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Characteristic Jacobi Operator on Almost Kenmotsu 3-manifolds

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ABSTRACT

The Ricci tensor field, φ-Ricci tensor field and the characteristic Jacobi operator on almost Kenmotsu 3-manifolds are investigated. We give a classification of locally symmetric almost Kenmotsu 3-manifolds.

Keywords: *Almost Kenmotsu manifolds, characteristic Jacobi operator, locally symmetric, Lie groups.*
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Contents

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Dedicated to professor Hiroyuki Tasaki on the occasion of his retirement of University of Tsukuba

1. Introduction

This article is the third one in our series [26, 54] on the study of characteristic Jacobi operators on almost contact 3-manifolds.

Tanno [103] classified complete almost contact Riemannian manifolds with automorphism group of maximum dimension. Those almost contact Riemannian manifolds are classified as (1) Sasakian space forms, (2) cosymplectic space forms and (3) warped products of the form $\mathbb{R} \times_{ce^t} \mathbb{C}^n$ (see Example 5.1). Kenmotsu clarified the almost contact Riemannian structures of $\mathbb{R} \times_{ce^t} \mathbb{C}^n$ (the warped product model of the hyperbolic space $\mathbb{H}^{2n+1} = \mathbb{H}^{2n+1}(-1)$ of constant curvature -1). Kenmotsu introduce a class of almost contact Riemannian structure modeled on $\mathbb{R} \times_{ce^t} \mathbb{C}^n$ [69]. The structure introduced in [69] is named as "Kenmotsu structure" by Janssens and Vanhecke [68]. Janssens and Vanhecke introduced the notion of almost Kenmotsu manifold in [68]. Kenmotsu manifolds are characterized as normal almost Kenmotsu manifolds. Here we recall that Sasakian manifolds (resp. cosymplectic manifolds) are characterized as normal contact Riemannian manifolds (resp. normal almost cosymplectic manifolds). In this sense, almost Kenmotsu manifolds constitute a class of almost contact Riemannian manifolds which should be compared with contact Riemannian manifolds and almost cosymplectic manifolds.

After the publication of the seminal paper [69], a huge number of articles concerning on Kenmotsu manifolds have been published. As is well known the unit sphere $\mathbb{S}^{2n+1} = \mathbb{S}^{2n+1}(1)$ is the standard model of Sasakian manifolds. In other words, the notion of Sasakian manifold is modeled on $\mathbb{S}^{2n+1}$. Thus Kenmotsu manifolds are regarded as "opposite correspondents" of Sasakian manifolds.

In our previous papers [26, 54], we studied characteristic flow invariance of the characteristic Jacobi operator ℓ on contact Riemannian 3-manifolds and cosymplectic 3-manifolds.

The present work has three aspects. The first aspect is to give an expository article on 3-dimensional almost Kenmotsu geometry (Section 2–Section 9).

Next, the second aspect is the classification of locally symmetric almost Kenmotsu 3-manifolds. Some literature claimed that locally symmetric almost Kenmotsu 3-manifolds are locally isomorphic to either hyperbolic 3-space $\mathbb{H}^3 = \mathbb{H}^3(-1)$ of curvature -1 equipped with a homogeneous Kenmotsu structure or the product space $\mathbb{H}^2(-4) \times \mathbb{R}$ equipped with a non-normal homogeneous Kenmotsu structure [22, 109]. Unfortunately, this claim has remained unverified and has been used as the correct classification in other studies. In this paper, we point out that this claim is incorrect. More precisely, this claim is correct under the assumption the characteristic vector field is an eigenvector field. There exit locally symmetric almost Kenmotsu 3-manifolds whose characteristic vector fields are *not* eigenvector fields of Ricci operators. Those examples are discovered by Perrone [93, Theorem 1.2 Case (IV)].

We give the correct classification theorem of locally symmetric almost Kenmotsu 3-manifolds (Theorem 10.1).

The third aspect concerns characteristic Jacobi operator. As a continuation of previous works [26, 54], we study almost Kenmotsu 3-manifolds whose characteristic Jacobi operator is invariant under the flows generated by the characteristic vector field. In addition we propose some unsolved problems.

Differential geometry of curves in almost Kenmotsu 3-manifolds is a very active research area. We do not discuss curves in almost Kenmotsu 3-manifolds in this article. For interested readers we refer to [59, 60, 61, 62].

This paper is organized as follows. Section 2 to Section 9 are devoted to the first aspect of this article.

In Section 2 we recall basic facts on local symmetry and semi-symmetry of Riemannian manifolds. The next section devotes to discuss fundamental properties of pseudo-symmetry of Riemannian manifolds. In addition we recall fundamental theory of harmonic maps (Section 3.3). One can consider the harmonicity of unit vector fields on a Riemannian manifold as maps into the unit tangent sphere bundles. The harmonicity of unit vector fields will be used to introduce the notion of H-almost Kenmotsu manifold (Definition 5.10). We prepare fundamental structure theory of 3-dimensional Lie groups in Section 4. We start our discussion on almost contact Riemannian manifolds from Section 5. We specialize our discussion to almost Kenmotsu 3-manifolds in Section 6. The H-almost Kenmotsu property is characterized by certain nullity condition. We discuss the generalized almost Kenmotsu (κ, μ, ν)-nullity condition in 7. We exhibit some explicit examples of generalized almost Kenmotsu (κ, μ, ν)-spaces. We also discuss φ-Einstein condition in 8. In Section 9 we study homogeneous almost Kenmotsu 3-manifolds. We give explicit models for all homogeneous almost Kenmotsu 3-manifolds.

The second aspect of this article is developed in Section 10. We study the system of local symmetry in Section 10.2. We give the correct classification theorem of locally symmetric almost Kenmotsu 3-manifolds (Theorem 10.1). In addition, we discuss (strong) η-parallelism (Section 10.4) and the characteristic flow invariance (Section 10.5) of the Ricci operator of almost Kenmotsu 3-manifolds.

As the third aspect of this article, in Section 11, we give our new results on characteristic Jacobi operator.
In the final section, we discuss relationship between harmonic maps and Ricci operator in almost Kenmotsu geometry.

Conventions

In this paper we use the following definition for exterior differentiation of differential forms:

- Let M be a manifold and η a 1-form on M. Then the exterior derivative $\mathrm{d}\eta$ is defined by

$$2\mathrm{d}\eta(X,Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y]), \quad X,Y \in \mathfrak{X}(M).$$

  Here $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on M.
- The exterior derivative $\mathrm{d}\Phi$ of a 2-form Φ is defined by

$$\mathrm{d}\Phi = X(\Phi(Y,Z)\,) + Y(\Phi(Z,X)\,) + Z(\Phi(X,Y)\,) - \Phi([X,Y],Z) - \Phi([Y,Z],X) - \Phi([Z,X],Y).$$

- On an oriented Riemannian manifold (M,g), $\mathrm{d}\eta$ and $\mathrm{d}\Phi$ are rewritten as

$$\mathrm{d}\eta(X,Y) = \frac{1}{2}\left((\nabla_X\eta)Y - (\nabla_Y\eta)X\right), \quad \mathrm{d}\Phi(X,Y,Z) = \frac{1}{3}\mathfrak{S}_{X,Y,Z}(\nabla_X\Phi)(Y,Z)$$

  in terms of Levi-Civita connection ∇.
- The codifferential $\delta_g\eta$ and $\delta_g\Phi$ are given respectively by

$$\delta_g\eta = -\mathrm{tr}(\nabla\eta), \quad (\delta_g\Phi)X = -\mathrm{tr}(\nabla.\Phi)(\cdot,X).$$

- The Lie differential operator by a vector field X is denoted by $\pounds_X$.
- Throughout this paper we denote the space of all smooth sections of a vector bundle E by $\Gamma(E)$.

2. The local symmetry and semi-symmetry

2.1. *The Riemannian curvature*

Let (M,g) be a Riemannian manifold with its Levi-Civita connection ∇. Then the curvature tensor field R of ∇ is called the *Riemannian curvature* of $M=(M,g)$. In this article we use the sign convention:

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Throughout this article we denote by R_1 the curvature-like tensor field defined by

$$R_1(X,Y)Z = (X \wedge Y)Z = g(Y,Z)X - g(Z,X)Y.$$

On a Riemannian manifold (M,g), the sectional curvature K of a tangent plane $\Pi = X \wedge Y$ is measured by the formula

$$K(\Pi) = \frac{g(R(X,Y)Y,X)}{g(R_1(X,Y)Y,X)},$$

As is well known, sectional curvature functions determine the Riemannian curvature. In other words, the knowledge of the full curvature tensor R is equivalent to the knowledge of the sectional curvatures K. For instance M is of constant curvature c if and only if $R = cR_1$.

2.2. *The Ricci tensor field and the tidal force*

The *Ricci tensor field* ρ of (M,g) is a symmetric tensor field defined by

$$\rho(X,Y) = \mathrm{tr}\,(Z \longmapsto R(Z,Y)X).$$

The Ricci operator S is a self-adjoint endomorphism field metrically equivalent to ρ, that is

$$\rho(X,Y) = g(SX,Y) = g(X,SY).$$

The smooth function $\mathrm{s} = \mathrm{tr}\,\rho = \mathrm{tr}\,S$ is called the *scalar curvature* of (M,g).

Example 2.1. Let $\overline{M} = (\overline{M}, \bar{g})$ be a Riemannian 2-manifold of constant curvature $\bar{c}$. Consider the Riemannian product $M = \overline{M} \times \mathbb{R}$ equipped with the product metric $g = \pi^*\bar{g} + dt^2$. Here $\pi : M \to \overline{M}$ is the projection. Then the Riemannian curvature R of M has the form

$$R(X,Y)Z = \bar{c}R_1(X,Y)Z - \bar{c}\{dt(Z)R_1(X,Y)\partial_t + dt(X)g(Y,Z)\partial_t - dt(Y)g(Z,X)\partial_t\}.$$

The Ricci operator is represented as

$$S = \bar{c}\,\mathrm{I} - \bar{c}\,dt \otimes \partial_t.$$

A Riemannian manifold (M, g) of dimension $\dim M \geq 3$ is said to be *Einstein* if $\rho = c\,g$ for some constant c. One can see that on an Einstein manifold, $\rho = (\mathrm{s}/n)g$ and s is constant. Riemannian manifolds of constant curvature are Einstein.

Let (M, g) be a Riemannian manifold. For a nonzero tangent vector $v \in T_pM$ at a point p, the *tidal force operator* F_v associated to v is a linear endomorphism on $(\mathbb{R}v)^\perp$ defined by $F_v(w) := -R(w,v)v$ for $w \perp v$ ([83, p. 219]). One can see that F_v is self-adjoint on $(\mathbb{R}v)^\perp$ and has the trace $\mathrm{tr}\, F_v = -\rho(v,v)$. For a geodesic γ in (M,g), a vector field X along γ is said to be a *Jacobi field* along γ if it satisfies the *Jacobi equation*:

$$\nabla_{\gamma'}\nabla_{\gamma'}X = -F_{\gamma'}(X).$$

2.3. The semi-parallelism

Every curvature-like tensor field acts on the space $\mathcal{T}_r{}^1(M) = \Gamma(\otimes^r T^*M \otimes TM)$ of tensor fields of type $(1,r)$ as a derivation. For instance, the derivative $F \cdot P$ of $P \in \mathcal{T}_1{}^1(M) = \Gamma(\mathrm{End}(TM))$ by a curvature-like tensor filed F is given by

$$(F \cdot P)(Z;Y,X) = F(X,Y)(PZ) - P(F(X,Y)Z), \quad X,Y,Z \in \mathfrak{X}(M).$$

The derivative $F \cdot R$ of the Riemannian curvature R by F is given by

$$\begin{aligned}(F \cdot R)(U,V,W;Y,X) = \;& (F(X,Y)R)(U,V,W)\\ = \;& F(X,Y)R(U,V)W - R(F(X,Y)U,V)W\\ & - R(U,F(X,Y)V)W - R(U,V)F(X,Y)W.\end{aligned}$$

As is well known, Riemannian manifolds with parallel Riemannian curvature are called *locally symmetric spaces*. Riemannian manifolds of constant curvature are locally symmetric.

As generalizations of parallelism, semi-parallelism and pseudo-parallelism are introduced. In this section we discuss the semi-parallelism. The pseudo-parallelism will be discussed in the next section.

Definition 2.1. Let (M, g) be a Riemannian manifold. A tensor field P on M of type $(1,r)$ is said to be *semi-parallel* if it satisfies

$$R \cdot P = 0.$$

A tensor field P of type $(1,r)$ is said to be *properly semi-parallel* if it is semi-parallel but not parallel.

A Riemannian manifold (M, g) is said to be *semi-symmetric* if $R \cdot R = 0$. The tensorial equation $R \cdot R = 0$ has a clear differential geometric meaning. At a point $p \in M$ of a Riemannian manifold M, denote by $\mathfrak{h}_p$ the linear subspace of $\mathfrak{so}(T_pM)$ spanned by the set $\{R_p(X,Y) \mid X,Y \in T_pM\}$. Then the semi-symmetry condition $R \cdot R = 0$ is equivalent to that $\mathfrak{h}_p$ is a Lie subalgebra of $\mathfrak{so}(T_pM)$. The connected Lie group H_p with Lie algebra $\mathfrak{h}_p$ is called the *primitive holonomy group* at p.

The semi-symmetry condition $R \cdot R = 0$ for Riemannian manifolds was recognized by E. Cartan [17, p. 265]. At the time of Cartan, the only known examples of semi-symmetric spaces are locally symmetric spaces and Riemannian 2-manifolds. The name "semi-symmetric space" was introduced by Sinjukov.

On a semi-symmetric Riemannian manifold M, at each point p, the Riemannian curvature R_p is the same as the Riemannian curvature of a Riemannian symmetric space which may vary with the point. For instance, every Riemannian 2-manifolds are semi-symmetric. Szabo clarified the local structure of semi-symmetric spaces [100] . Kowalski made a systematic study of foliated semi-symmetric 3-manifolds [76]. For more information on semi-symmetric spaces, we refer to [9].

The semi-symmetric Riemannian manifolds are precisely the Riemannian manifolds for which, up to second order, all the sectional curvatures are invariant under their parallel transports fully around all infinitesimal coordinate parallelograms.

Definition 2.2. Let (M, g) be a Riemannian manifold. The *nullity vector space* of the Riemannian curvature R at a point $p \in M$ is a linear subspace

$$T_p^0 M = \{X \in T_p M \mid R(X, Y)Z = 0, \ \forall Y, Z \in T_p M\}$$

of $T_p M$. The *index of nullity at* p is $n_M(p) := \dim T_p^0 M$. The *index of conullity at* p is $u_M(p) := \dim M - n_M(p)$.

Example 2.2. On a Riemannian product $M = \overline{M} \times \mathbb{R}$, where $\overline{M}$ is a Riemannian 2-manifold of constant curvature $\bar{c}$. Then its tangent space $T_p M$ at $p = (\bar{p}, t) \in M$ is decomposed as

$$T_p M = \mathcal{H}_p M \oplus \mathcal{V}_p M,$$

where $\mathcal{H}_p M$ [resp. $\mathcal{V}_p M$] is called the *horizontal space* [resp. vertical space] at p. Denote by $\pi_1 : M \to \bar{M}$ and $\pi_2 : M \to \mathbb{R}$ the projections, then

$$\pi_{1*p} : \mathcal{H}_p M \to T_{\bar{p}} \overline{M}, \quad \pi_{2*p} : \mathcal{V}_p M \to T_t \mathbb{R}$$

are linear isomorphisms. Thus we may identify $T_p M$ with the direct sum $T_{\bar{p}} \overline{M} \oplus T_t \mathbb{R}$.

The nullity space $T_p^0 M$ is $\mathcal{V}_p M$ and hence identified with $T_{\pi_2(p)} \mathbb{R}$. The index of nullity is $n_M = 1$ on M. The index of conullity is $u_M = 2$ on M.

2.4. Three dimensional semi-symmetric spaces

On a Riemannian 3-manifold (M, g), the Riemannian curvature R is described by the Ricci tensor field ρ and corresponding Ricci operator S by

$$R(X, Y)Z = \rho(Y, Z)X - \rho(Z, X)Y + g(Y, Z)SX - g(Z, X)SY - \frac{\mathrm{s}}{2} R_1(X, Y)Z \tag{2.1}$$

for all vector fields X, Y and Z on M. It should be remarked that for Riemannian 3-manifolds, Einstein property is equivalent to constancy of sectional curvature.

The covariant derivative ∇R is computed as

$$\begin{aligned}(\nabla_W R)(X, Y)Z = \ & (\nabla_W \rho)(Y, Z)X - (\nabla_W \rho)(Z, X)Y \\ & + g(Y, Z)(\nabla_W S)X - g(Z, X)(\nabla_W S)Y - \frac{\mathrm{ds}}{2}(W) R_1(X, Y)Z.\end{aligned} \tag{2.2}$$

Hence the covariant derivative ∇R satisfies the following formula:

$$\begin{aligned}g((\nabla_W R)(X, Y)Z, V) = \ & g((\nabla_W S)Y, Z)g(X, V) - g((\nabla_W S)Z, X)g(Y, V) \\ & + g(Y, Z)g((\nabla_W S)X, V) - g(Z, X)g((\nabla_W S)Y, V) \\ & - \frac{\mathrm{ds}}{2}(W) g(R_1(X, Y)Z, V).\end{aligned} \tag{2.3}$$

We know that the local symmetry ($\nabla R = 0$) implies the constancy of the scalar curvature, thus we confirm the following well-known fact:

Proposition 2.1. *A Riemannian* 3*-manifold* M *is locally symmetric if and only if its Ricci operator is parallel.*

Take a local orthonormal frame field $\{e_1, e_2, e_3\}$ and express the Riemannian cuvature as

$$R(e_i, e_j)e_k = \sum_{l=1}^{3} R_{lijk}\, e_l.$$

Next we denote the sectional curvature functions of the plane section field $e_i \wedge e_j$ spanned by e_i and e_j by

$$K_{ij} = K(e_i \wedge e_j).$$

Then

$$K_{ij} = R_{ijij}, \quad i \neq j.$$

The components $\rho_{ij} = \rho(e_i, e_j)$ of the Ricci tensor field ρ are related to the components of R by

$$K_{12} = \frac{1}{2}(\rho_{11} + \rho_{22} - \rho_{33}), \quad K_{13} = \frac{1}{2}(\rho_{11} - \rho_{22} + \rho_{33}), \quad K_{23} = \frac{1}{2}(-\rho_{11} + \rho_{22} + \rho_{33}),$$
$$R_{1213} = \rho_{23}, \quad R_{1223} = -\rho_{13}, \quad R_{1323} = \rho_{12}.$$

Now let (M^3, g) be a Riemannian 3-manifold with *principal Ricci curvatures* ρ_1, ρ_2 and ρ_3 (eigenvalues of the Ricci tensor field). Then the nullity vector space of R at p is rewritten as

$$T_p^0 M = \{X \in T_p M \mid S_p X = 0\}.$$

A Riemannian 3-manifold M is is semi-symmetric if and only if [99]:

$$(\rho_i - \rho_j)\{2(\rho_i + \rho_j) - \mathrm{s}\} = 0, \quad i, j = 1, 2, 3, \ i \neq j.$$

Thus at each point $p \in M$, there are three possibilities

1. $\rho_1 = \rho_2 = \rho_3 \neq 0$.
2. $\rho_1 = \rho_2 \neq 0$ and $\rho_3 = 0$ up to numeration.
3. $\rho_1 = \rho_2 = \rho_3 = 0$.

If $\rho_1 = \rho_2 = \rho_3 \neq 0$ at p, then this relation holds on a neighborhood U of p. On U, all of principal Ricci curvatures are non-zero constant, hence U is of constant nonzero curvature. In particular, U is locally symmetric.

Next, assume that S is of rank ≤ 2 on M and put $U = \{p \in M \mid \operatorname{rank} S_p = 2\}$.

Theorem 2.1. *Let (M^3, g) be a Riemannian 3-manifold. Then at each point $p \in M$, the index of nullity is either $n_M(p) = 0$, 1 or 3.*

Since we are interested in *non-symmetric* semi-symmetric spaces, we concentrate our attention to semi-symmetric 3-manifolds with $n_M = 1$ on M.

Denote by $T_p^1 M$ the orthogonal complement of $T_p^0 M$ in $T_p M$. Then the correspondences $T^0 M : p \longmapsto T_p^0 M$ and $T^1 M : p \longmapsto T_p^1 M$ define distributions on M. The former distribution is called the *nullity distribution* of M.

The Riemannian curvature R satisfies:

$$R(X, Y)Z = \rho_1(p)(X \wedge Y)Z, \quad X, Y, Z \in T_p^1 M,$$
$$R(X, Y)Z = 0, \quad X \in T_p^0 M, \quad Y, Z \in T_p M.$$

For any vector field V tangent to $T^0 M$, the integral curves of V are geodesics.

Moreover one can check that R has the same form of the Riemannian curvature of the product space $\overline{M} \times \mathbb{R}$ of a Riemannian 2-manifold $\overline{M}$ of constant curvature and the real line. In deed, the Riemannian curvature R of the Riemannian product $M = \overline{M} \times \mathbb{R}$ where $\overline{M}$ is a Riemannian 2-manifold of constant curvature $\bar{c}$ satisfies

$$R(X, Y)Z = \bar{c}(X \wedge Y), \quad X, Y, Z \in T_p^1 M,$$
$$R(X, Y)Z = 0, \quad X \in T_p^0 M, \quad Y, Z \in T_p M.$$

Example 2.3 (Real cones)**.** Let $(\overline{M}, \bar{g})$ be a Riemannian $(n-1)$-manifold $(n \geq 3)$. Set $m(t) := (t + m_0^{-1})^{-1}$ and $\mathbb{R}_{m_0} := \{t \in \mathbb{R} \mid t > -1/m_0\}$, where m_0 is a positive constant. Then the warped product $M^n(\overline{M}; m_0) := \mathbb{R}_{m_0} \times_{1/m} \overline{M}$ is called the *real cone* over $\overline{M}$. The unit vector field $T = \partial_t$ is regarded as a unit normal vector field to each fiber. Then the shape operator $\mathcal{B}_0$ of a fiber derived from $-T$ is given explicitly by

$$\mathcal{B}_0(X) = \nabla_X T = m(X - g(X, T)T).$$

We extend $\mathcal{B}_0$ to whole TM.

Let us denote by π the projection of the second factor. Then the Riemannian curvature R is described as:

$$R(X, Y)Z = g(\mathcal{B}_0(X), Z)\mathcal{B}_0 Y - g(\mathcal{B}_0(Y), Z)\mathcal{B}_0(X) + (\pi^* \overline{R})(X, Y)Z.$$

Any semi-symmetric real cone is locally isometric to some maximal cone $M_{\bar{c}}(\overline{M}; m_0)$ over a Riemannian $(n-1)$-manifold $\overline{M}$ of constant curvature $\bar{c}$. Note that if the real cone is irreducible and $\bar{c} \neq 0$, then at each point p, the index of nullity is 1 and index of conullity is $n-1$.

The real cone $M_{\bar{c}}(\overline{M}; m_0)$ is conformally flat. The Ricci tensor field and the scalar curvature are given by

$$\rho(X,Y) = (n-2)(\bar{c} - m^2)\{g(X,Y) - g(X,T)g(Y,T)\}, \quad \mathrm{s} = (n-1)(n-2)(\bar{c} - m^2).$$

Hence $M_{\bar{c}}(\overline{M}; m_0)$ is never locally symmetric.

Remark 2.1. The above scalar curvature formula corrects that in [13, 14].

A real cone $M_{\bar{c}}(\overline{M}; m_0)$ is said to be a *Euclidean cone*, *elliptic cone* or *hyperbolic cone* according as $\bar{c} = 0$, $\bar{c} > 0$ and $\bar{c} < 0$, respectively [9, §2.2.1]. The Riemannian curvature R of a semi-symmetric real cone $M_{\bar{c}}(\overline{M}; m_0)$ satisfies

$$R(X,Y) = (\bar{c} - m^2)(X \wedge Y), \quad X, Y \in T^1_p M,$$
$$R(X,Z) = 0, \quad X \in T^0_p M, \quad Z \in T^1_p M.$$

When $\dim \overline{M} = 2$, then the principal Ricci curvatures of the semi-symmetric real cone $M_{\bar{c}}(\overline{M}; m_0)$ are

$$\bar{c} - m(t)^2, \quad \bar{c} - m(t)^2, \quad 0.$$

Example 2.4. A Riemannian $(n+2)$-manifold is said to be of *conullity two* if the index of nullity is n on M. Each tangent space of a Riemannian manifold of conullity two has splitting

$$T_p M = T^0_p M + T^1_p M, \quad \dim T^1_p M = 2.$$

Decompose tangent vectors X, Y and $Z \in T_p M$ as $X = X_0 + X_1$, $Y = Y_0 + Y_1$ and $Z = Z_0 + Z_1$ along this spliiting, then we have

$$R(X,Y)Z = k(p)(g_p(Y_1, Z_1)X_1 - g_p(Z_1, X_1)Y_1),$$

where $k(p) = K(T^1_p M)$ is the sectional curvature of $T^1_p M$. Hence R satisfies

$$R(X,Y)Z = k(p)(X \wedge Y)Z, \quad X, Y, Z \in T^1_p M,$$
$$R(X,Y)Z = 0, \quad X \in T^0_p M, \quad Y, Z \in T_p M.$$

Every Riemannian manifold of conullity two is semi-symmetric and foliated by Euclidean and totally geodesic leaves of codimension 2 ([9, Theorem 2.11]). Those semi-symmetric spaces are called *foliated semi-symmetric spaces* [9, p. 20].

Let $\overline{M}$ be a Riemannian 2-manifold of constant curvature $\bar{c}$, then the direct product space $\overline{M} \times \mathbb{R}$ and a semi-symmetric real cone $M_{\bar{c}}(\overline{M}; m_0)$ are Riemannian 3-manifolds of conullity two.

On a semi-symmetric space M, every tangent space $T_p M$ has the irreducible and pairwise orthogonal decomposition (*V-decomposition* in the sense of Szabó [100])

$$T_p M = T^0_p M + V^{(1)}_p + \cdots + V^{(r)}_p$$

under the action of primitive holonomy group at p. The primitive holonomy group at p acts trivially on $T^0_p M$ and irreducibly on $V^{(i)}_p$ for $i > 0$.

A semi-symmetric space M is called a *simple leaf* if its V-decomposition has the form

$$T_p M = T^0_p M + V^{(1)}_p$$

at every point $p \in M$. In addition, a simple semi-symmetric leaf is said to be *infinitesimally irreducible* if at least one point the infinitesimal holonomy group acts irreducibly. Those semi-symmetric spaces are classified as follows (see *e.g.* [9, Theorem 2.8]).

Theorem 2.2. *Let M be an infinitesimally irreducible simple semi-symmetric leaf and $p \in M$. Then one of the following cases occurs*:

1. *a Riemannian symmetric space when $n_M = 0$ on M, or*

2. *a real cone when* $n_M = 1$ *and* $u_M = \dim M - 1 > 2$ *on* M*, or*
3. *a Kähler cone when* $n_M = 2$ *and* $u_M = \dim M - 2 > 2$ *on* M*, or*
4. *a Riemannian manifold foliated by Euclidean leaves of codimesnion two when* $n_M = \dim M - 2$ *and* $u_M = 2$ *at each point* p *of an open dense subset* U *of* M.

The following local structure theorem is due to Szabó [100] (see also [9, Theorem 2.6]).

Theorem 2.3. *For every semi-symmetric space* M*, there exists an open dense subset* $U \subset M$ *such that around any points of* U*, the semi-symmetric space is locally a Riemannian product of the form*

$$\mathbb{R}^k \times M_1 \times \cdots M_r, \quad k \geq 0, \quad r \geq 0$$

where $M_1, \ldots M_r$ *are infinitesimally irreducible simple semi-symmetric leaves or Riemannian* 2*-manifolds.*

Note that the direct product decomposition may vary in the different connnected components of U.
Special attention should be paid for semi-symmetric 3-spaces. Indeed, as pointed out in [13, p. 29], 3-dimensional semi-symmetric real cones exist as cones over Riemannian 2-manifolds of constant curvature. Those semi-symmetric spaces are not appeared explicitly in the classification due to Szabó, since those are special examples of foliated semi-symmetric 3-spaces.

3. The pseudo-symmetry

3.1. The pseudo-parallelism

The derivative $R_1 \cdot R$ is called the *Tachibana tensor field*. Take two tangent planes $\Pi_1 = X \wedge Y$ and $\Pi_2 = U \wedge V$, the *Deszcz-sectional curvature* $L(\Pi_1, \Pi_2)$ is defined by (see [45, 46]):

$$L(\Pi_1, \Pi_2) = \frac{(R \cdot R)(X, Y, Y, X; U, V)}{(R_1 \cdot R)(X, Y, Y, X; U, V)}.$$

One can see that $L(\Pi_1, \Pi_2)$ is independent of the choice of basis of Π_1 and Π_2.
As like the determination of R through the sectional curvatures, one can show that at any point of M, $R \cdot R$ is completely determined by the knowledge of Deszcz sectional curvatures. Motivated by these observations, the notion of pseudo-parallelism was introduced in the following mannar.

Definition 3.1. Let (M, g) be a Riemannian manifold. A tensor field P on M of type $(1, r)$ is said to be *pseudo-parallel* if there exits a smooth function L such that

$$R \cdot P = L\, R_1 \cdot P.$$

More precisely,

$$R(X, Y) \cdot P = L(X \wedge Y) \cdot P$$

holds for all vector fields X and Y on M.

A tensor field P of type $(1, r)$ is said to be *properly pseudo-parallel* if it is pseudo-parallel but not semi-parallel.
The notion of *pseudo-symmetry* is introduced by Deszcz as follows:

Definition 3.2. A Riemannian manifold (M, g) is said to be *pseudo-symmetric* if there exists a function L such that

$$R \cdot R = L\, R_1 \cdot R.$$

In particular, a pseudo-symmetric Riemannian manifold is called a *pseudo-symmetric space of constant type* if L is constant [77].

In particular, pseudo-symmetric Riemannian manifolds of constant type with $L = 0$ are called *semi-symmetric Riemannian manifolds*.
Obviously, locally symmetric Riemannian manifolds are pseudo-symmetric. A Riemannian manifold is said to be a *proper pseudo-symmetric space* if its Riemannian curvature is properly pseudo-parallel. Deszcz initiated studies on pseudo-symmetric Riemannian manifolds [2, 28]. Note that the tensor field $R_1 \cdot R$ is called the *Tachibana tensor field* [101].

3.2. Three dimensional pseudo-symmetry

The pseudo-symmetry is introduced as a generalization of local symmetry as well as semi-symmetry. The following characterization of pseudo-symmetry for Riemannian 3-manifolds is deduced.

Proposition 3.1. *A Riemannian 3-manifold (M^3, g) is a pseudo-symmetric space with $R \cdot R = L\, R_1 \cdot R$ if and only if the principal Ricci curvatures (eigenvalues of the Ricci tensor) locally satisfy the following relations (up to numeration):*

$$\rho_1 = \rho_2, \quad \rho_3 = 2L.$$

Note that when $\rho_1 = \rho_2 = \rho_3$, (M^3, g) is Einstein, *i.e.*, it is of constant curvature.

Corollary 3.1. *A Riemannian 3-manifold (M^3, g) of non-constant curvature is semi-symmetric if and only if the principal Ricci curvatures locally satisfy the following relations (up to numeration):*

$$\rho_1 = \rho_2, \quad \rho_3 = 0.$$

3.3. Harmonic maps

Let (M, g) and (N, g_N) be Riemannian manifolds and $f : M \to N$ a smooth map. We denote by f^*TN the pull-back bundle of TN by f, that is,

$$f^*TN = \bigcup_{p \in M} T_{f(p)}N.$$

A section of f^*TN is called a *vector field along* f. The Levi-Civita connection ${}^N\nabla$ induces a connection ∇^f on f^*TN.

The *second fundamental form* $\nabla \mathrm{d}f$ of f is defined by

$$(\nabla \mathrm{d}f)(Y; X) = \nabla^f_X \mathrm{d}f(Y) - \mathrm{d}f(\nabla_X Y), \quad X, Y \in \Gamma(TM).$$

The *tension field* $\tau(f)$ of f is a vector field along f defined by

$$\tau(f) = \mathrm{tr}_g\, (\nabla \mathrm{d}f) \in \Gamma(f^*TN).$$

Let $\{e_i\}_{i=1}^m$ be a local orthonormal frame field of M ($m = \dim M$) and tr_g is the metrical trace operator with respect to g. Namely the tension field $\tau(f)$ is computed as

$$\tau(f) = \sum_{i=1}^m (\nabla \mathrm{d}f)(e_i; e_i) = \sum_{i=1}^m \left\{ {}^N\nabla_{\mathrm{d}f(e_i)}(\mathrm{d}f(e_i)) - \mathrm{d}f(\nabla_{e_i} e_i) \right\}. \tag{3.1}$$

Definition 3.3. A smooth map $f : M \to N$ between Riemannian manifolds is said to be a *harmonic map* if it is a critical point of the Dirichlet energy

$$E(f; D) = \int_D e(f)\, \mathrm{d}v_g, \quad e(f) := \frac{1}{2}\|\mathrm{d}f\|^2 = \frac{1}{2}\sum_{i=1}^m g_N(\mathrm{d}f(e_i), \mathrm{d}f(e_i))$$

over any compact region D of M. Here $\mathrm{d}v_g$ is the volume element of M.

As is well known a smooth map f is harmonic if and only if its tension field vanishes.

3.4. Harmonic and minimal vector fields

Let (M, g) be a Riemannian m-manifold with unit tangent sphere bundle UM. We equip the Sasaki-lift metric g^{s} on UM. Denote by $\mathfrak{X}_1(M)$ the space of all smooth unit vector fields on M. Every unit vector field $V \in \mathfrak{X}_1(M)$ is regarded as an immersion of M into UM.

A unit vector field $V \in \mathfrak{X}_1(M)$ is said to be *minimal* if it is a critical point of the volume functional on $\mathfrak{X}_1(M)$. It is known that V is a minimal unit vector field if and only if it is a minimal immersion with respect to the pull-backed metric V^*g^{s}.

On the other hand, one can consider the Dirichlet energy

$$E(X; D) = \int_D \frac{1}{2}\|\mathrm{d}X\|^2\, \mathrm{d}v_g$$

of a unit vector field X over a compact region D of M. When M is compact we have

$$E(X;M)=\frac{m}{2}\mathrm{Vol}(M)+\int_M \frac{1}{2}\|\nabla X\|^2 \mathrm{d}v_g.$$

A unit vector field is said to be a *harmonic unit vector field* if it is a critical point of the Dirichlet energy through compactly supported variations. The corresponding critical point condition is (*cf.* [38]):

$$\overline{\Delta}_g\, X=\|\nabla X\|^2 X,$$

where $\overline{\Delta}_g$ is the *rough Laplacian* defined by

$$\overline{\Delta}_g=-\sum_{i=1}^{m}\left(\nabla_{e_i}\nabla_{e_i}-\nabla_{\nabla_{e_i}e_i}\right).$$

Here $\{e_1,e_2,\dots,e_m\}$ is a local orthonormal frame field of M as before.

A harmonic unit vector field is a harmonic map if is satisfies, in addition,

$$\mathrm{tr}_g R(\nabla X,X)=\sum_{i=1}^{m}R(\nabla_{e_i}X,X)e_i=0. \tag{3.2}$$

For more information on harmonic unit vector fields, see [38].

4. Three dimensional Lie groups

4.1. Unimodularity

Let G be a Lie group with a Lie algebra $\mathfrak{g}$ and a left invariant Riemannian metric $\langle\cdot,\cdot\rangle$. Then the Levi-Civita connection ∇ of $(G,\langle\cdot,\cdot\rangle)$ is described by the *Koszul formula*:

$$2\langle\nabla_X Y,Z\rangle=-\langle X,[Y,Z]\rangle+\langle Y,[Z,X]\rangle+\langle Z,[X,Y]\rangle,\quad X,Y,Z\in\mathfrak{g}.$$

A Lie group G is said to be *unimodular* if its left invariant Haar measure is right invariant. Milnor gave an infinitesimal reformulation of unimodularity for 3-dimensional Lie groups. We recall it briefly here.

Let $\mathfrak{g}$ be a 3-dimensional *oriented* Lie algebra with an inner product $\langle\cdot,\cdot\rangle$. Denote by $\times$ the *vector product operation* of the oriented inner product space $(\mathfrak{g},\langle\cdot,\cdot\rangle)$. The vector product operation is a skew-symmetric bilinear map $\times:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ which is uniquely determined by the following conditions:

(i) $\langle X,X\times Y\rangle=\langle Y,X\times Y\rangle=0$,
(ii) $\langle X\times Y,X\times Y\rangle=\langle X,X\rangle\langle Y,Y\rangle-\langle X,Y\rangle^2$,
(iii) if X and Y are linearly independent, then $\det(X,Y,X\times Y)>0$

for all $X,Y\in\mathfrak{g}$. On the other hand, the Lie-bracket $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ is a skew-symmetric bilinear map. Comparing these two operations, we get a linear endomorphism $L_{\mathfrak{g}}$ which is uniquely determined by the formula

$$[X,Y]=L_{\mathfrak{g}}(X\times Y),\quad X,Y\in\mathfrak{g}.$$

Now let G be an oriented 3-dimensional Lie group equipped with a left invariant Riemannian metric. Then the metric induces an inner product on the Lie algebra $\mathfrak{g}$. With respect to the orientation on $\mathfrak{g}$ induced from G, the endomorphism field $L_{\mathfrak{g}}$ is uniquely determined. The unimodularity of G is characterized as follows.

Proposition 4.1. ([78]) *Let G be an oriented 3-dimensional Lie group with a left invariant Riemannian metric. Then G is unimodular if and only if the endomorphism $L_{\mathfrak{g}}$ is self-adjoint with respect to the metric.*

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Denote by ad the *adjoint representation* of $\mathfrak{g}$,

$$\mathrm{ad}:\mathfrak{g}\to\mathrm{End}(\mathfrak{g});\quad \mathrm{ad}(X)Y=[X,Y].$$

Then one can see that $\mathrm{tr}\,\mathrm{ad}$;

$$X\longmapsto \mathrm{tr}\,\mathrm{ad}(X)$$

is a Lie algebra homomorphism into the commutative Lie algebra $\mathbb{R}$. The kernel

$$\mathfrak{u} = \{X \in \mathfrak{g} \mid \mathrm{tr}\,\mathrm{ad}(X) = 0\}$$

of $\mathrm{tr}\,\mathrm{ad}$ is an ideal of $\mathfrak{g}$ which contains the ideal $[\mathfrak{g}, \mathfrak{g}]$.

Now we equip a left invariant Riemannian metric $\langle \cdot, \cdot \rangle$ on G. Denote by $\mathfrak{u}^\perp$ the orthogonal complement of $\mathfrak{u}$ in $\mathfrak{g}$ with respect to $\langle \cdot, \cdot \rangle$. Then the homomorphism theorem implies that $\dim \mathfrak{u}^\perp = \dim \mathfrak{g}/\mathfrak{u} \leq 1$.

The following criterion for unimodularity is known (see [78, p. 317]).

Lemma 4.1. *A Lie group G with a left invariant metric is unimodular if and only if $\mathfrak{u} = \mathfrak{g}$.*

Based on this criterion, the ideal $\mathfrak{u}$ is called the *unimodular kernel* of $\mathfrak{g}$.

4.2. *Non-unimodular Lie groups*

For a 3-dimensional non-unimodular Lie group G, its unimodular kernel $\mathfrak{u}$ is commutative and of 2-dimension. On a non-unimodular Lie algebra $\mathfrak{g}$, we can take an orthonormal basis $\{E_1, E_2, E_3\}$ such that E_3 is *orthogonal* to $\mathfrak{u}$. The representation matrix A of $\mathrm{ad}(E_3) : \mathfrak{u} \to \mathfrak{u}$ relative to the basis $\{E_1, E_2\}$ is expressed as

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \neq O.$$

Then the commutation relations of the basis are given by

$$[E_3, E_1] = a_{11}E_1 + a_{21}E_2, \quad [E_3, E_2] = a_{12}E_1 + a_{22}E_2, \quad [E_1, E_2] = 0$$

with $\mathrm{tr}\,A = a_{11} + a_{22} \neq 0$. These commutation relations imply that $\mathfrak{g}$ is solvable.

Remark 4.1. Milnor [78] chose the following orthonormal basis $\{E_1, E_2, E_3\}$ for a non-unimodular Lie group G with left invariant Riemannian metric.

$$E_3 \in \mathfrak{u}^\perp, \quad \langle \mathrm{ad}(E_3)E_1, \mathrm{ad}(E_3)E_2 \rangle = 0.$$

Under this choice, the representation matrix $A = (a_{ij})$ satisfies

$$\mathrm{tr}\,A = a_{11} + a_{22} \neq 0, \quad a_{11}a_{12} + a_{21}a_{22} = 0.$$

Moreover $\{E_1, E_2, E_3\}$ diagonalises the Ricci operator (*cf.* [49]). In this paper we do not assume this orthogonality condition $\langle \mathrm{ad}(E_3)E_1, \mathrm{ad}(E_3)E_2 \rangle = 0$ (*cf.* [50]).

Non-unimodular Lie algebras $\mathfrak{g} = \mathfrak{g}_A$ are classified by the *Milnor invariant* $\mathsf{D} := \det A$. More precisely we know the following result (see [78, Lemma 4.10, p.320]).

Proposition 4.2 ([78]). *For any pair of real 2 by 2 matrices A and A' which are not scalar matrices, the non-unimodular Lie algebras $\mathfrak{g}_A$ and $\mathfrak{g}_{A'}$ are isomorphic if and only if $\mathrm{tr}\,A = \mathrm{tr}\,A'$ and their Milnor invariants D and D' agree.*

Corollary 4.1. *Let G_A and $G_{A'}$ be 3-dimensional non-unimodular Lie groups equipped with left invariant Riemannian metrics. Assume that both A and A' are not scalar matrices. Then G_A and $G_{A'}$ are isometric and isomorphic as Lie groups each other if and only if $\mathrm{tr}\,A = \mathrm{tr}\,A'$, $\mathsf{D} = \mathsf{D}'$ and the sets of principal Ricci curvatures are coincide.*

Remark 4.2. In [105] Tasaki and Umehara introduced an invariant of 3-dimensional Lie algebras equipped with inner products. Their invariant $\chi(\mathfrak{g}_A)$ for the non-unimodular Lie algebra $\mathfrak{g}_A$ is $4/\mathsf{D}$. Note that in case $\mathsf{D} = 0$, $\chi(\mathfrak{g}_A)$ is regarded as ∞.

The Levi-Civita connection ∇ is described as

$$\begin{aligned} \nabla_{E_1}E_1 &= a_{11}\,E_3, \ \nabla_{E_1}E_2 = \frac{a_{12}+a_{21}}{2}\,E_3, \ \nabla_{E_1}E_3 = -a_{11}E_1 - \frac{a_{12}+a_{21}}{2}\,E_2, \\ \nabla_{E_2}E_1 &= \frac{a_{12}+a_{21}}{2}\,E_3, \ \nabla_{E_2}E_2 = a_{22}\,E_3, \ \nabla_{E_2}E_3 = -\frac{a_{12}+a_{21}}{2}E_1 - a_{22}\,E_2, \\ \nabla_{E_3}E_1 &= \frac{a_{21}-a_{12}}{2}\,E_2, \ \nabla_{E_3}E_2 = \frac{a_{12}-a_{21}}{2}\,E_1, \ \nabla_{E_3}E_3 = 0. \end{aligned} \tag{4.1}$$

The Riemannian curvature R is described as

$$
\begin{aligned}
R(E_1,E_2)E_1 =& -K_{12}E_2, \quad R(E_1,E_2)E_2 = K_{12}E_2, \quad , R(E_1,E_2)E_3 = 0,\\
R(E_2,E_3)E_1 =&(a_{11}a_{12}+a_{21}a_{22})E_3, \quad R(E_2,E_3)E_2 = -K_{23}E_3,\\
R(E_2,E_3)E_3 =& -(a_{11}a_{12}+a_{21}a_{22})E_1 + K_{23}E_2,\\
R(E_3,E_1)E_1 =&K_{13}E_3, \quad R(E_3,E_1)E_2 = -(a_{11}a_{12}+a_{21}a_{22})E_3,\\
R(E_3,E_1)E_3 =& -K_{13}E_1 + (a_{11}a_{12}+a_{21}a_{22})E_2,
\end{aligned}
$$

where

$$
\begin{aligned}
K_{12} = K(E_1\wedge E_2) =& -a_{11}a_{22} + \frac{1}{4}(a_{12}+a_{21})^2,\\
K_{13} = K(E_1\wedge E_3) =& -a_{11}^2 + \frac{1}{4}(a_{12}^2 - a_{21}^2) - \frac{a_{21}}{2}(a_{12}+a_{21}),\\
K_{23} = K(E_1\wedge E_3) =& -a_{22}^2 - \frac{1}{4}(a_{12}^2 - a_{22}^2) - \frac{a_{12}}{2}(a_{12}+a_{21}).
\end{aligned}
$$

The Ricci tensor field ρ has components $\rho_{ij} = \rho(E_i,E_j)$;

$$
\begin{aligned}
\rho_{11} =& -a_{11}(a_{11}+a_{22}) + \frac{1}{2}(a_{12}^2 - a_{21}^2), \quad \rho_{12} = -a_{11}a_{12} + a_{21}a_{22}, \quad \rho_{13} = 0,\\
\rho_{22} =& -a_{22}(a_{11}+a_{22}) - \frac{1}{2}(a_{12}^2 - a_{21}^2), \quad \rho_{23} = 0,\\
\rho_{33} =& -(a_{11}^2 + a_{22}^2) - \frac{1}{2}(a_{12}+a_{21})^2.
\end{aligned}
$$

Remark 4.3. If we choose $\{E_1,E_2,E_3\}$ satisfying the orthogonality condition $\langle \mathrm{ad}(E_1)E_1, \mathrm{ad}(E_1)E_2\rangle = 0$, then the Ricci operator S is diagonalized by $\{E_1,E_2,E_3\}$.

The Lie algebra $\mathfrak{g}_A$ is realized as a Lie subalgebra of $\mathfrak{gl}_3\mathbb{R}$ spanned by the basis

$$
E_1 = \begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}, \quad E_3 = \begin{pmatrix} a_{11} & a_{12} & 0\\ a_{21} & a_{22} & 0\\ 0 & 0 & 0\end{pmatrix}.
$$

The simply connected Lie group $\widetilde{G}_A$ corresponding to the non-unimodular Lie algebra $\mathfrak{g}_A$ is given explicitly by

$$
\widetilde{G}_A = \left\{ \begin{pmatrix} \alpha_{11}(z) & \alpha_{12}(z) & x\\ \alpha_{21}(z) & \alpha_{22}(z) & y\\ 0 & 0 & 1\end{pmatrix} \;\middle|\; x,y,z\in\mathbb{R} \right\}, \tag{4.2}
$$

where $\alpha_{ij}(z)$ is the (i,j)-entry of $\exp(zA)$. This shows that $\widetilde{G}_A$ is the semi-direct product $\mathbb{R}^2 \rtimes \mathbb{R}$ with multiplication

$$
(x,y,z)\cdot(x',y',z') = (x+\alpha_{11}(z)x' + \alpha_{12}(z)y', y + \alpha_{21}(z)x' + \alpha_{22}(z)y', z+z'). \tag{4.3}
$$

These vectors E_1, E_2 and E_3 are regarded as left invariant vector fields

$$
E_1 = \alpha_{11}(z)\frac{\partial}{\partial x} + \alpha_{21}(z)\frac{\partial}{\partial y}, \quad E_2 = \alpha_{12}(z)\frac{\partial}{\partial x} + \alpha_{22}(z)\frac{\partial}{\partial y}, \quad E_1 = \frac{\partial}{\partial z}.
$$

The normal subgroup $U = \mathbb{R}^2 \rtimes \{0\}$ of $\tilde{G}_A$ with Lie algebra $\mathfrak{u}$ will be called the *canonical normal subgroup* of $\tilde{G}_A$ [67].

Let us choose E_3 as a unit normal vector field of U in $\tilde{G}_A$. Then the shape operator $\mathcal{A}_U$ derived from E_3 has components

$$
\mathcal{A}_U = \begin{pmatrix} a_{11} & (a_{12}+a_{21})/2\\ (a_{12}+a_{21})/2 & a_{22}\end{pmatrix}
$$

relative $\{E_1,E_2\}$. The extrinsic curvature $K_{\text{ext}}(U)$ of U is

$$
K_{\text{ext}}(U) = \det \mathcal{A}_U = a_{11}a_{22} - (a_{12}+a_{21})^2/4 = \mathsf{D} - \frac{1}{4}(a_{12}-a_{21})^2 \leq \mathsf{D}.
$$

Hence the Gauß curvature K_U is 0. The canonical normal subgroup U is a flat surface with constant mean curvature $\operatorname{tr} A/2$.

Example 4.1 (Hyperbolic 3-space)**.** When A is a nonzero scalar matrix $A = kI$, then $\tilde{G}_A$ is isometric to the hyperbolic 3-space of curvature $-k^2$. The matrix A has $\operatorname{tr} A = 2k$ and $\mathsf{D} = k^2 > 0$. The Lie group $A = -I$ will be appeared as $\widetilde{G}(0,0)$ in Example 9.1 and Example 9.2 and $S(-1)$ in Example 9.5.

On the other hand, when $a_{11} = a_{22} = k$ and $a_{21} = -a_{12} = \beta$, then $\operatorname{tr} A = 2k$ and $\mathsf{D} = k^2 + \beta^2 > 0$. The Lie group G_A is also isometric to hyperbolic 3-space of curvature $-k^2$. This Lie group with $k = -1$ will appear in Example 9.1 as $\widetilde{G}(\beta,\beta)$. The canonical normal subgroup of the hyperbolic 3-space $\mathbb{H}^3(-1)$ is nothing but the *horosphere*.

Example 4.2 (Product space)**.** If A is a diagonal matrix of the form $a_{11} = -k \neq 0$ and $a_{22} = 0$, then $K_{12} = K_{23} = 0$ and $K_{13} = -k^2 < 0$. The Lie group $\widetilde{G}_A$ is isometric to the product space $\mathbb{H}^2(-k^2) \times \mathbb{R}$. Analogously if A is a diagonal matrix of the form $a_{11} = 0$ and $a_{22} = -k \neq 0$, then $K_{12} = K_{13} = 0$ and $K_{23} = -k^2 < 0$. The Lie group $\widetilde{G}_A$ is isometric to the product space $\mathbb{H}^2(-k^2) \times \mathbb{R}$. In these cases, $\operatorname{tr} A = -k \neq 0$, $\mathsf{D} = 0$ and principal Ricci curvatures are $-k^2$, $-k^2$ and 0. Note that $\mathbb{H}^2(-k^2) \times \mathbb{R}$ is locally symmetric.

5. Almost contact Riemannian manifolds

In this section we recall fundamental ingredients of almost contact Riemannian geometry. In addition we recall some curvatures of our interest. For general information on almost contact Riemannian geometry, we refer to [6].

5.1. *Almost contact structures*

An *almost contact Riemannian structure* of a $(2n+1)$-manifold M is a quartet (φ, ξ, η, g) of structure tensor fields which satisfies:

$$\eta(\xi) = 1, \tag{5.1}$$

$$\varphi^2 = -\mathrm{I} + \eta \otimes \xi, \quad \varphi\xi = 0, \tag{5.2}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{5.3}$$

A $(2n+1)$-manifold $M = (M, \varphi, \xi, \eta, g)$ equipped with an almost contact Riemannian structure is called an *almost contact Riemannian manifold*. The vector field ξ is called the *characteristic vector field* of M.

The 2-form

$$\Phi(X,Y) = g(X, \varphi Y)$$

is called the *fundamental* 2*-form* of M.

The foliation defined by the characteristic vector field ξ is called the *characteristic foliation*. The following result is known (see *e.g.* [12, p. 196], [15]).

Proposition 5.1. *On an almost contact Riemannian manifold $M = (M, \varphi, \xi, \eta, g)$, the following properties are mutually equivalent*:

- *The characteristic foliation is taut.*
- $\nabla_\xi \xi = 0$*, where ∇ is the Levi-Civita connection of g.*
- $\pounds_\xi \eta = 0$*, where $\pounds_\xi$ is the Lie-differentiation by ξ.*
- $\mathrm{d}\eta(\xi, \cdot) = 0$.

Definition 5.1. An almost contact Riemannian manifold M is said to be *weakly η-Einstein* if its Ricci operator S has the form

$$S = \mathsf{A}\,\mathrm{I} + \mathsf{B}\eta \otimes \xi$$

for some functions A and B. When both A and B are constant, M is said to be *η-Einstein*.

Here we recall an auxiliary endomorphism fields h and ℓ which are highly useful for the study of almost contact Riemannian manifolds. The endomorphism field h is defined by $h = \pounds_\xi \varphi/2$. Next, we introduce a self-adjoint endomorphism field ℓ by

$$\ell(X) = R(X, \xi)\xi, \quad X \in \mathfrak{X}(M).$$

One can see that $\ell = -F_\xi$ on the distribution $\mathcal{D}$. The self-adjoint operator ℓ is called the *characteristic Jacobi operator* of M.

5.2. Normality and CR-structures

On an almost contact Riemannian manifold M, we define a complex vector subbundle $\mathcal{S}$ of the complexified tangent bundle $T^{\mathbb{C}}M$ by

$$\mathcal{S} = \{X - \sqrt{-1}\varphi X \mid X \in \Gamma(\mathcal{D})\},$$

where $\mathcal{D}$ is the hyperplane field

$$\mathcal{D} = \{X \in TM \mid \eta(X) = 0\}.$$

The complex vector bundle $\mathcal{S}$ is called an *almost* CR-*structure* associated to M. If $\mathcal{S}$ satisfies the integrability condition

$$[\Gamma(\mathcal{S}), \Gamma(\mathcal{S})] \subset \Gamma(\mathcal{S}),$$

then M is said to be *integrable*. To avoid the confusion with "integrability of $\mathcal{D}$" , we often say that M is *CR-integrable* if $\mathcal{S}$ is integrable. It is known that M is CR-integrable if and only if

$$[\varphi, \varphi](X, Y) + 2\mathrm{d}\eta(X, Y)\xi = 0$$

for all $X, Y \in \Gamma(\mathcal{D})$. Here $[\varphi, \varphi]$ is the Nijenhuis torsion of φ.

More strongly, an almost contact Riemannian manifold M is said to be *normal* if

$$N(X, Y) := [\varphi, \varphi](X, Y) + 2\mathrm{d}\eta(X, Y)\xi = 0$$

for all $X, Y \in \Gamma(TM)$.

5.3. Some curvature tensor fields

Definition 5.2. Let $(M, \varphi, \xi, \eta, g)$ be an almost contact Riemannian manifold. A tangent plane Π_p at $p \in M$ is said to be *holomorphic* if it is invariant under φ_p.

It is easy to see that a tangent plane Π_p is holomorphic if and only if ξ_p is orthogonal to Π_x.

In case $\dim M = 3$, the only holomorphic plane at p is $\mathcal{D}_p = \{X \in T_pM \mid \eta(X) = 0\}$. Hence we obtain a smooth function H on M defined by $H_p = K(\mathcal{D}_p)$. The smooth function H is called the *holomorphic sectional curvature* of M.

Next, we define a tensor field ρ^* on M by (*cf*. [81]):

$$\rho^*(X, Y) := \frac{1}{2}\operatorname{tr} R(X, \varphi Y)\varphi.$$

One can see that $\rho^*(X, \xi) = 0$ for all $X \in \mathfrak{X}(M)$. It should be remarked that ρ^* is *not* symmetric, in general. Next we denote by ρ^φ the symmetric part of ρ^*, that is,

$$\rho^\varphi(X, Y) = \frac{1}{2}\{\rho^*(X, Y) + \rho^*(Y, X)\}.$$

We call ρ^φ the *φ-Ricci tensor field* of M [24].

Definition 5.3. An almost contact Riemannian manifold M is said to be a *weakly φ-Einstein manifold* if

$$\rho^\varphi(X, Y) = \lambda g^\varphi(X, Y), \quad X, Y \in \mathfrak{X}(M)$$

for some function λ. Here the symmetric tensor field g^φ is defined by

$$g^\varphi(X, Y) = g(\varphi X, \varphi Y), \quad X, Y \in \mathfrak{X}(M).$$

When λ is a constant, then M is said to be a *φ-Einstein manifold*. The function $\mathrm{s}^\varphi = \operatorname{tr} \rho^\varphi$ is called the *φ-scalar curvature* of M.

When M is weakly φ-Einstein, then we have $\rho^\varphi = \{\mathrm{s}^\varphi/(2n)\}g^\varphi$.

Remark 5.1. An almost contact Riemannian manifold M is said to be *weakly $*$-Einstein* if

$$\rho^*(X, Y) = \lambda g(X, Y), \quad X, Y \in \Gamma(\mathcal{D})$$

for some function λ. The function $\mathrm{s}^* = \operatorname{tr} \rho^*$ is called the *$*$-scalar curvature* of M. A weakly $*$-Einstein manifold of constant $*$-scalar curvature is called a *$*$-Einstein manifold*. Clearly $\mathrm{s}^\varphi = \mathrm{s}^*$.

To close this subsection we recall the following definition (*cf*. Kimura-Maeda [72]):

Definition 5.4. Let M be an almost contact Riemannian manifold. A tensor field P on M of type $(1, r)$ is said to be *η-parallel* if

$$g((\nabla_X P)(Y_1, Y_2, \cdots, Y_r), Z) = 0$$

for all vector fields $X, Y_1, Y_2 \cdots, Y_r, Z \in \Gamma(\mathcal{D})$.

The following notion was introduced by the present author [53] (see also Kon [75]).

Definition 5.5 ([53])**.** Let M be an almost contact Riemannian manifold. A tensor field P on M of type $(1, r)$ is said to be *strongly η-parallel* if

$$g((\nabla_X P)(Y_1, Y_2, \cdots, Y_r), Z) = 0$$

for all vector fields $X \in \Gamma(TM)$ and $Y_1, Y_2 \cdots, Y_r, Z \in \Gamma(\mathcal{D})$.

In addition we introduce the following notion:

Definition 5.6 ([65])**.** Let M be an almost contact Riemannian manifold. A tensor field P on M of type $(1, r)$ is said to be *dominantly η-parallel* if

$$g((\nabla_X P)(Y_1, Y_2, \cdots, Y_r), Z) = 0$$

for all vector fields $X, Y_1, Y_2 \cdots, Y_r \in \Gamma(TM)$ and $Z \in \Gamma(\mathcal{D})$.

We extend the notion of η-parallelism to scalar fields:

Definition 5.7 ([65])**.** A scalar field f on an almost contact Riemannian manifold M is said to be *η-parallel* if

$$\mathrm{d}f(X) = 0$$

for all vector field $X \in \Gamma(\mathcal{D})$.

Here we introduce the η-parallelism for endomorphism fields and scalar fields:

Definition 5.8. An endomorphism field F on an almost contact metric manifold M is said to be

- *η-parallel* if it satisfies $g((\nabla_X F)Y, Z) = 0$ for all vector fields X, Y and Z on M orthogonal to ξ.
- *strongly η-parallel* if it satisfies $g((\nabla_X F)Y, Z) = 0$ for all vector field X on M and any vector fields Y and Z on M orthogonal to ξ.
- *dominantly η-parallel* if it satisfies $g((\nabla_X F)Y, Z) = 0$ for all vector fields X and Y on M and any vector field Z on M orthogonal to ξ.

Here we mention a notion which is related to the η-parallelism. According to Blair [5], an endomorphism field F on an almost contact Riemannian manifold M is said to be *Killing* if it satisfies $(\nabla_X F)X = 0$ for all vector fields on M. More generally F is said to be *transversally Killing* if $(\nabla_X F)X = 0$ for all vector field X on M orthogonal to ξ [23]. In [87, Remark 1.2], the authors claimed that the transversal Killing property for the Ricci operator S of an almost contact Riemannian 3-manifold is much weaker than the η-parallelism of S. However from the table (10.1)–(10.18) given in Section 10.2, "transversal-Killing S" is stronger than "η-parallel S".

5.4. *The local φ-symmetry*

An almost contract Riemannian manifold $M = (M, \varphi, \xi, \eta, g)$ is said to be a *contact Riemannian manifold* if $\Phi = \mathrm{d}\eta$. On a contact Riemannian manifold M, the 1-form η is a contact form.

Remark 5.2. A 1-form η on a $(2n+1)$-manifold M is called a *contact form* if it satisfies $(\mathrm{d}\eta)^n \wedge \eta \neq 0$ on whole M. A $(2n+1)$-manifold M equipped with a contact form η is called a *contact manifold* (in the strict sense). On a contact manifold (M, η), there exits a unique vector field ξ satisfying $\eta(\xi) = 1$ and $\mathrm{d}\eta(\xi, \cdot) = 0$. The vector field ξ is called the *Reeb vector field* of (M, η). Moreover there exist an endomorphism field φ and a Riemannian metric g so that the quartet $(M, \varphi, \xi, \eta, g)$ is a contact Riemannian manifold.

A contact Riemannian manifold M is said to be a *K-contact manifold* if its Reeb vector field ξ is a Killing vector field. A *Sasakian* manifold is a normal contact Riemannian manifold. One can see that Sasakian manifolds are K-contact. Only in dimension 3, the converse holds.

The η-parallelism of the Riemannian curvature was investigated first by Takahashi [102] for Sasakian manifolds. Buken and Vanhecke pointed out that if a K-contact manifold M has η-parallel Riemannian curvature, then M is Sasakian [11]. A K-contact manifold M is said to be a (Sasakian) *φ-symmetric space* (in the sense of Takahashi) if its Riemannian curvature R is η-parallel. A K-contact manifold M is locally φ-symmetric if and only if all the characteristic reflections are isometric [11, 102]. There are two directions to generalize the local φ-symmetry to general almost contact Riemannian manifolds. Blair, Koufogiorgos and Sharma [7] introduced the notion of local φ-symmetry for general contact Riemannian manifolds by the η-parallelism of R. On the other hand, Boeckx and Vanhecke [10] defined the local φ-symmetry of contact Riemannian manifolds by the property "all the characteristic reflections are isometric". To distinguish these two classes, Boeckx, Buken and Vanhecke [8] proposed the terminologies "strongly locally φ-symmetric space" and "weakly locally φ-symmetric space". According to [8], a contact Riemannian manifold is said to be a *weakly locally φ-symmetric space* if its Riemannian curvature R is η-parallel. On the other hand, a contact Riemannian manifold is said to be a *strongly locally φ-symmetric space* if all the characteristic reflections are isometric. They showed that strongly locally φ-symmetric spaces are weakly locally φ-symmetric. For more information on weakly locally φ-symmetric contact Riemannian 3-manifolds, we refer to [89].

Now let us consider almost contact Riemannian 3-manifolds. From the formula of ∇R, we deduce the following facts:

Proposition 5.2. *On an almost contact Riemannian 3-manifold M, if the Ricci operator S and scalar curvature s are η-parallel, then so is the Riemannian curvature R.*

Conversely, if the Riemannian curvature R is η-parallel, then the Ricci operator S is η-parallel if and only if

$$\eta((\nabla_W R)(\xi, X)Y) = 0 \tag{5.4}$$

holds for all vector fields W, X and Y orthogonal to ξ.

Proof. ($\Rightarrow$): Assume that both S and s are η-parallel, then from (2.2) R is η-parallel.

($\Leftarrow$): Conversely, take a local orthonormal frame field of the form $\{e_1, e_2 = \varphi e_1, e_3 = \xi\}$, then under the assumption R is η-parallel, from (2.3), we obtain

$$g((\nabla_W S)X, Y) = \eta((\nabla_W R)(\xi, X)Y)$$

for all W, X and Y orthogonal to ξ. Thus the η-parallelism of S is equivalent to (5.4). □

Remark 5.3. Obviously, η-parallelism of R together with the η-parallelism of S implies that of s.

While the local symmetry is equivalent to the parallelism of the Ricci operator on arbitrary Riemannian 3-manifolds, especially almost contact metric 3-manifolds, the η-parallelism of R is not equivalent to that of S on almost contact metric 3-manifolds.

Remark 5.4. In [29, 30], De and Pathak defined local φ-symmetry and φ-symmetry in the following manner:

- An almost contact Riemannian manifold M is said to be *locally φ-symmetric* if

$$\varphi^2((\nabla_W R)(X, Y)Z) = 0$$

  for all vector fields X, Y, Z and W orthogonal to M.
- An almost contact Riemannian manifold M is said to be *φ-symmetric* if

$$\varphi^2((\nabla_W R)(X, Y)Z) = 0$$

  for all vector fields X, Y, Z and W on M.

Obviously the local φ-symmetry in the sense of [29, 30] coincides with the η-parallelism of R, *i.e.*, the weak local φ-symmetry.

On the other hand, the φ-symmetry in the sense of [29, 30] is equivalent to

$$g((\nabla_W R)(X, Y)Z, V) = 0$$

for all vector fields X, Y, Z and W on M and vector field V orthogonal to ξ. This is equivalent to the dominant η-parallelism of R.

5.5. *Almost Kenmotsu manifolds*

Now we turn our attention to almost Kenmotsu manifolds.

Definition 5.9 ([68]). An almost contact Riemannian manifold M is said to be *almost Kenmotsu* if $\mathrm{d}\eta = 0$ and $\mathrm{d}\Phi = 2\eta \wedge \Phi$. An almost Kenmotsu manifold is said to be *Kenmotsu* if it is normal.

A non-normal almost Kenmotsu manifold is called a *strictly almost Kenmotsu manifold*.

Remark 5.5. An almost contact Riemannian manifold M is said to be *almost α-Kenmotsu* if $\mathrm{d}\eta = 0$ and $\mathrm{d}\Phi = 2\alpha\eta \wedge \Phi$ for some non-zero constant α [71]. On the other hand, M is said to be an *α-cosymplectic manifold* (or *α-coKähler manifold*) if $\mathrm{d}\eta = 0$ and $\mathrm{d}\Phi = 2\alpha\eta \wedge \Phi$ for some constant α.

Remark 5.6. Let M be an almost contact Riemannian manifold. Consider the Riemannian product $M \times \mathbb{R}$ of M and the real line. Then one can extend the almost contact structure to an almost complex structure J on $M \times \mathbb{R}$. Then the normality of M is equivalent to the integrability of J (see [6]). In particular if M is Kenmotsu, then $M \times \mathbb{R}$ is a locally conformal Kähler manifold with parallel anti Lee field (see [55]).

It should be remarked that every almost Kenmotsu manifold satisfies $\mathrm{div}\, \xi = 2n$. Hence almost Kenmotsu manifolds can not be compact. On an almost Kenmotsu manifold, we have [34, 36]:

$$\varphi \circ \ell \circ \varphi - \ell = 2(h^2 - \varphi^2), \quad \nabla_\xi h = -\varphi - 2h - \varphi \circ h^2 - \varphi \circ \ell, \quad \mathrm{tr}\, \ell = -2n - \mathrm{tr}(h^2).$$

Since $\mathrm{d}\eta = 0$ on an almost Kenmotsu manifold M, we have $\pounds_\xi \eta = 0$ (*cf.* Proposition 5.1). In addition, the hyperplane field $\mathcal{D}$ satisfies $[\mathcal{D}, \xi] \subset \mathcal{D}$. The Levi-Civita connection ∇ satisfies [71]:

$$\nabla_\xi \xi = 0, \quad \nabla_\xi \varphi = 0, \quad \nabla_\xi \mathcal{D} \subset \mathcal{D}.$$

The distribution $\mathcal{D}$ on an almost Kenmotsu manifold M is integrable and hence it defines a foliation $\mathcal{F}$ on M. This foliation is called the *canonical foliation* of M [71].

Let us consider a leaf L of the canonical foliation of an almost Kenmotsu manifold M. Then we choose ξ as a unit normal vector field of L in M. The *shape operator* $\mathcal{A}$ of L derived from ξ is introduced by the *Weingarten formula*

$$\mathcal{A}X = -\nabla_X \xi = -X + \varphi h X$$

for any vector field X tangents to L. We extend $\mathcal{A}$ to an endomorphism field on M by $\mathcal{A}X = -\nabla_X \xi$ for all $X \in \mathfrak{X}(M)$.

Kim and Park showed the following fact:

Proposition 5.3 ([71]). *The leaves of the canonical foliation are almost Kähler manifolds with mean curvature vector field $-\xi$. Those leaves are totally umbilical if and only if $h = 0$.*

An almost Kenmotsu manifold M is said to be an *almost Kenmotsu manifold with Kähler leaves* if leaves of the canonical foliation are Kähler manifolds.

Theorem 5.1 ([71, 36]). *An almost Kenmotsu manifold M has Kähler leaves if and only if*

$$(\nabla_X \varphi)Y = g((\varphi + h)X, Y)\xi - \eta(Y)(\varphi + h)X. \tag{5.5}$$

In such a case we have

$$\nabla_X \xi = (\mathrm{I} + h\varphi)X - \eta(X)\xi.$$

Dileo and Pastore obtained the following result.

Proposition 5.4 ([34]). *Let M be an almost Kenmotsu manifold with Kähler leaves. Then M is Kenmotsu if and only if $\nabla \xi = -\varphi^2$.*

Remark 5.7 (para-Sasakian structure). The Kenmotsu structure is closely related to the para-Sasakian structure. A quintet (ψ, ξ, η, g) of tensor fields on a manifold M is said to be an *almost paracontact Riemannian structure* if it satisfies

$$\psi^2 = \mathrm{I} - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\psi X, \psi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta = g(\xi, \cdot).$$

One can see that $g(\psi X, Y) = g(X, \psi Y)$. The *type* (p, q) is the signature of the symmetric tensor field $\Psi(X, Y) = g(\psi X, Y)$.

A manifold M equipped with an almost paracontact Riemannian structure is called an *almost paracontact Riemannian manifold*.

An almost paracontact Riemannian manifold M is said to be *para-Sasakian* if $\psi = \nabla \xi$ and satisfies

$$(\nabla_X \psi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi.$$

Corollary 5.1 ([56]). *Let $M = (M, \varphi, \xi, \eta, g)$ be a Kenmotsu manifold. Then $(M, \nabla\xi, \xi, \eta, g)$ is a para-Sasakian manifold of type $(2n, 0)$.*

To rewrite Proposition 5.4 here we recall the operator $h' := h \circ \varphi$.

Proposition 5.5 (*cf.* [71]). *The operator h' on an almost Kenmotsu manifold M satisfies*

$$h' = \nabla\xi + \varphi^2, \quad h' = -\varphi \circ h, \quad h' \circ \varphi + \varphi \circ h' = 0, \quad h'\xi = 0, \quad \operatorname{tr} h' = 0.$$

Thus $h' = 0$ if and only if $\nabla\xi = -\varphi^2$. The Kenmotsu property is characterized as follows:

Proposition 5.6. *Let M be an almost Kenmotsu manifold with Kähler leaves. Then M is Kenmotsu if and only if $h' = 0$.*

Example 5.1 (Warped products). Let $(\overline{M}, \bar{g}, J)$ be an almost Kähler manifold of dimension $2n \geq 2$. We consider the warped product manifold $M = \mathbb{R} \times_{ce^t} \overline{M}$ with structure

$$g = \mathrm{d}t^2 + c^2 e^{2t}\bar{g}, \quad \xi = \partial_t, \quad \eta = \mathrm{d}t,$$

where c is a positive constant. Then we introduce an endomorphism field φ by

$$\varphi X = JX, \quad \varphi\xi = 0, \quad X \in \Gamma(T\overline{M}).$$

Then one can check that $(M, \varphi, \xi, \eta, g)$ is an almost Kenmotsu manifold satisfying $h = h' = 0$. Moreover M is Kenmotsu if and only if $\overline{M}$ is Kähler.

In particular the warped product model

$$\mathbb{H}^{2n+1}(-1) = \mathbb{R} \times_{ce^t} \mathbb{C}^n$$

of the hyperbolic space $\mathbb{H}^{2n+1}(-1)$ of curvature -1 is a Kenmotsu manifold.

Local structure of almost Kenmotsu manifolds with $h = 0$ is described as follows:

Proposition 5.7 ([34]). *If an almost Kenmotsu manifold M satisfies $h = 0$, then M is locally isomorphic to the warped product $I \times_f \overline{M}$, where I is an interval, $\overline{M}$ is an almost Kähler manifold and $f(t) = ce^t$ for some positive constant c.*

Example 5.2 (Product manifold). Let us consider the upper half plane model

$$\mathbb{H}^2(-k^2) = (\{(u, v) \in \mathbb{R}^2 \mid v > 0\}, \bar{g}), \quad \bar{g} = \frac{\mathrm{d}u^2 + \mathrm{d}v^2}{k^2 v^2}$$

of the hyperbolic plane of curvature $-k^2 < 0$. On the Riemannian product $M = \mathbb{H}^2(-k^2) \times \mathbb{R}$, we introduce a strictly almost Kenmotsu structure on M. The product manifold M is realized as the following linear Lie group

$$G = \left\{ \begin{pmatrix} v & u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & w \end{pmatrix} \;\middle|\; u, v, w \in \mathbb{R},\ v > 0 \right\}.$$

The Lie algebra $\mathfrak{g}$ is spanned by the basis

$$\epsilon_1 = \begin{pmatrix} 0 & k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \epsilon_2 = \begin{pmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \epsilon_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The left invariant vector fileds induced from this basis are

$$\epsilon_1 = (kv)\frac{\partial}{\partial u}, \quad \epsilon_2 = (kv)\frac{\partial}{\partial v}, \quad \epsilon_3 = \frac{\partial}{\partial t}.$$

The Levi-Civita connection ∇ is described as

$$\nabla_{\epsilon_1}\epsilon_1 = k\,\epsilon_2, \quad \nabla_{\epsilon_1}\epsilon_2 = -k\,\epsilon_1, \quad \nabla_{\epsilon_i}\epsilon_j = 0 \text{ for other } i, j.$$

The Riemannian curvature R is given by

$$R(\epsilon_1, \epsilon_2)\epsilon_1 = k^2\epsilon_2, \quad R(\epsilon_1, \epsilon_2)\epsilon_2 = -k^2\epsilon_1, \quad R(\epsilon_i, \epsilon_j)\epsilon_k = 0 \text{ for other } i, j, k.$$

Hence the sectional curvatures $K(\epsilon_i \wedge \epsilon_j)$ are computed as

$$K(\epsilon_1 \wedge \epsilon_2) = -k^2, \quad K(\epsilon_1 \wedge \epsilon_3) = K(\epsilon_2 \wedge \epsilon_3) = 0.$$

Here we compare G with $\widetilde{G}_A$ given in (4.2). Set

$$E_1 = \epsilon_1, \quad E_2 = \epsilon_3, \quad E_3 = -\epsilon_2.$$

Then the unimodular kenel $\mathfrak{u}$ is spanned by $\{E_1, E_2\}$. Then the representation matrix A of $\mathrm{ad}(E_3)$ is

$$A = \begin{pmatrix} -k & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.6}$$

Hence G is isometric and isomorphic to the non-unimodular Lie group

$$\widetilde{G}_A = \left\{ \begin{pmatrix} e^{-kz} & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \; x, y, z \in \mathbb{R} \right\}$$

with representation matrix (5.6). The left invariant metric is expressed as

$$e^{2kz}\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

In fact G and $\widetilde{G}_A$ are isometric under the isometry

$$u := kx, \quad v := e^{-kz}, \quad t := y. \tag{5.7}$$

Define a left invariant almost contact structure (φ, ξ, η) compatible to the product metric $g = \bar{g} + dt^2$ by

$$\xi = -\epsilon_2, \quad \eta = -\frac{\mathrm{d}v}{kv}, \quad \varphi\epsilon_1 = -\epsilon_3, \quad \varphi\epsilon_2 = 0, \quad \varphi\epsilon_3 = \epsilon_1.$$

Then we obtain

$$\mathrm{d}\Phi = k\,\eta \wedge \Phi.$$

Hence if we choose $k = 2$, M is a strictly almost Kenmotsu 3-manifold. This example will appear in Example 9.7.

Every almost Kenmotsu manifold satisfies

$$2g((\nabla_X \varphi)Y, Z) = 2\eta(Z)g(\varphi X, Y) - 2\eta(Y)g(\varphi X, Z) + g(N(Y, Z), \varphi X).$$

From this we can deduce that M is CR-integrable if and only if φ is η-parallel, that is

$$g((\nabla_X \varphi)Y, Z) = 0$$

for all X, Y and $Z \in \Gamma(\mathcal{D})$.

Thus we arrive at the following fundamental facts (see *e.g.*, [32, 36]).

Theorem 5.2. *Let M be an almost Kenmotsu manifold. Then the following properties are mutually equivalent*:

- *M has Kähler leaves.*
- *The associated almost* CR*-structure is integrable.*
- *The endomorphism field φ is η-parallel.*
- *The covariant derivatie $\nabla\varphi$ is given by* (5.5)*.*

Theorem 5.3. *Let M be an almost Kenmotsu manifold. Then the following properties are mutually equivalent*:

- *$h = 0$.*
- *$h' = 0$.*
- *The canonical foliation is totally umbilical.*
- *M is locally isomorphic to a warped product of a real line and an almost Kähler manifold.*

Proposition 5.8 ([34])**.** *An almost Kenmotsu manifold is of constant curvature, then it is a Kenmotsu manifold of constant curvature* -1*.*

On the other hand, Dileo and Pastore [35, Proposition 4] showed the following fact:

Proposition 5.9. *Let M be an almost Kenmotsu manifold. If h' is η-parallel and* 0 *is a simple eigenvalue of it, then the associated almost* CR*-structure is integrable.*

Dileo and Pastore obtained a local classification of almost Kenmotsu manifolds with η-parallel h' and satisfying $\nabla_\xi h' = 0$ [35] (see also [32, 33]).

Theorem 5.4. *Let M^{2n+1} be an almost Kenmotsu manifold with η-parallel h'. Assume that $\nabla_\xi h' = 0$. Denote by $\{0, \lambda_1, -\lambda_1, \dots, \lambda_r, -\lambda_r\}$ the spectrum of h' with $\lambda_i > 0$. Let m_i and $2m_0 + 1$ be the multiplicity of λ_i and* 0*, respectively. Then M is locally isomorphic to a multi-warped product*

$$I \times_{f_0} M_0 \times_{f_1^+} M_{\lambda_1} \times_{f_1^-} M_{-\lambda_1} \times_{f_2^+} \cdots \times_{f_r^+} M_{\lambda_r} \times_{f_r^-} M_{-\lambda_r},$$

where I is an open interval, M_0 is an almost Kähler manifold of dimension $2m_0$, $M_{\pm\lambda_i}$ are flat Riemannian manifolds of dimension m_i. The warping functions are $f_0(t) = ce^t$, $f_i^+(t) = c_{1i}e^{(1+\lambda_i)t}$ and $f_i^-(t) = c_{2i}e^{(1-\lambda_i)t}$ with positive constants c_0, c_{1i} and c_{2i}.

5.6. Kenmotsu manifolds

From Proposition 5.5 we have the following curvature formula for almost Kenmotsu manifolds (*cf.* [34, 36]):

$$R(X,Y)\xi = \eta(X)(\mathrm{I}+h')Y - \eta(Y)(\mathrm{I}+h')X + (\nabla_X h')Y - (\nabla_Y h')X, \quad X, Y \in \Gamma(TM) \tag{5.8}$$

Hence we obtain

$$\ell(X) = R(X,\xi)\xi = \eta(X)\xi - (\mathrm{I}+h')X + (\nabla_X h')\xi - (\nabla_\xi h')X.$$

Kenmotsu manifolds have particular curvature properties. For instance [69]:

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \quad X, Y \in \Gamma(TM) \tag{5.9}$$

and

$$K(X \wedge \xi) = -1, \quad X \in \Gamma(\mathcal{D}).$$

The characteristic Jacobi operator has the form

$$\ell(X) = \eta(X)\xi - X = \varphi^2 X, \quad X \in \Gamma(TM).$$

Kenmotsu showed that semi-symmetric Kenmotsu manifolds are of constant curvature -1.

In the class of Sasakian manifolds [resp. cosymplectic manifolds], there is a particularly nice subclass, the class of *Sasakian space forms* [resp. *cosymplectic space forms*]. In the class of Kenmotsu manifolds, constancy of holomorphic sectional curvature is a too strong restriction. In fact, Kenmotsu [69] showed

Proposition 5.10. *Let M be a Kenmotsu manifold of dimension greater than* 3*. Then M is of constant holomorphic sectional curvature if and only f it is of constant curvature* -1*.*

Three dimensional case will be discussed in Proposition 6.6.

5.7. H-almost Kenmotsu manifolds

The harmonicity of ξ is characterized in terms of Ricci opeartor as follows.

Theorem 5.5 ([91, 94])**.** *On an almost Kenmotsu manifold M, ξ is a harmonic unit vector field if and only if ξ is an eigenvector field of the Ricci operator S.*

Here we introduce the following notion.

Definition 5.10. An almost Kenmotsu manifold whose characteristic vector field ξ is a harmonic unit vector field is called an *H-almost Kenmotsu manifold*.

5.8. Canonical connections

Let $M = (M, \varphi, \xi, \eta, g)$ be an almost contact Riemannian manifold. Define a tensor field $A = A^t$ of type $(1, 2)$ by

$$A^t_X Y = -\frac{1}{2}\varphi(\nabla_X \varphi)Y - \frac{1}{2}\eta(Y)\nabla_X\xi - t\eta(X)\varphi Y + (\nabla_X \eta)(Y)\xi, \tag{5.10}$$

for all vector fields X and Y. Here t is a real constant. We define a linear connection $\widetilde{\nabla}^t$ on M by

$$\widetilde{\nabla}^t_X Y = \nabla_X Y + A^t_X Y.$$

The connection $\widetilde{\nabla}^t$ is called the *generalized Tanaka-Webster-Okumura connection* of M (gTWO-connection, in short) [58]. The gTWO-connection $\widetilde{\nabla}^t$ on an almost contact Riemannian manifold satisfies the following properties:

$$\widetilde{\nabla}\varphi = 0, \quad \widetilde{\nabla}\xi = 0, \quad \widetilde{\nabla}\eta = 0, \quad \widetilde{\nabla}g = 0.$$

Note that the connection ∇^0 is the (φ, ξ, η)-connection introduced by Sasaki and Hatakeyama in [98]. Moreover $\widetilde{\nabla}^1$ was introduced by Cho [20]. When M is a strongly pseudo-convex CR-manifold, the gTWO-connection has the form:

$$\widetilde{\nabla}^t_X Y = \nabla_X Y - t\eta(X)\varphi Y + \eta(Y)\varphi(\mathrm{I} + h)X - g(\varphi(I + h)X, Y)\xi$$

This formula shows that when M is a strongly pseudo-convex CR-manifold, $\widetilde{\nabla}^t|_{t=-1}$ coincides with the *Tanaka-Webster connection*. In case M is a Sasakian manifold, $\{\widetilde{\nabla}^t\}_{t\in\mathbb{R}}$ coincides with the 1-parameter family of linear connections introduced by Okumura [80]. In particular, on Sasakian manifolds, $\widetilde{\nabla}^1$ is the so-called *Okumura connection*.

Remark 5.8 (Tanno's generalized Tanaka-Webster connection). Let M be a contact Riemannian manifold. Tanno [104] introduced the following linear connection on M:

$$^{\mathsf{T}}\nabla_X Y := \nabla_X Y + \eta(X)\varphi Y - \eta(Y)\nabla_X\xi + \{(\nabla_X\eta)Y\}\xi. \tag{5.11}$$

Since on contact Riemannian manifolds, the covariant derivative $\nabla\xi$ is given by $\nabla\xi = -\varphi(\mathrm{I} + h)$, (5.11) is rewritten as

$$^{\mathsf{T}}\nabla_X Y = \nabla_X Y + \eta(X)\varphi Y + \eta(Y)\varphi(\mathrm{I} + h)X - g(\varphi(\mathrm{I} + h)X, Y)\xi. \tag{5.12}$$

This linear connection is called the *generalized Tanaka-Webster connection*. Note that Tanno called it the *generalized Tanaka connection* in [104]. In case, the associated almost CR-structure $\mathcal{S}$ is integrable, the generalized Tanaka-Webster connection coincides with our gTWO-connection $\widetilde{\nabla}^t|_{t=-1}$. It should be remarked that the generalized Tanaka-Webster connection $^{\mathsf{T}}\nabla$ does not coincide with $\widetilde{\nabla}^t|_{t=-1}$ if $\mathcal{S}$ is non-integrable. In fact, ξ, η and g are parallel with respect to $^{\mathsf{T}}\nabla$ but for φ, $^{\mathsf{T}}\nabla$ satisfies

$$\left(^{\mathsf{T}}\nabla_X\varphi\right)Y = \mathcal{Q}(Y, X)$$

holds. Here $\mathcal{Q}$ is the *Tanno tensor field* defined by

$$\mathcal{Q}(X, Y) = (\nabla_Y\varphi)X + \{(\nabla_Y\eta)(\varphi X)\}\xi + \eta(X)\varphi\nabla_Y\xi.$$

Hence we notice that on a contact Riemannian manifold M, $^{\mathsf{T}}\nabla = \widetilde{\nabla}^t|_{t=-1}$ if and only if its associated CR-structure is integrable.

Dileo and Pastore proved the following characterization of CR-integrability for almost Kenmotsu manifolds.

Proposition 5.11 ([36]). *Let M be an almost Kenmotsu manifold. Then its associated CR-structure is integrable if and only if there exits a linear connection $\overline{\nabla}$ satisfying*

- *The structure tensor fields φ, ξ and g and η are parallel with respect to it.*
- *The torsion $\overline{T}$ of $\overline{\nabla}$ satisfies*

$$\overline{T}(X, Y) = 0, \quad X, Y \in \Gamma(\mathcal{D}),$$

$$2\overline{T}(\xi, X) = X + h'X, \quad X \in \Gamma(\mathcal{D}),$$

The endomorphism field $X \longmapsto \bar{T}(\xi, X)$ is self-adjoint with respect to g.

The linear connection $\overline{\nabla}$ is given explicitly by the formula:

$$\overline{\nabla}_X Y = \nabla_X Y + g((\mathrm{I}+h')X, Y)\xi - \eta(Y)(\mathrm{I}+h')X, \quad X, Y \in \Gamma(TM).$$

From Theorem 5.1 and Theorem 5.2, the gTWO-connection on an CR-integrable almost Kenmotsu manifold M has the form

$$\tilde{\nabla}^t_X Y = \nabla_X Y = g((\mathrm{I}+h')X, Y)\xi - \eta(Y)(\mathrm{I}+h')X - t\eta(X)\varphi Y.$$

Proposition 5.12. *On an almost Kenmotsu manifold M with Kähler leaves, Dileo-Pastore's connection $\overline{\nabla}$ coincides with the gTWO-connection $\tilde{\nabla}^0$.*

Dileo proved the following characterization for Kenmotsu manifolds of constant curvature.

Proposition 5.13 ([33])**.** *Let M be a Kenmotsu manifold. Then the following conditions are mutually equivalent*:

- *The curvature tensor field $\tilde{R}$ of $\tilde{\nabla}^0$ vanishes.*
- *The curvature tensor field $\tilde{R}$ of $\tilde{\nabla}^0$ is parallel with respect to $\tilde{\nabla}^0$.*
- *M is locally symmetric.*
- *M is of constant curvature -1.*

Some authors use the connection $\tilde{\nabla}^t$ on Kenmotsu manifolds (see *e.g.*, [1, 44, 73]). Acet, Perktaş and Kılıç used the connection $\tilde{\nabla}^t|_{t=-1}$ on Kenmotsu manifolds. More precisely they considered the connection defined by (5.11) even if Kenmotsu manifold is not contact Riemannian. The connection is called the *generalized Tanaka-Webster connection* in [1].

Next, Ghosh and De [44] stated that the following connection

$$\nabla_X Y - \eta(X)\varphi Y - \eta(Y)\nabla_X \xi + \{(\nabla_X \eta)Y\}\xi.$$

was introduced by Tanno [103] and call it the *generalized Tanaka-Webster connection* even if the almost contact Riemmanian manifolds under consideration are not necessarily contact Riemannian. However this connection does not appear neither in [103] nor [104]. As we saw before, the connection ${}^{\mathsf{T}}\nabla$ defined by (5.11) coincides with $\tilde{\nabla}^t|_{t=-1}$ on Kenmotsu manifolds.

On Kenmotsu manifolds, the connection considered in [44] coincides with $\tilde{\nabla}^1$. They showed that the curvature tensor field $\tilde{R}^1$ of $\tilde{\nabla}^1$ vanishes when and only when $\tilde{\nabla}^1\tilde{R}^1 = 0$ holds. In such a case the Kenmotsu manifolds under consideration are of constant curvature -1.

$${}^{\mathsf{T}}\nabla_X Y := \nabla_X Y + \eta(X)\varphi Y + \eta(Y)\varphi(\mathrm{I}+h)X - g(\varphi(\mathrm{I}+h)X, Y)\xi.$$

Kiran Kumar, Nagaraja, Manjulamma and Shashidhar also used $\tilde{\nabla}^1$ in [73].

5.9. Nullity distributions

Motivated by the formula (5.9) and the notion of (generalized) contact (κ, μ, ν)-space, the following notion was introduced:

Definition 5.11. An almost Kenmotsu manifold M is said to be a *generalized almost Kenmotsu (κ, μ, ν)-space* if

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) + \nu(\eta(Y)\varphi hX - \eta(X)\varphi hY) \tag{5.13}$$

for some smooth functions κ, μ and ν. generalized almost Kenmotsu $(\kappa, \mu, 0)$-spaces are called *generalized almost Kenmotsu (κ, μ)-spaces.*

Since $\varphi \circ h = -h \circ \varphi = -h'$ holds on any almost Kenmotsu manifolds, the generalized almost Kenmotsu (κ, μ, ν)-condition (5.13) is rewritten as

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) - \nu(\eta(Y)h'X - \eta(X)h'Y) + \mu(\eta(Y)\varphi h'X - \eta(X)\varphi h'Y).$$

The notion of generalized almost Kenmotsu $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})'$-space can be defined as follows:

Definition 5.12. An almost Kenmotsu manifold M is said to be a *generalized almost Kenmotsu $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})'$-space* if

$$R(X, Y)\xi = \tilde{\kappa}(\eta(Y)X - \eta(X)Y) + \tilde{\mu}(\eta(Y)h'X - \eta(X)h'Y) + \tilde{\nu}(\eta(Y)\varphi h'X - \eta(X)\varphi h'Y). \tag{5.14}$$

Comparing (5.13) and (5.14), we notice that a generalized almost Kenmotsu $(\kappa,\mu,\nu)'$-space is a generalized almost Kenmotsu $(\kappa,\nu,-\mu)$-space. In other words, an almost Kenmotsu (κ,μ,ν)-space is an almost Kenmotsu $(\kappa,-\nu,\mu)'$-space.

Definition 5.13. Let M be a generalized almost Kenmotsu (κ,μ,ν)-space. If all the functions κ, μ and ν are constants, then M is called an *almost Kenmotsu (κ,μ,ν)-space*. A generalized almost Kenmotsu (κ,μ,ν)-space is said to be *proper* if $|\mathrm{d}\kappa|^2+|\mathrm{d}\mu|^2+|\mathrm{d}\nu|^2\neq 0$.

Remark 5.9. In [85] the following additional condition:

$$\mathrm{d}\kappa\wedge\eta=0,\quad \mathrm{d}\mu\wedge\eta=0,\ \text{and}\quad \mathrm{d}\nu\wedge\eta=0 \tag{5.15}$$

is assumed for generalized almost Kenmotsu (κ,μ,ν)-spaces. Öztürk, Aktan and Murathan [85] showed that if the dimension of a generalized almost Kenmotsu (κ,μ,ν)-space is greater than 3, then κ, μ and ν satisfy this additional condition.

Proposition 5.14 ([16, 85])**.** *Let M be a generalized almost Kenmotsu (κ,μ,ν)-space. Then*

$$h^2=(\kappa+1)\varphi^2,$$

holds. Hence $\kappa\leqslant-1$ and $\kappa=-1$ if and only if $h=0$. Moreover, M satisfies

$$\xi(\kappa)=2(\kappa+1)(\nu-2).$$

5.10. Generalized almost Kenmotsu (κ,μ)-spaces

generalized almost Kenmotsu (κ,μ)-spaces satisfy [88, 97]:

$$h^2=(\kappa+1)\varphi^2,\quad S\xi=2n\kappa\,\xi,$$

$$\pounds_\xi h=2\lambda^2\varphi-2h+\mu h',\quad \pounds_\xi h'=-\mu h-2h'$$
$$\nabla_\xi h=-2h-\mu\varphi h,\quad \mathrm{d}\lambda(\xi)=-2\lambda,\quad \mathrm{d}\kappa(\xi)=-4(\kappa+1).$$

On the other hand, generalized almost Kenmotsu $(\kappa,\mu)'$-spaces satisfy

$$(h')^2=(\kappa+1)\varphi^2,\quad S\xi=2n\kappa\,\xi,$$

$$\pounds_\xi h'=-(\mu+2)h',\quad \pounds_\xi h=-2\lambda^2\varphi-(\mu+2)h,$$
$$\nabla_\xi h'=-(2+\mu)h',\quad \mathrm{d}\lambda(\xi)=-2\lambda(\mu+2),\quad \mathrm{d}\kappa(\xi)=-2(\kappa+1)(\mu+2).$$

Here we collect some fundamental results on generalized almost Kenmotsu (κ,μ)-spaces.

Proposition 5.15 ([36])**.** *Let M be an almost Kenmotsu (κ,μ)-space, then $\kappa=-1$ and $h=0$. Moreover M is locally isomorphic to a warped product $I\times_f\overline{M}$ with almost Kähler fiber.*

Proposition 5.16 ([36])**.** *Let M be an almost Kenmotsu $(\kappa,\mu)'$-space with $h'\neq0$, then $\kappa<-1$ and $\mu=-2$ and hence M is an almost Kenmotsu $(\kappa,0,2)$-space. The eigenvalue λ of h satisfies $\lambda^2=-(\kappa+1)$. Moreover M is* CR*-integrable.*

Theorem 5.6 ([36])**.** *Let M be an almost Kenmotsu $(\kappa,-2)'$-space with $h'\neq0$, then M is locally isomorphic to a warped product $B^{n+1}\times_f\mathbb{R}^n$, where the base space*

1. *when $\kappa-2\lambda<0$, the base space B^{n+1} is the hyperbolic space $\mathbb{H}^{n+1}(\kappa-2\lambda)$ of curvature $\kappa-2\lambda<0$. The warping function is expressed as $f(t)=ce^{(1-\lambda)t}$ for some positive constant c.*
2. *when $\kappa+2\lambda<0$, the base space B^{n+1} is the hyperbolic space $\mathbb{H}^{n+1}(\kappa+2\lambda)$ of curvature $\kappa+2\lambda<0$. The warping function is expressed as $f(t)=ce^{(1+\lambda)t}$ for some positive constant c.*
3. *when $\kappa+2\lambda=0$, the base space B^{n+1} is the Euclidean space $\mathbb{E}^{n+1}$. The warping function is expressed as $f(t)=ce^{(1+\lambda)t}$ for some positive constant c.*

In 3-dimensional case Dileo and Pastore obtained the following result.

Theorem 5.7 ([36])**.** *Let M be a 3-dimensional almost Kenmotsu $(\kappa,\mu)'$-space then either*

- *M is a Kenmotsu manifold if $\kappa = -1$ or*
- *M is locally isomorphic to a non-unimodular Lie group whose Lie algebra is determined by the commutation relations:*

$$[e_1, e_2] = 0, \quad [e_2, e_3] = (1-\lambda)e_2, \quad [e_3, e_1] = -(1+\lambda)e_2 \tag{5.16}$$

and $\mu = -2$. Here $\{e_1, e_2, e_3\}$ is a left invariant orthonormal frame field satisfying $he_1 = \lambda e_1$ with $e_3 = \xi$.

Note that $\kappa = -1 - \lambda^2$.

The non-unimodular Lie group with Lie algebra (5.16) will be explained in Example 9.2.

To compare Theorem 5.6 and Theorem 5.7, here we give explicit models of the warped products given in Theorem 5.6.

Corollary 5.2. *Let M be an almost Kenmotsu $(\kappa, -2)'$-space with $h' \neq 0$, then M is locally isomorphic to one of the following warped products:*

1. *when $\kappa - 2\lambda < 0$, $\mathbb{R}^{2n+1}(t, x_1, \dots, x_n, y_1, \dots, y_n)$ equipped with the warped product metric*
$$g = \mathrm{d}t^2 + e^{-2\sqrt{-\kappa+2\lambda}t}(\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2) + c^2 e^{2(1-\lambda)t}(\mathrm{d}y_1^2 + \cdots + \mathrm{d}y_n^2).$$
In this case $(\mathbb{R}^{2n+1}, g)$ is the warped product $\mathbb{H}^{n+1}(\kappa - 2\lambda) \times_f \mathbb{R}^n$.
2. *when $\kappa + 2\lambda < 0$, $\mathbb{R}^{2n+1}(t, x_1, \dots, x_n, y_1, \dots, y_n)$ equipped with the warped product metric*
$$g = \mathrm{d}t^2 + e^{-2\sqrt{-\kappa-2\lambda}t}(\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2) + c^2 e^{2(1+\lambda)t}(\mathrm{d}y_1^2 + \cdots + \mathrm{d}y_n^2).$$
In this case $(\mathbb{R}^{2n+1}, g)$ is the warped product $\mathbb{H}^{n+1}(\kappa + 2\lambda) \times_f \mathbb{R}^n$.
3. *when $\kappa + 2\lambda = 0$, $\mathbb{R}^{2n+1}(t, x_1, \dots, x_n, y_1, \dots, y_n)$ equipped with the warped product metric*
$$g = \mathrm{d}t^2 + \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2 + c^2 e^{2(1+\lambda)t}(\mathrm{d}y_1^2 + \cdots + \mathrm{d}y_n^2).$$
In this case $(\mathbb{R}^{2n+1}, g)$ is the direct product $\mathbb{H}^{n+1}(-1-\lambda^2) \times \mathbb{R}^n$ where $\mathbb{H}^{n+1}(-1-\lambda^2)$ is understood as the warped product
$$\mathbb{H}^{n+1}(-1-\lambda^2) = \left(\mathbb{R}^{n+1}(t, y_1, \dots, y_n), \mathrm{d}t^2 + c^2 e^{2(1+\lambda)t}(\mathrm{d}y_1^2 + \cdots + \mathrm{d}y_n^2)\right).$$

In particular, for 3-dimensional cases, we have

Corollary 5.3. *Let M be a 3-dimensional almost Kenmotsu $(\kappa, -2)'$-space with $h' \neq 0$, then M is locally isomorphic to one of the following warped products:*

1. *when $\kappa - 2\lambda < 0$, $\mathbb{R}^3(t, x, y)$ equipped with the warped product metric*
$$g = \mathrm{d}t^2 + e^{-2\sqrt{-\kappa+2\lambda}t}\,\mathrm{d}x + c^2 e^{2(1-\lambda)t}\,\mathrm{d}y.$$
In this case $(\mathbb{R}^3, g)$ is the warped product $\mathbb{H}^2(\kappa - 2\lambda) \times_f \mathbb{R}$.
2. *when $\kappa + 2\lambda < 0$, $\mathbb{R}^3(t, x, y)$ equipped with the warped product metric*
$$g = \mathrm{d}t^2 + e^{-2\sqrt{-\kappa-2\lambda}t}\,\mathrm{d}x + c^2 e^{2(1+\lambda)t}\,\mathrm{d}y.$$
In this case $(\mathbb{R}^3, g)$ is the warped product $\mathbb{H}^2(\kappa + 2\lambda) \times_f \mathbb{R}$.
3. *when $\kappa + 2\lambda = 0$, $\mathbb{R}^3(t, x, y)$ equipped with the warped product metric*
$$g = \mathrm{d}t^2 + \mathrm{d}x^2 + c^2 e^{2(1+\lambda)t}\,\mathrm{d}y^2.$$
In this case $(\mathbb{R}^3, g)$ is the direct product $\mathbb{H}^2(-1-\lambda^2) \times \mathbb{R}$ where $\mathbb{H}^2(-1-\lambda^2)$ is understood as the warped product
$$\mathbb{H}^2(-1-\lambda^2) = \left(\mathbb{R}^2(t, y), \mathrm{d}t^2 + c^2 e^{2(1+\lambda)t}\,\mathrm{d}y^2\right).$$

All of these warped products are realized as non-unimodular Lie groups equipped with left invariant almost Kenmotsu structure. See Example 9.2 and Section 9.5.

Theorem 5.8 ([91])**.** *If the endomorphism field h' on an almost Kenmotsu manifold M is η-parallel and satisfies $\nabla_\xi h' = 0$, then ξ is a harmonic unit vector field but never a harmonic map. In particular, the characteristic vector field of an almost Kenmotsu $(\kappa, \mu)'$-space is a harmonic unit vector field.*

5.11. Locally conformal almost cosymplectic structures

An almost contact Riemannian manifold (M,φ,ξ,η,g) is said to be a *locally conformal almost cosymplectic manifold* [resp.emphlocally conformal cosymplectic manifold] if there exists an open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ together with smooth functions $\sigma_\lambda\in C^\infty(U_\lambda)$ such that the structure $(U_\lambda,\varphi|_{U_\lambda},e^{\sigma_\lambda}\xi|_{U_\lambda},e^{\sigma_\lambda}\eta|_{U_\lambda},e^{\sigma_\lambda}g|_{U_\lambda})$ is almost cosymplectic [resp. cosymplectic].

Proposition 5.17 ([107])**.** *An almost contact Riemannian manifold M is locally conformal almost cosymplectic if and only if there exists a closed 1-form ω satisfying*

$$\mathrm{d}\eta=\omega\wedge\eta,\quad \mathrm{d}\Phi=2\omega\wedge\Phi.$$

On the other hand, the notion of almost f-cosymplectic manifold was proposed in [3] (see also Pak [86]):

Definition 5.14. An almost contact Riemannian manifold M is said to be an almost *f-cosymplectic manifold* if

$$\mathrm{d}\eta=0,\quad \mathrm{d}\Phi=2f\,\eta\wedge\Phi.$$

Here f is a smooth function satisfying

$$\mathrm{d}f\wedge\eta=0. \tag{5.17}$$

Comparing these two classes of almost contact Riemannian manifolds, we notice that any almost f-cosymplectic manifold M is locally conformal almost cosymplectic with $\omega=f\eta$. An almost f-cosymplectic manifold M is an almost Kenmotsu manifold when f is a constant function 1.

6. Almost Kenmotsu 3-manifolds

Hereafter we concentrate on almost Kenmotsu 3-manifolds.

6.1. Kenmotsu 3-manifolds

Let $M=(M,\varphi,\xi,\eta,g)$ be an almost contact Riemannian 3-manifold. Then the covariant derivative of φ is given by the following *Olszak formula* [82]:

$$(\nabla_X\varphi)Y=g(\varphi\nabla_X\xi,Y)\xi-\eta(Y)\varphi\nabla_X\xi.$$

Olszak formula implies that an almost contact Riemannian 3-manifold is normal if and only if $\nabla\xi$ commutes with φ.

Moreover the exterior derivatives of η and Φ are given by

$$\mathrm{d}\eta=-\eta\wedge\nabla_\xi\eta+\frac{1}{2}\mathrm{tr}_g(\varphi\nabla\xi)\,\Phi,\quad \mathrm{d}\Phi=(\mathrm{div}\,\xi)\eta\wedge\Phi.$$

These formulas imply the following fundamental fact.

Proposition 6.1. *An almost contact Riemannian 3-manifold M is almost Kenmotsu if and only if*

$$\nabla_\xi\xi=0,\quad \mathrm{tr}_g(\varphi\nabla\xi)=0,\quad \mathrm{div}\,\xi=2$$

holds.

Proposition 6.2 ([90])**.** *An almost contact Riemannian 3-manifold M is almost Kenmotsu if and only if $\nabla\xi$ is self-adjoint and $\mathrm{div}\,\xi=2$.*

Proposition 6.3. *Let M be an almost Kenmotsu 3-manifold. Then M is Kenmotsu and only if $h'=0$.*

Here we recall curvature properties of *Kenmotsu 3-manifolds*. The Riemannian curvature R of a Kenmotsu 3-manifold has the form

$$R(X,Y)Z=\frac{\mathrm{s}+4}{2}(X\wedge Y)Z+\frac{\mathrm{s}+6}{2}[\xi\wedge\{(X\wedge Y)\xi\}]Z. \tag{6.1}$$

In particular, every Kenmotsu 3-manifold satisfies

$$R(X,Y)\xi=(-1)\{\eta(Y)X-\eta(X)Y\}.$$

Thus every Kenmotsu 3-manifolds is a normal almost Kenmotsu $(-1,0)$-space as well as $(-1,0)'$-space.

Proposition 6.4. *The Ricci operator S of a Kenmotsu 3-manifold M has the form*

$$S = \frac{\mathrm{s}+2}{2}\,\mathrm{I} - \frac{\mathrm{s}+6}{2}\,\eta\otimes\xi.$$

The principal Ricci curvatures are $(\mathrm{s}+2)/2$, $(\mathrm{s}+2)/2$ and -2.

The characteristic Jacobi operator ℓ is given by

$$\ell(X) = \eta(X)\xi - X = \varphi^2 X.$$

For a unit vector $X \in \mathcal{D}_p$, the sectional curvatures of planes $X\wedge\varphi X$ and $X\wedge\xi$ are given by

$$H = K(X\wedge\varphi X) = \frac{\mathrm{s}}{2}+2, \quad K(X\wedge\xi) = -1.$$

Remark 6.1. The characteristic Jacobi operators of Sasakian, cosymplectic and Kenmotsu manifolds are given as follows:

Structure	Sasakian	Cosymplectic	Kenmotsu
Characteristic Jacobi opeartor	$\ell = -\varphi^2$	$\ell = 0$	$\ell = \varphi^2$

Example 6.1 (Warped products)**.** Here we restudy Example 5.1. Let $(\overline{M},\bar{g},J)$ be a Riemannian 2-manifold together with a compatible orthogonal complex structure J. Take a smooth function on the Euclidean line $\mathbb{E}^1$ with coordinate t and consider the warped product $M = \mathbb{E}^1(t)\times_f \overline{M}$. We denote π_1 and π_2 the natural projections onto the first and second factors,

$$\pi_1 : M \to \mathbb{E}^1, \quad \pi_2 : M \to \overline{M},$$

respectively. On the warped product $M = \mathbb{E}^1\times_f \overline{M}$, we define the vector field ξ by $\xi = \frac{\partial}{\partial t}$. Then the Levi-Civita connection ∇ of M is given by

$$\nabla_{\overline{X}^{\mathrm{v}}}\overline{Y}^{\mathrm{v}} = (\overline{\nabla}_{\overline{X}}\overline{Y})^{\mathrm{v}} - \frac{1}{f}g(\overline{X}^{\mathrm{v}},\overline{Y}^{\mathrm{v}})f'\,\xi, \quad \nabla_\xi \overline{X}^{\mathrm{v}} = \nabla_{\overline{X}^{\mathrm{v}}}\xi = \frac{f'}{f}\overline{X}^{\mathrm{v}}, \quad \nabla_\xi\xi = 0.$$

Here the superscript v means the vertical lift operation of vector fields from $\overline{M}$ to M. The prime means the differentiation by t. Define φ by $\varphi X = \{J(\pi_{2*}X)\}^{\mathrm{v}}$. Then we get

$$\nabla_X\xi = \beta(X-\eta(X)\xi), \quad (\nabla_X\varphi)Y = \beta\{g(\varphi X,Y)-\eta(Y)\varphi X\}, \quad \beta = f'/f.$$

One can see that M is normal almost contact Riemannian 3-manifold. In particular M is a Kenmotsu manifold if and only if $f(t) = ce^t$ for some positive constant c. Take a local orthonormal frame field $\{\bar{e}_1,\bar{e}_2\}$ of $(\overline{M},\bar{g})$ such that $\bar{e}_2 = J\bar{e}_1$. Then we obtain a local orthonormal frame field $\{e_1,e_2,e_3\}$ by

$$e_1 = \frac{1}{f}\bar{e}_1^{\mathrm{v}}, \quad e_2 = \frac{1}{f}\bar{e}_2^{\mathrm{v}} = \varphi\,e_1, \quad e_3 = \xi.$$

Then sectional curvatures of M are given by

$$K(e_1\wedge e_2) = \frac{1}{f^2}\{\overline{K}\circ\pi_2 - (f')^2\}, \quad K(e_1\wedge e_3) = K(e_2\wedge e_3) = -\frac{f''}{f},$$

where $\overline{K}$ is the Gaussian curvature of $\overline{M}$. The Ricci tensor components $\rho_{ij} = \rho(e_i,e_j)$ are given by

$$\rho_{11} = \rho_{22} = \frac{\kappa}{f^2} - \frac{f''}{f} - \left(\frac{f'}{f}\right)^2, \quad \rho_{33} = -\frac{2f''}{f}$$

and other components are 0.

Remark 6.2. The hyperbolic 3-space $\mathbb{H}^3(-1)$ is represented as a warped product (see [4, p. 28]):

$$\mathbb{R}\times_{\cosh z}\mathbb{H}^2(-1) = \left(\mathbb{R}^3(z,x,y), \mathrm{d}z^2 + \cosh^2 z(e^{2y}\mathrm{d}x^2 + \mathrm{d}y^2)\right).$$

If we apply the construction described in Example 6.1 to $\mathbb{R}\times_{\cosh z}\mathbb{H}^2(-1)$, then we obtain f-Kenmotsu structure on $\mathbb{H}^3(-1)$ with $\mathrm{f}(z) = \tanh z$. For more information on f-Kenmotsu 3-manifolds, especially on curves in those manifolds, see [60, 61].

The local structure of Kenmotsu 3-manifolds is rephrased as:

Lemma 6.1. ([69]) *A Kenmotsu* 3*-manifold* M *is locally isomorphic to a warped product* $I \times_f \overline{M}$ *whose base* $I \subset \mathbb{E}^1$ *is an open interval,* $\overline{M}$ *is a surface and warping function* $f(t) = ce^t$*,* $c > 0$*. The unit vector field* ξ *is* $\xi = \partial/\partial t$*.*

Proposition 6.5. *Every Kenmotsu* 3*-manifold has the following properties*:

- *pseudo-symmetric, especially weakly* η*-Einstein.*
- *The Ricci operator commutes with* φ*.*
- *The Ricci operator satisfies* $\pounds_\xi S = 0$*.*
- *The characteristic Jacobi opeartor satisfies* $\pounds_\xi \ell = 0$*.*

The local symmetry for Kenmotsu 3-manifolds is described as follows:

Proposition 6.6 ([51, 53])**.** *The following properties of a Kenmotsu* 3*-manifold* M *are mutually equivalent.*

- M *is locally symmetric.*
- M *is* η*-Einstein.*
- *the scalar curvature* s *is constant.*
- *the holomorphic sectional curvature* H *is constant.*
- M *is locally isomorphic to the hyperbolic* 3*-space* $\mathbb{H}^3(-1)$ *of curvature* -1*.*

Here we generalize Proposition 6.6 to the following Proposition.

Proposition 6.7. *Every Kenmotsu* 3*-manifold is a pseudo-symmetric space of constant type.*
The following properties of a Kenmotsu 3*-manifold* M *are mutually equivalent.*

- M *is semi-symmetric.*
- M *is locally symmetric.*
- M *is* η*-Einstein.*
- *the scalar curvature* r *is constant.*
- *the holomorphic sectional curvature* H *is constant.*
- M *is locally isomorphic to the hyperbolic* 3*-space* $\mathbb{H}^3(-1)$ *of curvature* -1*.*

Proof. The derivative $R \cdot S$ is computed as follows:

$$(R(X,Y)S)Z = -\frac{\mathrm{s}+6}{2}\left\{\eta(X)(\eta(Z)Y + g(Y,Z)\xi) - \eta(Y)(\eta(Z)X + g(Z,X)\xi)\right\}.$$

In particular for any vector field X orthogonal to ξ, we have

$$(R(X,\xi)S)X = \frac{\mathrm{s}+6}{2}\,g(X,X)\xi, \quad (R(X,\xi)S)\xi = \frac{\mathrm{s}+6}{2}\,X.$$

Assume that M is semi-symmetric, then $\mathrm{s} = -6$. Conversely if $\mathrm{s} = -6$ then M is locally symmetric by Proposition 6.6. Hence the semi-symmetry of M is equivalent to $\mathrm{s} = -6$. □

6.2. Fundamental quantities of almost Kenmotsu 3-manifolds

Let M be an almost Kenmotsu 3-manifold. Denote by $\mathcal{U}_1$ the open subset of M consisting of points p such that $h \neq 0$ around p. Next let $\mathcal{U}_0$ the open subset of M consisting of points $p \in M$ such that $h = 0$ around p. Since h is smooth, $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_0$ is an open dense subset of M. So any property satisfied in $\mathcal{U}$ is also satisfied in whole M. For any point $p \in \mathcal{U}$, there exits a local orthonormal frame field $\mathcal{E} = \{e_1, e_2 = \varphi e_2, e_3 = \xi\}$ around p, where e_1 is an eigenvector field of h.

Lemma 6.2 (*cf.* [90])**.** *Let* M *be an almost Kenmotsu* 3*-manifold. Then there exists a local orthonormal frame field* $\mathcal{E} = \{e_1, e_2, e_3\}$ *such that*

$$he_1 = \lambda e_1, \quad e_2 = \varphi e_1, \quad e_3 = \xi$$

for some locally defined smooth function λ*. The Levi-Civita connection* ∇ *is described as*

$$\nabla_{e_1} e_1 = -be_2 - e_3, \quad \nabla_{e_1} e_2 = be_1 + \lambda e_3, \quad \nabla_{e_1} e_3 = e_1 - \lambda e_2,$$

$$\nabla_{e_2} e_1 = ce_2 + \lambda e_3, \quad \nabla_{e_2} e_2 = -ce_1 - e_3, \quad \nabla_{e_2} e_3 = -\lambda e_1 + e_2,$$

$$\nabla_{e_3}e_1 = \delta e_2, \quad \nabla_{e_3}e_2 = -\delta e_1, \quad \nabla_{e_3}e_3 = 0,$$

where δ, b and c are locally defined smooth functions.

The commutation relations are

$$[e_1, e_2] = be_1 - ce_2, \quad [e_2, e_3] = (\delta - \lambda)e_1 + e_2, \quad [e_3, e_1] = -e_1 + (\delta + \lambda)e_2.$$

The Jacobi identity is described as

$$e_1(\lambda - \delta) - \xi(b) + c(\lambda - \delta) - b = 0, \quad e_2(\lambda + \delta) - \xi(c) + b(\lambda + \delta) - c = 0. \tag{6.2}$$

The covariant derivative $\nabla_\xi h$ of h by ξ is given by

$$\nabla_\xi h = -2\delta\, h\varphi + \frac{\xi(\lambda)}{\lambda}h.$$

An local orthonormal frame field $\{e_1, e_2, e_3\}$ as in Lemma 6.2 will be called a local *h-eigenframe field*.

Remark 6.3. It should be remarked on an almost Kenmotsu 3-manifold M,

$$\delta = g(\nabla_\xi W, \varphi W)$$

is independent of the choice of unit eigenvector field W of h on $\mathcal{U}$.

Every Kenmotsu 3-manifold satisfies the commutativity $S\varphi = \varphi S$ and $S\xi = -2\xi$ and hence it is H-almost Kenmotsu (Theorem 5.5). Thus "H-almost Kenmotsu" is an intermediate notion between "almost Kenmotsu" and "Kenmotsu" for almost contact Riemannian 3-manifolds.

6.3. *Curvatures of almost Kenmotsu 3-manifolds*

The Riemannian curvature R is computed by the table of Levi-Civita connection in Lemma 6.2.

$$\begin{aligned}
&R(e_1, e_2)e_1 = -He_2 + (e_1(\lambda) + 2c\lambda)\xi, \quad R(e_1, e_2)e_2 = He_1 - (e_2(\lambda) + 2b\lambda)\xi,\\
&R(e_1, e_2)e_3 = -(e_1(\lambda) + 2c\lambda)e_1 + (e_2(\lambda) + 2b\lambda)e_2,\\
&R(e_2, e_3)e_1 = -(e_2(\lambda) + 2b\lambda)e_2 - (\xi(\lambda) + 2\lambda)\xi,\\
&R(e_2, e_3)e_2 = (e_2(\lambda) + 2b\lambda)e_1 - K_{23}\xi,\\
&R(e_2, e_3)e_3 = (\xi(\lambda) + 2\lambda)e_1 + K_{23}e_2,\\
&R(e_3, e_1)e_1 = (e_1(\lambda) + 2c\lambda)e_2 + K_{13}\xi,\\
&R(e_3, e_1)e_2 = -(e_1(\lambda) + 2c\lambda)e_1 + (\xi(\lambda) + 2\lambda)\xi,\\
&R(e_3, e_1)e_3 = -K_{13}e_1 - (\xi(\lambda) + 2\lambda)e_2,
\end{aligned}$$

where the sectional curvatures $K_{ij} = K(e_i \wedge e_j)$ are given by

$$H = K_{12} = K(e_1 \wedge e_2) = -(e_1(c) + e_2(b) + 1 - \lambda^2 + b^2 + c^2),$$

$$K_{13} = -(\lambda^2 + 2\lambda\delta + 1), \quad K_{23} = -(\lambda^2 - 2\lambda\delta + 1).$$

The Ricci operator S is given by

$$\begin{aligned}
Se_1 =& \rho_{11}e_1 + (\xi(\lambda) + 2\lambda)e_2 - (e_2(\lambda) + 2\lambda b)\xi,\\
Se_2 =& (\xi(\lambda) + 2\lambda)e_1 + \rho_{22}e_2 - (e_1(\lambda) + 2\lambda c)\xi,\\
Se_3 =& -(e_2(\lambda) + 2\lambda b)e_1 - (e_1(\lambda) + 2\lambda c)e_2 - 2(1 + \lambda^2)\xi,
\end{aligned}$$

where

$$\rho_{11} = -(e_1(c) + e_2(b) + b^2 + c^2 + 2\lambda\delta + 2), \quad \rho_{22} = -(e_1(c) + e_2(b) + b^2 + c^2 - 2\lambda\delta + 2).$$

From these, the scalar curvature is computed as

$$\mathrm{s} = -2(e_1(c) + e_2(b) + b^2 + c^2 + \lambda^2 + 3). \tag{6.3}$$

By using the scalar curvature, ρ_{11} and ρ_{22} are rewritten as

$$\rho_{11} = \frac{\mathrm{s}}{2} + \lambda^2 - 2\lambda\delta + 1, \quad \rho_{22} = \frac{\mathrm{s}}{2} + \lambda^2 + 2\lambda\delta + 1.$$

The holomorphic sectional curvature H is rewritten as

$$K_{12} = \frac{\mathrm{s}}{2} + 2(\lambda^2 + 1).$$

The characteristic Jacobi operator ℓ is given by

$$\ell e_1 = -(1 + 2\lambda\delta + \lambda^2)e_1 + (\xi(\lambda) + 2\lambda)e_2, \quad \ell e_2 = (\xi(\lambda) + 2\lambda)e_1 - (1 - 2\lambda\delta + \lambda^2)e_2. \tag{6.4}$$

The covariant derivative $\nabla_\xi h$ is given by

$$\nabla_\xi h = -2\delta h\varphi + \xi(\lambda)\sigma, \tag{6.5}$$

where σ is an endomorphism defined by

$$\sigma(e_1) = e_1, \quad \sigma(e_2) = -e_2, \quad \sigma(e_3) = 0$$

Thus we have $\nabla_\xi h = 0$ if and only if $h = 0$ or $\delta = 0$ and $\xi(\mathrm{tr}\, h^2) = 0$. Note that (6.5) is rewritten as

$$\nabla_\xi h = -2\delta h\varphi + \frac{\xi(\lambda)}{\lambda} h$$

on $\mathcal{U}_1$. From (6.4) one can deduce the following fact.

Proposition 6.8. *There exists no almost Kenmotsu 3-manifold with vanishing characteristic Jacobi operator.*

This makes a sharp contrast with contact Riemannian 3-manifolds. In fact the class of contact Riemannian 3-manifolds with vanishing characteristic Jacobi operator is pretty large (see *e.g.*, [6, 25]).

6.4. Submanifold geometry of leaves

Let us consider again a leaf L of the canonical foliation. Take a local h-eigenframe field $\{e_1, e_2, e_3\}$ as in Lemma 6.2, then the representation matrix of $\mathcal{A}$ relative to $\{e_1, e_2\}$ is

$$\begin{pmatrix} -1 & \lambda \\ \lambda & -1 \end{pmatrix}.$$

The principal curvatures of L are $-1 \pm \lambda$. Hence the mean curvature of L is -1 (see [71]). Moreover L is totally umbilical if and only if $h = 0$ (Proposition 5.3).

The Gauß curvature K_L of L is given by the *Gauß equation*

$$K_L = K_{12} + \det \mathcal{A} = \frac{\mathrm{s}}{2} + \lambda^2 + 3 = \frac{\mathrm{s}}{2} + \frac{1}{2}\mathrm{tr}(h^2) + 3.$$

Thus L is flat if and only if $\mathrm{s} = -3 - \mathrm{tr}(h^2)/2$.

6.5. Weakly η-Einstein almost Kenmotsu 3-manifolds

The weakly η-Einstein property for an almost Kenmotsu 3-manifold is the following system:

$$\delta\lambda = 0, \quad \xi(\lambda) + 2\lambda = 0, \quad e_1(\lambda) + 2c\lambda = e_2(\lambda) + 2b\lambda = 0. \tag{6.6}$$

From this system we obtain the following result.

Corollary 6.1 ([63]). *An almost Kenmotsu 3-manifold is weakly η-Einstein if and only if it is an H-almost Kenmotsu 3-manifold and satisfies $\delta\, \mathrm{tr}\, h^2 = 0$ and $\xi(\mathrm{tr}\, h^2) + 4\mathrm{tr}\, h^2 = 0$. In particular, if M is non-Kenmotsu, then M is weakly η-Einstein if and only if it is an H-almost Kenmotsu 3-manifold satisfying $\delta = 0$ and $\xi(\mathrm{tr}\, h^2) + 4\mathrm{tr}\, h^2 = 0$.*

6.6. Commutativity of φ and S

Here we consider the commutativity condition $S\varphi = \varphi S$. If $M = \mathcal{U}_0$, then M is Kenmotsu and satisfies $S\varphi = \varphi S$. Hereafter we assume that $\mathcal{U}_1$ is non-empty. Take a local h-eigenframe field $\{e_1, e_2, e_3\}$ as in Lemma 6.2, then one can see that $S\varphi = \varphi S$ holds if and only if M satisfies the system

$$\delta = 0, \quad \xi(\lambda) + 2\lambda = 0, \quad e_1(\lambda) + 2c\lambda = e_2(\lambda) + 2b\lambda = 0.$$

This system coincides with (6.6) over $\mathcal{U}_1$. Hence we obtain

Proposition 6.9 ([63]). *For an almost Kenmotsu 3-manifold M, the following three properties are mutually equivalent*:

- *M satisfies $S\varphi = \varphi S$.*
- *M is weakly η-Einstein.*
- *M is H-almost Kenmotsu and satisfies*

$$\delta \operatorname{tr} h^2 = 0, \quad \xi(\operatorname{tr} h^2) + 4 \operatorname{tr} h^2 = 0.$$

The following can be deduced from the second Bianchi identity (see [63]).

Proposition 6.10. *Let M be a strictly almost Kenmotsu 3-manifold satisfying $S\varphi = \varphi S$. Then the eigenvalue λ of h only varies in the direction of ξ, that is, $\mathrm{d}\lambda \wedge \eta = 0$.*

Remark 6.4. Let M be a contact Riemannian 3-manifold satisfying $S\varphi = \varphi S$, then the eigenvalues of h is constant by virtue of the second Bianchi identity. This fact provides another evidence for the differences between contact Riemannian geometry and almost Kenmotsu geometry.

A weakly η-Einstein almost Kenmotsu 3-manifold M satisfies $\nabla_\xi h = -2h$.

7. Generalized almost Kenmotsu (κ, μ, ν)-spaces

7.1. The H-almost Kenmotsu property

Öztürk, Aktan and Murathan showed the following fact ([84, Theorem 4.5], [85, Theorem 7]).

Proposition 7.1. *Let M be an almost Kenmotsu 3-manifold. If $S\xi$ is colinear to ξ, then M satisfies the generalized (κ, μ, ν)-condition on an open dense subset. In such a case we have*

$$\kappa = -(\lambda^2 + 1), \quad \mu = -2\delta, \quad \lambda\nu = 2\lambda + \xi(\lambda).$$

Combining this with Theorem 5.5, we obtain

Theorem 7.1. *Let M be an almost Kenmotsu 3-manifold. If M is a generalized almost Kenmotsu (κ, μ, ν)-space, then M is an H-almost Kenmotsu manifold. Conversely if M is an H-almost Kenmotsu manifold, then M satisfies the generalized (κ, μ, ν)-condition on an open dense subset. In such a case we have*

$$\kappa = -(\lambda^2 + 1), \quad \mu = -2\delta, \quad \lambda\nu = 2\lambda + \xi(\lambda).$$

The Ricci operator has the form

$$S = \left(\frac{\mathrm{s}}{2} - \kappa\right) \mathrm{I} - \left(\frac{\mathrm{s}}{2} - 3\kappa\right) \eta \otimes \xi + \mu h + \nu \varphi h.$$

Moreover, we have

$$S\xi = 2\kappa\xi = (\operatorname{tr} \ell)\xi, \quad \operatorname{tr}(h^2) = -2(\kappa + 1).$$

On a 3-dimensional generalized almost Kenmotsu (κ, μ, ν)-space, we can take a local h-eigenframe field $\{e_1, e_2, e_3\}$ as in Lemma 6.2, then the sectional curvatures are expressed as

$$H = \frac{\mathrm{s}}{2} - 2\kappa, \quad K_{13} = \kappa + \lambda\mu, \quad K_{23} = \kappa - \lambda\mu.$$

the Ricci operator S has the form

$$S = \begin{pmatrix} \mathrm{s}/2 - \kappa + \lambda\mu & \lambda\nu & 0 \\ \lambda\nu & \mathrm{s}/2 - \kappa - \lambda\mu & 0 \\ 0 & 0 & 2\kappa \end{pmatrix}$$

relative to $\{e_1, e_2, e_3\}$. Hence the principal Ricci curvatures are given by

$$\rho_1 = \frac{\mathrm{s}}{2} - \kappa + \lambda\sqrt{\mu^2 + \nu^2}, \quad \rho_2 = \frac{\mathrm{s}}{2} - \kappa - \lambda\sqrt{\mu^2 + \nu^2}, \quad \rho_3 = 2\kappa. \tag{7.1}$$

Theorem 7.1 together with Proposition 6.9 implies the following result which improves [97, Proposition 31.].

Corollary 7.1. *For an almost Kenmotsu 3-manifold M, the following three properties are mutually equivalent*:

- *M satisfies $S\varphi = \varphi S$.*
- *M is weakly η-Einstein.*
- *M is a generalized almost Kenmotsu $(\kappa, 0)$-space satisfying $\mathrm{d}\kappa \wedge \eta = 0$ (see Example 7.1).*

In such a case M is a pseudo-symmetric space.

Remark 7.1. If M is a Kenmotsu 3-manifold, then it is weakly η-Einstein. Now, we assume that M is a strictly almost Kenmotsu 3-manifold. From Theorem 7.1, the Ricci operator of an H-almost Kenmotsu manifold has the form

$$S = \left(\frac{\mathrm{s}}{2} - \kappa\right) \mathrm{I} - \left(\frac{\mathrm{s}}{2} - 3\kappa\right) \eta \otimes \xi + \mu h + \nu\varphi h.$$

An H-almost Kenmotsu manifold M is weakly η-Einstein if and only if $\mu = -2\delta = 0$ and $\nu = 0 = 2\lambda + \xi(\lambda)$. Since $\mathrm{tr}\, h^2 = 2\lambda^2$, we have

$$0 = \xi(\mathrm{tr}\, h^2) + 4\mathrm{tr}\, h^2 = 4\lambda\xi(\lambda) + 8\lambda^2 = 4\lambda(\xi(\lambda) + 2\lambda).$$

Moreover we have

Proposition 7.2 ([63])**.** *Let M be an almost Kenmotsu 3-manifolds satisfying the commutativity $S\varphi = \varphi S$, then the characteristic Jacobi operator ℓ is pseudo-parallel. More precisely ℓ satisfies $R \cdot \ell = \kappa R_1 \cdot \ell$. Here $\kappa = \mathrm{tr}\, \ell/2$.*

Here we mention the following result (see [97, Lemma 4.1]).

Lemma 7.1. *Let M be a 3-dimensional generalized almost Kenmotsu (κ, μ)-space satisfying $\mathrm{d}\kappa \wedge \eta = 0$. If $\kappa = -1$ at a certain point of M, then $\kappa = -1$ on whole of M and h vanishes identically.*

The Riemannian curvature R of a generalized almost Kenmotsu (κ, μ, ν)-space with $\kappa < -1$ is described as follows ([16, Theorem 3.25]).

$$R = \left(\frac{\mathrm{s}}{2} - 2\kappa\right) R_1 + \left(\frac{\mathrm{s}}{2} - 3\kappa\right) R_3 + \mu R_4 + \nu R_7, \tag{7.2}$$

where

$$\begin{aligned}
R_3(X,Y)Z =& \eta(Z)\eta(X)Y - \eta(Y)\eta(Z)X + \{g(Z,X)\eta(Y) - g(Y,Z)\eta(X)\}\xi,\\
R_4(X,Y)Z =& g(Y,Z)hX - g(Z,X)hY + g(hY,Z)X - g(Z,hX)Y,\\
R_7(X,Y)Z =& g(Y,Z)\varphi hX - g(Z,X)\varphi hY + g(\varphi hY,Z)X - g(Z,\varphi hX)Y.
\end{aligned}$$

Corollary 7.2 ([16])**.** *Let M be a generalized almost Kenmotsu (κ, μ, ν)-space. If $\kappa < -1$ is a function that only varies in the direction of ξ, i.e., $\mathrm{d}\kappa \wedge \eta = 0$, then its curvature can be written as*

$$R = -\left(\kappa + 2\right) R_1 - 2\left(\kappa + 1\right) R_3 + \mu R_4 + \nu R_7. \tag{7.3}$$

Corollary 7.3 ([85])**.** *Let M be a generalized almost Kenmotsu (κ, μ, ν)-space. Then*

$$S\varphi - \varphi S = 2\mu h\varphi + 2\nu h$$

for a vector field on M. Moreover, $S\varphi = \varphi S$ if and only if $h = 0$ or $\mu = 0$ and $\nu = 0$.

From (7.1), pseudo-symmetry of generalized almost Kenmotsu (κ, μ, ν)-spaces are described as follows:

Proposition 7.3. *A 3-dimensional generalized almost Kenmotsu (κ, μ, ν)-space. Then M is pseudo-symmetric if and only if*

- *M is a Kenmotsu 3-manifold ($\kappa = -1$ and $L = -1$) or*
- *M is a generalized almost Kenmotsu (κ, μ, ν)-space of scalar curvature $\mathrm{s} = 6\kappa \pm 2\lambda\sqrt{\mu^2 + \nu^2}$ and $L = H = \kappa \pm \lambda\sqrt{\mu^2 + \nu^2}$.*

The second class includes weakly η-Einstein strictly almost Kenmotsu 3-manifolds as well as homogeneous almost Kenmotsu Lie groups $G_{\mathrm{II}}(\beta, \beta^{-1})$ for $\beta \neq 0$ (see Example 9.3).

Proof. From (7.1) $\rho_1 = \rho_2$ holds if and only if $\lambda = 0$ or $\mu = \nu = 0$. In the former case M is a Kenmotsu 3-manifold and hence $\kappa = -1$ and $h = 0$. In the latter case with $\lambda \neq 0$, M is weakly η-Einstein.

Next, $\rho_1 = \rho_3$ holds if and only if

$$\mathrm{s} = 6\kappa - 2\lambda\sqrt{\mu^2 + \nu^2}.$$

Finally $\rho_2 = \rho_3$ holds if and only if

$$\mathrm{s} = 6\kappa + 2\lambda\sqrt{\mu^2 + \nu^2}.$$

Note that weakly η-Einstein strictly almost Kenmotsu 3-manifold M is a 3-dimensional generalized almost Kenmotsu (κ, μ, ν)-spaces of scalar curvature $\mathrm{s} = 6\kappa \pm 2\lambda\sqrt{\mu^2 + \nu^2}$ with $\mu = \nu = 0$. It should be remarked that if $\lambda = 0$ on a 3-dimensional generalized almost Kenmotsu (κ, μ, ν)-spaces of scalar curvature $\mathrm{s} = 6\kappa \pm 2\lambda\sqrt{\mu^2 + \nu^2}$, then M is a Kenmotsu 3-manifold of scalar curvature -6. Thus it is of constant curvature -1. □

7.2. Examples

Here we exhibit some examples of generalized almost Kenmotsu (κ, μ, ν)-spaces constructed by Pastore and Saltarelli.

Example 7.1 ([88]). Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$ be the upper half space. We introduce an almost contact Riemannian structure on M by

$$\xi = \frac{\partial}{\partial z}, \quad \eta = \mathrm{d}z, \quad g = ze^{2z}\mathrm{d}x^2 + \frac{e^{2z}}{z}\mathrm{d}y^2 + \mathrm{d}z^2,$$

$$\varphi\frac{\partial}{\partial x} = z\frac{\partial}{\partial y}, \quad \varphi\frac{\partial}{\partial y} = -\frac{1}{z}\frac{\partial}{\partial x}, \quad \varphi\frac{\partial}{\partial z} = 0.$$

Then $M = (M, \varphi, \xi, \eta, g)$ is a strictly almost Kenmotsu 3-manifold. We can take a global h-eigenframe field

$$e_1 = \frac{e^{-z}}{\sqrt{2}}\left(\frac{1}{\sqrt{z}}\frac{\partial}{\partial x} + \sqrt{z}\frac{\partial}{\partial y}\right), \quad e_2 = -\frac{e^{-z}}{\sqrt{2}}\left(\frac{1}{\sqrt{z}}\frac{\partial}{\partial x} - \sqrt{z}\frac{\partial}{\partial y}\right), \quad e_3 = \xi.$$

Then $\{e_1, e_2, e_3\}$ satisfies

$$[e_1, e_2] = 0, \quad [e_2, e_3] = -\lambda\, e_1 + e_2, \quad [e_3, e_1] = -e_1 + \lambda\, e_2.$$

Thus $\{e_1, e_2, e_3\}$ is a global h-eigenframe field satisfying $b = c = 0$, $\delta = 0$ and $\lambda = 1/(2z)$. Note that the coordinate vector fields ∂_x and ∂_y are eigenvector fields of h' corresponding to $\lambda = 1/(2z)$ and $-\lambda$, respectively. The sectional curvatures are given by

$$H = -(1 - \lambda^2), \quad K_{13} = K_{23} = -(1 + \lambda^2).$$

The Ricci tensor field and the scalar curvature are computed as

$$\rho = -\frac{e^{2z}(4z^2 + 2z - 1)}{2z}\mathrm{d}x^2 - \frac{e^{2z}(4z^2 - 2z + 1)}{2z^3}\mathrm{d}y^2 - \frac{4z^2 + 1}{2z^2}\mathrm{d}z^2,$$

$$\mathrm{s} = -\frac{12z^2 + 1}{2z^2} = -2(\lambda^2 + 3) = 2(\kappa - 2).$$

The components of S relative to $\{e_1, e_2, e_3\}$ are given by

$$\begin{pmatrix} -2 & 1/z - 1/(2z^2) & 0 \\ 1/z - 1/(2z^2) & -2 & 0 \\ 0 & 0 & -2(1 + \lambda^2) \end{pmatrix}.$$

The principal Ricci curvatures are

$$\rho_1 = -2 + \frac{2z - 1}{2z^2}, \quad \rho_2 = -2 - \frac{2z - 1}{2z^2}, \quad \rho_3 = -2 + \frac{1}{2z^2}.$$

Hence M is not pseudo-symmetric. One can check that $(M, \varphi, \xi, \eta, g)$ is a generalized almost Kenmotsu $(\kappa, 0, \nu)$-space with

$$\kappa = -1 - \frac{1}{4z^2} < -1, \quad \nu = 2 - \frac{1}{z}.$$

In particular κ and ν satisfy $\mathrm{d}\kappa \wedge \eta = \mathrm{d}\nu \wedge \eta = 0$. Moreover the Ricci operator S is η-parallel.

Example 7.2 ([88]). On the Cartesian 3-space $\mathbb{R}^3(x,y,z)$, we define an almost contact Riemannian structure (φ,ξ,η,g) by

$$\xi=\frac{\partial}{\partial z},\quad \eta=\mathrm{d}z,\quad g=e^{f(z)+2z}\mathrm{d}x^2+e^{2z-f(z)}\mathrm{d}y^2+\mathrm{d}z^2,$$
$$\varphi\frac{\partial}{\partial x}=e^{f(z)}\frac{\partial}{\partial y},\quad \varphi\frac{\partial}{\partial y}=-e^{-f(z)}\frac{\partial}{\partial x},\quad \varphi\frac{\partial}{\partial z}=0,$$

where $f(z)=-e^{-2z}$. The non-zero eigenvalues of h are $\lambda=e^{-2z}$ and $-\lambda$. Then resulting almost contact Riemannian 3-manifold $M=(\mathbb{R}^3,\varphi,\xi,\eta,g)$ is a generalized almost Kenmotsu $(\kappa,0)$-space with $\kappa=-1-e^{-4z}$. In particular M satisfies $\mathrm{d}\kappa\wedge\eta=0$. From Corollary 7.1, M is weakly η-Einstein. Here we confirm this by computing the Ricci operator explicitly.

Let us take a global h-eigenframe field

$$\begin{aligned}
e_1&=\frac{1}{\sqrt{2}}\left(\ \exp\left(-z-\frac{f(z)}{2}\right)\frac{\partial}{\partial x}+\exp\left(-z+\frac{f(z)}{2}\right)\frac{\partial}{\partial y}\right),\\
e_2&=\frac{1}{\sqrt{2}}\left(-\exp\left(-z-\frac{f(z)}{2}\right)\frac{\partial}{\partial x}+\exp\left(-z+\frac{f(z)}{2}\right)\frac{\partial}{\partial y}\right),\\
e_3&=\xi.
\end{aligned}$$

This h-eigenframe field satisfies $b=c=\delta=0$. By using Lemma 6.2, the sectional curvatures are computed as

$$H=-(1-\lambda^2),\quad K_{13}=K_{23}=-(1+\lambda^2).$$

The Ricci operator S and the scalar curvature s are computed as

$$S=-2\,\mathrm{I}-2\lambda^2\eta\otimes\xi,\quad \mathrm{s}=-2e^{-4z}-6=-2(\lambda^2+3)=2(\kappa-2).$$

Thus M is really weakly η-Einstein and hence it is pseudo-symmetric. One can verify that S is η-parallel.

Example 7.3 ([97]). We construct a generalized almost Kenmotsu (κ,μ)-space with prescribed function μ.

Let M be an open submanifold of $\mathbb{R}^3(x,y,z)$ defined by $z<-1$. Take arbitrary smooth functions $\mu(z)$, $q(z)$ and $r(z)$ of z. We put

$$f_1(x,y,z)=x-\frac{y(\mu(z)+2\sqrt{-1-z}\ )}{2}+q_1(z),\quad f_2(x,y,z)=y+\frac{x(\mu(z)-2\sqrt{-1-z}\ )}{2}+q_2(z),$$
$$f_3(z)=-4(1+z).$$

We introduce an almost contact Riemannian structure (φ,ξ,η,g) by

$$\xi=f_1(x,y,z)\frac{\partial}{\partial x}+f_2(x,y,z)\frac{\partial}{\partial y}+f_3(z)\frac{\partial}{\partial z},\quad \eta=\frac{\mathrm{d}z}{f_3(z)},$$
$$g=\mathrm{d}x^2+\mathrm{d}y^2+(1+f_1^2+f_2^2)\eta\otimes\eta-f_1(\mathrm{d}x\otimes\eta+\eta\otimes\mathrm{d}x)-f_2(\mathrm{d}y\otimes\eta+\eta\otimes\mathrm{d}y),$$
$$\varphi\frac{\partial}{\partial x}=\frac{\partial}{\partial y},\quad \varphi\frac{\partial}{\partial y}=-\frac{\partial}{\partial x},\quad \varphi\frac{\partial}{\partial z}=\frac{f_2}{f_3}\frac{\partial}{\partial x}-\frac{f_1}{f_3}\frac{\partial}{\partial y}.$$

Then $M(\mu,q,r)=(\mathbb{R}^3,\varphi,\xi,\eta,g)$ is a strictly almost Kenmotsu 3-manifold. The orthonormal frame field

$$e_1=\frac{\partial}{\partial x},\quad e_2=\frac{\partial}{\partial y},\quad e_3=\xi$$

is a global h-eigenframe field satisfying $b=c=0$, $\delta=-\mu(z)/2$ and $\lambda=\sqrt{-1-z}$. By using Lemma 6.2 one can see that

$$R(X,Y)\xi=z\{\eta(Y)X-\eta(X)Y\}+\mu(z)\{\eta(Y)hX-\eta(X)hY\}.$$

Thus $M(\mu,q,r)$ is a generalized almost Kenmotsu (κ,μ)-space. Note that $\kappa=z$ and $\mu=\mu(z)$ satisfy $\mathrm{d}\kappa\wedge\eta=0$ and $\mathrm{d}\mu\wedge\eta=0$.

Hence the Ricci operator S, the scalar curvature s and the characteristic Jacobi operator are computed as

$$Se_1=(-2+\lambda\mu)e_1,\quad Se_2=(-2-\lambda\mu)e_2,\quad Se_3=-2(1+\lambda^2)e_3,\quad \mathrm{s}=-2(\lambda^2+3)=2(\kappa-2).$$

Thus $M(\mu, q, r)$ is pseudo-symmetric if and only if

$$\mu(z) = 0, \quad \text{or} \quad \mu(z) = \pm 2\lambda(z) = \pm 2\sqrt{-1-z}.$$

The characteristic Jacobi operator is given by

$$\ell(e_1) = -(\lambda^2 - \lambda\mu + 1)e_1, \quad \ell(e_2) = -(\lambda^2 + \lambda\mu + 1)e_2.$$

In particular, when $\mu(z) = 0$, then $M(\mu, q, r)$ is weakly η-Einstein with

$$S = -2\mathrm{I} - 2\lambda^2\eta \otimes \xi, \quad \ell(e_1) = -(\lambda^2 + 1)e_1, \quad \ell(e_2) = -(\lambda^2 + 1)e_2.$$

Example 7.4 ([97]). Here we construct generalized almost Kenmotsu $(\kappa, 0, \nu)$-space with prescribed function ν.

Let M be an open submanifold of $\mathbb{R}^3(x, y, z)$ defined by $z < -1$. Take arbitrary smooth functions $\nu(z)$, $q(z)$ and $r(z)$ of z. We assume that $\nu(z) \neq 2$ for all $z \in \mathbb{R}$. We put

$$\begin{aligned} f_1(x, z) =& x(1 + \sqrt{-1-z}) + q(z), \quad f_2(y, z) = y(1 - \sqrt{-1-z}) + r(z), \\ f_3(z) =& -2(2 - \nu)(1 + z). \end{aligned}$$

We introduce an almost contact Riemannian (φ, ξ, η) by

$$\xi = f_1(x, z)\frac{\partial}{\partial x} + f_2(y, z)\frac{\partial}{\partial y} + f_3(z)\frac{\partial}{\partial z}, \quad \eta = \frac{dz}{f_3(z)},$$

$$\varphi\frac{\partial}{\partial x} = \frac{\partial}{\partial y}, \quad \varphi\frac{\partial}{\partial y} = -\frac{\partial}{\partial x}, \quad \varphi\frac{\partial}{\partial z} = \frac{f_2}{f_3}\frac{\partial}{\partial x} - \frac{f_1}{f_3}\frac{\partial}{\partial y}.$$

Introduce a Riemannian metric g by the condition

$$e_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \quad e_2 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right), \quad e_3 = \xi$$

is orthonormal with respect to g. Then $M(\nu, q, r) = (M, \varphi, \xi, \eta, g)$ is a strictly almost Kenmotsu 3-manifold. The orthonormal frame field $\{e_1, e_2, e_3\}$ is a global h-eigenframe field satisfying $b = c = 0$, $\delta = 0$ and $\lambda = \sqrt{-1-z}$. By using Lemma 6.2, one can see that

$$R(X, Y)\xi = z\{\eta(Y)X - \eta(X)Y\} + \nu(z)\{\eta(Y)\varphi hX - \eta(X)\varphi hY\}.$$

Thus we confirmed that $M(\nu, q, r)$ is a generalized almost Kenmotsu $(\kappa, 0, \nu)$-space with $\kappa = z$. Note that κ and ν satisfy $d\kappa \wedge \eta = 0$ and $d\nu \wedge \eta = 0$. The Ricci operator S, the scalar curvature s and the characteristic Jacobi operator ℓ of $M(\nu, q, r)$ are described as

$$Se_1 = -2e_1 + \lambda\nu e_2, \quad Se_2 = \lambda\nu e_1 - 2e_2, \quad Se_3 = -2(\lambda^2 + 1)\xi, \quad \mathrm{s} = -2(\lambda^2 + 3) = 2(\kappa - 2),$$

$$\ell e_1 = -(\lambda^2 + 1)e_1 + \lambda\nu e_2, \quad \ell e_2 = \lambda\nu e_1 - (\lambda^2 + 1)e_2.$$

When we choose $\nu(z) = 0$ then $M(\nu, q, r)$ is weakly η-Einstein and

$$S = -2\mathrm{I} - 2\lambda^2\eta \otimes \xi, \quad \ell(e_1) = -(\lambda^2 + 1)e_1, \quad \ell(e_2) = -(\lambda^2 + 1)e_2.$$

Saltarelli proved the following local classifications.

Theorem 7.2 ([97]). *Let M be a 3-dimensional generalized almost Kenmotsu (κ, μ)-space satisfying $d\kappa \wedge \eta = 0$ and $\kappa < -1$. Then M is locally isomorphic to the $M(\mu, q, r)$ in Example 7.3.*

Theorem 7.3 ([36, 97]). *Let M be a 3-dimensional generalized almost Kenmotsu $(\kappa, 0, \nu)$-space satisfying $d\kappa \wedge \eta = 0$ and $\kappa < -1$. Then*

1. *If $\nu = 2$, then κ is constant. In this case, M has a local orthonormal frame field $\{e_1, e_2, e_3\}$ as in Lemma 6.2 with $b = c = \delta = 0$ and $\rho(e_1, \xi) = \rho(e_2, \xi) = 0$. Hence M is locally isomorphic to a non-unimodular Lie group whose Lie algebra is generated by the commutation relations:*

$$[e_1, e_2] = 0, \quad [e_2, e_3] = (1 - \lambda)e_2, \quad [e_3, e_1] = -(1 + \lambda)e_2. \tag{7.4}$$

   The almost Kenmotsu $(\kappa, 0, 2)$-space is locally isomorphic to the non-unimodular Lie group $G_{\mathrm{II}}(\lambda, \lambda)$ in Example 9.2.

2. *If $\nu \neq 2$, then M is locally isomorphic to $M(\nu, q, r)$ in Example 7.4.*

Theorem 7.3 is a generalization of Theorem 5.7.

Remark 7.2. Local coordinate changes between Example 7.2 and Example 7.3 with $\mu = 0$ can be seen in [97].

7.3. Almost Kenmotsu generalized (κ,μ)-spaces

Saltarelli obtained the following results.

Proposition 7.4 ([97]). *Let M be an almost Kenmotsu 3-manifold.*

- *If M is an almost Kenmotsu generalized (κ,μ)-space. Then*

$$S=\left(\frac{\mathrm{s}}{2}-\kappa\right)\mathrm{I}+\left(3\kappa-\frac{\mathrm{s}}{2}\right)\eta\otimes\xi+\mu\,h,\quad h(\mathrm{grad}\,\mu)=\mathrm{grad}\,\kappa-\mathrm{d}\kappa(\xi)\xi.$$

- *If M is an almost Kenmotsu generalized $(\kappa,\mu)'$-space. Then*

$$S=\left(\frac{\mathrm{s}}{2}-\kappa\right)\mathrm{I}+\left(3\kappa-\frac{\mathrm{s}}{2}\right)\eta\otimes\xi+\mu\,h',\quad h'(\mathrm{grad}\,\mu)=\mathrm{grad}\,\kappa-\mathrm{d}\kappa(\xi)\xi.$$

Proposition 7.5 ([97]). *Let M be an almost Kenmotsu generalized (κ,μ)-space or $(\kappa,\mu)'$ with $\mathrm{d}\kappa\wedge\eta=0$ and $\kappa<-1$, then M has flat Kähler leaves.*

8. φ-Einstein almost Kenmotsu 3-manifolds

Here we compute the φ-Ricci tensor field ρ^{φ} of almost Kenmotsu 3-manifolds. Take a local orthonormal frame field of of the form $\{e_1,e_2=\varphi e_1,e_3=\xi\}$. First we compute $2\rho^*=\mathrm{tr}R(X,\varphi Y)\varphi$. Note that $\rho^*(e_1,e_3)=\rho^*(e_2,e_3)=0$. Next we notice that $\rho^*(e_1,e_1)=\rho^*(e_2,e_2)=K_{12}=H$. Thus we only have to compute $\rho^*(e_3,e_1)$ and $\rho^*(e_3,e_2)$.

$$\begin{aligned}
2\rho^*(e_3,e_1)=&\sum_{i=1}^{3}g(R(e_3,\varphi e_1)\varphi e_i,e_i)=\sum_{i=1}^{3}g(R(e_3,e_2)\varphi e_i,e_i)=g(R(e_3,e_2)\varphi e_1,e_1)+g(R(e_3,e_2)\varphi e_2,e_2)\\
=&-2g(R(e_2,e_3)e_2,e_1)=2\rho_{13}=2(e_2(\lambda)+2b\lambda),\\
2\rho^*(e_3,e_2)=&\sum_{i=1}^{3}g(R(e_3,\varphi e_2)\varphi e_i,e_i)=-\sum_{i=1}^{3}g(R(e_3,e_1)\varphi e_i,e_i)=-g(R(e_3,e_1)\varphi e_1,e_1)-g(R(e_3,e_1)\varphi e_2,e_2)\\
=&-2g(R(e_3,e_1)e_2,e_1)=2\rho_{23}=2(e_1(\lambda)+2c\lambda).
\end{aligned}$$

Hence the φ-Ricci tensor field has the components

$$\rho^{\varphi}=\begin{pmatrix} H & 0 & (e_2(\lambda)+2b\lambda)/2\\ 0 & H & (e_1(\lambda)+2c\lambda)/2\\ (e_2(\lambda)+2b\lambda)/2 & (e_1(\lambda)+2c\lambda)/2 & 0\end{pmatrix}$$

relative to $\{e_1,e_2,e_3\}$. On the other hand, g^{φ} has the components

$$g^{\varphi}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}.$$

From this formula we get the following results.

Proposition 8.1. *An almost Kenmotsu 3-manifold M is weakly φ-Einstein if and only if ξ is an eigenvector field of S. In particular M is φ-Einstein if and only if M is an H-almost Kenmotsu and of constant holomorphic sectional curvature.*

It should be remarked that "η-Einstein" and "φ-Einstein" are not in inclusion relation. In fact, there exist φ-Einstein almost Kenmotsu 3-manifolds which are *not* η-Einstein (see the Lie group $G_{\mathrm{II}}(\beta,\gamma)$ in Example 9.2.

Note that the symmetric property of ρ^* is characterized as follows (*cf.* [31]).

Proposition 8.2. *An almost Kenmotsu 3-manifold M has symmetric ρ^* if and only if M is H-almost Kenmotsu.*

Next we give the following result.

Corollary 8.1 ([31]). *An almost Kenmotsu 3-manifold M satisfying $\nabla_\xi h = 0$ and symmetric ρ^* is locally isomorphic to a non-unimodular Lie group.*

Proof. Let us consider a non-Kenmotsu almost Kenmotsu 3-manifold M with symmetric ρ^*. Then M is H-almost Kenmotsu. Take a local h-eigenframe field as in Lemma 6.2, then we have $R(e_1, e_2)\xi = 0$ and

$$R(e_2, \xi)\xi = (\xi(\lambda) + 2\lambda)e_1 - (\lambda^2 - 2\lambda\delta + 1)e_2, \quad R(\xi, e_1)\xi = (\lambda^2 + 2\lambda\delta + 1)e_1 - (\xi(\lambda) + 2\lambda)e_2.$$

Assume that $\nabla_\xi h = 0$. Then from $\delta = 0$ and $\xi(\lambda) = 0$, we have

$$R(e_1, e_2)\xi = 0, \quad R(e_2, \xi)\xi = -(\lambda^2 + 1)\xi, \quad R(\xi, e_1)\xi = (\lambda^2 + 1)e_1 - 2\lambda e_2.$$

These formulas imply that M is a generalized almost Kenmotsu (κ, μ, ν)-space with

$$\kappa = -1 - \lambda^2, \quad \mu = 0, \quad \nu = 2.$$

Then from Theorem 7.3, M is locally isomorphic to the non-unimodular Lie group $G_{\mathrm{II}}(\lambda, \lambda)$ which will be exhibited in Example 9.2. □

Corollary 8.2 ([31]). *Let M be an almost Kenmotsu 3-manifold M satisfying $\nabla_\xi h = 0$. If ρ^* is η-parallel, then it is locally isomorphic to a non-unimodular Lie group.*

9. Homogeneous almost Kenmotsu 3-manifolds

9.1. Homogeneity

In this section we study homogeneous almost Kenmotsu 3-manifolds as the model cases.

Definition 9.1 (*cf.* [90]). An almost contact Riemannian manifold $M = (M, \varphi, \xi, \eta, g)$ is said to be a *homogeneous almost contact Riemannian manifold* if there exists a Lie group G of isometries which acts transitively on M such that every element f of G preserves η, that is

$$f^*\eta = \eta.$$

Calvaruso and A. Perrone showed that 3-dimensional Lie groups which admit left invariant almost Kenmotsu structure are non-unimodular [15].

9.2. Non-unimodular Lie groups

9.2.1. The standard almost Kenmotsu structure First of all we explain that every 3-dimensional non-unimodular Lie group admits a left invariant almost Kenmotsu structure.

Let G be a 3-dimensional non-unimodular Lie group equipped with a left invariant Riemannian metric. On the Lie algebra $\mathfrak{g}$ of G, we take an orthonormal basis $\{E_1, E_2, E_3\}$ as in Section 4.2. We define a left invariant endomorphism field φ and a left invariant 1-form η by

$$\varphi E_1 = E_2, \quad \varphi E_2 = -E_1, \quad \varphi E_3 = 0, \quad \eta = \langle \cdot, \xi \rangle, \quad \xi = E_3.$$

Then (φ, ξ, η) is a left invariant almost contact structure compatible to the metric. Now let $A = (a_{ij})$ be the representation matrix of $\mathrm{ad}(\xi)$ on the unimodular kernel $\mathfrak{u}$. By using the table (4.1) of the Levi-Civita connection, one can see that $(G, \varphi, \xi, \eta, \langle \cdot, \cdot \rangle)$ is almost Kenmotsu if and only if $\mathrm{tr}\, A = -2$.

Theorem 9.1 ([36, 93, 94]). *On a 3-dimensional non-unimodular Lie group G equipped with a left invariant Riemannian metric, there exists a left invariant almost contact structure compatible to the metric and satisfies*

- *the resulting structure is almost Kenmotsu and*
- *the left invariant distribution $\mathcal{D}$ defined by the Pfaff equation $\eta = 0$ coincides with the distribution generated by the unimodular kernel $\mathfrak{u}$.*

The almost Kenmotsu structure is called the standard almost Kenmotsu structure of G.

The operator h and h' with respect to the standard almost Kenmotsu structure are computed as

$$hE_1 = \frac{1}{2}\left((a_{12}+a_{21})E_1 + (a_{22}-a_{11})E_2\right), \quad hE_2 = \frac{1}{2}\left((a_{22}-a_{11})E_1 - (a_{12}+a_{21})E_2\right),$$

$$h'E_1 = \frac{1}{2}\left((a_{22}-a_{11})E_1 - (a_{12}+a_{21})E_2\right), \quad h'E_2 = -\frac{1}{2}\left((a_{12}+a_{21})E_1 + (a_{22}-a_{11})E_2\right).$$

The eigenvalues of h are 0 and

$$\frac{\pm\sqrt{(a_{12}+a_{21})^2+(a_{22}-a_{11})^2}}{2} = \frac{\pm\sqrt{(a_{12}+a_{21})^2+4(1+a_{11})^2}}{2},$$

since $a_{11}+a_{22}=-2$.

9.3. Normalization

Let G be a (simply connected) 3-dimensional non-unimodular Lie group equipped with a left invariant almost Kenmotsu structure. Then there are two possibilities:

- $\xi \perp \mathfrak{u}$.
- ξ is transversal to $\mathfrak{u}$ but it is not orthogonal to $\mathfrak{u}$.

It should be remarked that under almost Kenmotsu assumption, the case $\xi \in \mathfrak{u}$ does not occur (see [90]).

9.3.1. The case $\xi \perp \mathfrak{u}$ In this case we may assume that the structure is the standard almost Kenmotsu structure. Thus the Lie algebra $\mathfrak{g}$ is $\mathfrak{g}_A$ as in Section 4.2.

Since h is self-adjoint with respect to g, we can take an orthonormal basis $\{e_1, e_2\}$ of $\mathfrak{u}$ which diagonalizes h. We may assume that $\{e_1, e_2\}$ is related to $\{E_1, E_2\}$ by a rotation.

$$e_1 = \cos\theta\, E_1 + \sin\theta\, E_2, \quad e_2 = -\sin\theta\, E_1 + \cos\theta\, E_2$$

for some θ. Then

$$\varphi e_1 = \cos\theta\, E_2 - \sin\theta\, E_1 = e_2, \quad \varphi e_2 = -\sin\theta\, E_2 - \cos\theta\, E_1 = -e_1.$$

Hence with respect to the new orthonormal frame field $\{e_1, e_2, e_3 = E_3\}$, we have

$$\varphi e_1 = e_2, \quad \varphi e_2 = -e_1$$

and

$$he_1 = \lambda e_1, \quad he_2 = -\lambda e_2.$$

The matrix A is transformed as

$$\begin{pmatrix} \tilde{\alpha} & \gamma \\ \beta & \tilde{\alpha} \end{pmatrix}$$

with respect to $\{e_1, e_2, e_3\}$. Since $\operatorname{tr} A = -2$ is invariant, we have $\tilde{\alpha} = -1$.

Proposition 9.1 ([15, 94])**.** *Let G be a 3-dimensional non-unimodular Lie group equipped with a left invariant almost Kenmotsu structure. If the characteristic vector field is orthogonal to the unimodular kernel, then G is locally isomorphic to the Lie group G_A given by* (4.2) *with*

$$A = \begin{pmatrix} -1 & \gamma \\ \beta & -1 \end{pmatrix}$$

for some β, $\gamma \in \mathbb{R}$. Namely, the Lie algebra $\mathfrak{g}$ is generated by an orthonormal basis $\{e_1, e_2 = \varphi e_2, e_3 = \xi\}$ satisfying $he_1 = \lambda e_1$, $he_2 = -\lambda e_2$, $he_3 = 0$ and the commutation relations:

$$[e_1, e_2] = 0, \quad [e_2, e_3] = -\gamma e_1 + e_2, \quad [e_3, e_1] = -e_1 + \beta e_2,$$

where $\lambda = (\beta+\gamma)/2$. The Lie algebra is referred as to a non-unimodular Lie algebra of type II in [15].

9.3.2. The case ξ is transversal to $\mathfrak{u}$ and $\xi \notin \mathfrak{u}^{\perp}$ Next, we consider the case ξ is transversal to $\mathfrak{u}$ but not orthogonal to $\mathfrak{u}$. We express ξ as

$$\xi = \xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3, \quad \xi_1^2 + \xi_2^2 + \xi_3^2 = 1.$$

By the assumption, $\xi_1^2 + \xi_2^2 \neq 0$. Let us choose a basis $\{E_1, E_2, E_3\}$ as in Section 4.2 and we represent $\mathfrak{g}$ as $\mathfrak{g}_A$ for some A. Since the almost contact structure is almost Kenmotsu, $\nabla\xi$ is self-adjoint with respect to g and $\operatorname{div}\xi = 2$ (Proposition 6.2). One can see that $\operatorname{div}\xi = -\xi_3 \operatorname{tr} A$. Hence

$$\xi_3 = -\frac{2}{\operatorname{tr} A}. \tag{9.1}$$

Perrone [94] showed that the self-adjointness of $\nabla\xi$ is equivalent to $\det A = 0$. The coefficients ξ_1 and ξ_2 satisfy

$$a_{11}\xi_1 + a_{21}\xi_2 = a_{12}\xi_1 + a_{22}\xi_2 = 0.$$

Let us take an orthonormal basis $\{e_1, e_2 = \varphi e_1, e_3 = \xi\}$ of $\mathfrak{g}_A$ satisfying $he_1 = \lambda e_1$. Calvaruso and Perrone proved that $e_1 \notin \mathfrak{u}$, $e_2 \notin \mathfrak{u}$ and $\xi \notin \mathfrak{u}$. Then Lemma 6.2 and the proof of [15, Theorem 4.3], we have

$$\nabla_{e_1} e_1 = -be_2 - e_3, \quad \nabla_{e_1} e_2 = be_1 + \lambda e_3, \quad \nabla_{e_1} e_3 = e_1 - \lambda e_2,$$

$$\nabla_{e_2} e_1 = ce_2 + \lambda e_3, \quad \nabla_{e_2} e_2 = -ce_1 - e_3, \quad \nabla_{e_2} e_3 = -\lambda e_1 + e_2,$$

$$\nabla_{e_3} e_1 = \delta e_2, \quad \nabla_{e_3} e_2 = -\delta e_1, \quad \nabla_{e_3} e_3 = 0,$$

where $b \neq 0$, $c \neq 0$ and δ are constants. From this table we confirm that $\operatorname{div}\xi = 2$. From the Jacobi identity, we have

$$b = c(\lambda - \delta), \quad c = b(\lambda + \delta).$$

According to [15], we set

$$\bar{e}_1 := \frac{be_1 - ce_2}{\sqrt{b^2 + c^2}}, \quad \bar{e}_2 := \varphi\bar{e}_2, \quad \bar{e}_3 = \xi, \quad \beta := -2\delta, \quad \gamma := \sqrt{b^2 + c^2} > 0. \tag{9.2}$$

For simplicity of notation, we denote this new basis by $\{e_1, e_2, e_3\}$. Then the new basis satisfies the commutation relations:

$$[e_1, e_2] = \gamma e_1, \quad [e_2, e_3] = -\beta e_1, \quad [e_3, e_1] = -2e_1.$$

Moreover we have

$$\lambda = \frac{\sqrt{\beta^2 + 4}}{2}, \quad \lambda^2 = 1 + \delta^2 \geq 1.$$

The unimodular kernel is spanned by

$$e_1, \quad \xi + \frac{2}{\gamma}(e_1 - e_2).$$

The Lie algebra determined by these commutation relations is referred as to a non-unimodular Lie algebra of type IV in [15]. Type IV Lie algebras will be studied in Section 9.6. For the classification of homogeneous almost Kenmotsu 3-manifolds in terms of the submanifold geometry of canonical foliation. See [93, Theorem 5.1].

9.4. The type II Lie algebra

Let $G_{\mathrm{II}}(\beta, \gamma)$ be a 3-dimensional non-unimodular Lie group equipped with a left invariant almost Kenmotsu structure with Lie algebra $\mathfrak{g}_{\mathrm{II}}(\beta, \gamma)$ of type II. The Lie algebra $\mathfrak{g}_{\mathrm{II}}(\beta, \gamma)$ coincides with the Lie algebra $\mathfrak{g}_A$ described in Section 4.2 with

$$A = \begin{pmatrix} -1 & \gamma \\ \beta & -1 \end{pmatrix}$$

and $e_1 = E_1$, $e_2 = E_2$ and $\xi = E_3$. Note that $\{e_1, e_2, e_3\}$ is regarded as a global h-eigenframe field as in Lemma 6.2 under the choice $b = c = 0$.

The Levi-Civita connection is described as

$$\nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_2 = \lambda e_3, \quad \nabla_{e_1} e_3 = e_1 - \lambda e_2,$$

$$\nabla_{e_2} e_1 = \lambda e_3, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_3 = -\lambda e_1 + e_2,$$

$$\nabla_{e_3}e_1 = \delta e_2, \quad \nabla_{e_3}e_2 = -\delta e_1, \quad \nabla_{e_3}e_3 = 0,$$

where

$$\lambda = \frac{\beta+\gamma}{2}, \quad \delta = \frac{\beta-\gamma}{2}.$$

The Milnor invariant is $\mathsf{D} = 1 - \beta\gamma$.

If $\beta\gamma \neq 0$, the simply connected Lie group $\tilde{G}_A = \tilde{G}_{\mathrm{II}}(\beta,\gamma)$ is given by (see (4.2)):

$$\tilde{G}_{\mathrm{II}}(\beta,\gamma) = \left\{ \left(\begin{array}{ccc} e^{-z}\cosh(\sqrt{\beta\gamma}z) & \frac{\gamma}{\sqrt{\beta\gamma}}e^{-z}\sinh(\sqrt{\beta\gamma}z) & x \\ \frac{\beta}{\sqrt{\beta\gamma}}e^{-z}\sinh(\sqrt{\beta\gamma}z) & e^{-z}\cosh(\sqrt{\beta\gamma}z) & y \\ 0 & 0 & 1 \end{array} \right) \middle| \; x,y,z \in \mathbb{R} \right\}$$

In case $\beta = 0$, we have

$$\tilde{G}_{\mathrm{II}}(0,\gamma) = \left\{ \left(\begin{array}{ccc} e^{-z} & \gamma z e^{-z} & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{array} \right) \middle| \; x,y,z \in \mathbb{R} \right\}.$$

In addition, we have

$$\tilde{G}_{\mathrm{II}}(\beta,0) = \left\{ \left(\begin{array}{ccc} e^{-z} & 0 & x \\ \beta z e^{-z} & e^{-z} & y \\ 0 & 0 & 1 \end{array} \right) \middle| \; x,y,z \in \mathbb{R} \right\}.$$

The Riemannian curvature R of $G_{\mathrm{II}}(\beta,\gamma)$ is given by

$$\begin{array}{lll} R(e_1,e_2)e_1 = -K_{12}e_2, & R(e_1,e_2)e_2 = K_{12}e_1, & R(e_1,e_2)e_3 = 0, \\ R(e_2,e_3)e_1 = -2\lambda\xi, & R(e_2,e_3)e_2 = -K_{23}\xi, & R(e_2,e_3)e_3 = 2\lambda e_1 + K_{23}e_2, \\ R(e_3,e_1)e_1 = K_{13}\xi, & R(e_3,e_1)e_2 = 2\lambda\xi, & R(e_3,e_1)e_3 = -K_{13}e_1 - 2\lambda e_2, \end{array}$$

where

$$K_{12} = -(1-\lambda^2), \quad K_{13} = -(\lambda^2 + 2\lambda\delta + 1), \quad K_{23} = -(\lambda^2 - 2\lambda\delta + 1).$$

The Ricci operator S is given by

$$Se_1 = -2(1+\lambda\delta)e_1 + 2\lambda e_2, \quad Se_2 = 2\lambda e_1 - 2(1-\lambda\delta)e_2, \quad Se_3 = -2(1+\lambda^2)\xi.$$

The scalar curvature is a negative constant

$$\mathrm{s} = -2(3+\lambda^2).$$

The principal Ricci curvatures are

$$\begin{aligned} \rho_1 &= -2 + 2\lambda\sqrt{1+\delta^2} = -2 + \frac{\beta+\gamma}{2}\sqrt{(\beta-\gamma)^2+4}, \\ \rho_2 &= -2 - 2\lambda\sqrt{1+\delta^2} = -2 - \frac{\beta+\gamma}{2}\sqrt{(\beta-\gamma)^2+4}, \\ \rho_3 &= -2(1+\lambda^2) = -\frac{1}{2}(4+(\beta+\gamma)^2). \end{aligned}$$

From these we conclude that $G_{\mathrm{II}}(\beta,\gamma)$ is pseudo-symmetric if and only if $\beta = -\gamma$ (locally symmetric) or $\mathsf{D} = 0$ (equivalently $\beta\gamma = 1$).

The covariant derivative ∇S is computed as

$$(\nabla_{e_1}S)e_1 = 2\delta\lambda\xi, \quad (\nabla_{e_1}S)e_2 = 2\lambda(\lambda^2 + \alpha\lambda - 1)\xi,$$

$$(\nabla_{e_1}S)e_3 = 2\delta\lambda e_1 + 2\lambda(\lambda^2+\delta\lambda-1)e_2, \quad (\nabla_{e_2}S)e_1 = 2\lambda(\lambda^2-\delta\lambda-1)\xi,$$

$$(\nabla_{e_2}S)e_2 = -2\delta\lambda\xi, \quad (\nabla_{e_2}S)e_3 = 2\lambda(\lambda^2-\delta\lambda-1)e_1 - 2\delta\lambda e_2,$$

$$(\nabla_{e_3}S)e_1 = -4\delta\lambda e_1 - 4\delta^2\lambda e_2, \quad (\nabla_{e_3}S)e_2 = -4\delta^2\lambda e_1 + 4\delta\lambda e_2, \quad (\nabla_{e_3}S)e_3 = 0.$$

From this table, one can see that $G_{\mathrm{II}}(\beta,\gamma)$ is locally symmetric if and only if $\lambda = 0$ (equivalently $\beta = -\gamma$, see Example 9.1) or $\delta = 0$ (equivalently $\beta = \gamma$, see Example 9.2).

The characteristic Jacobi operator ℓ is given by

$$\ell e_1 = -(1+2\lambda\delta+\lambda^2)e_1 + 2\lambda e_2, \quad \ell e_2 = 2\lambda e_1 - (1-2\lambda\delta+\lambda^2)e_2.$$

Let L be a leaf of the canonical foliation. The Gauß curvature K_L of the leaf is identically zero. Note that the leaf through the origin is the canonical normal subgroup U.

Let us investigate commutativity of S and φ. Since

$$[S,\varphi]e_1 = 4\lambda(e_1+\delta e_2), \quad [S,\varphi]e_2 = 4\lambda(\delta e_1 - e_2), \quad [S,\varphi]e_3 = 0,$$

$G_{\mathrm{II}}(\beta,\gamma)$ satisfies $[S,\varphi]=0$ when and only when $\lambda=0$ (compare with [21, Proposition 6]).

Next

$$(\pounds_\xi S)e_1 = -4\lambda\delta(e_1+(\lambda+\delta)e_2), \quad (\pounds_\xi S)e_2 = 4\lambda\delta((\lambda-\delta)e_1+e_2).$$

Thus $\pounds_\xi S = 0$ holds if and only if $\lambda=0$ or $\delta=0$ (compare with [21, Proposition 7]). Note that one can check that $\pounds_\xi S=0$ is equivalent to $\nabla_\xi S=0$ on $G(\beta,\gamma)$ (see [21]). On can check that $G_{\mathrm{II}}(\beta,\gamma)$ is almost Kenmotsu $(-1-\lambda^2,-2\delta,2)$-space and $(-1-\lambda^2,-2,-2\delta)'$-space.

Proposition 9.2. *A non-unimodular Lie group $G_{\mathrm{II}}(\beta,\gamma)$ with Lie algebra $\mathfrak{g}_{\mathrm{II}}(\beta,\gamma)$ has the following properties.*

- *it is φ-Einstein.*
- *it is η-Einstein if and only if $\beta=-\gamma$. In such a case $G(\beta,-\beta)$ is Kenmotsu and Einstein.*
- *it is H-almost Kenmotsu.*
- *it is an almost Kenmotsu $(-1-(\beta+\gamma)^2/4, \gamma-\beta, 2)$-space.*
- *it is pseudo-symmetric if and only if $\beta=-\gamma$ or $\beta\gamma=1$ (equivalently $\mathsf{D}=0$). In the former case $G(\beta,-\beta)$ is Kenmotsu.*
- *it is semi-symmetric if and only if $\beta=-\gamma$ or $\beta=\gamma=\pm1$. In the former case $G_{\mathrm{II}}(\beta,-\beta)$ is Kenmotsu. In the latter case $G_{\mathrm{II}}(1,1)$ and $G_{\mathrm{II}}(-1,-1)$ are isometric to $\mathbb{H}^2(-4)\times\mathbb{R}$ (see Example 9.2).*
- *it has η-parallel Ricci operator.*
- *it has transversally Killing Ricci operator when and only when $\beta=\pm\gamma$.*
- *it satisfies $\pounds_\xi S=0$ when and only when $\beta=\pm\gamma$.*
- *it has η-parallel Riemannian curvature.*
- *The canonical normal subgroup U is intrinsically flat and constant mean curvature -1.*

In particular, $G_{\mathrm{II}}(\beta,\gamma)$ is semi-symmetric if and only if it is locally symmetric.

Remark 9.1. The type II Lie group $G_{\mathrm{II}}(\beta,\gamma)$ in this article and $G(\lambda,\alpha)$ in [64, 65] are related by

$$G(\lambda,\alpha) = G_{\mathrm{II}}(\lambda+\alpha,\lambda-\alpha), \quad G_{\mathrm{II}}(\beta,\gamma) = G((\beta+\gamma)/2,(\beta-\gamma)/2).$$

Note that $G_{\mathrm{II}}(\beta,\gamma)$ is denoted by $G(\beta,\gamma)$ in [63].

Example 9.1 ($\lambda=0$)**.** Assume that $\lambda=0$, *i.e.,* $\beta+\gamma=0$. In this case, $G_{\mathrm{II}}(\beta,-\beta)$ is Kenmotsu and $\delta=\beta$. The Milnor invariant is $\mathsf{D}=1+\beta^2\geq 1$. The simply connected Lie group $\tilde{G}_{\mathrm{II}}(\beta,-\beta)$ is given by

$$\widetilde{G}_{\mathrm{II}}(\beta,-\beta) = \left\{ \left(\begin{array}{ccc} e^{-z}\cos(\beta z) & -e^{-z}\sin(\beta z) & x \\ e^{-z}\sin(\beta z) & e^{-z}\cos(\beta z) & y \\ 0 & 0 & 1 \end{array} \right) \middle| \ x,y,z\in\mathbb{R} \right\}.$$

Note that $\widetilde{G}_{\mathrm{II}}(\beta,-\beta)$ is a Lie subgroup of the universal covering of the orientation preserving similarity transformation group

$$\widetilde{\mathrm{Sim}}(2) = \left\{ \left(\begin{array}{ccc} e^t\cos z & -e^t\sin z & x \\ e^t\sin z & e^t\cos z & y \\ 0 & 0 & 1 \end{array} \right) \middle| \ x,y,z,t\in\mathbb{R} \right\}$$

of Euclidean plane. The left invariant metric is expressed as the warped product metric

$$e^{2z}(\mathrm{d}x^2+\mathrm{d}y^2)+\mathrm{d}z^2.$$

Hence $\tilde{G}_{\mathrm{II}}(\beta,-\beta)$ is isometric to the hyperbolic 3-space $\mathbb{H}^3(-1)=\mathbb{R}\times_{e^z}\mathbb{C}$ of constant curvature -1 (see Example 5.1). The canonical normal subgroup U is nothing but the *horosphere* of $\mathbb{H}^3(-1)$. As is well known $\mathbb{H}^3(-1)$ is expressed as $\mathbb{H}^3(-1)=\mathrm{SL}_2\mathbb{C}/\mathrm{SU}_2$ as a Riemannian symmetric space. The Lie group

$$\tilde{G}_{\mathrm{II}}(0,0) = \left\{ \left(\begin{array}{ccc} e^{-z} & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{array} \right) \middle| \ x,y,z\in\mathbb{R} \right\}$$

is isomorphic to the solvable part $\mathcal{S}$ of $\mathrm{SL}_2\mathbb{C}$ according to the *Iwasawa decomposition* $\mathrm{SL}_2\mathbb{C} = \mathcal{S}\cdot\mathrm{SU}_2$. Note that $\widetilde{G}_{\mathrm{II}}(0,0)$ is nothing but the non-unimodular Lie group $\widetilde{G}_A$ in Example 4.1 with $A=-I$. The Lie group $\widetilde{G}_{\mathrm{II}}(0,0)$ coincides with the solvable Lie group $S(-1)$ in Example 9.5. The Lie group $\widetilde{G}_{\mathrm{II}}(\beta,\beta)$ with $\beta\neq 0$ coincides with the Lie group $\widetilde{G}_A$ in Example 4.1 with $a_{11}=a_{22}=-1$ and $a_{21}=-a_{21}=\beta\neq 0$.

It should be remarked that the automorphism group of the Kenmotsu structure of $\mathbb{H}^3(-1)$ is 4-dimensional ([103]). The hyperbolic 3-space $\mathbb{H}^3(-1)$ has several homogeneous Kenmotsu manifold representtaions. For instance we know $\mathbb{H}^3(-1)=\tilde{G}_{\mathrm{II}}(\beta,-\beta)/\{\mathrm{Id}\}$. On the other hand, $\mathbb{H}^3(-1)$ admits a homogeneous Kenmotsu manifold representation $\mathbb{H}^3(-1)=(\mathcal{S}\cdot\mathrm{U}_1)/\mathrm{U}_1$ (see [57]). This phenomena has Sasakian analogue. The 3-sphere $\mathbb{S}^3$ is represanted by $\mathrm{SO}_4/\mathrm{SO}_3=(\mathrm{SU}_2\times\mathrm{SU}_2)/\mathrm{SU}_2$ as a Riemannian symmetric space. On the other hand $\mathbb{S}^3$ has homogeneous Sasakian manifold representations $\mathrm{SU}(2)/\{\mathrm{Id}\}$ as well as $\mathrm{U}_2/\mathrm{U}_1=(\mathrm{SU}_2\times\mathrm{U}_1)/\mathrm{U}_1$.

Example 9.2 ($\delta=0$)**.** When $\delta=0$, we have $\lambda=\beta$ and $\widetilde{G}_{\mathrm{II}}(\beta,\beta)$ is given by

$$\widetilde{G}_{\mathrm{II}}(\beta,\beta)=\left\{\left(\begin{array}{ccc} e^{-z}\cosh(\beta z) & e^{-z}\sinh(\beta z) & x\\ e^{-z}\sinh(\beta z) & e^{-z}\cosh(\beta z) & y\\ 0 & 0 & 1\end{array}\right)\ \middle|\ x,y,z\in\mathbb{R}\right\}.$$

The Milnor invariant is $\mathsf{D}=1-\beta^2\leq 1$. In particular when $\beta=\pm1$, $\mathsf{D}=0$. Note that $\widetilde{G}_{\mathrm{II}}(\beta,\beta)$ is a Lie subgroup of the identity component

$$\left\{\left(\begin{array}{ccc} e^{t}\cosh z & e^{t}\sinh z & x\\ e^{t}\sinh z & e^{t}\cosh z & y\\ 0 & 0 & 1\end{array}\right)\ \middle|\ x,y,z,t\in\mathbb{R}\right\}$$

of the similarity transformation group $\mathrm{Sim}(1,1)$ of Minkowski plane.

The almost Kenmotsu manifold $G_{\mathrm{II}}(\beta,\beta)$ is non-Kenmotsu unless $\beta=0$. The left invariant metric is

$$e^{2z}\cosh(2\beta z)(\mathrm{d}x^2+\mathrm{d}y^2)+2e^{2z}\sinh(2\beta z)\mathrm{d}x\,\mathrm{d}y+\mathrm{d}z^2.$$

The sectional curvatures are given by

$$K_{12}=-1+\beta^2,\quad K_{13}=K_{23}=-1-\beta^2.$$

The Ricci operator S is given by

$$Se_1=-2e_1+2\beta e_2,\quad Se_2=2\beta e_1-2e_2,\quad Se_3=-2(1+\beta^2)\xi.$$

The scalar curvature is computed as

$$\mathrm{s}=-2(3+\beta^2).$$

The Ricci eigenvalues are

$$-2+2\beta,\quad -2-2\beta,\quad -2-2\beta^2.$$

Thus $G_{\mathrm{II}}(\beta,\beta)$ is pseudo-symmetric if and only if $\beta=0$ (locally symmetric and hence of constant curvature -1) or $\beta=\pm1$. In the latter case, the principal Ricci curvatures are $\{-4,-4,0\}$. As we saw before $\mathbb{H}^2(-4)\times\mathbb{R}$ in Example 4.2 and Example 5.2 has $\mathrm{tr}\,A=-2$, $\mathsf{D}=0$ and principal Ricci curvatures $\{-4,-4,0\}$. Hence we conclude that $G_{\mathrm{II}}(1,1)$ and $G_{\mathrm{II}}(-1,-1)$ are isometric to $\mathbb{H}^2(-4)\times\mathbb{R}$ as well as isomorphic to $\mathbb{H}^2(-4)\times\mathbb{R}$ as Lie groups. Thus for $G_{\mathrm{II}}(\beta,\beta)$, pseudo-symmetry is equivalent to local symmetry.

The characteristic Jacobi operator ℓ is given by

$$\ell e_1=-(1+\beta^2)e_1+2\beta e_2,\quad \ell e_2=2\beta e_1-(1+\beta^2)e_2.$$

We notice that $G_{\mathrm{II}}(\beta,\beta)$ is an almost Kenmotsu $(-1-\beta^2,0,2)$-space and an almost Kenmotsu $(-1-\beta^2,-2)'$-space (see Theorem 5.7). The Lie group $G_{\mathrm{II}}(\beta,\beta)$ satisfies $\pounds_\xi S=0$. The Lie group given in [27, Theorem] is $G(\beta,\beta)$.

Here we give an explicit models for pseudo-symmetric almost Kenmotsu Lie group of type II.

Example 9.3 (Pseudo-symmetric Lie groups)**.** Let $G_{\mathrm{II}}(\beta,\gamma)$ be an almost Kenmotsu Lie group of type II. As we saw before, $G_{\mathrm{II}}(\beta,\gamma)$ is pseudo-symmetric if and only if $\beta+\gamma=0$ or $\beta\gamma=1$. In the former case, $G_{\mathrm{II}}(\beta,-\beta)$ is Kenmotsu. In the latter case, the universal covering $\widetilde{G}_{\mathrm{II}}(\beta,\beta^{-1})$ is given by

$$\widetilde{G}_{\mathrm{II}}(\beta,\beta^{-1})=\left\{\left(\begin{array}{cc} e^{-z}\cosh z & \beta^{-1}e^{-z}\sinh z\\ \beta e^{-z}\cosh z & e^{-z}\cosh z\end{array}\right)\ \middle|\ x,y,z\in\mathbb{R}\right\}.$$

This Lie group is an example of the second class in Proposition 7.3. Indeed $\widetilde{G}_{\mathrm{II}}(\beta,\beta^{-1})$ is an almost Kenmotsu $(\kappa,\mu,2)$-space with $\kappa=-1-(\beta+\beta^{-1})^2/4$ and $\mu=\beta^{-1}-\beta$. Thus $\widetilde{G}_{\mathrm{II}}(\beta,\beta^{-1})$ is proper pseudo-symmetric if $\beta\neq\pm1$.

Let us reexamine the almost Kenmotsu $(\kappa,0,2)$-spaces appeared in Theorem 7.3. Dileo and Pastore proved the following Lemma.

Lemma 9.1 ([36])**.** *Let M be a strictly almost Kenmotsu 3-manifold. Assume that M is an almost Kenmotsu $(\kappa,0,2)$-space. Then for any unit eigenvector field E_1 of h' corresponding to an eigenvalue $\lambda\neq0$, the Levi-Civita connection is described as*

$$\begin{gathered}\nabla_{E_1}E_1=-(1+\lambda)E_1,\quad \nabla_{E_1}E_2=0,\quad \nabla_{E_1}E_3=(1+\lambda)E_1\\ \nabla_{E_2}E_1=0,\quad \nabla_{E_2}E_2=-(1-\lambda)E_3,\quad \nabla_{E_2}E_3=(1-\lambda)E_2\\ \nabla_{E_3}E_1=0,\quad \nabla_{E_3}E_2=0,\quad \nabla_{E_3}E_3=0,\end{gathered}\tag{9.3}$$

where $E_2=\varphi E_1$ and $E_3=\xi$. In particular $[\xi,E_1]=-(1+\lambda)E_1$ and $[\xi,E_2]=-(1-\lambda)E_2$.

Now let M be 3-dimensional strictly almost Kenmotsu $(\kappa,0,\nu)$-space. Then Dileo and Pastore proved that $\nu=2$. Then we can apply the preceding Lemma. Since $\kappa=-1-\lambda^2$, the eigenvalue λ is constant. The orthonormal frame field satisfies the commutation relation (5.16)

$$[E_1,E_2]=0,\quad [E_2,E_3]=(1-\lambda)E_2,\quad [E_3,E_1]=-(1+\lambda)E_1.$$

Hence M is locally isomorphic to a non-unimodular Lie group G_A with representation matrix

$$A=\begin{pmatrix}-1-\lambda & 0\\ 0 & -1+\lambda\end{pmatrix}.\tag{9.4}$$

The matrix A has $\operatorname{tr}A=-2$ and $\mathsf{D}=\det A=1-\lambda^2<1$. On the other hand, we know that the non-unimodular Lie group $\widetilde{G}_{\mathrm{II}}(\lambda,\lambda)$ has $\mathsf{D}=1-\lambda^2<1$. Hence G_A is locally isomorphic to $\widetilde{G}_{\mathrm{II}}(\lambda,\lambda)$ given in Example 9.2.

Remark 9.2. In $\widetilde{G}_{\mathrm{II}}(\beta,\gamma)$, if $\lambda=-1$, then the representation matrix A is rewritten as

$$A=\begin{pmatrix}-1 & -1-\delta\\ -1+\delta & -1\end{pmatrix}.$$

In this case $\beta=-1+\delta$ and $\gamma=-1-\delta$. The Lie group $G_{\mathrm{II}}(-1+\delta,-1-\delta)$ is an almost Kenmotsu $(-2,-2\delta,2)$-space.

Example 9.4 (Pseudo-symmetric Lie groups)**.** When $\beta\gamma=1$, the simply connected Lie group $\widetilde{G}_{\mathrm{II}}(\beta,\beta^{-1})$ is realized as

$$\widetilde{G}_{\mathrm{II}}(\beta,\beta^{-1})=\left\{\left.\begin{pmatrix}e^{-z}\cosh z & \beta^{-1}e^{-z}\sinh z & x\\ \beta e^{-z}\sinh z & e^{-z}\cosh z & y\\ 0 & 0 & 1\end{pmatrix}\,\right|\, x,y,z\in\mathbb{R}\right\}.$$

The principal Ricci curvatures are

$$-2-\frac{1}{2\beta^2}(\beta^2+1)^2,\quad -2-\frac{1}{2\beta^2}(\beta^2+1)^2,\quad -2+\frac{1}{2\beta^2}(\beta^2+1)^2.$$

In particular, as we saw before, $\widetilde{G}_{\mathrm{II}}(1,1)$ and $\widetilde{G}_{\mathrm{II}}(-1,-1)$ are locally symmetric and isometric to $\mathbb{H}^2(-4)\times\mathbb{R}$.

Remark 9.3. The simply connected almost Kenmotsu Lie group $\widetilde{G}_{\mathrm{II}}(\beta,\gamma)$ has negative constant scalar curvature $\mathrm{s}=-2(3+\lambda^2)$ and constant square norm $\|S\|^2=4(\lambda^2+(\delta+4)\lambda^2+3)$ of the Ricci operator S. Thus we have the inequality

$$\frac{\mathrm{s}^2}{3}\leq\|S\|^2\leq\frac{\mathrm{s}^2}{2}.$$

One can confirm that

- $\mathrm{s}^2/3\leq\|S\|^2$ holds when and only when $\lambda=0$, *i.e.*, $\widetilde{G}_{\mathrm{II}}(\beta,\gamma)$ is isometric to $\mathbb{H}^3(-1)$.
- $\mathrm{s}^2/2\leq\|S\|^2$ holds when and only when $\delta=0$, *i.e.*, $\widetilde{G}_{\mathrm{II}}(\beta,\gamma)$ is isometric to $\mathbb{H}^2(-4)\times\mathbb{R}$.

- $\widetilde{G}_{\mathrm{II}}(\beta,\gamma)$ is conformally flat when and only when $\lambda = 0$ or $\delta = 0$.

In [18], Cheng, Ishikawa and Shiohama proved that a complete conformally flat Riemannian 3-manifold M of constant negative scalar curvature and constant square norm of the Ricci operator is isometric to a space form or else the square norm $\|S\|^2$ satisfies

$$\frac{\mathrm{s}^2}{3} < \|S\|^2 \leq \frac{\mathrm{s}^2}{2}.$$

The almost Kenmotsu Lie group $\widetilde{G}_{\mathrm{II}}(\beta,\gamma)$ with $\lambda = (\beta+\gamma)/2 \neq 0$ satisfies this inequality. But it is *not* conformally flat unless $\delta = 0$.

9.5. Solvable Lie group models

In this subsection we study simply connected non-unimodular Lie group $\widetilde{G}_A$ with representation matrix

$$A = \begin{pmatrix} -2-c & 0 \\ 0 & c \end{pmatrix}, \quad c \in \mathbb{R}. \tag{9.5}$$

Note that $\operatorname{tr} A = -2$ and $\mathsf{D} = -c(c+2)$. We equip the standard left invariant almost Kenmotsu structure (φ,ξ,η,g) on $\widetilde{G}_A$. we denote the resulting almost Kenmotsu manifold by $S(c)$. The metric on $S(c)$ is

$$g = e^{2(c+2)z}\mathrm{d}x^2 + e^{-2cz}\mathrm{d}y^2 + \mathrm{d}z^2.$$

Since $\mathsf{D} = 0$ if and only if $c = 0$ or $c = -2$, $S(0)$ and $S(-2)$ are isometric to $\mathbb{H}^2(-4)\times\mathbb{R}$.

Note that this 1-parameter family of homogeneous spaces can be seen in [106]. The Lie algebra $\mathfrak{s}(c)$ of $S(c)$ is spanned by the orthonormal basis

$$E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; E_3 = \begin{pmatrix} -2-c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We denote by the left invariant vector field on $S(c)$ which is obtained by left translation of E_i ($i = 1,2,3$) by the same letter. Then we have

$$E_1 = e^{-(c+2)z}\frac{\partial}{\partial x}, \quad E_2 = e^{cz}\frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z}. \tag{9.6}$$

The sectional curvatures are given by

$$H = K_{12} = K(E_1\wedge E_2) = c(c+2), \quad K_{13} = K(E_1\wedge E_3) = -(c+2)^2, \quad K_{23} = K(E_2\wedge E_3) = -c^2.$$

The operators h and h' are computed as

$$hE_1 = (c+1)E_2, \quad hE_2 = -(c+1)E_1, \quad hE_3 = 0,$$
$$h'E_1 = (c+1)E_1, \quad h'E_2 = -(c+1)E_2, \quad h'E_3 = 0.$$

From these we notice that the matrix (9.4) and (9.5) are related by $\lambda = 1+c$. Hence $S(c)$ is an almost Kenmotsu $(\kappa,0,2)$-space (equivalently $(\kappa,-2)'$-space) with $\kappa = -1-\lambda^2 = -1-(c+1)^2$. Moreover $S(c)$ is isomorphic to $G_{\mathrm{II}}(c+1,c+1)$ in Example 9.2.

Example 9.5 (Hyperbolic space)**.** For $c = -1$, then $S(c)$ is the warped product model $\mathbb{R}(z)\times_{e^z}\mathbb{E}^2(x,y)$;

$$(\mathbb{R}^3(x,y,z), e^{2z}(\mathrm{d}x^2+\mathrm{d}y^2)+\mathrm{d}z^2)$$

of the hyperbolic 3-space $\mathbb{H}^3(-1)$ of constant curvature -1 (Example 5.1). The almost Kenmotsu structure satisfies

$$(\nabla_X\varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X$$

for all vector fields X and Y on $S(-1)$. Hence $S(-1)$ is a Kenmotsu manifold. The solvable Lie group $S(-1)$ coincides with the non-unimodular Lie group $\widetilde{G}_{\mathrm{II}}(0,0)$ in Example 9.1.

Example 9.6 ($\mathbb{H}^2(-4)\times\mathbb{R}$)**.** Choose $c = -2$. Then $S(c)$ is the Riemannian product of $\mathbb{R}(x)$ and the warped product model $\mathbb{R}(z)\times_{e^{2z}}\mathbb{R}(y)$ of the hyperbolic plane $\mathbb{H}^2(-4)$. The structure is strictly almost Kenmotsu. The metric is expressed as $\mathrm{d}x^2 + e^{4z}\mathrm{d}y^2 + \mathrm{d}z^2$.

Example 9.7 **($\mathbb{H}^2(-4)\times\mathbb{R}$).** Choose $c=0$. Then $S(c)$ is the Riemannian product of $\mathbb{R}(y)$ and the warped product model $\mathbb{R}(z)\times_{e^{2z}}\mathbb{R}(x)$ of the hyperbolic plane $\mathbb{H}^2(-4)$. The structure is strictly almost Kenmotsu. The metric is expressed as $e^{4z}\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2$. The homogenous almost Kenmotsu 3-manifolds $S(0)$ is isomorphic to the example $\mathbb{H}^2(-4)\times\mathbb{R}$ in Example 5.2 as an almost Kenmotsu manifold under the isometry:

$$u:=2x, \quad v:=e^{-2z}, \quad t:=y.$$

Note that this is the isometry given in (5.7) with $k=2$.

Kenmotsu [70] generalized the Weierstrass-Enneper's representation formula for minimal surfaces in Euclidean 3-space $\mathbb{E}^3$ to surfaces in $\mathbb{E}^3$ with prescribed mean curvature function. Kenmotsu's representation formula inspired differential geometers to obtain such a formula for surfaces in hyperbolic 3-space $\mathbb{H}^3(-1)$. In 1987, Góes and Simões [43] obtained integral representation formulas for minimal surfaces in $\mathbb{H}^3(-1)$ and $\mathbb{H}^4(-1)$. Góes and Simões used the upper half space model of the hyperbolic spaces. In 1997, Kokubu [74] obtained an integral representation formula for minimal surfaces in the hyperbolic space $\mathbb{H}^n(-1)$ of arbitrary dimension $n\geq 3$ by using the solvable Lie group model $S(-1)$. Kokubu's formula for $S(-1)$ is generalized to general $G(c)$ in [47, 48, 66]. Nistor [79] studied constant angle surfaces in $S(c)$.

9.6. The type IV Lie algebra

Let us consider a 3-dimensional non-unimodular Lie group $G_{\mathrm{IV}}[\beta,\gamma]$ of type IV equipped with a left invariant almost Kenmotsu structure. The Lie algebra $\mathfrak{g}_{\mathrm{IV}}[\beta,\gamma]$ is determined by the commutation relations:

$$[e_1,e_2]=\gamma e_1, \quad [e_2,e_3]=-\beta e_1, \quad [e_3,e_1]=-2e_1.$$

Then the Levi-Civita connection is described as

$$\begin{gathered}\nabla_{e_1}e_1=-\gamma e_2-2e_3, \quad \nabla_{e_1}e_2=\gamma e_1+\frac{\beta}{2}e_3, \quad \nabla_{e_1}e_3=2e_1-\frac{\beta}{2}e_2,\\ \nabla_{e_2}e_1=\frac{\beta}{2}e_3, \quad \nabla_{e_2}e_2=0, \quad \nabla_{e_2}e_3=-\frac{\beta}{2}e_1,\\ \nabla_{e_3}e_1=-\frac{\beta}{2}e_2, \quad \nabla_{e_3}e_2=\frac{\beta}{2}e_1, \quad \nabla_{e_3}e_3=0,\end{gathered} \tag{9.7}$$

The Lie group $G_{\mathrm{IV}}[\beta,\gamma]$ is strictly almost Kenmotsu. The unimodular kernel is spanned by

$$e_1, \quad \xi+\frac{2}{\gamma}(e_1-e_2).$$

The operators h and h' are given by

$$he_1=\frac{\beta}{2}e_1+e_2, \quad he_2=e_1-\frac{\beta}{2}e_2,$$

$$h'e_1=e_1-\frac{\beta}{2}e_2, \quad h'e_2=-\frac{\beta}{2}e_1-e_2.$$

The eigenvalues of h are 0, λ and $-\lambda$ where

$$\lambda=\frac{\sqrt{\beta^2+4}}{2}.$$

The covariant derivative $\nabla_\xi h$ is computed as

$$\nabla_\xi h=\beta h\varphi.$$

This formula implies that $\delta=-\beta/2$. Hence $\nabla_\xi h=0$ holds when and only when $\beta=0$.

Remark 9.4 (h-eigenframe). To look for h-eigenframes, one need to rotate $\{e_1,e_2\}$ as

$$u_1=\cos\theta\, e_1+\sin\theta\, e_2, \quad u_2=-\sin\theta\, e_1+\cos\theta\, e_2$$

with angle θ determined by

$$\cos(2\theta)=\frac{\beta}{\sqrt{\beta^2+4}}, \quad \sin(2\theta)=\frac{-2}{\sqrt{\beta^2+4}}.$$

Then $\{u_1, u_2, u_3 = e_3\}$ is a left invariant h-eigenframe satisfying

$$hu_1 = \lambda u_1, \quad hu_2 = -\lambda u_2, \quad hu_3 = 0.$$

Note that $\{u_1, u_2, u_3\}$ and $\{e_1, e_2, e_3\}$ are related by (9.2). More precisely in (9.2), replace e_i [resp. $\bar{e}_i$] by u_i [resp. e_i], then we get the present $\{e_1, e_2, e_3\}$ and $\{u_1, u_2, u_3\}$.

Moreover $\{u_1, u_2, u_3\}$ is a global h-eigenframe as in Lemma 6.2 with

$$b = \gamma \cos\theta, \quad c = \gamma \sin\theta, \quad \delta = -\frac{\beta}{2}.$$

Remark 9.5. The Lie group $G[\alpha, \gamma]$ of type IV in [64, 65] is related to $G_{\mathrm{IV}}[\beta, \gamma]$ by

$$G[\alpha, \gamma] = G_{\mathrm{IV}}[-2\alpha, \gamma], \quad G_{\mathrm{IV}}[\beta, \gamma] = G[-\beta/2, \gamma].$$

The Riemannian curvature R, Ricci operator S and the the scalar curvature are described as

$$R(e_1, e_2)e_1 = \left(\gamma^2 - \frac{\beta^2}{4}\right) e_2 + 2\gamma e_3, \quad R(e_1, e_2)e_2 = \left(\frac{\beta^2}{4} - \gamma^2\right) e_1 - \beta\gamma e_3, \quad R(e_1, e_2)e_3 = -\gamma(2e_1 - \beta e_2),$$

$$R(e_1, e_3)e_1 = 2\gamma e_2 + \left(4 - \frac{\beta^2}{4}\right) e_3, \quad R(e_1, e_3)e_2 = -2\gamma e_1 - 2\beta e_3, \quad R(e_1, e_3)e_3 = \left(\frac{\beta^2}{4} - 4\right) e_1 + 2\beta e_2,$$

$$R(e_2, e_3)e_1 = -\beta(\gamma e_2 + 2e_3), \quad R(e_2, e_3)e_2 = \beta\gamma e_1 + \frac{3\beta^2}{4} e_3, \quad R(e_2, e_3)e_3 = 2\beta e_1 - \frac{3\beta^2}{4} e_2.$$

$$\rho_{11} = \frac{\beta^2}{2} - \gamma^2 - 4, \quad \rho_{12} = 2\beta, \quad \rho_{13} = -\beta\gamma,$$

$$\rho_{22} = -\frac{\beta^2}{2} - \gamma^2, \quad \rho_{23} = -2\gamma, \quad \rho_{33} = -\frac{\beta^2}{2} - 4,$$

$$\mathrm{s} = -\frac{\beta^2}{2} - 2\gamma^2 - 8.$$

The principal Ricci curvatures are

$$-4 - \frac{\beta^2}{2} - \gamma^2, \quad -4 - \frac{\beta^2}{2} - \gamma^2, \quad \frac{\beta^2}{2}.$$

These formulas show that this Lie group is never H-almost Kenmotsu (see also [94, Theorem 8]). Moreover $G_{\mathrm{IV}}[\beta, \gamma]$ is always pseudo-symmetric. One can check that $\pounds_\xi S = 0$ holds when and only when $\beta = 0$.

Let us consider a leaf of the canonical foliation $\mathcal{D}$. Then the shape operator $\mathcal{A}$ has the components

$$\mathcal{A} = \begin{pmatrix} -2 & \beta/2 \\ \beta/2 & 0 \end{pmatrix}$$

relative to $\{e_1, e_2\}$. Thus L has extrinsic curvature $-\beta^2/4$ and hence $K_L = -\gamma^2 < 0$.

The characteristic Jacobi operator is given by

$$\ell(e_1) = \left(\frac{\beta^2}{4} - 4\right) e_1 + 2\beta e_2, \quad \ell(e_2) = 2\beta e_1 - \frac{3\beta^2}{4} e_2.$$

Here we compute the Lie derivative $\pounds_\xi \ell$:

$$(\pounds_\xi \ell)e_1 = 2\beta(\beta e_1 + 2e_2), \quad (\pounds_\xi \ell)e_2 = -\beta^2(\beta e_1 + 2e_2).$$

Proposition 9.3. *Let G be a 3-dimensional non-unimodular Lie group equipped with a left invariant almost Kenmotsu structure. Assume that the Lie algebra is of type IV. Then the following properties are mutually equivalent*:

- $\beta = 0$.
- $\pounds_\xi \ell = 0$ *holds.*
- $\pounds_\xi S = 0$ *holds.*
- $(\nabla_\xi S)\xi = 0$ *holds.*

- $\nabla_{\xi} S = 0$ *holds.*
- S *is* η*-parallel.*
- R *is* η*-parallel.*
- G *is locally symmetric.*
- G *is semi-symmetric.*

Proof. The equivalence of $\beta = 0$ and $\nabla S = 0$ can be checked by direct computation. The covariant derivative ∇S is computed as

$$\begin{aligned}
(\nabla_{e_1} S)e_1 &= -\frac{\beta^2}{2}(\gamma e_2 + 2e_3), \quad (\nabla_{e_1} S)e_2 = -\frac{\beta}{2}(\beta\gamma e_1 + 4\gamma e_2 - (\gamma^2 - 4)e_3),\\
(\nabla_{e_1} S)e_3 &= -\frac{\beta}{2}(2\beta e_1 - (\gamma^2 - 4)e_2 - 4\gamma e_3),\\
(\nabla_{e_2} S)e_1 &= \frac{\beta}{2}(2\beta\gamma e_1 + 2\gamma e_2 + (\beta^2 - \gamma^2)e_3), \quad (\nabla_{e_2} S)e_2 = \beta(\gamma e_2 + \beta e_3),\\
(\nabla_{e_3} S)e_1 &= \frac{\beta}{2}\left(4\beta e_1 - (\beta^2 - 4)e_2 - 2\gamma e_3\right), \quad (\nabla_{e_3} S)e_2 = -\frac{\beta}{2}\left((\beta^2 - 4)e_1 + 4\beta e_2 - \beta\gamma e_3,\right)\\
(\nabla_{e_3} S)e_3 &= -\frac{\beta\gamma}{2}(2e_2 - \beta e_3).
\end{aligned}$$

From these we deduce that

$$\nabla S = 0 \iff \nabla_{\xi} S = 0 \iff (\nabla_{\xi} S)\xi = 0 \iff \beta = 0.$$

In a similar manner one can check that $R \cdot S = 0$ if and only if $\beta = 0$. For example $(R(e_1, e_2)S)e_1$ is computed as

$$(R(e_1, e_2)S)e_1 = \beta^2 \left\{\beta e_1 - \frac{1}{4}(\beta^2 - 4)e_2 - \frac{\gamma}{2}e_3\right\}.$$

□

Since $\xi \notin \mathfrak{u}^{\perp}$, the leaf of the canonical foliation through the origin is different from the canonical normal subgroup U. Here we study the canonical normal subgroup U. It is known that U is intrinsically flat. Here we compute the extrinsic curvature of U.

Take a basis

$$X_1 = e_1, \quad X_2 = 2(e_1 - e_2) + \gamma\xi$$

of the unimodular kernel $\mathfrak{u}$. Then we can take a unit normal vector field of U as

$$N = \frac{1}{\sqrt{\gamma^2 + 4}}(\gamma\, e_2 + 2\xi).$$

The shape operator derived from N is

$$\mathcal{A}_U = \frac{1}{\sqrt{\gamma^2 + 4}} \begin{pmatrix} \gamma^2 + 4 - \beta & 2\gamma^2 + \frac{\beta}{2}\gamma^2 + 8 \\ \frac{\beta}{2} & \beta \end{pmatrix}.$$

Let us compute the extrinsic curvature $K_{\text{ext}}(U)$ of U. Direct computations show that

$$\begin{aligned}
\langle \mathcal{A}_U X_1, X_1\rangle &= \sqrt{\gamma^2 + 4}, \quad \langle \mathcal{A}_U X_1, X_2\rangle = \frac{1}{2}\sqrt{\gamma^2 + 4}(\beta + 4),\\
\langle \mathcal{A}_U X_2, X_2\rangle &= 2\sqrt{\gamma^2 + 4}(\beta + 2).\\
Q(X_1, X_2) :&= \langle X_1, X_1\rangle\langle X_2, X_2\rangle - \langle X_1, X_2\rangle^2 = \gamma^2 + 4.
\end{aligned}$$

Hence

$$K_{\text{ext}}(U) = -\frac{\beta^2}{4}.$$

On the other hand, we have

$$\langle R(X_1, X_2)X_1, X_2\rangle = -\frac{\beta^2}{4}(\gamma^2 + 4).$$

Hence we get

$$K(X_1 \wedge X_2) = -\frac{\langle R(X_1, X_2)X_1, X_2\rangle}{Q(X_1, X_2)} = \frac{\beta^2}{4}.$$

This confirms that the Gauss curvature K_U of U is 0.

In the Lie algebra $\mathfrak{g}_{\mathrm{IV}}[\beta,\gamma]$, we can take an orthonormal basis $\{E_1, E_2, E_3\}$ as

$$E_1 := e_1, \quad E_2 := \frac{-2e_2 + \gamma\xi}{\sqrt{\gamma^2+4}}, \quad E_3 = N.$$

Then one can confirm that $\mathrm{tr}\,\mathrm{ad}(E_1) = \mathrm{tr}\,\mathrm{ad}(E_2) = 0$ and $[E_1, E_2] = 0$. Hence $\{E_1, E_2\}$ spans $\mathfrak{u}$. The representation matrix $A_{\beta,\gamma}$ of $\mathrm{ad}(E_3)$ relative to $\{E_1, E_2\}$ is

$$A_{\beta,\gamma} = \begin{pmatrix} -\sqrt{\gamma^2+4} & -\beta \\ 0 & 0 \end{pmatrix}.$$

This representation matrix implies that the above definition of $\{E_1, E_2, E_3\}$ is consistent with (9.1).

Remark 9.6. The Lie algebra $\mathfrak{g}$ with $A = A_{\beta,\gamma}$ coincides with $\mathcal{G}_{a,b}$ in [93, Example 4.6] under the choice $a = -\sqrt{\gamma^2+4}$ and $b = -\beta$.

Note that $\mathrm{tr}\,A_{\beta,\gamma} = -\sqrt{\gamma^2+4}$ and $\mathsf{D} = 0$. The Ricci operator has the components

$$S = \begin{pmatrix} \frac{\beta^2}{2} - (\gamma^2+4) & -\beta\sqrt{\gamma^2+4} & -\frac{4\beta\gamma}{\sqrt{\gamma^2+4}} \\ -\beta\sqrt{\gamma^2+4} & -\frac{\beta^2}{2} & 0 \\ -\frac{4\beta\gamma}{\sqrt{\gamma^2+4}} & 0 & -\frac{\beta^2}{2} - (\gamma^2+4) \end{pmatrix} \tag{9.8}$$

relative to $\{E_1, E_2, E_3\}$. Note that the orthogonality condition $\langle [E_3, E_1], [E_3, E_2]\rangle = 0$ is equivalent to $\beta = 0$. In case $\beta = 0$, S is diagonalized.

On the other hand, we know that $\mathbb{H}^2(-k^2) \times \mathbb{R}$ is realized as a simply connected non-unimodular Lie group $\widetilde{G}_A$ with

$$A = \begin{pmatrix} -k & 0 \\ 0 & 0 \end{pmatrix}.$$

(see Example 5.2). Hence the universal cover $\widetilde{G}_{\mathrm{IV}}[\beta,\gamma]$ of $G_{\mathrm{IV}}[\beta,\gamma]$ is isomorphic to $\mathbb{H}^2(-4-\gamma^2) \times \mathbb{R}$ as a Lie group. However the spectrum set of the Ricci operator S is not identical to $\{-4-\gamma^2, -4-\gamma^2, 0\}$ in general. One can see that $\widetilde{G}_{\mathrm{IV}}[\beta,\gamma]$ is isometric to $\mathbb{H}^2(-4-\gamma^2) \times \mathbb{R}$ when and only when $\beta = 0$, *i.e.*, L is extrinsically flat (*cf.* [93, Theorem 5.1-III]).

Corollary 9.1. *Let G be a 3-dimensional non-unimodular Lie group equipped with a left invariant almost Kenmotsu structure. Assume that the Lie algebra $\mathfrak{g}$ of G is of type IV, then the following properties are mutually equivalent*:

- *The leaves of the canonical foliation are extrinsically flat.*
- *The characteristic Jacobi operator ℓ is invariant under the flow of ξ.*
- *The Ricci operator S is invariant under the flow of ξ.*
- *G is semi-symmetric.*
- *G is locally symmetric.*
- *G is locally isomorphic as a Lie group and isometric to $\mathbb{H}^2(-4-\gamma^2) \times \mathbb{R}$ for some $\gamma \neq 0$.*

Here we give an explicit model for $\widetilde{G}_{\mathrm{IV}}[\beta,\gamma]$.

Example 9.8 (Simply connected Lie group of type IV)**.** The simply connected non-unimodular Lie group $\widetilde{G}_{\mathrm{IV}}[\beta,\gamma]$ determined by $A = A_{\beta,\gamma}$ is given explicitly by

$$\widetilde{G}_{\mathrm{IV}}[\beta,\gamma] = \left\{ \begin{pmatrix} \exp(-\sqrt{\gamma^2+4}z) & \frac{\beta}{\gamma^2+4}(\exp(-\sqrt{\gamma^2+4}z) - 1) & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y, z \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{g}_{\mathrm{IV}}[\beta,\gamma]$ of $\widetilde{G}_{\mathrm{IV}}[\beta,\gamma]$ is spanned by the orthonormal basis

$$E_1 = \exp(-\sqrt{\gamma^2+4}z)\frac{\partial}{\partial x}, \quad E_2 = \frac{\beta}{\gamma^2+4}(\exp(-\sqrt{\gamma^2+4}z) - 1)\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z}.$$

The global orthonormal frame field $\{e_1, e_2 = \varphi e_1, e_3 = \xi\}$ is computed as

$$\begin{aligned}
e_1 =& \exp(-\sqrt{\gamma^2+4}z)\frac{\partial}{\partial x},\\
e_2 =& -\frac{2\beta}{(\gamma^2+4)^{3/2}}(\exp(-\sqrt{\gamma^2+4}z)-1)\frac{\partial}{\partial x}+\frac{1}{\sqrt{\gamma^2+4}}\left(-2\frac{\partial}{\partial y}+\gamma\frac{\partial}{\partial z}\right),\\
e_3 =& \frac{\beta\gamma}{(\gamma^2+4)^{3/2}}(\exp(-\sqrt{\gamma^2+4}z)-1)\frac{\partial}{\partial x}+\frac{1}{\sqrt{\gamma^2+4}}\left(\gamma\frac{\partial}{\partial y}+2\frac{\partial}{\partial z}\right).
\end{aligned}$$

The left invariant metric g is expressed as

$$\begin{aligned}
g =\ & \exp(2\sqrt{\gamma^2+4}z)\,\mathrm{d}x^2+\left\{1+\frac{\beta^2}{(\gamma^2+4)^2}(\exp(\sqrt{\gamma^2+4}z)-1)^2\right\}\mathrm{d}y^2\\
&+\frac{2\beta\exp(\sqrt{\gamma^2+4}z)}{\gamma^2+4}(\exp(\sqrt{\gamma^2+4}z)-1)\mathrm{d}x\,\mathrm{d}y+\mathrm{d}z^2.
\end{aligned}$$

The Ricci operator is given by (9.8) relative to $\{E_1, E_2, E_3\}$.

Example 9.9 ($\mathbb{H}^2(-4-\gamma^2)\times\mathbb{R}$)**.** In case $\beta = 0$, the Ricci operator has components

$$\begin{pmatrix} -(\gamma^2+4) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -(\gamma^2+4)\end{pmatrix}.$$

Hence $\widetilde{G}_{\mathrm{IV}}[0,\gamma]$ is semi-symmetric and isometric to $\mathbb{H}^2(-4-\gamma^2)\times\mathbb{R}$. The canonical normal subgroup U is extrinsically flat.

The Lie group $\widetilde{G}_{\mathrm{IV}}[0,\gamma]$ is realized as the following non-unimodular Lie group

$$\widetilde{G}_{\mathrm{IV}}[0,\gamma]=\left\{\left.\begin{pmatrix}\exp(-\sqrt{\gamma^2+4}\,z) & 0 & x\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\right|\ x,y,z\in\mathbb{R}\right\}$$

equipped with a left invariant Riemannian metric

$$g=\exp(2\sqrt{\gamma^2+4}\,z)\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2.$$

This shows that $\widetilde{G}[0,\gamma]$ is interpreted as a warped product $\mathbb{R}^2(y,z)\times_f\mathbb{R}(x)$ with warping function $f(z)=\exp(\sqrt{\gamma^2+4}\,z)$.

Take a global orthonormal frame field:

$$e_1=\exp(-\sqrt{\gamma^2+4}\,z)\frac{\partial}{\partial x},\quad e_2=\frac{1}{\sqrt{\gamma^2+4}}\left(-2\frac{\partial}{\partial y}+\gamma\frac{\partial}{\partial z}\right),\quad e_3=\frac{1}{\sqrt{\gamma^2+4}}\left(\gamma\frac{\partial}{\partial y}+2\frac{\partial}{\partial z}\right).$$

Define an endomorphism field φ by

$$\varphi e_1=e_2,\quad \varphi e_2=-e_1,\quad \varphi e_3=0.$$

Equivalently,

$$\begin{aligned}
\varphi\frac{\partial}{\partial x} =& \frac{\exp(\sqrt{\gamma^2+4}\,z)}{\sqrt{\gamma^2+4}}\left(-2\frac{\partial}{\partial y}+\gamma\frac{\partial}{\partial z}\right),\\
\varphi\frac{\partial}{\partial y} =& -\frac{2\exp(-\sqrt{\gamma^2+4}\,z)}{\sqrt{\gamma^2+4}}\frac{\partial}{\partial x},\\
\varphi\frac{\partial}{\partial z} =& \frac{\gamma\exp(-\sqrt{\gamma^2+4}\,z)}{\sqrt{\gamma^2+4}}\frac{\partial}{\partial x}.
\end{aligned}$$

Set $\xi = e_3$. Then the 1-form η metrically dual to ξ is

$$\eta = \frac{1}{\sqrt{\gamma^2+4}}(\gamma\, \mathrm{d}y + 2\, \mathrm{d}z).$$

The fundamental 2-form is

$$\Phi = \frac{\exp(\sqrt{\gamma^2+4}\, z)}{\sqrt{\gamma^2+4}}(2\, \mathrm{d}x \wedge \mathrm{d}y - \gamma\, \mathrm{d}x \wedge \mathrm{d}z).$$

One can check that $\mathrm{d}\eta = 0$ and $\mathrm{d}\Phi = 2\eta \wedge \Phi$. Under the isometry (5.7), the metric of $\widetilde{G}_{\mathrm{IV}}[0,\gamma]$ is transformed as

$$\frac{\mathrm{d}u^2 + \mathrm{d}v^2}{\gamma^2+4} + \mathrm{d}t^2.$$

The orthonormal frame field is transformed as

$$e_1 = -\sqrt{\gamma^2+4}v\, \frac{\partial}{\partial u}, \quad e_2 = \gamma\, v\, \frac{\partial}{\partial v} - \frac{2}{\sqrt{\gamma^2+4}}\frac{\partial}{\partial t}, \quad e_3 = 2v\, \frac{\partial}{\partial v} + \frac{\gamma}{\sqrt{\gamma^2+4}}\frac{\partial}{\partial t}.$$

Hence

$$\eta = \frac{1}{\sqrt{\gamma^2+4}}\left(\frac{2\mathrm{d}v}{\sqrt{\gamma^2+4}v} + \gamma\, \mathrm{d}t\right)$$

Under the limit $\gamma \to 0$, η converges to $\mathrm{d}v/(2v)$. This is the (opposite sign of the) 1-form of the standard almost Kenmotsu structure of $\mathbb{H}^2(-4) \times \mathbb{R}$ given in Example 5.2.

10. Ricci curvatures of almost Kenmotsu 3-manifolds

Now we start our investigation on Ricci tensor field and related tensor fields on almost Kenmotsu 3-manifolds.

10.1. The parallelism

As is well known, locally symmetric Kenmotsu manifolds are of constant curvature -1. In [34], Dileo and Pastore proposed the following question:

> *Is a locally symmetric almost Kenmotsu manifold either Kenmotsu of constant curvature -1 or locally isometric to the product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$?*

As a partial affirmative answer, they proved that locally symmetric almost Kenmotsu manifolds of dimension greater than 3 satisfying $R(X,Y)\xi = 0$ for all vector fields X and Y orthogonal to ξ are Kenmotsu of constant curvature -1 or locally isometric to the product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

Next, Wang and Liu [111] proved that locally symmetric CR-integrable almost Kenmotsu manifolds of dimension greater than 3 are Kenmotsu of constant curvature -1 or locally isometric to the product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. It should be remarked that 3-dimensional almost Kenmotsu manifolds are automatically CR-integrable. In this section we give a complete classification of (simply connected and complete) locally symmetric almost Kenmotsu 3-manifolds.

10.2. The system of local symmetry

We compute the covariant derivative ∇S and the derivative $R \cdot S$ over $\mathcal{U}_1$. Take a local orthonormal frame field $\{e_1, e_2, e_3\}$ as in Lemma 6.2, then we have

$$\begin{aligned}
(\nabla_{e_1}S)e_1 = {} & \{e_1(\rho_{11}) + 2b\rho_{12} + 2\rho_{13}\}e_1 + \{e_1(\rho_{12}) - b(\rho_{11}-\rho_{22}) - \lambda\rho_{13} + \rho_{23}\}e_2 \\
& + \{e_1(\rho_{13}) - (\rho_{11}-\rho_{33}) + \lambda\rho_{12} + b\rho_{23}\}e_3, \\
(\nabla_{e_1}S)e_2 = {} & \{e_1(\rho_{12}) - b(\rho_{11}-\rho_{22}) - \lambda\rho_{13} + \rho_{23}\}e_1 + \{e_1(\rho_{22}) - 2b\rho_{12} - 2\lambda\rho_{23}\}e_2 \\
& + \{e_1(\rho_{23}) - \rho_{12} - b\rho_{13} + \lambda(\rho_{22}-\rho_{33})\}e_3, \\
(\nabla_{e_1}S)e_3 = {} & \{e_1(\rho_{13}) - (\rho_{11}-\rho_{33}) + \lambda\rho_{12} + b\rho_{23}\}e_1 + \{e_1(\rho_{23}) - \rho_{12} - b\rho_{13} + \lambda(\rho_{22}-\rho_{33})\}e_2 \\
& + \{e_1(\rho_{33}) - 2\rho_{13} + 2\lambda\rho_{23}\}e_3.
\end{aligned}$$

Hence $\nabla_{e_1} S = 0$ if and only if

$$e_1(\rho_{11}) + 2b\rho_{12} + 2\rho_{13} = 0, \tag{10.1}$$
$$e_1(\rho_{12}) - b(\rho_{11} - \rho_{22}) - \lambda\rho_{13} + \rho_{23} = 0, \tag{10.2}$$
$$e_1(\rho_{13}) - (\rho_{11} - \rho_{33}) + \lambda\rho_{12} + b\rho_{23} = 0, \tag{10.3}$$
$$e_1(\rho_{22}) - 2b\rho_{12} - 2\lambda\rho_{23} = 0, \tag{10.4}$$
$$e_1(\rho_{23}) - \rho_{12} - b\rho_{13} + \lambda(\rho_{22} - \rho_{33}) = 0, \tag{10.5}$$
$$e_1(\rho_{33}) - 2\rho_{13} + 2\lambda\rho_{23} = 0. \tag{10.6}$$

$$\begin{aligned}
(\nabla_{e_2} S)e_1 = &\quad \{e_2(\rho_{11}) - 2c\rho_{12} - 2\lambda\rho_{13}\}e_1 + \{e_2(\rho_{12}) + c(\rho_{11} - \rho_{22}) + \rho_{13} - \lambda\rho_{23}\}e_2, \\
&+ \{e_2(\rho_{13}) + \lambda(\rho_{11} - \rho_{33}) - \rho_{12} - c\rho_{23}\}e_3, \\
(\nabla_{e_2} S)e_2 = &\quad \{e_2(\rho_{12}) + c(\rho_{11} - \rho_{22}) + \rho_{13} - \lambda\rho_{23}\}e_1 + \{e_2(\rho_{22}) + 2c\rho_{12} + 2\rho_{23}\}e_2 \\
&+ \{e_2(\rho_{23}) + \lambda\rho_{12} + c\rho_{13} - (\rho_{22} - \rho_{33})\}e_3, \\
(\nabla_{e_2} S)e_3 = &\quad \{e_2(\rho_{13}) + \lambda(\rho_{11} - \rho_{33}) - \rho_{12} - c\rho_{23}\}e_1 + \{e_2(\rho_{23}) + \lambda\rho_{12} + c\rho_{13} - (\rho_{22} - \rho_{33})\}e_2 \\
&+ \{e_2(\rho_{33}) + 2\lambda\rho_{13} - 2\rho_{23}\}e_3.
\end{aligned}$$

Hence $\nabla_{e_2} S = 0$ if and only if

$$e_2(\rho_{11}) - 2c\rho_{12} - 2\lambda\rho_{13} = 0, \tag{10.7}$$
$$e_2(\rho_{12}) + c(\rho_{11} - \rho_{22}) + \rho_{13} - \lambda\rho_{23} = 0, \tag{10.8}$$
$$e_2(\rho_{13}) + \lambda(\rho_{11} - \rho_{33}) - \rho_{12} - c\rho_{23} = 0, \tag{10.9}$$
$$e_2(\rho_{22}) + 2c\rho_{12} + 2\rho_{23} = 0, \tag{10.10}$$
$$e_2(\rho_{23}) + \lambda\rho_{12} + c\rho_{13} - (\rho_{22} - \rho_{33}) = 0, \tag{10.11}$$
$$e_2(\rho_{33}) + 2\lambda\rho_{13} - 2\rho_{23} = 0. \tag{10.12}$$

$$\begin{aligned}
(\nabla_{e_3} S)e_1 = &\quad \{e_3(\rho_{11}) - 2\alpha\rho_{12}\}e_1 + \{e_3(\rho_{12}) + \alpha(\rho_{11} - \rho_{22})\}e_2 + \{e_3(\rho_{13}) - \alpha\rho_{23}\}e_3, \\
(\nabla_{e_3} S)e_2 = &\quad \{e_3(\rho_{12}) + \alpha(\rho_{11} - \rho_{22})\}e_1 + \{e_3(\rho_{22}) + 2\alpha\rho_{12})\}e_2 + \{e_3(\rho_{23}) + \alpha\rho_{13}\}e_3 \\
(\nabla_{e_3} S)e_3 = &\quad \{e_3(\rho_{13}) - \alpha\rho_{23}\}e_1 + \{e_3(\rho_{23}) + \alpha\rho_{13}\}e_2 + \xi(\rho_{33})e_3.
\end{aligned}$$

Thus $\nabla_\xi S = 0$ if and only if

$$e_3(\rho_{11}) - 2\alpha\rho_{12} = 0, \tag{10.13}$$
$$e_3(\rho_{12}) + \alpha(\rho_{11} - \rho_{22}) = 0, \tag{10.14}$$
$$e_3(\rho_{13}) - \alpha\rho_{23} = 0, \tag{10.15}$$
$$e_3(\rho_{22}) + 2\alpha\rho_{12} = 0, \tag{10.16}$$
$$e_3(\rho_{23}) + \alpha\rho_{13} = 0, \tag{10.17}$$
$$e_3(\rho_{33}) = 0. \tag{10.18}$$

10.3. Classification

Now let us assume that M is a locally symmetric almost Kenmotsu 3-manifold. Then the scalar curvature s is constant on M. If $M = \mathcal{U}_0$, then M is a Kenmotsu manifold of constant curvature -1. Hence M is locally isometric and to $\mathbb{H}^3(-1)$. Moreover M is locally isomorphic to $G_{\mathrm{II}}(\beta, -\beta)$ for some β as a Kenmotsu 3-manifold.

Hereafter we assume that $\mathcal{U}_1$ is non-empty. On $\mathcal{U}$ we take a local orthonormal frame field $\{e_1, e_2, e_3\}$ as in Lemma 6.2.

From $g((\nabla_{e_3} S)e_3, e_3) = 0$, that is, (10.18), we get $\xi(\lambda) = 0$. Thus we have $\rho_{12} = 2\lambda$. Thus we obtain

$$g((\nabla_{e_3} S)e_1, e_2) = \alpha(\rho_{11} - \rho_{22}) = 0.$$

and Since $\rho_{11} - \rho_{22} = -4\alpha\lambda$, we get $\alpha^2\lambda = 0$. Thus $\alpha = 0$ on $\mathcal{U}_1$. This implies that $\nabla_\xi h = 0$ on $\mathcal{U}_1$. Moreover

$$\rho_{11} = \rho_{22} = \frac{\mathrm{s}}{2} + 1 + \lambda^2$$

holds on $\mathcal{U}_1$.

Proposition 10.1. *If an almost Kenmotsu 3-manifold M is locally symmetric then M satisfies $\nabla_\xi h = 0$.*

The converse statement of this Proposition does not hold. In fact the Lie group $G_{\mathrm{II}}(\lambda, \lambda)$ with $\lambda^2 \neq 1$ in Example 9.2 satisfies $\nabla_\xi h = 0$ but it is not locally symmetric.

Proposition 10.2. *On $\mathcal{U}_1$, λ is constant.*

Proof. From the condition $g((\nabla_{e_1} S)e_1, e_1) = 0$, we get $e_2(\lambda) = \lambda e_1(\lambda)$. Analogously from $g((\nabla_{e_2} S)e_2, e_2) = 0$, we get $e_1(\lambda) = \lambda e_2(\lambda)$. Since $\xi(\lambda) = 0$, λ is constant. □

Hence the Ricci operator has components

$$\begin{pmatrix} \mathrm{s}/2 + \lambda^2 + 1 & 2\lambda & -2\lambda b \\ 2\lambda & \mathrm{s}/2 + \lambda^2 + 1 & -2\lambda c \\ -2\lambda b & -2\lambda c & -2(1+\lambda^2) \end{pmatrix}$$

Since λ is constant $g((\nabla_{e_1} S)e_1, e_2) = 0$ reduces to $\rho_{23} = \lambda\rho_{13}$. In addition $g((\nabla_{e_2} S)e_2, e_1) = 0$ reduces to $\rho_{13} = \lambda\rho_{23}$. Hence

$$(\lambda^2 - 1)\rho_{13} = 0, \quad (\lambda^2 - 1)\rho_{23} = 0$$

hold on $\mathcal{U}_1$. Since λ is constant on $\mathcal{U}_1$, we have two possibilities: (1) $\lambda^2 \neq 1$ or (2) $\lambda^2 = 1$.

First we discuss the case $\lambda^2 \neq 1$. In this case $\rho_{13} = \rho_{23} = 0$ on $\mathcal{U}_1$, hence $\mathcal{U}_1$ is H-almost Kenmotsu. Thus we obtain $b = c = 0$ on $\mathcal{U}_1$. Next, since $b = c = 0$, the scalar curvature is $\mathrm{s} = -2(\lambda^2 + 3)$ because of (6.3). Hence the components of the Ricci operator are

$$\begin{pmatrix} -2 & 2\lambda & 0 \\ 2\lambda & -2 & 0 \\ 0 & 0 & -2(1+\lambda^2) \end{pmatrix}.$$

The principal Ricci curvatures are

$$-2 + 2\lambda, \quad -2 - 2\lambda, \quad -2 - 2\lambda.$$

The orthonormal frame field satisfies the commutation relations:

$$[e_1, e_2] = 0, \quad [e_2, e_3] = -\lambda e_1 + e_2, \quad [e_3, e_1] = -e_1 + \lambda e_2.$$

These show that $\mathcal{U}_1$ is locally isometric to the non-unimodular Lie group $G_{\mathrm{II}}(\lambda, \lambda)$ exhibited in Example 9.2 and locally isomorphic to $G(\lambda, 0)$ as an almost Kenmotsu mannifold. However $G_{\mathrm{II}}(\lambda, \lambda)$ is locally symmetric when and only when $\lambda^2 = 1$. This contradicts to the assumption $\lambda^2 \neq 1$ on $\mathcal{U}_1$. Thus we proved the following result.

Proposition 10.3. *If an almost Kenmotsu 3-manifold M is locally symmetric then M is a Kenmotsu manifold of constant curvature -1 or strictly almost Kenmotsu manifold satisfying $\mathrm{tr}\, h^2 = 2$.*

Thus we know that $\lambda = 1$ or $\lambda = -1$ on $\mathcal{U}_1$. Since $\rho_{13} = \lambda\rho_{23}$ and $\rho_{23} = \lambda\rho_{13}$, we have $c = \lambda b$ and hence the Ricci operator has components:

$$\begin{pmatrix} \mathrm{s}/2 + 2 & 2\lambda & -2\lambda b \\ 2\lambda & \mathrm{s}/2 + 2 & -2b \\ -2\lambda b & -2b & -4 \end{pmatrix}.$$

Next from the Jacobi identity, we have $\xi(b) = 0$. We prove the constancy of b. First, from $g((\nabla_{e_1} S)e_1, e_3) = 0$, we have

$$2\lambda e_1(b) = -\rho_{11} - 2 - 2b^2.$$

Next, from $g((\nabla_{e_2} S)e_1, e_3) = 0$, we have

$$2e_2(b) = \rho_{11} + 2 + 2b^2.$$

Thus we get

$$e_2(b) = -\lambda e_1(b).$$

and

$$\rho_{11} = -2 - 2b^2 - 2\lambda e_1(b) = -2 - 2b^2 - 2e_2(b).$$

From $g((\nabla_{e_1}S)e_2, e_3) = 0$, we have

$$\rho_{22} = 2\lambda\, e_1(b) - 2 - 2b^2 = \frac{r}{2} + 2.$$

By using $\rho_{11} = \rho_{22}$, we obtain $e_1(b) = 0$ and hence $e_2(b) = 0$. Thus we conclude that b is constant. Here we set $\gamma := b\sqrt{2}$. Then the scalar curvature is $-2(\gamma^2+4)$. Hence the Ricci operator is

$$\begin{pmatrix} -\gamma^2-2 & 2\lambda & -\sqrt{2}\lambda\gamma \\ 2\lambda & -\gamma^2-2 & -\sqrt{2}\gamma \\ -\sqrt{2}\lambda\gamma & -\sqrt{2}\gamma & -4 \end{pmatrix}, \quad \lambda = \pm 1.$$

The commutation relations are

$$[e_1, e_2] = \frac{\gamma}{\sqrt{2}}(e_1 - \lambda e_2), \quad [e_2, e_3] = -\lambda e_1 + e_2, \quad [e_3, e_1] = -e_1 + \lambda e_2, \quad \lambda = \pm 1.$$

It should be remarked that $\mathcal{U}_1$ is H-almost Kenmotsu if and only if $\gamma = 0$. In case $\gamma = 0$, we notice that $\mathcal{U}_1$ is locally isomorphic to $G_{\mathrm{II}}(1,1)$ or $G_{\mathrm{II}}(-1,-1)$ in Example 9.2. In this case $\mathcal{U}_1$ is locally isometric to $\mathbb{H}^2(-4)\times\mathbb{R}$.

In case $\lambda = 1$, then we rotate the orthonormal frame field and get a new one $\{E_1, E_2, E_3\}$ as

$$E_1 = \frac{1}{\sqrt{2}}(e_1 - e_2), \quad E_2 = \frac{1}{\sqrt{2}}(e_1 + e_2), \quad E_3 = e_3.$$

Then the new frame field satisfies

$$[E_1, E_2] = \gamma\, E_1, \quad [E_2, E_3] = 0, \quad [E_3, E_1] = -2E_1, \quad h'E_1 = E_1, \quad h'E_2 = -E_2, \quad h'E_3 = 0. \tag{10.19}$$

Hence $\mathcal{U}_1$ is locally isomorphic to $G_{\mathrm{IV}}[0,\gamma]$ in Example 9.9. Thus $\mathcal{U}_1$ is locally isometric to $\mathbb{H}^2(-4-\gamma^2)\times\mathbb{R}$.

In case $\lambda = -1$, then set

$$E_1 = \frac{1}{\sqrt{2}}(e_1 + e_2), \quad E_2 = \frac{1}{\sqrt{2}}(-e_1 + e_2), \quad E_3 = e_3.$$

This new frame field satisfies (10.19) and hence $\mathcal{U}_1$ is is locally isomorphic to $G_{\mathrm{IV}}[0,\gamma]$.

Henceforth we proved that $\mathcal{U}_1$ is locally isomorphic to $\mathbb{H}^2(-4-\gamma^2)\times\mathbb{R}$ as an almost Kenmotsu manifold.

Conversely we know that $\mathbb{H}^3(-1)$, $\mathbb{H}^2(-4)\times\mathbb{R}$ and $\mathbb{H}^2(-4-\gamma^2)$ are locally symmetric.

Thus we arrive at the following classification theorem.

Theorem 10.1. *Let M be an almost Kenmotsu 3-manifold. Then M is locally symmetric if and only if M is one of the following spaces*:

1. *If M is H-almost Kenmotsu, then M is a Kenmotsu manifold of constant curvature -1 or locally isomorphic to $\mathbb{H}^2(-4)\times\mathbb{R}$ or*
2. *If M is non H-almost Kenmotsu, then M is locally locally isomorphic to $\mathbb{H}^2(-4-\gamma^2)\times\mathbb{R}$ for some $\gamma\neq 0$.*

This classification result was announced in [21] (see also [22, 109]).

The third example $\mathbb{H}^2(-4-\gamma^2)\times\mathbb{R}$ was discovered in [93, Theorem 1.2 Case (IV)]. This classification gives a negative answer to the question posed by Dileo and Pastore in dimension 3 (see also [21, 22, 109]).

From Theorem 10.1, the following corollary is deduced (see also [110]).

Corollary 10.1. *Let M be an almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = 0$. Then M has harmonic curvature, that is,*

$$(\nabla_X S)Y = (\nabla_Y S)X$$

for all vector fields on X and Y if and only if M is one of the following spaces:

1. *If M is H-almost Kenmotsu, then M is a Kenmotsu manifold of constant curvature -1 or locally isomorphic to $\mathbb{H}^2(-4)\times\mathbb{R}$ or*
2. *If M is non H-almost Kenmotsu, then M is locally locally isomorphic to $\mathbb{H}^2(-4-\gamma^2)\times\mathbb{R}$ for some $\gamma\neq 0$.*

Theorem 10.1 says that local symmetry (equivalently the parallelism of S) is a very strong assumption for almost Kenmotsu 3-manifolds. We are interested in more mild condition for S. First we are interested in the semi-symmetry. The full classification of proper semi-symmetric strictly almost Kenmotsu 3-manifolds is still open. Some partial classifications are obtained.

Theorem 10.2 ([64]). *Let M be an almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = -2\delta\, h\varphi$ for some constant δ. Then M is semi-symmetric if and only if M is locally symmetric.*

Theorem 10.3 ([64]). *Let M be a proper semi-symmetric H-almost Kenmotsu 3-manifold, then it is locally a generalized almost Kenmotsu (κ,μ,ν)-space satisfying*

$$\kappa^2 = \lambda^2(\mu^2+\nu^2), \quad \mu^2+\nu^2 \geq 4, \quad \mathrm{s} = 4\kappa < -4, \quad \mu \neq 0.$$

Theorem 10.4 ([64]). *There does not exist a proper semi-symmetric H-almost Kenmotsu 3-manifold satisfying $\mathrm{d}(\mathrm{tr}(h^2)) \wedge \eta = 0$.*

Here we propose the following problem:

Problem 1. *Classify pseudo-symmetric almost Kenmotsu 3-manifolds (including semi-symmetric ones).*

As a partial answer, pseudo-symmetric homogeneous almost Kenmotsu 3-manifolds are classified as follows:

Proposition 10.4. *Let M be a pseudo-symmetric homogeneous almost Kenmotsu 3-manifold. Then M is locally isomorphic to the following simply connected Lie groups:*

- *The Kenmotsu Lie group $\widetilde{G}_{\mathrm{II}}(\beta,-\beta)$ for any $\beta \in \mathbb{R}$ (locally symmetric).*
- *The strictly almost Kenmotsu Lie group $\widetilde{G}_{\mathrm{II}}(\beta,\beta^{-1})$ for $\beta \neq 0$. It is proper if and only if $\beta \neq \pm 1$. The Lie group $\widetilde{G}_{\mathrm{II}}(1,1)$ and $\widetilde{G}_{\mathrm{II}}(-1,-1)$ are isomorphic to $\mathbb{H}^2(-4)\times\mathbb{R}$ as a homogeneous almost Kenmotsu 3-manifold (locally symmetric).*
- *The strictly almost Kenmotsu Lie group $\widetilde{G}_{\mathrm{IV}}[\beta,\gamma]$ for any $\beta,\gamma \in \mathbb{R}$, $\gamma \neq 0$. In particular, $\widetilde{G}_{\mathrm{IV}}[0,\gamma]$ is isomorphic to $\mathbb{H}^2(-4-\gamma^2)\times\mathbb{R}$ as a homogeneous almost Kenmotsu 3-manifold (locally symmetric).*

It should be emphasize that there exist non-homogeneous proper pseudo-symmetric almost Kenmotsu 3-manifolds, see Example 7.2 and Example 7.3.

Theorem 10.5 ([64]). *Let M be an H-almost Kenmotsu 3-manifold. If M is proper pseudo-symmetric then*

- *M is a Kenmotsu 3-manifold of non-constant curvature and $L = -1$,*
- *M is locally a generalized almost Kenmotsu $(\kappa,0,0)$-space with $\kappa < -1$, $\mathrm{s} \neq 6\kappa$ and $L = \kappa$, or*
- *M is locally a generalized almost Kenmotsu (κ,μ,ν)-space with $\kappa < -1$, $H = \mathrm{s}/2 - 2$ and $L = H = \kappa$.*

10.4. The strong η-parallelism

One of the relaxation of local symmetry (parallelism of S) is the η-parallelism of S. In our previous paper [53], the present author investigated η-parallelism of S of Kenmotsu 3-manifolds.

Theorem 10.6 ([53]). *A Kenmotsu 3-manifold M has η-parallel Ricci operator if and only if it is locally isomorphic to the warped product $I \times_{ce^t} \overline{M}$, where I is an interval and $\overline{M}$ is of constant curvature.*

For Kenmotsu 3-manifolds, η-parallelism and strong η-parallelism of S are *not* equivalent.

Theorem 10.7 ([53]). *Let M be a Kenmotsu 3-manifold. Then the following properties are mutually equivalent:*

- *The scalar curvature is constant along the trajectories of ξ.*
- *The scalar curvature is constant.*
- *The scalar curvature is -6.*
- *The holomorphic sectional curvature is constant.*
- *The Ricci operator is strongly η-parallel.*
- *M is locally symmetric.*
- *M is of constant curvature -1.*

Thus the warped products

$$\mathbb{E}^1 \times_{ce^t} \mathbb{S}^2(k^2), \quad \mathbb{E}^1 \times_{ce^t} \mathbb{H}^2(-k^2)$$

are Kenmotsu 3-manifolds whose Ricci operator is η-parallel but not strongly η-parallel.

Up to now, classification of almost Kenmotsu 3-manifolds with η-parallel Ricci operator is still open.

Problem 2. *Classify almost Kenmotsu 3-manifolds with η-parallel Ricci operator.*

Here we quote the following result.

Proposition 10.5 ([65]). *An almost Kenmotsu 3-manifold M has η-parallel Riemannian curvature if and only if*

$$dH(X) + 2\rho(X,\xi) + 2\rho(h\varphi X,\xi) = 0 \tag{10.20}$$

for all vector field X orthogonal to ξ.

Moreover M has strongly η-parallel Riemannian curvature if R is η-parallel and $dH(\xi) = 0$ holds.

In particular if M is H-almost Kenmotsu, then R is η-parallel if and only if the holomorphic sectional curvature H is η-parallel. In addition, an H-almost Kenmotsu 3-manifold M has strongly η-parallel Riemannian curvature if and only if H is constant.

We know some examples of almost Kenmotsu 3-manifolds whose Ricci operator is η-parallel but not parallel. See Example 7.1 and Example 7.2. Some partial classification of almost Kenmotsu 3-manifolds with η-parallel Ricci operator were obtained in [65]. Here we only quote the following results.

Proposition 10.6 ([65]). *Every almost Kenmotsu Lie group $G_{\mathrm{II}}(\beta,\gamma)$ has η-parallel Ricci operator.*

1. *The Ricci operator is strongly η-parallel when and only when $\beta = \pm\gamma$. In case $\beta = -\gamma$, $G_{\mathrm{II}}(\beta,-\beta)$ is locally isometric to $\mathbb{H}^3$. In case $\beta = \gamma$, $G_{\mathrm{II}}(\beta,\beta)$ is an almost Kenmotsu $(-1-\beta^2, 0, 2)$-space and $(-1,-\beta^2,-2)'$-space.*
2. *The Ricci operator is dominantly η-parallel if and only if $\beta = -\gamma$ or $\beta = \gamma = \pm 1$. In the former case $G_{\mathrm{II}}(\beta,-\beta)$ is Kenmotsu and of constant curvature -1. In the latter case, $G_{\mathrm{II}}(1,1)$ and $G_{\mathrm{II}}(-1,-1)$ are locally isometric to $\mathbb{H}^2(-4)\times\mathbb{R}$.*

Proposition 10.7 ([65]). *For a homogeneous almost Kenmotsu 3-manifold M, the dominant η-parallelism of the Ricci operator is equivalent to the local symmetry.*

Proposition 10.8 ([65]). *Let M be an almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = 0$. Then the dominant η-parallelism of the Ricci operator S is equivalent to the parallelism of S.*

In [29, Theorem 4.1], De claimed that if a Kenmotsu 3-manifold M has η-parallel Riemannian curvature, then M is of constant scalar curvature. However the conclusion of [29, Theorem 4.1] should be corrected as "M has η-parallel scalar curvature". This is a special case of Proposition 5.2. One can see that the conclusion of [29, Theorem 4.1] is true under the assumption R is strongly η-parallel. Hence we obtain the following result.

Proposition 10.9. *A Kenmotsu 3-manifold satisfies the condition*

$$\varphi^2\{(\nabla_W R)(X,Y)Z\} = 0$$

for all vector fields X, Y, Z orthogonal to ξ and any vector field W on M if and only if M is of constant curvature -1.

This Proposition corrects [51, Corollary 4] and [53, Corollary 2.2].

10.5. The characteristic flow invariance

It is known that every Kenmotsu 3-manifold satisfies the commutativity condition $[S,\varphi] = 0$. More generally, we know that the commutativity $[S,\varphi] = 0$ is equivalent to the generalized almost Kenmotsu $(\kappa, 0)$-condition (see Corollary 7.1).

Next candidate is the invariance under the flows of ξ. The following result is due to Cho, Kimura and D. Perrone.

Theorem 10.8 ([27, 95]). *Let M be an H-almost Kenmotsu 3-manifold. Then M satisfies $\pounds_\xi S = 0$ if and only if M is either Kenmotsu 3-manifold of constant curvature -1 or locally isomorphic to a non-unimodular Lie group $G_{\mathrm{II}}(\beta,\beta)$ with $\beta \neq 0$ given in Example 9.2.*

Up to now full classification of almost Kenmotsu 3-manifolds satisfying $\pounds_\xi S = 0$ is still open.

Problem 3. *Classify almost Kenmotsu 3-manifolds satisfying $\pounds_\xi S = 0$.*

Perrone obtained a classification of those spaces under the homogeneity (see [95, Theorem 1.2]).

Theorem 10.9. *Let M be a homogenous almost Kenmotsu 3-manifold. If M satisfies $\pounds_\xi S = 0$, then M is a locally isometric to and locally isomorphic as an almost Kenmotsu 3-manifold to one of the following spaces:*

1. *The hyperbolic 3-space $\mathbb{H}^3(-1)$ realized as $\widetilde{G}_{\mathrm{II}}(0,0)$ in Example 9.1. The structure is Kenmotsu.*

2. *The hyperbolic 3-space $\mathbb{H}^3(-1)$ realized as $\widetilde{G}_{\mathrm{II}}(\beta,-\beta)$ in Example 9.1 with $\beta \neq 0$. The structure is Kenmotsu.*

3. *The Lie group $\widetilde{G}_{\mathrm{II}}(\beta,\beta)$ in Example 9.2 with $\beta \neq 0$. It is H-almost Kenmotsu.*

4. *The non-unimodular Lie group $\widetilde{G}_{\mathrm{IV}}[\beta,\gamma]$ with type IV Lie algebra. It is never H-almost Kenmotsu.*

Generally speaking, the condition $\pounds_\xi \ell = 0$ is weaker that $\pounds_\xi S = 0$. In Section 11 we shall study this property for almost Kenmotsu 3-manifolds under the assumption that ξ is an eigenvector field of S. It should be remarked that for $\widetilde{G}_{\mathrm{IV}}[\beta,\gamma]$, $\pounds_\xi \ell = 0$ is equivalent to $\pounds_\xi S = 0$.

11. Characteristic flow invariant characteristic Jacobi operator

11.1. *The Lie derivative $\pounds_\xi \ell$*

In our previous paper [26] we have investigated contact Riemannian 3-manifolds satisfying $\pounds_\xi \ell = 0$ under the assumption that ξ is an eigenvector field of S (*i.e.*, H-contact assumption). Corresponding classification for almost cosymplectic 3-manifolds was given in [54]. In this section we study almost Kenmostu 3-manifolds satisfying $\pounds_\xi \ell = 0$ under the assumption that ξ is an eigenvector field of S.

Let M be an almost Kenmotsu 3-manifold. Take the open sets $\mathcal{U}_0$ and $\mathcal{U}_1$ as before. In case $M = \mathcal{U}_0$, then M is Kenmotsu and $\pounds_\xi \ell = 0$ holds. Hereafter we assume that $\mathcal{U}_1$ is non-empty. We take a local h-eigenframe field $\{e_1, e_2, e_3\}$ on $\mathcal{U} = \mathcal{U}_0 \cup \mathcal{U}_1$ as in Lemma 6.2.

Let M be an H-almost Kenmotsu 3-manifold. Then as we have seen before, M is locally a generalized almost Kenmotsu (κ,μ,ν)-space. In this case $S\xi = 2\kappa\xi$.

In case $M = \mathcal{U}_0$, then $S\xi = 2\kappa\xi$ and $\pounds_\xi \ell = 0$ holds on whole M, since M is Kenmotsu. Thus hereafter we assume that $\mathcal{U}_1$ is non-empty and take a local orthonormal frame field $\mathcal{E} = (e_1, e_2, e_3)$ as in Lemma 6.2.

We put $\ell(e_1) = \ell_{11}e_1 + \ell_{21}e_2$ and $\ell(e_2) = \ell_{12}e_1 + \ell_{22}e_2$. Then

$$\begin{aligned}(\pounds_\xi \ell)e_1 =& [\xi, \ell(e_1)] - \ell[\xi, e_1] = [\xi, \ell_{11}e_1 + \ell_{21}e_2] - \ell[\xi, e_1]\\ =& \{-\xi(1+\lambda^2+2\lambda\delta) - 2\delta(\xi(\lambda)+2\lambda)\}e_1 + \{\xi(\xi(\lambda)+2\lambda) - 4\lambda\delta(\lambda+\delta)\}e_2.\end{aligned}$$

Analogously we get

$$(\pounds_\xi \ell)e_2 = \{\xi(\xi(\lambda)+2\lambda) + 4\lambda\delta(\lambda-\delta)\}e_1 + \{-\xi(1-\lambda^2-2\lambda\delta) + 2\delta(\xi(\lambda)+2\lambda)\}e_2.$$

Since $\lambda \neq 0$ on $\mathcal{U}_1$, $(\pounds_\xi \ell)e_1 = 0$ holds if and only if

$$-\xi(1+\lambda^2+2\lambda\delta) - 2\delta(\xi(\lambda)+2\lambda) = 0, \quad \xi(\xi(\lambda)+2\lambda) - 4\lambda\delta(\lambda+\delta) = 0. \tag{11.1}$$

On $\mathcal{U}_1$, $(\pounds_\xi \ell)e_2 = 0$ holds if and only if

$$-\xi(1-\lambda^2-2\lambda\delta) + 2\delta(\xi(\lambda)+2\lambda) = 0, \quad \xi(\xi(\lambda)+2\lambda) + 4\lambda\delta(\lambda-\delta) = 0. \tag{11.2}$$

From (11.1)-(11.2) we deduce that $\delta = 0$ on $\mathcal{U}_1$. Hence $\xi(\lambda) = 0$. Conversely if $\delta = \xi(\lambda) = 0$, then (11.1) and (11.2) hold on $\mathcal{U}_1$. It should be remarked that the system $\delta = \xi(\lambda) = 0$ is equivalent to $\nabla_\xi h = 0$. Thus we obtain the following result.

Proposition 11.1. *Let M be an almost Kenmotsu 3-manifold. Then M satisfies $\pounds_\xi \ell = 0$ if and only if M satisfies $\nabla_\xi h = 0$.*

Now let us assume that M is H-almost Kenmotsu. Then M is locally a generalized almost Kenmotsu (κ,μ,ν)-space with $\kappa = -1-\lambda^2$, $\mu = -2\delta$ and $\lambda\nu = \xi(\lambda) + 2\lambda$.

The characteristic Jacobi operator ℓ is computed as

$$\ell(X) = R(X,\xi)\xi = (\kappa\mathrm{I} + \mu h + \nu\varphi h)(X - \eta(X)\xi) = \kappa(X - \eta(X)\xi) + \mu hX + \nu\varphi hX.$$

Hence we get

$$\ell(e_1) = (\kappa + \lambda\mu)e_1 + \lambda\nu e_2, \quad \ell(e_2) = \lambda\nu e_1 + (\kappa - \lambda\mu)e_2.$$

On a generalized almost Kenmotsu (κ,μ,ν)-space we know that (Proposition 5.14)

$$\xi(\kappa) = 2(\kappa+1)(\nu-2).$$

From Proposition 11.1 we obtain

Corollary 11.1. *Let M be a 3-dimensional generalized almost Kenmotsu (κ,μ,ν)-space. Then M satisfies $\pounds_\xi \ell = 0$ if and only if M is a Kenmotsu manifold or a generalized almost Kenmotsu $(\kappa, 0, 2)$-space.*

If in addition M is strictly almost Kenmotsu and satisfies $d\kappa \wedge \eta = 0$, then M is locally isomorphic to a non-unimodular Lie group exhibited in Example 9.4.
On the other hand we know that non-unimodular Lie groups whose Lie algebras are type IV and $\beta = 0$ satisfy $\pounds_\xi \ell = 0$ but not H-almost Kenmotsu.
Up to now the full classification of strictly almost Kenmotsu 3-manifolds satisfying $\pounds_\xi \ell = 0$ is still open.

Problem 4. *Classify strictly almost Kenmotsu 3-manifolds satisfying $\pounds_\xi \ell = 0$ which are non H-almost Kenmotsu.*

11.2. Pseudo-parallelism of ℓ

In [63], we studied almost Kenmotsu 3-manifolds with pseudo-parallel characteristic Jacobi operator. For instance the following results are obtained:

Proposition 11.2 ([63])**.** *There are no almost Kenmotsu 3-manifolds with semi-parallel characteristic Jacobi operator. In particular, the characteristic Jacobi operator of an almost Kenmotsu 3-manifold can not be parallel.*

Proposition 11.3 ([63])**.** *If the characteristic Jacobi operator of an almost Kenmotsu 3-manifold is pseudo-parallel, then M is H-almost Kenmotsu.*

Proposition 11.4 ([63])**.** *Let M be an almost Kenmotsu 3-manifold. Assume that $\operatorname{tr} \ell$ only varies in the direction of ξ. Then M has pseudo-parallel characteristic Jacobi operator if and only if M is weakly η-Einstein.*

Proposition 11.5 ([63])**.** *Homogeneous almost almost Kenmotsu 3-manifolds with pseudo-parallel characteristic Jacobi operator are locally isomorphic to the Kenmotsu Lie group $\tilde{G}_{\mathrm{II}}(\beta, -\beta)$ for some β.*

12. Harmonic maps

In this section we discuss harmonic maps and Ricci operators on Kenmotsu 3-manifolds.

12.1. Holomorphic maps

Let $(M, \varphi, \xi, \eta, g)$ be a Kenmotsu manifold and (N, J, h) an almost Hermitian manifold. A smooth map $f : N \to M$ is said to be a *holomorphic map* if it satisfies

$$df \circ \varphi = J \circ df.$$

On the other hand, a smooth map $f : (N, h) \to (M, \varphi, \xi, \eta, g)$ of a Riemannian manifold into a Kenmotsu manifold is *vertically harmonic* if it satisfies

$$\tau(f) - \eta(\tau(f))\xi = 0.$$

Assume that N is a Kähler manifold, then the tension field $\tau(f)$ of a holomorphic map $f : N \to M$ into a Kenmotsu manifold is given by [52]:

$$\tau(f) = -2e(f)\xi.$$

Theorem 12.1 ([52])**.** *Holomorphic maps from a Kähler manifold into a Kenmotsu manifold are vertically harmonic.*

Corollary 12.1 ([52],[96])**.** *Holomorphic maps from a Kähler manifold into a Kenmotsu manifold are harmonic if and only if they are constants.*

Gherghe [41] obtained the following result (*cf.* Rehman [96]).

Proposition 12.1. *Let $f : M \to N$ be a smooth map from a Kenmotsu manifold M into a Kähler manifold. If f is holomorphic, i.e., $df \circ \varphi = J \circ df$, then f is harmonic.*

Voicu [108] studied holomorphic horizontally conformal submersions from a Kenmotsu 3-manifold $(M, \varphi, \xi, \eta, g)$ onto an oriented Riemannian 2-manifold (N, g_N, J) equipped with a compatible complex structure J.

Proposition 12.2 ([108]). *Let $\pi : M \to N$ be a holomorphic horizontally conformal submersion with dilation Λ. Then $\xi(\log \Lambda) = -1$ and the Ricci operator S of M is described as*

$$S = (\Lambda^2 \pi^* K_N + \mathbf{\Delta}_g(\log \Lambda)\} (\mathrm{I} - \eta \otimes \xi) - 2\eta \otimes \xi.$$

Here K_N is the Gauß curvature of N and $\mathbf{\Delta}_g$ is the Laplace-Beltrami operator of (M, g).

By using the formula of S, we obtain the following result.

Corollary 12.2. *If there exist a holomorphic horizontally conformal submersion from a Kenmotsu 3-manifold M onto a complex 1-dimensional Kähler manifold N, then M has the scalar curvature*

$$\mathrm{s} = 2(\Lambda^2 \pi^* K_N + \mathbf{\Delta}_g(\log \Lambda) - 1).$$

Remark 12.1. In [108, Corollary 3.8], Voicu stated that under the assumption of 12.2, M is weakly η-Einstein. It should be remarked that every Kenmotsu 3-manifold is weakly η-Einstein.

Note that Chinea computed the tension field of holomorphic maps between general almost contact Riemannian manifolds.

Proposition 12.3 ([19]). *Let $(M, \varphi, \xi, \eta, g)$ and $(N, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ be almost contact manifolds. For a holomorphic map $f : M \to N$, its tension field is*

$$\tau(f) = \tilde{\varphi}\{\mathrm{tr}_g(f^*(\tilde{\nabla}\tilde{\varphi})) - f_*(\varphi(\delta_g \varphi) + (\delta_g \eta)\xi) + \{(f_*\xi)\tilde{\eta}(f_*\xi) - \mathrm{tr}_g(f^*(\tilde{\nabla}\tilde{\eta}))\}\tilde{\xi}.$$

Here $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{g}$. The vector field $\delta_g \varphi$ is defined by

$$\delta_g \varphi = \mathrm{tr}_g(\nabla \varphi)$$

and related to $\delta_g \Phi$ by

$$\delta_g \Phi(X) = g(\delta_g \Phi, X), \quad X \in \mathfrak{X}(M).$$

Gherghe and Viĩcu [42] studied harmonicity of holomorphic maps from locally conformal almost cosymplectic manifolds into cosymplectic manifolds [42]. Moreover they studied stability of identity maps of compact locally conformal almost cosymplectic manifolds. Erdem has done more systematic study of harmonicity of holomorphic maps between almost contact Riemannian manifolds [40].

12.2. The φ-condition

During their studies on Levi-harmonic maps, Dragomir and D. Perrone introduced the notion of φ-condition for almost contact Riemannian manifolds [39, Definition 3.8].

Definition 12.1. An almost contact Riemannian manifold $(M, \varphi, \xi, \eta, g)$ satisfies the *φ-condition* if

$$\nabla_{\varphi X}(\varphi X) + \nabla_X X = \varphi[\varphi X, X]$$

for all $X, Y \in \Gamma(\mathcal{D})$.

Perrone introduced the following tensor field [92]:

$$P(X, Y) = (\nabla_X \varphi)(\varphi Y) - (\nabla_{\varphi X} \varphi)Y, \quad X, Y \in \Gamma(TM).$$

Proposition 12.4 ([92]). *An almost contact Riemannian manifold M satisfies the φ-condition if and only if $P(X, X) = 0$ for all $X \in \Gamma(\mathcal{D})$.*

Proposition 12.5 ([92]). *On an almost contact Riemannian manifold M, the following three conditions are mutually equivalent*:

- *Msatisfies Rawnsley's condition* (A) :

$$(\nabla_{\bar{Z}} \varphi)W = 0, \quad Z, W \in \Gamma(\mathcal{S}).$$

- $P(X, Y) = 0$ *for all* $X, Y \in \Gamma(\mathcal{D})$.
- $P(X, X) = 0$ *and* $P(X, \varphi X) = 0$ *for all* $X \in \Gamma(\mathcal{D})$.

Now let us compute the $(1,1)$-part of the covariant exterior derivative $\mathrm{d}^{\nabla}\varphi$ of φ (see [52, p. 359]). For any $\bar{Z} = X + \sqrt{-1}\varphi X \in \Gamma(\bar{\mathcal{S}})$ and $W = Y - \sqrt{-1}\varphi Y \in \Gamma(\mathcal{S})$, we get

$$(\mathrm{d}^{\nabla}\varphi)(\bar{Z}, W) = (P(\varphi X, Y) - P(\varphi Y, X)) - \sqrt{-1}(P(X,Y) + P(Y,X)).$$

Thus $(\mathrm{d}^{\nabla}\varphi)^{(1,1)} = 0$ if and only if φ satisfies φ-condition. Thus we obtain

Proposition 12.6. *An almost contact Riemannian manifold satisfies φ-condition if and only if $(\mathrm{d}^{\nabla}\varphi)^{(1,1)} = 0$ holds.*

As mentioned in [92], contact Riemannian manifolds satisfy φ-condition but not the condition (A). On the other hand, one can check that almost Kenmotsu 3-manifolds never satisfy φ-condition.

Remark 12.2. According to Dragomir and Kamishima [37] a smooth map $f : M \to N$ of a strongly pseudo-convex CR-manifold M into a Riemannian manifold N is said to be $\bar{\partial}_b$*-pluriharmonic* if

$$(\hat{\nabla}\mathrm{d}\varphi)(X,Y) + (\hat{\nabla}\mathrm{d}\varphi)(JX, JY) = 0.$$

Here $\hat{\nabla}$ is the Tanaka-Webster connection of M. When M is a normal strongly pseudo-convex CR-manifold (*i.e.*, Sasakian manifold), $\bar{\partial}_b$-pluriharmonicity coincides with the CR-pluriharmonicity in the sense of [52].

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