SAKARYA ÜNİVERSİTESİ FEN BİLİMLERİ ENSTİTÜSÜ DERGİSİ
SAKARYA UNIVERSITY JOURNAL OF SCIENCE

e-ISSN: 2147-835X
Dergi sayfası: http://dergipark.gov.tr/saufenbilder

Geliş/Received
01.11.2016
Kabul/Accepted
04.04.2017

Doi
10.16984/saufenbilder.306867

On the differential geometric elements of bertrandian darboux ruled surface in E^3

Şeyda Kılıçoğlu [1*], Süleyman Şenyurt[2*]

ABSTRACT

In this paper, we consider two special ruled surfaces associated to a Bertrand curve α and Bertrand mate α^*. First, Bertrandian Darboux Ruled surface with the base curve α has been defined and examined in terms of the Frenet-Serret apparatus of the curve α, in E^3. Later, the differential geometric elements such as, Weingarten map S, Gaussian curvature K and mean curvature H, of Bertrandian Darboux Ruled the surface and Darboux ruled surface has been examined relative to each other. Further, first, second and third fundamental forms of Bertrandian Darboux Ruled surface have been investigated in terms of the Frenet apparatus of Bertrand curve α, too.

Keywords: Ruled surface, Darboux vector, Bertrand curves

Öklid uzayında bertrandian darboux regle yüzeyin diferensiyel geometrik elemanlar

ÖZ

Bu çalışmada Bertrand eğrisi ve Bertrand eşi olan eğriler üzerinde Darboux vektörleri ile üretilen iki özel regle yüzeyi gözönüne alındı. İlk olarak, α eğrisinin Bertrand Darboux regle yüzeyi, Bertrand eğrisinin Frenet-Serret aparatlar cinsinden tanımlandı ve araştırıldı. Daha sonra, Bertrand Darboux regle yüzeyi ile Darboux regle yüzeyinin Weingarten dönüşümü, Gauss eğriliği ve ortalama eğriliği gibi diferensiyel geometrik değişmezleri birbirleri ile ilişkili olarak incelendi. Son olarak, Bertrand Darboux regle yüzeyinin birinci, ikinci ve üçüncü temel formlar α Bertrand eğrisinin Frenet-Serret aparatlar cinsinden ifadeleri verildi.

Anahtar Kelimeler: Regle yüzey, Darboux vektörü, Bertrand eğrileri

1. (INTRODUCTION AND PRELIMINARIES)

The set, whose elements are frame vectors and curvatures of a curve α, is called Frenet-Serret apparatus of the curve. Let Frenet-Serret apparatus of the curve $\alpha(s)$ be $\{V_1,V_2,V_3,k_1,k_2\}$, collectively. The Darboux vector is the areal velocity vector of the Frenet frame of a space curve. It was named after Gaston Darboux who discovered it. It is also called angular momentum vector, because it is directly proportional to angular momentum. For any unit speed curve α, in terms of the Frenet-Serret apparatus the Darboux vector D can be expressed as

$$D(s)=k_2(s)V_1(s)+k_1(s)V_3(s),[2]. \tag{1}$$

Let a vector field be

$$\tilde{D}(s)=\frac{k_2}{k_1}(s)V_1(s)+V_3(s), \tag{2}$$

along $\alpha(s)$ under the condition that $k_1(s)\neq 0$ and it is called the modified Darboux vector field of α, [4]. The Bertrand mate of a given curve is a well- known concept in Euclidean 3-space. Bertrand curves have the following fundamental properties; which are given in more detail in, [4]. Two curves which have a common principal normal vector at any point are called Bertrand curves. If $k_2(s)\neq 0$ along $\alpha(s)$, then $\alpha(s)$ is a Bertrand curve if and only if $\lambda\neq 0,\beta\neq 0\in R$

$$\lambda k_1(s)+\beta k_2(s)=1 \tag{3}$$

where λ and β are constants for any $s\in I$. From the fact that

$$\frac{k_1(s)}{k_2(s)}=\frac{1-\lambda\beta}{\lambda k_2(s)},$$

if k_1 and $k_2\neq 0$ are constants then it is easily seen that a circular helix is a Bertrand curve. Let α, $\alpha^*\subset \mathbf{E}^3$ and α be a unit-speed curve with the position vector $\alpha(s)$, where s is the arc length parameter, if the curve α^* is Bertrand mate of α, then we may write that

$$\alpha^*(s)=\alpha(s)+\lambda V_2(s) \tag{4}$$

and $|\lambda|=\left|\frac{1-\beta k_2}{k_1}\right|$ gives the distance between the unit-speed curves α and α^*. Also, it is known that

$$\frac{ds}{ds^*}=\pm\frac{\sin\theta}{\lambda k_2}=\frac{\cos\theta}{(1-\lambda k_1)}, \tag{5}$$

or

$$\frac{ds^*}{ds}=k_2\sqrt{\lambda^2+\beta^2}. \tag{6}$$

The following result shows that we can write the Frenet apparatus of the Bertrand mate based on the Frenet apparatus of the Bertrand curve, [3]. Let α^* be the Bertrand mate of the curve α. The quantities $\{V_1^*,V_2^*,V_3^*,k_1^*,k_2^*\}$ are, collectively, Frenet-Serret apparatus of the Bertrand mate α^* and they satisfy the relations,

$$\begin{aligned} V_1^*&=\frac{\beta V_1}{\sqrt{\lambda^2+\beta^2}}+\frac{\lambda V_3}{\sqrt{\lambda^2+\beta^2}},\\ V_2^*&=V_2,\\ V_3^*&=\frac{-\lambda V_1}{\sqrt{\lambda^2+\beta^2}}+\frac{\beta V_3}{\sqrt{\lambda^2+\beta^2}}. \end{aligned} \tag{7}$$

The first and second curvatures of the offset curve α^* are given by

$$\begin{cases} k_1^*=\dfrac{\beta k_1-\lambda k_2}{(\lambda^2+\beta^2)k_2}\\ k_2^*=\dfrac{1}{(\lambda^2+\beta^2)k_2}. \end{cases} \tag{8}$$

Here λ and β are constants such that $\lambda k_1+\beta k_2=1$ for any $s\in I$. The product of the torsions of Bertrand curves is a constant, that is, $k_2k_2^*=\frac{1}{(\lambda^2+\beta^2)}$ is non-negative constant, where k_2^* is the torsion of α^*. The offset curve constitutes another Bertrand curve, since $\lambda^*k_1^*+\beta^*k_2^*=1,\ \lambda^*=-\lambda,\ \beta^*=\beta,$ it is trivial.

A ruled surface is one which can be generated by the motion of a straight line in Euclidean 3-space, [1]. Choosing a directrix on the surface, i.e. a smooth unit speed curve $\alpha(s)$ orthogonal to the straight lines and then choosing $v(s)$ to be unit vectors along the curve in the direction of the lines, the velocity vector α_s and v satisfy $\langle\alpha',v\rangle=0$. A Frenet ruled surface is a ruled surfaces generated by Frenet vectors of the base curve.

For more detail [5]. The differential see geometric elements of Mannheim Darboux ruled surface are examined in [7].

Definition 1.1 The ruled surface

$$\varphi(s,u) = \alpha(s) + u\tilde{D}(s)$$
$$= \alpha(s) + u\frac{k_2(s)}{k_1(s)}V_1(s) + uV_3(s) \qquad (9)$$

is the parametrization of the ruled surface which is called rectifying developable surface of the curve α, in [4].

2. BERTRANDIAN DARBOUX VECTOR

In this section, we will define and study on Bertrandian Darboux ruled surface, which is known as rectifying developable ruled surface $\tilde{D}$-scroll. First, we will find Darboux vector field of the Bertrand mate α^*.

Theorem 2.1 Let α be a Bertrand curve with the Bertrand mate α^*. The modified Darboux vector fields of a curve α and Bertrand mate α^* are lineary dependent.

Proof. In the same point of view of the equation (2) Darboux vector field of the Bertrand mate is given as follows;

$$\tilde{D}^* = \frac{k_2^*}{k_1^*}V_1^* + V_3^*,$$

and from the equation (8) it is easily seen that

$$\frac{k_2^*}{k_1^*} = \frac{1}{\beta k_1 - \lambda k_2}.$$

From the last two equation, we obtain

$$\tilde{D}^* = \frac{\beta V_1 + \lambda V_3}{(\beta k_1 - \lambda k_2)\sqrt{\lambda^2+\beta^2}} + \frac{-\lambda V_1 + \beta V_3}{\sqrt{\lambda^2+\beta^2}}. \qquad (10)$$

From the offset property of Bertrand curves $\lambda k_1 + \beta k_2 = 1,$ the modified Darboux vector field of the Bertrand mate α^* is

$$\tilde{D}^* = \frac{k_1\sqrt{\lambda^2+\beta^2}}{(\beta k_1 - \lambda k_2)}\tilde{D}. \qquad (11)$$

This complete the proof.

Corollary 2.1 The angle between the modified Darboux vector fields of a cylindrical helix α and Bertrand mate α^* is the function

Proof.

$$\theta = \arccos\left(\frac{\sqrt{\lambda^2+\beta^2}}{\beta - \lambda d}\right)(d^2+1)$$

where the constant d is the Lancret invariant of α. Considering the equations (2) and (11) completes the proof.

Definition 2.1 Let the curve α^* be Bertrand mate of α, the parametrization of the Bertrandian Darboux ruled surface with base curve α, according to the Frenet-Serret apparatus of the curve α is

$$\varphi^*(s,v) = \alpha(s) + k_2(s)v\frac{\sqrt{\lambda^2+\beta^2}}{\beta k_1(s) - \lambda k_2(s)}V_1(s) + \lambda V_2(s) + k_1(s)v\frac{\sqrt{\lambda^2+\beta^2}}{\beta k_1(s) - \lambda k_2(s)}V_3(s), \qquad (12)$$

or

$$\varphi^*(s,v) = \alpha(s) + \lambda V_2(s) + v\left(\frac{k_1\sqrt{\lambda^2+\beta^2}}{(\beta k_1 - \lambda k_2)}\tilde{D}\right). \qquad (13)$$

Corollary 2.2 A Darboux ruled surface and a Bertrandian Darboux ruled surface are not intersect each other, excluding the case of $\lambda = 0$ and $u = \mp v.$

Proof. The common solution of the equation (9) and (12), we have

$$\lambda = 0 \text{ and } u = vk_1\frac{\sqrt{\lambda^2+\beta^2}}{\beta k_1 - \lambda k_2}.$$

Hence,

$$u = vk_1\frac{\mp\beta}{\beta k_1} \Rightarrow u = \mp v.$$

Theorem 2.2 The normal vector field N of a Darboux ruled surface with base curve α is parallel to the normal vector field N^* of a Bertrandian Darboux ruled surface with the same base curve α.

Proof. Since the normal vector field N of a Darboux ruled surface of curve α is

$$N = \frac{\varphi_s \Lambda \varphi_u}{\|\varphi_s \Lambda \varphi_u\|} = V_2,$$

and the normal vector field N^* of a Bertrandian Darboux ruled surface of the curve α is

$$N^* = \frac{\varphi_s^* \Lambda \varphi_v^*}{\left\| \varphi_s^* \Lambda \varphi_v^* \right\|} = V_2^*.$$

Since the principal normal vector of α and α^* is common desired result is trivial.

Theorem 2.3 The matrix corresponding to the Weingarten map (Shape Operator) S^* of a Bertrandian Darboux ruled surface of curve α is

$$S^* = \begin{bmatrix} \dfrac{-(\beta k_1 - \lambda k_2)^3}{k_2(\beta k_1 - \lambda k_2)^2 - v(\beta k_1 - \lambda k_2)'} & 0 \\ 0 & 0 \end{bmatrix}. \tag{14}$$

Proof. The matrix form of the Weingarten map (Shape Operator) S of a Darboux ruled surface of curve α is given [6] as

$$S = \begin{bmatrix} \dfrac{-k_1}{\left(1 + u\left(\dfrac{k_2}{k_1}\right)'\right)}. & 0 \\ 0 & 0 \end{bmatrix}.$$

In a similar manner, the matrix form of the Weingarten map S^* of a Bertrandian Darboux ruled surface with base curve α is

$$S^* = \begin{bmatrix} \dfrac{-k_1^*}{\left(1 + u\left(\dfrac{k_2^*}{k_1^*}\right)'\right)} & 0 \\ 0 & 0 \end{bmatrix}.$$

If we substitute the equations (6) and (8), into the last equation, we obtain

$$S^* = \begin{bmatrix} \dfrac{\dfrac{-\beta k_1 + \lambda k_2}{(\lambda^2 + \beta^2) k_2}}{1 + u\left(\dfrac{1}{\beta k_1 - \lambda k_2}\right)' \dfrac{1}{k_2\sqrt{\lambda^2 + \beta^2}}} & 0 \\ 0 & 0 \end{bmatrix}.$$

Corollary 2.3 The Gaussian curvature and mean curvature of a Bertrandian Darboux ruled surface of curve α are, respectively

$$K = \det S^* = 0, \tag{15}$$

$$H = \frac{-(\beta k_1 - \lambda k_2)^3}{k_2(\beta k_1 - \lambda k_2)^2 - v(\beta k_1 - \lambda k_2)'}. \tag{16}$$

The fundamental forms are extremely important and useful in determining the metric properties of a surface, such as line element, area element, normal curvature, Gaussian curvature and mean curvature. The third fundamental form is given according to the first and second forms by $III - 2HII + KI = 0$ where [2]

$$I = dsds + dudu, \tag{17}$$

$$II = -k_1 dsds + k_2 dsdu, \tag{18}$$

$$III = (k_1^2 + k_2^2) dsds. \tag{19}$$

Theorem 2.4 The first fundamental form of a Bertrandian Darboux ruled surface with the base curve α, is given by

$$I^* = k_2^2(\lambda^2 + \beta^2) dsds + dvdv \tag{20}$$

Proof. In a similar way from the equation (17), we can write the first fundamental form of Bertrandian Darboux ruled surface as

$$I^* = ds^* ds^* + dvdv. \tag{21}$$

If we substitute the equation (6) into the last equation, we get

$$I^* = k_2^2(\lambda^2 + \beta^2) dsds + dvdv$$

Theorem 2.5 The second fundamental form of a Bertrandian Darboux ruled surface of the curve α is

$$II^* = (\lambda k_2^2 - \beta k_1 k_2) dsds + \frac{1}{\sqrt{\lambda^2 + \beta^2}} dsdv. \tag{22}$$

Proof. Considering the equation (18) we can write the second fundamental form of Bertrandian Darboux ruled surface as

$$II^* = -k_1^* ds^* ds^* + k_2^* ds^* dv,$$

here the equation (6) given as

$$II^* = (\lambda k_2^2 - \beta k_1 k_2) dsds + \frac{1}{\sqrt{\lambda^2 + \beta^2}} dsdv.$$

Theorem 2.6 The third fundamental form of a ruled surface a Bertrandian Darboux ruled surface is denoted by III^* and

$$III^* = \frac{1 + (\beta k_1 - \lambda k_2)^2}{(\lambda^2 + \beta^2)} dsds. \tag{23}$$

Proof. Taking the equation (19) we can write the third fundamental form of Bertrandian Darboux ruled surface as

$$III^{*} = \left(k_1^{*2} + k_2^{*2}\right) ds^{*}ds^{*}.$$

The equation (6) and the last equation completes the proof.

REFERENCES

[1] do Carmo, M. P., Differential geometry of curves and surfaces. Prentice-Hall, ISBN 0-13-212589-7, 1976.

[2] Gray, A., Modern Differential Geometry of Curves and Surfaces with Mathematica, 2nd ed. Boca Raton, FL: CRC Press, 1997.

[3] Hacısalihoğlu, H.H., Differential Geometry (in Turkish), vol.1, Inönü University Publications, 1994.

[4] Izumiya, S., Takeuchi, N., Special curves and ruled surfaces., Beitrage zur Algebra und Geometrie Contributions to Algebra and Geometry, 44(1), 203-212, 2003.

[5] Kılıçoğlu, S., Senyurt, S., Hacısalihoğlu, H. H., An examination on the positions of Frenet ruled surface along Bertrand pairs and according to their normal vector fields in E^3, Applied Mathematical Sciences, 9(142), 7095-7103, 2015.

[6] Senyurt, S., Kılıçoğlu, S., On the differential geometric elements of the involute $\tilde{D}$ scroll, Adv. Appl. Clifford Algebras, 25(4), 977-988, 2015, doi:10.1007/s00006-015-0535-z..

[7] Kılıçoğlu S., Senyurt S., On the Differential Geometric Elements of Mannheim Darboux Ruled surface in E^3, Applied Mathematical Sciences, 10(62), 3087-3094, 2016, doi.org/10.12988/ams.2016.67221.