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The Greatest Common Divisors and The Least Common Multiples in Neutrosophic Integers

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Abstract: As a continuation of previous studies, we give some results about the neutrosophic integers theory. We first stated that the neutrosophic real numbers are not closed according to the division operation. Later, we gave divisibility properties of neutrosophic integers. We have given properties such as the greatest common divisor for two neutrosophic integers being positive and unique. Then, we gave the Euclid's Theorem, Bezout's Theorem for neutrosophic ingers set Z[I]. It is known that these concepts are important for number theory in integers set Z. Finally, it is defined the least common multiple for neutrosophic integers. Finally, a theorem is given which enables one to easily find the least common multiple of neutrosophic integers and after a conclusion about the sign of the product of two neutrosophic integers, a theorem is given that shows the relationship of between the greatest common divisor with the least common multiple.

Nötrosofik Tamsayılarda En Büyük Ortak Bölen ve En Küçük Ortak Kat

Anahtar Kelimeler
Nötrosfik tam sayılar,
nötrosofik tamsayılarda ebob,
nötrosofik tam sayılarda ekok,
nötrosofik tam sayılar için Euclid Algoritması,
nötrosofik tam sayılar için Bezout Teoremi

Öz: Bu makalede önceki çalışmaların devamı olarak, nötrosofik tam sayılar teorisi ile ilgili bazı sonuçlar verilecektir. İlk olarak nötrosofik reel sayıların bölme işlemi altında kapalı olmadığı ifade edilmiştir. Daha sonra nötrosofik tam sayıların bölünebilme özellikleri verilmiş, iki nötrosofik tam sayının en büyük ortak böleninin pozitif ve tek olduğu gösterilmiştir. Sayılar teorisi kuramında tam sayılar için verilen Euclid ve Bezout teoremlerinin nötrosofik tam sayılar için karşılığı incelenmiştir. Son olarak iki nötrosofik tam sayının en küçük ortak katı tanımlanmış ve bu sayının nasıl bulunacağı ile ilgili sonuçlar verilmiştir. İki nötrosofik tam sayının çarpımının işareti hakkındaki incelemeden sonra en küçük ortak kat ile en büyük ortak bölen arasındaki ilişki verilmiştir.

1. Introduction

Neutrosophy is a concept that presented by F. Smarandache to deal with indeterminacy in nature and science [1]. This concept has many applications in various fields and many studies have been done in this field. The first studies in which algebraic structures were applied in neutrosophy theory is given by Kandasamy and Smarandache in [2,3]. One of the fields in which neutrosophy theory is applied is neutrosophic number theory. Neutrophic number theory is the science that studies the properties of neutrophic integers. Neutrosophic number theory was introduced in [4]. Also, in [5] and [6], the authors examined some properties of neutrosophic integers. Studies on neutrophic integers have inspired many articles. For some of these, see [7-12]. The source of inspiration for our work is the studies in [13].

In this paper, firstly, it is given divisibility properties of neutrosophic integers. Then we have given some properties on the greatest of common divisor (gcd) of neutrosophic integers, Euclid's Theorem, Bezout's Theorem. Finally, it is defined the least of common multiple (lcm) of neutrosophic integers and given a result that shows the relationship of between the gcd and the lcm.

2. Material and Method

A well-known definition and a theorem of integers are given below.

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Definition 2.1 Let $u, v \in Z$ and $u \neq 0$. It is called u divides v iff $v = uk$ for any integer k. It is denoted by $u|v$.

Theorem 2.2 Let $\alpha, \beta, \delta \in Z$. Then

i) for $\alpha \in Z - \{0\}, \alpha|\alpha$,
ii) for $\alpha \in Z - \{0\}, \alpha|0$,
iii) for any $x \in Z$, if $\alpha|\beta$, then $\alpha|\beta x$,
iv) if $\alpha|\beta$ and $\beta|\delta$, then $\alpha|\delta$,
v) for all $x, y \in Z$, if $\alpha|\beta$ and $\alpha|\delta$, then $\alpha|\beta x + \delta y$,
vi) if $\alpha x|\beta x$ for $x \neq 0$, then $\alpha|\beta$,
vii) if $\alpha|\beta$ and $\beta \neq 0$, then $|\alpha| \leq |\beta|$,
viii) if $\alpha|\beta$ and $\alpha \neq 0$, then $\frac{\beta}{\alpha}|\beta$,
ix) if $\alpha|\beta$ and $\beta|\alpha$, then $\alpha = \pm\beta$.

The set $Z[I] = \{u + vI : u, v \in Z, I^2 = I\}$ is known as the ring of the neutrosophic integers and I is called an indeterminate element.

Definition 2.3 [4] For any $\alpha, \beta \in Z[I]$, we say that $\alpha|\beta$ if there exists a $k \in Z[I]$ such that $\beta = k\alpha$.

Theorem 2.4 [6] Let $\alpha = \alpha_1 + \alpha_2 I$ and $\beta = \beta_1 + \beta_2 I$ be any two elements in $Z[I]$. In this case, $\alpha|\beta$ iff $\alpha_1|\beta_1$ and $\alpha_1 + \alpha_2|\beta_1 + \beta_2$.

Definition 2.5 [6] Let $a + bI \in Z[I]$. $a + bI$ is a positive neutrosophic number if and only if $a > 0$, $a + bI > 0$.

Definition 2.6 [4] Let $\alpha = \alpha_1 + \alpha_2 I \in Z[I]$. The conjugate and norm for α is defined by

$$\overline{\alpha} = \alpha_1 + \alpha_2 - \alpha_2 I$$

and

$$N(\alpha) = \alpha.\overline{\alpha} = \alpha_1(\alpha_1 + \alpha_2)$$

respectively.

Theorem 2.7 [4] Let $\alpha, \beta \in Z[I]$. Then

$$N(\alpha\beta) = N(\alpha)N(\beta).$$

Proposition 2.8 [4] The elements $\pm 1, \pm(1 - 2I)$ of $Z[I]$ have inverses in $Z[I]$.

Definition 2.9 [6] We say that $\alpha = \gcd(\beta, \gamma)$ if $\alpha|\beta$ and $\alpha|\gamma$ and for each divisor $\delta|\beta$ and $\delta|\gamma$, then $\delta|\alpha$. Also if $\gcd(\beta, \gamma) = 1$, then it is called that β and γ are relatively prime in $Z[I]$.

Theorem 2.10 [6, Theorem 3.7] Let $\alpha = \alpha_1 + \alpha_2 I$ and $\beta = \beta_1 + \beta_2 I \in Z[I]$. Then $m + nI = \gcd(\alpha, \beta)$ if $m = \gcd(\alpha_1, \beta_1)$ and $m + n = \gcd(\alpha_1 + \beta_1, \alpha_2 + \beta_2)$.

Theorem 2.11 [4] (Division Theorem) Let x and $y \in Z[I]$ and $N(y) \neq 0$. In this case, there exist $b, r \in Z[I]$ such that $x = by + r$, where $|N(r)| < |N(y)|$.

3. Results

The set $R[I] = \{u + vI : u, v \in R\}$ are not closed under the division. For example; $\frac{4+I}{1-2I} = 4 - 9I \in R[I]$ but there do not exist any $a + bI \in R[I]$ such that $\frac{1+3I}{2I} = a + bI$. In the following, we will answer the question "for which neutrosophic real numbers division operation is closed".

Definition 3.1 Let $\alpha + \beta I$ and $\gamma + \delta I$ be two neutrosophic real numbers and $\alpha + \beta I \neq 0$. If $\gamma + \delta I = (k + tI).(\alpha + \beta I)$ for any $k + tI \in R[I]$, then we say $\alpha + \beta I$ divides $\gamma + \delta I$ and denote $\alpha + \beta I|\gamma + \delta I$. In this case $\frac{\gamma+\delta I}{\alpha+\beta I} = k + tI \in R[I]$.

Theorem 3.2 Let $\alpha, \beta, \gamma \in Z[I]$. Then

i) $\alpha|0$ and $\alpha|\alpha$ for any $\alpha \neq 0$,
ii) if $\alpha|\beta$, then $\alpha|\beta u$ for all $u \in Z[I]$,
iii) if $\alpha|\beta$ and $\beta|\gamma$, then $\alpha|\gamma$,
iv) if $\alpha|\beta$ and $\alpha|\gamma$, then $\alpha|\beta\gamma$,
v) if $\alpha|\beta$ and $\alpha|\gamma$, then $\alpha|\beta u + \gamma v$ for all $u, v \in Z[I]$.
vi) if $\gamma|\alpha$ and $\alpha \neq 0$, then $N(\gamma) \leq N(\alpha)$,
vii) if $\gamma|\alpha, \gamma \neq 0$, then $\frac{\alpha}{\gamma}|\alpha$,
viii) if $\gamma|\alpha$ and $\alpha|\gamma$, then $\alpha = u\gamma$ where u is a unit element.

Proof. The proofs of (i)-(v) are similar in Z.
(vi) [4], Theorem 3.10.
(vii) By Theorem 2.4, $x|y$ if and only if $x_1|y_1$ and $x_1 + x_2|y_1 + y_2$. We see that

$$\frac{y}{x} = \frac{y.\bar{x}}{x.\bar{x}} = \frac{(x_1 + x_2)y_1}{(x_1 + x_2)x_1} + \frac{x_1 y_2 - x_2 y_1}{(x_1 + x_2)x_1} I.$$

By hypothesis, we have $\frac{y_1}{x_1}|y_1$ and so

$$\frac{x_1(y_1 + y_2)}{x_1(x_1 + x_2)}|y_1 + y_2$$

by Theorem 2.2 (viii). Hence we obtain $\frac{y}{x}|y$.
(viii) if $x|y$ and $y|x$, then $x = ky$ and $y = tx$ for any $k, t \in Z[I]$. Hence we have $x = (kt)x$. So it should be $kt = 1$. Then we obtain $k = t = \mp 1$ or $k = t = \mp(1 - 2I)$. So we have $y = ux$ where u is an unit element.

Proposition 3.3 The gcd of two neutrosophic integers is positive.

Proof. Let $u = a + bI, v = c + dI \in Z[I]$ and $z = m + nI = gcd(u, v)$. Then, by Theorem 2.10, $m = gcd(a, c) \in Z$ and $m + n = gcd(a + b, c + d) \in Z$. It is known that gcd of two integer is positive integer. So we have $m > 0$ and $m + n > 0$. Then we obtain $z = m + nI > 0$ by Definition 2.5.

Proposition 3.4 Let $u = a + bI \in Z[I]$. If $u > 0$ or $u < 0$, then $N(u) > 0$.

Proof. If $a + bI > 0$ or $a + bI < 0$, then $(a > 0, a + b > 0)$ or $(a < 0, a + b < 0)$ by Definition 2.5. Hence we have $N(a + bI) = a(a + b) > 0$.

By Definition 2.6 and using Theorem 3.2 (vi), we can write the following Definition:

Definition 3.5 For non-zero α and β in $Z[I]$, the gcd of α and β is a common divisor which its norm is maximal.

If z is the gcd of α and β, we have $N(z) > 0$. Unit multiples of z are $z, -z, (1-2I)z, -(1-2I)z$. These are some common divisors of x and y. We see that $N(-z) = N((1-2I)z) = N(-(1-2I)z) = -N(z) < 0$ by Proposition 3.5 (vi) in [4].

Definition 3.6 We call x and y are relatively prime when they only have unit factors in common.

Lemma 3.7 Let $r, x \in Z[I]$. If $r|x$, then $ur|x$ where u is a unit element.

Proof. Let $r|x$. Then we have $x = kr$ for any $k \in Z[I]$. We know that $u \in \{\pm 1, \pm(1-2I)\}$ and $(1-2I)^2 = 1$. Since also we can write x as $x = (-k)(-r)$ or $x = k(1-2I)(1-2I)r$ or $x = k(-1+2I)(-1+2I)r$, we get that $-r$, $(1-2I)r$, $(-1+2I)r$ divide x. Hence $ur|x$ where u is a unit element.

Lemma 3.8 Let $x = a + bI \in Z[I]$. Then only one of the numbers $x, -x, (1-2I)x, (-1+2I)x$ is a positive neutrosophic integer.

Proof. Case 1: Let $x > 0$. Then we know that $a > 0$ and $a + b > 0$. In this case, since $-x = -a - bI$ and $-a < 0, -(a+b) < 0$, we have $-x < 0$. Since $(1-2I)x = (1-2I)(a+bI) = a + (-2a-b)I$ and $a > 0, a + (-2a-b) = -(a+b) < 0$, we have $(1-2I)x$ is neither positive nor negative. Similarly since $-a < 0$, $-a + (2a+b) = a + b > 0$, $-(1-2I)x = -a + (2a+b)I$ we have $-(1-2I)x$ is neither positive nor negative.

Case 2: Let $x < 0$. Then we know that $a < 0$ and $a + b < 0$. In this case, since $-x = -a - bI$ and $-a > 0$, $-(a+b) > 0$, we have $-x > 0$.
Since $a < 0, a + (-2a-b) = -(a+b) > 0$, $(1-2I)x = a + (-2a-b)I$, we have $(1-2I)x$ is neither positive nor negative. Similarly since $-a > 0, -a + (2a+b) = a + b < 0$, $-(1-2I)x = -a + (2a+b)I$ we have $-(1-2I)x$ is neither positive nor negative.

Case3: Suppose that $x = a + bI$ is neither positive nor negative. Then a and $a + b$ are opposite sign. If $a > 0, a + b < 0$, it can be easily seen that only $(1-2I)x > 0$. If $a < 0, a + b > 0$, we easily see that only $-(1-2I)x > 0$.

Proposition 3.9 The *gcd* of two non-zero neutrosophic integers is unique.

Proof. Let v and z be gcd of neutrosophic numbers x and y. Then it is clear that $v|z$ and $z|v$. By Theorem 3.2 (viii), we have $z = uv$ where u is a unit. Since *gcd* of two neutrosophic integers is positive by Proposition 3.3 and only one of the numbers $\pm v, \pm(1-2I)v$ is a positive by Lemma 3.8, $z = uv$ where $uv > 0$ is the *gcd* of neutrosophic numbers x and y.

Theorem 3.10 (Euclid's Algorithm) Let x and $y \in Z[I]$ be non-zero and $N(x) \neq 0, N(y) \neq 0$. Define the neutrosophic integers r_i and q_i for $i \geq 1$ by repeated application of the Division Algorithm to divisors and remainders. We have

$$y = xq_1 + r_1, |N(r_1)| < |N(x)| \text{ and } N(r_1) \neq 0,$$

$$x = r_1 q_2 + r_2, |N(r_2)| < |N(r_1)| \text{ and } N(r_2) \neq 0,$$

$$r_1 = r_2 q_3 + r_3, |N(r_3)| < |N(r_2| \text{ and } N(r_3) \neq 0,$$

$$\vdots$$

$$r_{j-2} = r_{j-1} q_j + r_j, |N(r_j)| < |N(r_{j-1})| \text{ and } N(r_j) \neq 0,$$

$$r_{j-1} = r_j q_{j+1}$$

Then, for any unit element u, ur_j is positive neutrosophic integer where r_j is the non-zero last remainder and it is the gcd of x and y.

Proof. We have a decreasing sequences of positive integers such that $|N(x)| > |N(r_1)| > |N(r_2)| > \ldots$. So this sequence is finite and $r_k = 0$ for any $k \in Z^+$. Now starting from the last equation to first equation, we have $r_j|r_{j-1}, r_j|r_{j-2}, r_j|r_{j-3}, \ldots, r_j|x$ and $r_j|y$. By Lemma 3.7, $ur_j|x$ and $ur_j|y$. So ur_j is a common divisor of x and y. If z is another common divisor of x and y, we have $z|x$ and $z|y$. Hence starting from the first equation to last equation, we have $z|r_1, z|r_2, \ldots, z|r_j$. By Theorem 3.2 (ii), $z|ur_j$. Then, by Lemma 3.8, for any unit element uur_j is positive neutrosophic integer where r_j is the non-zero last remainder and it is the gcd of x and y.

In the division operation in Z, the quotient and remainder are unique. But as we will see in the following example, the quotient and remainder are not unique in division in $Z[I]$.

Example 3.11 We apply the Division Theorem to the numbers $y = 4 + 5I$ and $x = 6 + 9I$. Since $N(4 + 5I) = 36$ and $N(6 + 9I) = 90$, we can write as $x = by + k$ such that $|N(k)| < |N(y)|$. Consider the ratio $\frac{x}{y}$ and rationalize the denominator:

$$\frac{x}{y} = \frac{x.\overline{y}}{y.\overline{y}} = \frac{54 + 6I}{36} = \frac{54}{36} + \frac{6}{36}I \cong 1{,}5 + (0{,}16)I$$

The neutrosophic integers around $1{,}5 + (0{,}16)I$ is the numbers $1 + 0I, 2 + 0I, 1 + I$ and $2 + I$ in the coordinate plane (see [5]). If we choose $q_1 = 1 + 0I, q_2 = 2 + 0I, q_3 = 1 + I$ and $q_4 = 2 + I$, then we can write

$$\underbrace{6+9I}_{x} = \underbrace{1}_{q_1}.\underbrace{(4+5I)}_{y} + \underbrace{2+4I}_{r_1}, \quad |N(r_1)| = 12 < |N(y)| = 36,$$

$$\underbrace{6+9I}_{x} = \underbrace{2}_{q_2}.\underbrace{(4+5I)}_{y} + \underbrace{-2-I}_{r_2},$$

$$|N(r_2)| = 6 < |N(y)| = 36,$$

$$\underbrace{6+9I}_{x} = \underbrace{(1+I)}_{q_3}.\underbrace{(4+5I)}_{y} + \underbrace{2-5I}_{r_3},$$

$$|N(r_3)| = 6 < |N(y)| = 36,$$

$$\underbrace{6+9I}_{x} = \underbrace{(2+I)}_{q_4}.\underbrace{(4+5I)}_{y} + \underbrace{-2-10I}_{r_4},$$

$$|N(r_4)| = 24 < |N(y)| = 36$$

Also it can be another equalities satisfying $x = qy + r$ such that $|N(r)| < |N(y)|$.

Example 3.12 Let us find gcd of $4 + 5I$ and $6 + 9I$. By Division Algorithm and Euclid Algorithm, we have

$$6 + 9I = (4 + 5I).2 - 2 - I,$$
$$|N(-2 - I)| = 6 < |N(4 + 5I)| = 36$$
$$4 + 5I = (-2 - I)(-2 - I) + 0.$$

Hence, since $-2 - I < 0$, we have $-(-2 - I) = 2 + I$ is gcd of $4 + 5I$ and $6 + 9I$.

Secondly, since

$$6 + 9I = (4 + 5I).1 + 2 + 4I,$$
$$|N(2 + 4I)| = 12 < |N(4 + 5I)| = 36$$
$$4 + 5I = (2 + 4I).1 + 2 + I,$$
$$|N(2 + I)| = 6 < |N(2 + 4I)| = 12$$
$$2 + 4I = (2 + I)(1 + I) + 0,$$

we get that $2 + I$ is gcd of $4 + 5I$ and $6 + 9I$.

Thirdly, since

$$6 + 9I = (4 + 5I).(1 + I) + 2 - 5I,$$
$$|N(2 - 5I)| = 6 < |N(4 + 5I)| = 36$$
$$4 + 5I = (2 - 5I)(2 - 5I) + 0,$$

and since $2 - 5I$ is not a positive neutrosophic integer, we obtain that $(1 - 2I)(2 - 5I) = 2 + I$ is gcd of $4 + 5I$ and $6 + 9I$.

Example 3.13 The conjugate of the number $4 + 5I$ is $9 - 5I$. Since

$$4 + 5I = (9 - 5I).2I + 4 - 3I,$$
$$|N(4 - 3I)| = 4 < |N(9 - 5I)| = 36$$
$$9 - 5I = (4 - 3I)(2 + I) + 1, |N(1)| = 1 < |N(4 - 3I)| = 4,$$
$$4 - 5I = (4 - 5I).1 + 0,$$

we get $gcd(4 + 5I, 9 - 5I) = 1$. So they are relatively prime in $Z[I]$.

Theorem 3.14 (Bezout's Theorem) For $0 \neq x$ and $0 \neq y \in Z[I]$ and $N(x) \neq 0, N(y) \neq 0$, if $gcd(x, y) = z$, then $z = xa + yb$ for some $a, b \in Z[I]$.

Proof. By back-substitution in Euclid Algorithm, we can find $a, b \in Z[I]$ such that $r_j = xa + yb$. If r_j is the gcd of x and y, proof is clear. If r_j is not gcd but ur_j where u is a unit element is gcd of x and y, then we get $ur_j = x(ua) + y(ub)$ multiplying the above equality by u. So proof is clear.

Corollary 3.15 For $0 \neq \alpha$ and $0 \neq \beta \in Z[I]$ and $N(\alpha) \neq 0, N(\beta) \neq 0$, α and β are relatively prime iff $\alpha a + \beta b = 1$ for some $a, b \in Z[I]$.

Proof. Since α and β are relatively prime, we know that $gcd(\alpha, \beta) = 1$. Hence, by Theorem 3.14, $\alpha a + \beta b = 1$ for some $a, b \in Z[I]$. Conversely, let $\alpha a + \beta b = 1$ for some $a, b \in Z[I]$. If w is a joint divisor of α and β, then, by Theorem 3.2 (v),we have$w|\alpha a + \beta b = 1$. Hence we get $gcd(\alpha, \beta) = 1$.

Theorem 3.16 For α, β and $\gamma \in Z[I]$, if $\alpha|\beta\gamma$ and $gcd(\alpha, \beta) = 1$, then $\alpha|\gamma$.

Proof. Let $\alpha|\beta\gamma$ and $gcd(\alpha, \beta) = 1$. By Corollary 3.15, we have $\alpha a + \beta b = 1$ for some $a, b \in Z[I]$. Multiplying γ, we have $\alpha\gamma a + \beta\gamma b = \gamma$. Then, by Theorem 3.2 (v), since $\alpha|\alpha\gamma a$ and $\alpha|\beta\gamma b$, we have $\alpha|\alpha\gamma a + \beta\gamma b = \gamma$.

Example 3.17 Consider $x = 4 + 5I$ and $y = 9 - 5I$. We know that $gcd(x, y) = 1$ from Example 3.13. Hence using the equalities in Example 3.13, we have

$$\begin{aligned} 1 &= 9 - 5I - (4 - 3I)(2 + I) \\ &= 9 - 5I - (4 + 5I - (9 - 5I).2I)(2 + I) \\ &= -(2 + I)x + (1 + 6I)y \end{aligned}$$

Example 3.18 For the neutrosophic numbers $x = 4 + 5I$ and $y = 6 + 9I$ in Example 3.12, if we use the equality $6 + 9I = (4 + 5I).(1 + I) + 2 - 5I$, we have $2 - 5I = 6 + 9I - (4 + 5I).(1 + I)$. Multiplying both sides by $1 - 2I$, we have, since $(1 - 2I)(2 - 5I) = 2 + I$,

$$\begin{aligned} gcd(4 + 5I, 6 + 9I) &= 2 + I \\ &= (6 + 9I)(1 - 2I) - (4 + 5I)(1 + I)(1 - 2I) \\ &= x.(1 - 2I) + y.(-1 + 3I) \end{aligned}$$

Definition 3.19 (LCM) For α and $\beta \in Z[I]$, if $\alpha|\gamma$ and $\beta|\gamma$, γ is called a joint multiple of α and β. The smallest of the positive joint multiples of α and β is called the lcm of α and β.

Theorem 3.20 For $x = x_1 + x_2I, y = y_1 + y_2I \in Z[I]$ and $x_1 \neq 0$, $x_1 + x_2 \neq 0$, $y_1 \neq 0$ and $y_1 + y_2 \neq 0$, $w = w_1 + w_2I = lcm(x, y)$ if and only if $w_1 = lcm(x_1, y_1)$ and $w_1 + w_2 = lcm(x_1 + x_2, y_1 + y_2)$.

Proof. Let $w = w_1 + w_2I = lcm(x, y)$. We know that $x|w$ and $y|w$. If $v = v_1 + v_2I$ is another common multiple of x and y then $w|v$. Then $x_1|w_1$, $x_1 + x_2|w_1 + w_2$, $y_1|w_1$, $y_1 + y_2|w_1 + w_2$, $w_1|v_1$, $w_1 + w_2|v_1 + v_2$ by Theorem 2.4. In this case, we have $w_1 = lcm(x_1, y_1)$ and $w_1 + w_2 = lcm(x_1 + x_2, y_1 + y_2)$. Conversely, let $w_1 = lcm(x_1, y_1)$ and $w_1 + w_2 =$

$lcm(x_1 + x_2, y_1 + y_2)$. Then we have $x_1|w_1$, $y_1|w_1$, $x_1 + x_2|w_1 + w_2, y_1 + y_2|w_1 + w_2$. Hence we get $x|w$ and $y|w$ and so w is a common multiple of x with y. Now let v be another common multiple of x with y. Then $x|v$ and $y|v$. Since $x_1|v_1, y_1|v_1, x_1 + x_2|v_1 + v_2$, $y_1 + y_2|v_1 + v_2, v_1$ is a common multiple of x_1 and y_1 and $v_1 + v_2$ is a common multiple of $x_1 + x_2$ and $y_1 + y_2$. Since $w_1 = lcm(x_1, y_1)$ and $w_1 + w_2 = lcm(x_1 + x_2, y_1 + y_2)$, we have $w_1|v_1, w_1 + w_2|v_1 + v_2$. So we obtain $w|v$ and $w = w_1 + w_2 I = lcm(x, y)$.

Example 3.21 Consider $x = 2 + 4I, y = 3 + I$. Since $lcm(2,3) = 6 = w_1$ and $lcm(2 + 4,3 + 1) = 12 = w_1 + w_2$, we have $w = w_1 + w_2 I = 6 + 6I = lcm(2 + 4I, 3 + I)$.

We remember the coordinate system for $Z[I]$ in [5]: Let $x \in Z[I]$. We know that $x > 0$ in Region 1, $x < 0$ in Region 3, x is neither positive nor negative in Region 2 and 4.

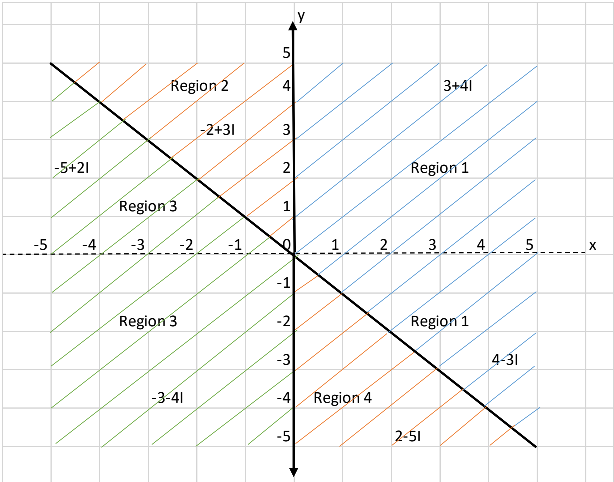

Figure 1. Neutrosophic integers

Theorem 3.22 Let $\alpha, \beta \in Z[I]$.
i) If α and β have the same sign, then $\alpha\beta > 0$,
ii) if α and β have the opposite sign, then $\alpha\beta < 0$
iii) if α and β are neither positive nor negative and are in the same region, then $\alpha\beta > 0$
iv) if α and β are neither positive nor negative and are in the different region, then $\alpha\beta < 0$
v) if only one of α and β is positive or negative and the other is neither positive nor negative, then $\alpha\beta$ is neither positive nor negative.

Proof. Let $\alpha = \alpha_1 + \alpha_2 I$ and $\beta = \beta_1 + \beta_2 I$. By Definition 2.5, we know that

- $\alpha = \alpha_1 + \alpha_2 I > 0$ iff $\alpha_1 > 0, \alpha_1 + \alpha_2 > 0$,
- $\alpha = \alpha_1 + \alpha_2 I < 0$ iff $\alpha_1 < 0, \alpha_1 + \alpha_2 < 0$,
- $\alpha = \alpha_1 + \alpha_2 I$ is neither negative nor positive iff $\alpha_1 > 0,\ \alpha_1 + \alpha_2 < 0$ or $\alpha_1 < 0,\ \alpha_1 + \alpha_2 > 0$.

Since $\alpha\beta = \alpha_1\beta_1 + (\alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_2\beta_2)I$, we should investigate the signs of $\alpha_1\beta_1$ and $\alpha_1\beta_1 + \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_2\beta_2 = (\alpha_1 + \alpha_2)(\beta_1 + \beta_2)$.

i) Let $\alpha = \alpha_1 + \alpha_2 I$ and $\beta = \beta_1 + \beta_2 I$ have the same sign. Then $\alpha_1, \alpha_1 + \alpha_2, \beta_1, \beta_1 + \beta_2$ has same sign. Hence we get $\alpha_1\beta_1 > 0, (\alpha_1 + \alpha_2)(\beta_1 + \beta_2) > 0$. So $\alpha\beta > 0$.

ii) Let $\alpha = \alpha_1 + \alpha_2 I$ and $\beta = \beta_1 + \beta_2 I$ have the opposite sign. Then we see that $\alpha_1, \alpha_1 + \alpha_2$ are positive and $\beta_1, \beta_1 + \beta_2$ are negative or α_1, $\alpha_1 + \alpha_2$ are negative and β_1, $\beta_1 + \beta_2$are positive . Hence we get $\alpha_1\beta_1 < 0, (\alpha_1 + \alpha_2)(\beta_1 + \beta_2) < 0$. So $\alpha\beta < 0$.

iii) Let $\alpha = \alpha_1 + \alpha_2 I$ and $\beta = \beta_1 + \beta_2 I$ are neither positive nor negative and are in the same region. Then we see that α_1, β_1 are positive and $\alpha_1 + \alpha_2, \beta_1 + \beta_2$ are negative or α_1, β_1 are negative and $\alpha_1 + \alpha_2, \beta_1 + \beta_2$ are positive. Hence we get $\alpha_1\beta_1 > 0$, $(\alpha_1 + \alpha_2)(\beta_1 + \beta_2) > 0$. So $\alpha\beta > 0$.

iv) Let $\alpha = \alpha_1 + \alpha_2 I$ and $\beta = \beta_1 + \beta_2 I$ are neither positive nor negative and are in the different region. Then we see that $\alpha_1, \beta_1 + \beta_2$ are positive and $\alpha_1 + \alpha_2$ and β_1 are negative or $\alpha_1 + \alpha_2$, β_1 are positive and α_1, $\beta_1 + \beta_2$ are negative. Hence we get $\alpha_1\beta_1 < 0$, $(\alpha_1 + \alpha_2)(\beta_1 + \beta_2) < 0$. So $\alpha\beta < 0$.

v) Let only one of $\alpha = \alpha_1 + \alpha_2 I$ and $\beta = \beta_1 + \beta_2 I$ be positive or negative and the other neither positive nor negative. Since $\alpha\beta = \alpha_1\beta_1 + (\alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_2\beta_2)I$ we should investigate the signs of $\alpha_1\beta_1$ and $\alpha_1\beta_1 + \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_2\beta_2 = (\alpha_1 + \alpha_2)(\beta_1 + \beta_2)$.

Then we have eight case:
. $\alpha_1 > 0, \beta_1 > 0,\ \alpha_1 + \alpha_2 > 0, \beta_1 + \beta_2 < 0$
$\Longrightarrow \alpha\beta$ is neither positive nor negative,
. $\alpha_1 > 0, \beta_1 < 0, \alpha_1 + \alpha_2 > 0, \beta_1 + \beta_2 > 0$
$\Longrightarrow \alpha\beta$ is neither positive nor negative,
. $\alpha_1 > 0, \beta_1 > 0, \alpha_1 + \alpha_2 < 0, \beta_1 + \beta_2 > 0$
$\Longrightarrow \alpha\beta$ is neither positive nor negative,
. $\alpha_1 > 0, \beta_1 < 0, \alpha_1 + \alpha_2 < 0, \beta_1 + \beta_2 < 0$
$\Longrightarrow \alpha\beta$ is neither positive nor negative,
. $\alpha_1 < 0, \beta_1 > 0,\ \alpha_1 + \alpha_2 > 0, \beta_1 + \beta_2 > 0$
$\Longrightarrow \alpha\beta$ is neither positive nor negative,
. $\alpha_1 < 0, \beta_1 < 0, \alpha_1 + \alpha_2 > 0, \beta_1 + \beta_2 < 0$
$\Longrightarrow \alpha\beta$ is neither positive nor negative,
. $\alpha_1 < 0, \beta_1 > 0, \alpha_1 + \alpha_2 < 0, \beta_1 + \beta_2 < 0$
$\Longrightarrow \alpha\beta$ is neither positive nor negative,
. $\alpha_1 < 0, \beta_1 < 0,\ \alpha_1 + \alpha_2 < 0, \beta_1 + \beta_2 > 0$
$\Longrightarrow \alpha\beta$ is neither positive nor negative.

Theorem 3.23 For neutrosophic integers α and β,

$$\gcd(\alpha, \beta).\mathrm{lcm}(\alpha, \beta) = \alpha\beta \text{ if } \alpha\beta > 0$$

and

$$\gcd(\alpha, \beta).\mathrm{lcm}(\alpha, \beta) = -\alpha\beta \text{ if } \alpha\beta < 0.$$

Proof. Let $\alpha\beta > 0$. Denote $\gcd(\alpha, \beta) = \delta$ and $\frac{\alpha\beta}{\delta} = m$. Then since $\delta|\beta$ and$\delta|\alpha$, we have $\beta = \delta t$ and $\alpha = \delta k$ for any $k, t \in Z[I]$. It is clear that $gcd(k, t) = 1$, otherwise $\gcd(\alpha, \beta) > \delta$. Hence since $m = \frac{\alpha\beta}{\delta} = \frac{\delta k\beta}{\delta} = k\beta$ and $m = \frac{\alpha\beta}{\delta} = \frac{\alpha\delta t}{\delta} = \alpha t$, we get $\alpha|m$ and $\beta|m$.

Hence m is a joint multiple of α and β. Now if another joint multiple of α and β is n, then since $\alpha|n$ and $\beta|n$, we have $n = \alpha r$ and $n = \beta s$ for any $r, s \in Z[I]$. In this case, since $\alpha r = \beta s$ and $\delta kr = \delta ts$, we have $kr = ts$. Hence we see that $k|ts$. Since $gcd(k,t) = 1$, by Theorem 3.16, we get $k|s$. Hence $s = kl$ for any $l \in Z[I]$. Then since $n = \beta s = \beta kl = ml$, we obtain $m|n$. Therefore $m = lcm(\alpha, \beta)$, by the equality $\frac{\alpha\beta}{\delta} = m$. We have $\gcd(\alpha,\beta).\mathrm{lcm}(\alpha,\beta) = \alpha\beta$. If $\alpha\beta < 0$, then since or $-\alpha\beta = \alpha(-\beta)$ or $-\alpha\beta = (-\alpha)\beta$, taking $-\alpha\beta = ab > 0$, the proof can be easily proved.

Example 3.24 Consider Example 3.21. We see that $gcd(x,y) = 1 + I$ and $xy = 6 + 18I > 0$. Since $lcm(x,y) = 6 + 6I$, we have

$$\begin{aligned} gcd(x,y).\,lcm(x,y) &= (1+I)(6+6I) \\ &= 6 + 18I \\ &= (2+4I)(3+I) \\ &= xy \end{aligned}$$

4. Discussion and Conclusion

In this study, as a continuation of previous studies [4, 5], we gave some results about the neutrosophic integers theory. We first stated that the neutrosophic real numbers are not closed according to the division operation. Using known properties on integers and properties given on Gaussian integers ([13]), we gave divisibility properties of neutrosophic integers. We defined the gcd of two neutrosophic integers and proved it is positive and unique. Then, we gave the Euclid's Theorem, Bezout's Theorem for neutrosophic ingers set Z[I] which it is an important concept in number theory in the integers set Z. Finally, it is defined the lcm of two neutrosophic integers. A theorem is given which enables one to easily find the lcm of two neutrosophic integers. After a conclusion about the sign of the product of two neutrosophic integers, a theorem is given that shows the relationship of between the gcd and the lcm.

Declaration of Ethical Code

In this paper, we undertake that all of the rules required to be followed within the scope of the "Higher Education Institutions Scientific Research and Publication Ethics Directive" are complied with, and that none of the actions stated under the heading "Actions Against Scientific Research and Publication Ethics" are not carried out.

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