD^{\frac{1}{4}}[{}_CD^{\frac{3}{4}}[{}_CD^{0.9}[{}_{RL}D^{\frac{5}{6}}u(0)]]]] = 0,\\
{}_CD^{\frac{1}{4}}[{}_CD^{\frac{3}{4}}[{}_CD^{0.9}[{}_{RL}D^{\frac{5}{6}}u(0)]]] = 0,\\
{}_CD^{\frac{3}{4}}[{}_CD^{0.9}[{}_{RL}D^{\frac{5}{6}}u(0)]] = 0,\\
{}_CD^{0.9}[{}_{RL}D^{\frac{5}{6}}u(0)] = \displaystyle\int_0^{\frac{1}{4}} u(s)ds,\\
{}_{RL}D^{\frac{5}{6}}u(1) = {}_{RL}\ D^{\frac{1}{2}}u(\frac{1}{5}),
\end{cases}
\tag{21}
$$

where,

$$
\begin{aligned}
&f(t,x_1,x_2) = \tfrac{\cos(x_1+x_2)}{e^{t^2+1}} + t^2,\\
&g(t,x_1,x_2) = \frac{1}{(t^2+1)}\left(\cos(\pi t) + \frac{1}{20}x_1 + \frac{1}{30}x_2\right),\\
&h_j(t,x) = \tfrac{x}{30j^2e^{jt}},\\
&l_j(t) = \tfrac{1}{j(t^2+1)}).
\end{aligned}
$$

For this example, we have:

$$\Theta 1 = 0.0133, \quad \Theta 2 = 0.0139,$$

$$\Theta = max\left\{0.0133, 0.0139\right\} = 0.0139.$$

Also, by Theorem 3.1, our example has a unique solution.

4. Conclusion

We have introduced a new sequential nonlinear differential problem with nonlocal integral conditions. The problem involves Riemann-Liouville integrals and convergent series on its right-hand side. It involves also n sequential Caputo derivatives combined to a Riemann Liouville derivative. An existence and uniqueness result has been established by means of Banach contraction principle. Then, by Leray-Schauder alternative fixed point theorem, another main result for the existence of one solution has been discussed. At the end, two illustrative examples have been presented to show the applicability of the main results.
In the future, we will be concerned with the stability analysis and numerical simulations of the problem.

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