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Araştırma Makalesi - Research Article

The Convergence of Some Spectral Characteristics on the Convergent Series

Yakınsak Seriler Üzerinde Bazı Spektral Karakteristiklerin Yakınsaması

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ABSTRACT

In this study, convergence properties of spectral, numerical and Crawford gap functions via convergences of Hilbert space operator series in difference and ratio cases are investigated. Obtained results have been applied to some classes continuous functions of the operators.

Keywords- Operator Norm, Spectral Radius, Numerical Radius, Crawford Number

ÖZ

Bu çalışmada, fark ve oran durumlarında yakınsak Hilbert uzay operatör serileri üzerinden spektral, sayısal ve Crawford boşluk fonksiyonlarının yakınsama özellikleri incelenmiştir. Elde edilen sonuçlar operatörlerin bazı sürekli fonksiyon sınıflarına uygulanmıştır.

Anahtar Kelimeler- Operatör Normu, Spektral Yarıçap, Sayısal Yarıçap, Crawford Sayısı

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I. INTRODUCTION

In spectral theory of linear operators, obtaining the spectrum set, the numerical range set of a given operator and calculating spectral radii, numerical radii and Crawford number are main questions. Generally, finding the set of spectrums and the numerical range of non-normal linear bounded operators is theoretically and technically quite difficult.

Throughout this paper, H and $L(H)$ denote any complex Hilbert space with $(\cdot,\cdot)$ is the inner product and $\|\cdot\|$ is its corresponding norm on H and the Banach algebra of linear bounded operators in H, respectively.

In the literature, Gelfand formula is the only one formula used to calculate the spectral radius $r(A) = \sup\{|\lambda|: \lambda \in \sigma(A)\}$ of linear bounded operator $A \in L(H)$. The following is the Gelfand formula:

$$r(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} \quad [1].$$

Note that for a linear bounded normal operator A in H the relation $r(A) = \|A\|$ is true (see [2]).

It is an easy consequence that if $A, B \in L(H)$ are commutative operators, then

$$r(A+B) \leq r(A) + r(B) \quad [2].$$

Recall that the numerical radius of $A \in L(H)$ is defined by

$$w(A) = \sup_{\|x\|=1} |(Ax, x)|.$$

It is known that

$$w(A) = \sup_{t\in\mathbb{R}} \|\mathrm{Re}(e^{it}A)\| = \sup_{t\in\mathbb{R}} \|\mathrm{Im}(e^{it}A)\|$$

(see, e.g. [3]). It is obvious that the function $w(\cdot)$ defines a norm on $L(H)$, which is equivalent to the usual operator norm $\|\cdot\|$. Indeed, for every $A \in L(H)$ the following inequality holds:

$$\frac{\|A\|}{2} \leq w(A) \leq \|A\|. \tag{1}$$

Moreover, for the linear normal bounded operator A the relation $w(A) = \|A\|$ is true (see [2]).

It is well known that for every two operators $A, B \in L(H)$

$$w(A+B) \leq w(A) + w(B) \tag{2}$$

is valid (see [2]).

We refer the reader to [2, 4] for the other basic information and results for the numerical radius. Furthermore, developments on the numerical radius inequalities (1) and (2) can be seen in [3, 5-9] and references there in.

Furthermore, remember that the following spectral inclusion holds $\sigma(A) \subset \overline{W(A)}$ for the spectrum set $\sigma(A)$ and numerical range $W(A)$ of any $A \in L(H)$ (see [2, 4] for more information).

For $A \in L(H)$ the Crawford number of A is defined by

$$c(A) = \inf\{|\lambda|: \lambda \in W(A)\}.$$

It is easily seen that the following inequality holds for every $A \in L(H)$:

$$0 \leq c(A) \leq r(A) \leq w(A) \leq \|A\|.$$

Throughout this paper, for $A \in L(H)$ the spectral gap, the numerical gap and the Crawford gap functions in difference cases will be denoted by

$$\mathrm{g}_r(A) := \|A\| - r(A), \mathrm{g}_r(A): L(H) \to [0,\infty),$$

$$\mathrm{g}_w(A) := \|A\| - w(A), \mathrm{g}_w(A): L(H) \to [0,\infty),$$

$$\mathrm{g}_c(A) := \|A\| - c(A), \mathrm{g}_c(A): L(H) \to [0,\infty),$$

respectively.

Similarly, for $A \in L(H)$ and $A \neq 0$, the spectral gap, the numerical gap and the Crawford gap functions in ratio cases will be denoted by $q_r(T) = \frac{r(T)}{\|T\|}, q_w(T) = \frac{w(T)}{\|T\|}$ and $q_c(T) = \frac{c(T)}{\|T\|}$, respectively [10]. The similar problems for square matrices have been investigated in [11].

Demuth's open problem in 2015 and the works of Kittaneh and his researcher group in this area had a significant impact on forming the subject discussed in this paper (see [8, 9, 12]).

Some studies related to this area can be found in [13-18].

This work is organized as follows: In Section 2, convergence properties of spectral, numerical and Crawford gap functions via convergences of Hilbert space operator series in difference and ratio cases have been investigated. Note that here a new inequality for difference Crawford numbers of two operator has been obtained. In Section 3, obtained results have been applied to some classes continuous functions of the operators.

II. ON THE CONVERGENCE OF SOME SPECTRAL CHARACTERISTICS ON THE CONVERGENCE OF OPERATOR SERIES

Firstly, define the uniform convergence of operator series from [19].

Definition 2.1. Let H be a Hilbert space and for any $n \geq 1$, $A_n \in L(H)$. The operator series $\sum_{n=1}^{\infty} A_n$ is said to converges uniformly to $A \in L(H)$ if for any $\varepsilon > 0$ there is some $n_0 \in \mathbb{N}$ such that for all $n > n_0$ it is true that

$$\|A - S_n\| \leq \varepsilon,$$

where $S_n = \sum_{m=1}^{n} A_m : H \to H, n \geq 1$.

Now give the following simple fact.

Remark 2.2. If the series $\sum_{n=1}^{\infty} \|A_n\|$ is convergent, then series $\sum_{n=1}^{\infty} A_n : H \to H$ uniformly converges in H.

Now we give results on the difference gaps repeatedly.

Theorem 2.3. Let $A_n \in L(H), n \geq 1$, the series $\sum_{n=1}^{\infty} A_n$ uniformly converges to some operator $A \in L(H)$ and for any $i, j \geq 1$ the operators A_i and A_j are commutative. Then

$$g_r(A) = \lim_{n\to\infty} g_r(S_n).$$

Proof. In this case it is clear that

$$AS_n = S_n A, n \geq 1.$$

Then from the subadditivity property of spectral radius

$$r(A) \leq r(A - S_n) + r(S_n),$$

$$r(S_n) \leq r(A - S_n) + r(A).$$

Since the series $\sum_{n=1}^{\infty} A_n$ is uniformly converges to A, then

$$|r(A) - r(S_n)| \leq r(A - S_n) \leq \|A - S_n\| \xrightarrow[n\to\infty]{} 0.$$

So, it is obtained that

$$r(A) = \lim_{n\to\infty} r(S_n).$$

Consequently, it is clear that

$$|g_r(A) - g_r(S_n)| = |(\|A\| - r(A)) - (\|S_n\| - r(S_n))| \leq \|A - S_n\| + |r(A) - r(S_n)| \leq 2\|A - S_n\|, n \geq 1.$$

Then, since the series $\sum_{n=1}^{\infty} A_n$ is uniformly converges to A, then we get

$$g_r(A) = \lim_{n\to\infty} g_r(S_n).$$

Theorem 2.4. If the operator series $\sum_{n=1}^{\infty} A_n$, $A_n \in L(H), n \geq 1$ uniformly converges to operator $A \in L(H)$, then

$$g_w(A) = \lim_{n\to\infty} g_w(S_n).$$

Proof. From the subadditivity property of numerical radius function it is clear that

$$|w(A) - w(S_n)| \leq w(A - S_n) \leq \|A - S_n\| \to 0, n \to \infty.$$

From this and uniform convergence of operator series $\sum_{n=1}^{\infty} A_n$ to operator A it is established that

$$w(A) = \lim_{n\to\infty} w(S_n).$$

Therefore, the following inequality

$$|g_w(A) - g_w(S_n)| \leq |\|A\| - \|S_n\|| + w(A - S_n) \leq \|A - S_n\| + \|A - S_n\| \leq 2\|A - S_n\|, n \geq 1$$

is hold. Consequently, since the series $\sum_{n=1}^{\infty} A_n$ is uniformly converges to A, then we have

$$g_w(A) = \lim_{n\to\infty} g_w(S_n).$$

Now prove the following proposition.

Lemma 2.5. For any $A, B \in L(H)$ the following relation

$$|c(A) - c(B)| \leq w(A \pm B)$$

is hold.

Proof. In this case, for any $x \in H$ with $\|x\| = 1$, the following relation

$$|(Ax, x)| = \left|\left((A + B)x, x\right) - (Bx, x)\right| \geq \left|\left((A + B)x, x\right)\right| - |(Bx, x)|$$

is true. Then from the last relation it is clear that

$$c(A) \geq c(A + B) - w(B). \tag{3}$$

Similarly, from the following inequality

$$\left|\left((A + B)x, x\right)\right| = |(Ax, x) - (Bx, x)| \geq |(Ax, x)| - |(Bx, x)|$$

satisfying for any $x \in H$ with $\|x\| = 1$, it implies that

$$c(A + B) \geq c(A) - w(B). \tag{4}$$

Consequently, from inequalities (3) and (4) it implies that

$$|c(A + B) - c(A)| \leq w(B).$$

In this case, if we take $B - A$ instead of B in the last inequality, we have

$$|c(A) - c(B)| \leq w(A - B).$$

Also, from the last relation if we take $-B$ instead of B, then we have

$$|c(A) - c(B)| \leq w(A + B).$$

In this way, the lemma's proof is complete.

Theorem 2.6. If the operator series $\sum_{n=1}^{\infty} A_n$, $A_n \in L(H), n \geq 1$ uniformly converges to operator $A \in L(H)$, then

$$c(A) = \lim_{n\to\infty} c(S_n),$$

$$g_c(A) = \lim_{n\to\infty} g_c(S_n).$$

Proof. Indeed, by Lemma 2.5, we have

$$|c(A) - c(S_n)| \leq w(A - S_n) \leq \|A - S_n\| \to 0, n \to \infty.$$

Hence the validity of claim

$$c(A) = \lim_{n\to\infty} c(S_n)$$

is clear. And also, since

$$|g_c(A) - g_c(S_n)| \leq \|A - S_n\| + |c(A) - c(S_n)| \leq 2\|A - S_n\|, n \geq 1,$$

the validity of second claim of theorem is established.

For the ratio gaps the following claim is true.

Theorem 2.7. If the operator series $\sum_{n=1}^{\infty} A_n$, $A_n \in L(H), n \geq 1$ uniformly converges to operator $A \in L(H)$ such that for any $n \geq 1$ $S_n \neq 0$ and $A \neq 0$, then the following conclusions are true

(a) If for any $n \geq 1$ $S_n A = A S_n$, then $q_r(A) = \lim_{n\to\infty} q_r(S_n)$,

(b) $q_w(A) = \lim_{n\to\infty} q_w(S_n)$,

(c) $q_c(A) = \lim_{n\to\infty} q_c(S_n)$.

Proof. In this case from Theorem 2.3, Theorem 2.4 and Theorem 2.6, it implies that

(a) $|q_r(A) - q_r(S_n)| \le \frac{r(A)+\|A\|}{\|S_n\|\|A\|}\|A - S_n\| \underset{n\to\infty}{\longrightarrow} 0$,

(b) $|q_w(A) - q_w(S_n)| \le \frac{w(A)+\|A\|}{\|S_n\|\|A\|}\|A - S_n\| \underset{n\to\infty}{\longrightarrow} 0$,

(c) $|q_c(A) - q_c(S_n)| \le \frac{c(A)+\|A\|}{\|S_n\|\|A\|}\|A - S_n\| \underset{n\to\infty}{\longrightarrow} 0$.

Example 2.8. Consider the following sequence of operators in Hilbert space of complex-valued functions $L^2(0,1)$ in form:

$$A_n f(t) := \frac{1}{n(n+1)}\int_0^x f(t)dt\,, f \in L^2(0,1), A_n: L^2(0,1) \to L^2(0,1), n \ge 1.$$

Then it is clear that $A_nA_m = A_mA_n, m,n \ge 1$, $S_n = \sum_{m=1}^{n} A_m = \left(1 - \frac{1}{n+1}\right)\int_0^x f(t)dt\,, n \ge 1$ and the sequence (S_n) uniformly converges to Volterra integration operator

$$Af(x) = \int_0^x f(t)dt\,, f \in L^2(0,1),\ \ A: L^2(0,1) \to L^2(0,1).$$

It is known that $\|A\| = \frac{2}{\pi}$ and $\sigma(A) = \{0\}$ [2]. Therefore, by Theorem 2.3 and Theorem 2.7, it implies that

$$\lim_{n\to\infty} g_r(S_n) = \frac{2}{\pi} \text{ and } \lim_{n\to\infty} q_r(S_n) = 0.$$

Example 2.9. Consider the following sequence of operators in real space $L^2(0,1)$ in form:

$$A_n f(x) := \int_0^x \frac{t}{(1+(n-1)t)(1+nt)} f(t)dt\,, f \in L^2(0,1), A_n: L^2(0,1) \to L^2(0,1), n \ge 1.$$

In this case, it is clear that

$$S_n f(x) := \int_0^x \left(1 - \frac{1}{1+nt}\right) f(t)dt\,, f \in L^2(0,1), n \ge 1.$$

Using the Lebesgue Dominated Convergence Theorem it can be proved that the series $\sum_{n=1}^{\infty} A_n$ uniformly converges to the Volterra integration operator

$$Af(x) = \int_0^x f(t)dt\,, f \in L^2(0,1), A: L^2(0,1) \to L^2(0,1).$$

It well known that $\|A\| = \frac{2}{\pi}$ and $\sigma(A) = \{0\}$ [2] and numerical radius $w(A) = \frac{1}{2}$ [20]. Then by Theorem 2.4, Theorem 2.6 and Theorem 2.7 we have

$$\lim_{n\to\infty} g_w(S_n) = \frac{2}{\pi} - \frac{1}{2} = \frac{4-\pi}{2\pi} \text{ and } \lim_{n\to\infty} q_w(S_n) = \frac{\pi}{4},$$

$$\lim_{n\to\infty} g_c(S_n) = \frac{2}{\pi} \text{ and } \lim_{n\to\infty} q_c(S_n) = 0.$$

III. APPLICATION

Now it will be given one important function class $(\Lambda_\omega)_+$ (see [21]). Let ω be a modulus of continuity, i.e., ω be a nondecreasing continuous function on $[0,\infty)$ such that $\omega(0) = 0$ and for $x > 0$ $\omega(x) > 0$ with property $\omega(x+y) \le \omega(x) + \omega(y), x, y \in [0,\infty)$. And also, it will be denoted by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ unit disc and $\mathbb{A}(\mathbb{D})$ class of all analytic functions on $\mathbb{D}$.

Let us denote one space of analytic functions

$$(\Lambda_\omega)_+ := \left\{ f \in \mathbb{A}(\mathbb{D}) : \|f\|_{\Lambda_\omega} = \sup_{\substack{u,v\in\mathbb{D} \\ u\neq v}} \frac{|f(u)-f(v)|}{\omega(|u-v|)} < \infty \right\}.$$

Given a modulus of continuity ω, it will be defined the function ω_* by

$$\omega_*(x) := x \int_x^\infty \frac{\omega(t)}{t^2} dt, x > 0.$$

Note that $\lim_{x \to 0^+} \omega_*(x) = 0$.

Recall that the following result has been proved in [21].

Theorem 3.1. There exists a constant $c > 0$ such that for every modulus continuity ω, for every $f \in (\Lambda_\omega)_+$ and for arbitrary contractions T and S, the following inequality holds

$$\|f(T) - f(S)\| \leq c\|f\|_{\Lambda_\omega} \omega_*(\|T - S\|).$$

Here we will investigate how the results obtained in the previous section will change for operator-functions.

Theorem 3.2. Let (A_n) be a sequence of bounded linear operators in H such that for any $n \geq 1$ the operator $S_n := \sum_{m=1}^n A_m$ is a contraction operator. If the series $\sum_{n=1}^\infty A_n$ uniformly converges to the $A: H \to H$, then for any $f \in (\Lambda_\omega)_+$, the following statements are correct:

(1) If for any $n \geq 1$ $S_n A = A S_n$, then $g_r\big(f(A)\big) = \lim_{n \to \infty} g_r(f(S_n))$,

(2) $g_w\big(f(A)\big) = \lim_{n \to \infty} g_w(f(S_n))$,

(3) $g_c\big(f(A)\big) = \lim_{n \to \infty} g_c(f(S_n))$,

(4) If for any $n \geq 1$ $S_n A = A S_n$ and $f(A) \neq 0$, then $q_r\big(f(A)\big) = \lim_{n \to \infty} q_r(f(S_n))$,

(5) $q_w\big(f(A)\big) = \lim_{n \to \infty} q_w(f(S_n))$, $f(A) \neq 0$,

(6) $q_c\big(f(A)\big) = \lim_{n \to \infty} q_c(f(S_n))$, $f(A) \neq 0$.

Proof. Let f is any function of $(\Lambda_\omega)_+$ and contraction operators sequences (S_n) in H which uniformly converges to operator $A : H \to H$. Then A is a contraction operator. Moreover, since $f \in (\Lambda_\omega)_+$, then by Theorem 3.1, there exists $c > 0$ such that $\|f(A) - f(S_n)\| \leq c\|f\|_{\Lambda_\omega} \omega_*(\|A - S_n\|), n \geq 1$.

Consequently, since $\lim_{n \to \infty} \omega_*(\|A - S_n\|) = 0$, then the operator sequences $(f(S_n))$ uniformly converges to $f(A)$. Thus, the validity of the claims of this theorem under corresponding conditions it is clear from Theorems 2.3, 2.4 and Theorems 2.6, 2.7.

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