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RESEARCH ARTICLE

ATTAINABLE SETS OF INTEGRAL CONSTRAINED SEIR CONTROL SYSTEM WITH NONLINEAR INCIDENCE

Ali. S. NAZLIPINAR[1,*], Farideh MOHAMMADIMEHR[2]

[1]Kütahya Dumlupınar Üniversity, Faculty of Science and Letters, Department of Mathematics, Kütahya,
ali.nazlipinar@dpu.edu.tr, ORCID: 0000-0002-5114-208X
[2]Kütahya Dumlupınar Üniversity, Faculty of Science and Letters, Department of Mathematics, Kütahya,
moh.mehr68@gmail.com, ORCID: 0000-0003-0122-7920

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ABSTRACT

In this survey, we consider the dynamics of a contagious disease spread by employing a nonlinear dynamical control system of differential equations. It considers treatment and vaccination as key control parameters to discern their influence on disease control. The study, approximate the attainable sets of a given control system and presents visual results, while also discussing potential biological applications of their findings.

Keywords: *Attainable set, nonlinear incidence, SEIR model.*

1. INTRODUCTION

One of the core issues in control theory is the determination or estimation of attainable sets. Attainable set or reachable set is the set of all possible phase states of a system at different points in time and occurs in various applications, e.g in the existence of disturbances of parameters in control problems, in terminal point estimations of all solutions of a control problem, optimization, differential inclusions and differential games [16], [17]. Also, with this notion, an optimal control problem can be reduced to the construction or estimation of the sets in which the phase vector of the system lies. Thus, having approximate or exact knowledge about attainable set of a control system allows one to observe the limited capabilities of the control system to determine an optimal or suboptimal control.

The approaches developed to estimate the attainable sets of a specific control system depend on whether the function representing the system is linear or not, as well as on the limits that are a part of the control functions. Geometric constraints and integral constraints are both possible for control functions.

While the integral limitation of the controls is explained in such a way that the system is restricted and depleted when it is utilized, the control functions of the geometrical constraint of the system mean that

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the effect of control is a kind of limited but not depleted amount. Thus, control systems with integral limitations on the control functions are used to represent the control problems involving finite and depleted sources.

Approximate computational techniques and some topological characteristics are present in attainable sets of affine control systems with integral limitations for control have been studied in [10], [11], [17]. In publications [12–15], these analyses are generalized for the fully nonlinear case, and [15] presents an approximation approach for computing the reachable sets in a specified terminal time.

The approximate calculation of attainable sets for control systems describing real physical or biological phenomena can be used for the in-depth study of such phenomena. For example, in biology, problems such as tumor development, changing populations of struggling species, and the spread of epidemics may need to calculate the points at which the state vector of the system can be brought with the use of limited resources.

In [18], the attainable sets of the SIR epidemic model with bilinear incidence and integral restriction of the control function are calculated approximately and shown graphically. The SIR (Susceptible-Infectious-Recovered) model is a basic compartmental epidemiological model widely used to understand the spread of infectious diseases. It assumes that individuals in a population can be classified into three compartments: Individuals who are susceptible to the disease and can become infected(S), individuals who are infected and capable of transmitting the disease to susceptible individuals(I), and people who have recovered from the disease and gained immunity, so they cannot be infected again(R).

Several infectious diseases can be reasonably modeled using the SIR system, especially those that exhibit a relatively straightforward transmission pattern and where immunity is acquired after infection. Some examples of diseases can be modeled as an SIR system include: Mumps, rubella, chickenpox, ifluenza etc.

The SEIR model is used instead of SIR model in diseases where there is an incubation period before the disease is contagious. By introducing the Exposed compartment(E), the SEIR model allows for a more detailed representation of disease transmission dynamics, making it more suitable for modeling diseases with incubation periods or other delays between infection and infectiousness. The SEIR model is particularly useful for diseases that have a significant latent period between exposure and becoming infectious. This includes diseases like COVID-19, where an individual may be exposed to the virus but may not show symptoms or be infectious immediately.

In this study, we will approximately calculate the points that a non-linear incidence SEIR control system can reach under the influence of limited and exhausted vaccination and treatment controls. The parameters of the system to be calculated are not produced from the actual data of the epidemics that have occurred before. The parameters have been chosen in accordance with the rapid spread of the epidemic in order to better show the points that the system can bring with the use of the control effect.

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The paper is organised as follows: In Section 2, the SEIR system to be examined has been introduced, and a control system has been created by adding vaccination and treatment controls to the system. In Section 3, the main theorem used in the approximate calculation of attainable sets and the calculation algorithm obtained by this theorem are given. Section 4, calculates the reachable set of the epidemiological control system for various control stocks and presents the graphical results for specified parameters.

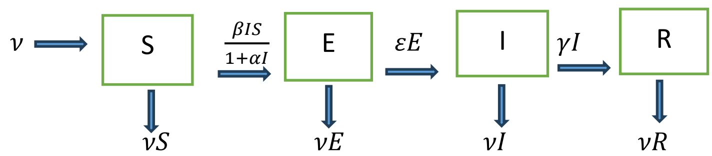
2. MODEL FORMULATION

There are numerous mathematical models that explain how infectious diseases spread and these models have been used to analyze a variety of diseases [2, 3, 4, 5, 6]. The 1927 publication of the Kermack-McKendrick model is one of the early epidemiology models [1]. Following this work, various mathematical models were created to study different types of infectious diseases. The basic rationale for constructing these models is to divide the population into various compartments and characterize the transitions from each compartment to the next over time. Therefore, models are named SI,SIS, SIR, SEIR, SEIRS and so on, considering the compartments in which the population is divided [2-6, 19,20].

In this work, we consider epidemic SEIR model with the nonlinear incidence rate $\beta SI/(1+\alpha I)$. To formulate our model let $S(t)$, $E(t), I(t)$ and $R(t)$ be the fractions of susceptible, exposed(infected but not yet infectious) and recovered individuals at time t. Also following assumptions are made:

1. Disease is assumed to transit horizontally that can be occurred by direct contact(licking, touching etc.) or indirect contact (vectors or fomites). All newborns are included in the susceptible class.
2. $\epsilon > 0$ is the rate of conversion ofexposed population to infectious, γ>0 is the rate of conversion of infectious to recovered, ν>0 represents the birth (and death) rate.
3. $\beta > 0$ is the contact rate and $\alpha \geq 0$ represents the half saturation constant.

The dynamical transfer of the population is depicted in the following schema:



Under these assumptions, the model can be expressed as

$$\begin{aligned} \dot{S}(t) &= \nu - \beta\frac{I(t)S(t)}{1+\alpha I(t)} - \nu S(t), \\ \dot{E}(t) &= \beta\frac{I(t)S(t)}{1+\alpha I(t)} - (\epsilon+\nu)\,E(t)\,, \\ \dot{I}(t) &= \varepsilon E - (\gamma+\nu)I(t)\,, \\ \dot{R}(t) &= \gamma I(t) - \nu R(t) \end{aligned} \tag{2.1}$$

(2.1)

where the derivative d/dt is denoted by •(dot).

Additionally, it observes that in the first three equations of (2.1), the compartment $R = R(t)$ is absent. The last equation of the system (2.1), $R = 1 - S - E - I$, can be used to determine R. Consequently, we can think about the sub-system provided by

$$\begin{aligned} \dot{S}(t) &= \nu - \beta\frac{I(t)S(t)}{1+\alpha I(t)} - \nu S(t), \\ \dot{E}(t) &= \beta\frac{I(t)S(t)}{1+\alpha I(t)} - (\epsilon+\nu)\,E(t)\,, \\ \dot{I}(t) &= \varepsilon E - (\gamma+\nu)I(t)\,. \end{aligned} \tag{2.2}$$

We set $\Omega = \{\boldsymbol{x} = (S, E, I) \in R^3 | 0 \leq S + E + I \leq 1\}$. It can be easily corrected that the set Ω is positively invariant for the system (2.2). As a result, the system is well presented from a mathematical and epidemiological perspective and we can focus only on the region Ω.

If high contact frequency ($\beta = 0.2$) and low recovery rate ($\gamma = 0.001$) is used in the system (2.1) and the system is solved numerically, the evaluation of the system with initial condition $S(0) = 0.7, E(0) = 0.1,\ I(0) = 0.2,\ R(0) = 0$, is shown in the figure below:

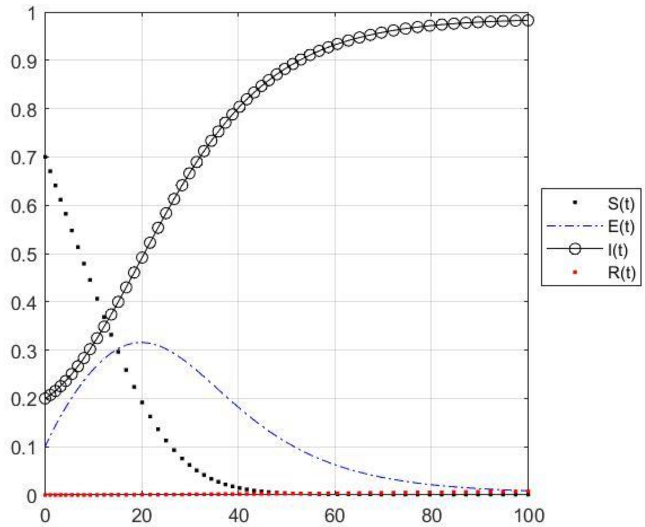


Figure 1. *Solution of the system (2.1) with parameters* $\beta = 0.2, \gamma = 0.001$ *and given initial conditions.*

The graphic above demonstrates how quickly the percentage of infected people is growing. In such circumstances some exterior efforts are needed to control the spread of the disease such like isolation, quarantine, raising awareness using the media, vaccination, treatment etc. Since all these have an economic value, it is aimed to get the best results by using the resources in the best way in case the resources are limited. There are various studies in the literature on the control of the spread of epidemic diseases. However, these studies generally appear in the form of optimal control problems [7, 8, 9].

Unlike optimal control problems, in this study, it is aimed to determine the points where the system can be brought at a certain final time by using existing control resources. Let's add two external measures that can be used to prevent the spread of disease:

$\pi_1(t)$: Vaccination of those who are susceptible at time t,
$\pi_2(t)$: Treatment operations for infected people.

By adding this control variables to the system (2.1), we obtain the control system as follows,

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$$
\begin{aligned}
\dot{S}(t) &= \nu - \beta \frac{I(t)S(t)}{1+\alpha I(t)} - \nu S(t) - S(t)\ \pi_1(t)\,, \\
\dot{E}(t) &= \beta \frac{I(t)S(t)}{1+\alpha I(t)} - (\epsilon + \nu)\, E(t)\,, \\
\dot{I}(t) &= \varepsilon E - (\gamma + \nu) I(t) - I(t)\ \pi_2(t)\,, \\
\dot{R}(t) &= \gamma I(t) - \nu R(t) + S(t)\ \pi_1(t) + I(t)\ \pi_2(t).
\end{aligned} \tag{2.3}
$$

It is considered that $\pi(\cdot) = (\pi_1(\cdot), \pi_2(\cdot)): [0,1] \rightarrow R^2$ satisfies the integral inequality:

$$\int_0^1 \left||\pi(t)|\right|^2 dt = \int_0^1 \left(\pi_1^2(t) + \pi_2^2(t)\right) dt \leq \mu^2 \tag{2.4}$$

which means in applications that the total stock for the controls to effect spreading of the disease is μ and the stock is depleted by using during the time period.

3. APPROXIMATE CALCULATION OF ATTAINABLE SETS

In this section, at first, attainable sets aspect will be denoted for a general control system whose control functions belong the L_p space and whose L_p norms bounded with a positive certain number μ. Then, we will give the main theorem used in the approximate calculation of attainable sets. This method's algorithm and comprehensive information are provided in [12, 13, 14, 15].

Let us consider the control system whose behaviour is investigated by the differential equations system

$$\dot{y}(t) = g(t, y(t), \pi(t)), \quad y(0) = y_0 \in \mathbb{R}^n \tag{3.1}$$

Here, $t \in [0,1]$ is time, $y \in \mathbb{R}^n$ *and* $\pi \in \mathbb{R}^m$ are the phase state vectors and control vectors of the system respectively.

Assume that $\mu > 0$ and $p > 1$. For all $\pi(\cdot) \in L_p([0,1]; \mathbb{R}^m)$ such that

$$\left(\int_0^1 \|\pi(t)\|^p dt\right)^{\frac{1}{p}} \leq \mu \tag{3.2}$$

is called an acceptable control function. Here, $L_p([0,1]; \mathbb{R}^m)$ denotes measurable $\pi(\cdot): [0,1] \rightarrow \mathbb{R}^m$ functions space such that $\|\pi(\cdot)\|_p < +\infty$, $\|\pi(\cdot)\|_p = \left(\int_0^1 \|\pi(t)\|^p dt\right)^{\frac{1}{p}}$.

Ω_p, is the symbol to denoting all admissible control functions set i.e.

$$\Omega_p = \{\pi(\cdot) \in L_p([0,1]; \mathbb{R}^m): \|\pi(\cdot)\|_p \leq \mu\},$$

which is the closed sphere with the radius μ and centered in the origin in $L_p([0,1]; \mathbb{R}^m)$.

We assume that the following conditions hold for the system (3.1) :

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i) $g(\cdot): [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous;

ii) For every finite set $D \subset [0,1] \times \mathbb{R}^n$, there are constants $C_1 = C_1(D) > 0$, $C_2 = C_2(D) > 0$ and $C_3 = C_3(D) > 0$ such that

$$\|g(t, y_1, \pi_1) - f(t, y_2, \pi_2)\| \leq [C_1 + C_2(\|\pi_1\| + \|\pi_2\|)]\|y_1 - y_2\| + C_3\|\pi_1 - \pi_2\|$$

holds for every $(t, y_1) \in D$, $(t, y_2) \in D$, $\pi_1 \in \mathbb{R}^m$ and $\pi_2 \in \mathbb{R}^m$;

iii) There exists a constant $K > 0$ such that $\|f(t, y, \pi)\| \leq K(1 + \|y\|)(1 + \|\pi\|)$

for every $(t, y, \pi) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^m$.

Let $\pi_*(\cdot) \in \Omega_p$. The trajectory of the system (3.1) produced by the acceptable function $\pi_*(\cdot)$ from the initial point $y_*(0) = y_0 \in \mathbb{R}^n$ is the absolutely continuous function $y_*(\cdot): [0,1] \to \mathbb{R}^n$ that holds the equation $\dot{y}_*(t) = g(t, y_*(t), \pi_*(t))$ a.a $t \in [0,1]$, is denoted by $y\big(\cdot; 0, y_0, \pi_*(\cdot)\big)$.

We set $Y_p(t; 0, y_0) = \big\{y\big(t; 0, y_0, \pi(\cdot)\big): \pi(\cdot) \in U_p\big\}$ for any given $t \in [0,1]$.

The reachable set of system (3.1) constrained by (3.2) at time t is the set $Y_p(t; 0, y_0)$, which trivially consists of all $y \in \mathbb{R}^n$ into which system (3.1) can be brought to the moment of time $t \in [0,1]$.

Hausdorff distance of sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ is symbolized by $h(U, V)$ and is defined as

$$h(U, V) = \max\left\{\sup_{u \in U} d(u, V), \sup_{v \in V} d(v, U)\right\},$$

where $d(u, V) = \inf\{\|u - v\|: v \in V\}$.

For given $\psi > 0$, let $N_\psi = \{n_0, n_1, n_2, \dots, n_K\}$ be a finite ψ-net of unit sphere $S = \{v \in \mathbb{R}^m: \|v\| = 1\}$.

Assume that $\xi = \{0 = t_0 < t_1 < \dots < t_N = 1\}$ and $\xi^* = \{0 = x_0 < x_1 < \dots < x_a = H\}$ are a uniform partition on the intervals $[0,1]$ and $[0, H]$ with diameters $\Delta = t_{i+1} - t_i$, $i = 0,1, \dots N - 1$, and $\Delta_* = x_{j+1} - x_j$, $j = 0,1, \dots a - 1$ respectively.

By setting

$$\Omega^H_{p,\Delta,\Delta_*,\psi} = \{\pi(\cdot) \in L_p([0,1]; \mathbb{R}^m): \pi(t) = x_{j_i} n_{l_i}, \qquad t \in [t_i, t_{i+1}),$$
$$x_{j_i} \in \xi^*, n_{l_i} \in N_\psi, i = 0,1, \dots, N - 1 \ \ and \ \ \Delta \cdot \sum_{i=0}^{N-1} x^p_{j_i} \leq \mu^p\}$$

we develop a new set of control functions. It is obvious that $\Omega^H_{p,\Delta,\Delta_*,\psi} \subset \Omega_p$.

Since the real numbers $x_{j_i} \in \xi^*$ can be written as

$$x_{j_i} = j_i \Delta_*, \tag{3.3}$$

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on the segment $[0, H]$ where $0 \le j_i \le a$ is an integer, considering the definition control functions set $\Omega^H_{p,\Delta,\Delta_*,\psi}$, the inequality

$$\sum_{i=0}^{N-1} (j_i)^p \le \frac{\mu^p}{\Delta(\Delta_*)^p} \tag{3.4}$$

holds. Taking into account (3.3) and (3.4), for $i = 0,1, \dots, N-1$, $t \in [t_i, t_{i+1})$, we can rewrite the set $\Omega^H_{p,\Delta,\Delta_*,\psi}$ as

$$\Omega^H_{p,\Delta,\Delta_*,\psi} = \{\pi(\cdot) \in L_p([0,1]; \mathbb{R}^m): \pi(t) = \Delta_* j_i n_{l_i}, 0 \le j_i \le a, n_{l_i} \in N_\psi, \sum_{i=0}^{N-1} j_i^p \le \frac{\mu^p}{\Delta(\Delta_*)^p}\}.$$

By $W^H_{p,\Delta,\Delta_*,\psi}(1; 0, y_0)$, denoting the collection of all points $w(1) = w(t_N)$ calculated by using the recurrence formula

$$w(t_{i+1}) = w(t_i) + (t_{i+1} - t_i) g\big(t_i, w(t_i), \Delta_* j_i n_{l_i}\big), \quad w(t_0) = y_0, \quad i = 0,1, \dots, N-1, \tag{3.5}$$

where $n_{l_i} \in N_\psi$ and the integers $0 \le j_i \le a$, satisfy the inequality (3.4).

The following theorem describes the Hausdorff distance between the sets $W^H_{p,\Delta,\Delta_*,\psi}(1; 0, y_0)$ and $Y_p(1; 0, y_0)$. Here, $W^H_{p,\Delta,\Delta_*,\psi}(1; 0, y_0)$ is the set of points containing finite elements and calculated with the recurrent formula (3.5), while $Y_p(1; 0, y_0)$ is the reachable set that satisfies the constraint of (3.2) of the system (3.1).

Theorem 3.1 *[14-15]* For given any $\varepsilon > 0$, there exists $\psi(\varepsilon) > 0$, $H(\varepsilon) > 0$, $\Delta^*(\varepsilon) > 0$, $\Delta_*(\varepsilon) > 0$ such that the inequality

$$h\left(Y_p(1; 0, y_0), W^{H(\varepsilon)}_{p,\Delta,\Delta_*(\varepsilon),\psi(\varepsilon)}(1; 0, y_0)\right) < \varepsilon \tag{3.6}$$

holds for every $\Delta \le \Delta^*(\varepsilon)$.

Remark 3.1 Theorem 3.1 allows for the creation of an approximate algorithm for computing the reachable set of the system with the restriction (3.2). For arbitrary $\varepsilon > 0$, the parameters in the theorem 3.1 can be predicted beforehand (see [14-15]). After the numbers $\Delta_*(\varepsilon), \Delta^*(\varepsilon)$, $H(\varepsilon)$, $\psi(\varepsilon) > 0$ have been determined, approximately calculation of attainable set $Y_p(1; 0, y_0)$ can be condensed to the computation of the set $W^H_{p,\Delta,\Delta_*,\psi}(1; 0, y_0)$ containing a finite number of points $w(1) = w(t_N)$ determined by the recursive formula (3.5).

Below, the steps of the algorithm to be used to approximate the set $W^H_{p,\Delta,\Delta_*,\psi}(1; 0, y_0)$ are summarized:

1. For given number $\psi > 0$, finite ψ-net $N_\psi = \{n_0, n_1, n_2, \dots, n_K\}$ of the unit sphere $S = \{v \in \mathbb{R}^m: \|v\| = 1\}$ is contructed (a method for this can be found in [15]).

2. Integers $j_0, j_1, \dots, j_{N-1}$ satisfy the inequality (3.4) are selected.

3. The set $W^{H}_{p,\Delta,\Delta_*,\psi}(1;0,y_0)$ is calculated using the formula (3.5) for all elements $\{n_{l_0}, n_{l_1}, \dots, n_{l_N}\}$ belonging to the set N_ψ, and for all integers $0 \le j_i \le a$ selected according to inequality (3.4).

4. APPROXIMATE CALCULATION OF THE ATTAINABLE SETS OF SEIR SYSTEM

Take into consideration the SEIR model, whose behavior is given by the system of equations (2.3). The r.h.s of the function's Lipschitz continuity makes it simple to confirm that the system complies with requirements 3.A, 3.B, and 3.C.

As we mentioned before that $R = R(t)$ does not appear in the first three equations of (2.3) and $S + E + R + I = 1$, we can consider the control system given by

$$\begin{aligned} \dot{S}(t) &= \nu - \beta \frac{I(t)S(t)}{1+\alpha I(t)} - \nu S(t) - S(t)\ \pi_1(t)\ , \\ \dot{E}(t) &= \beta \frac{I(t)S(t)}{1+\alpha I(t)} - (\varepsilon + \nu)\ E(t)\ , \\ \dot{I}(t) &= \varepsilon E(t) - (\gamma + \nu) I(t) - I(t)\ \pi_2(t)\,. \end{aligned} \tag{4.1}$$

Denote

$\widetilde{\Omega}_2 = \{\pi(\cdot) \in L_2([0,1];\mathbb{R}^2) : \|\pi(\cdot)\|_2 \le \mu\}$

Lebesgue-measurable functions $\pi(\cdot):[0,1] \to \mathbb{R}^2$ included in the set of control functions $\widetilde{\Omega}_2$ that are satisfying inequality (2.4). The symbol $\big(S(\cdot;0,S_0,\pi_*(\cdot)), E(\cdot;0,E_0,\pi_*(\cdot)), I(\cdot;0,I_0,\pi_*(\cdot))\big)$ designates the collection of system trajectories (4.1) which are generated by control functions $\pi_*(\cdot) \in \widetilde{\Omega}_2$ and satisfy initial condition $(S(0),E(0),I(0)) = (S_0,E_0,I_0)$.

Let

$$\tilde{Y}_2\big(t;0,(S_0,E_0,I_0)\big) = \big\{\big(S(t;0,S_0,\pi(\cdot)), E(t;0,E_0,\pi(\cdot)), I(t;0,I_0,\pi(\cdot))\big) : \pi(\cdot) \in \widetilde{\Omega}_2\big\}.$$

Thus, the set $\tilde{Y}_2(t;0,(S_0,E_0,I_0))$ is attainable set of the system (4.1) where control functions fulfill (2.4).

For given positive number ψ, a ψ-net in 2-dimensional euclidean space can be defined as

$$N_\psi = \{(\sin k\theta, \cos k\theta) : k = 0,1,\dots,r\} \tag{4.2}$$

where

$$\theta \le \frac{\psi^2}{2}, \quad r = \left[\left|\frac{2\pi}{\theta}\right|\right]. \tag{4.3}$$

Since $\theta > 0$, from (4.3) we have

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$$\begin{aligned}&\|(\sin(k+1)\theta,\cos(k+1)\theta)-(\sin k\theta,\cos k\theta)\|\\&\qquad=\sqrt{(\cos(k+1)\theta-\cos k\theta)^2+(\sin(k+1)\theta-\sin k\theta)^2}=\sqrt{2(1-\cos\theta)}\le\sqrt{2\theta}\\&\qquad\le\psi.\end{aligned}$$

Thus, N_ψ defined by (4.2) is really a ψ-net in $S=\{v=(v_1,v_2)\in\mathbb{R}^2:\|v\|=1\}$.

By $\widetilde{W}^H_{p,\Delta,\Delta_*,\sigma}\big(1;0,(S_0,E_0,I_0)\big)$, we denote the set of all points $(S(1),E(1),I(1))=(S(t_N),E(t_N),I(t_N))$ evaluated using the recursive formula

$$\begin{cases}\dot{S}(t_{i+1}) &= S(t_i)+\Delta\left[\nu-\beta\frac{I(t_i)S(t_i)}{1+\alpha I(t_i)}-\nu S(t_i)-\Delta_* j_i|\sin l_i\theta|S(t_i)\right], & S(0)=S_0,\\ \dot{E}(t_{i+1}) &= E(t_i)+\Delta\left[\beta\frac{I(t_i)S(t_i)}{1+\alpha I(t_i)}-(\varepsilon+\nu)\,E(t_i)\right], & E(0)=E_0,\\ \dot{I}(t_{i+1}) &= I(t_i)+\Delta[\varepsilon E(t_i)-(\gamma+\nu)I(t_i)-\Delta_* j_i|\cos l_i\theta|I(t_i)], & I(0)=I_0,\end{cases}$$

where for every $i=0,1,\dots,N-1$, the integers $0\le l_i\le r$, $0\le j_i\le a$. Here the integers j_i satisfy the inequality (3.4) and r is defined by (4.3).

The following theorem is true in accordance with Theorem 3.1.

Theorem 4.1 For arbitrariliy given >0 there exist numbers $\mathrm{H}(\varepsilon)>0$, $\Delta^*(\varepsilon)>0$, $\Delta_*(\varepsilon)>0$ and $\psi(\varepsilon)>0$ such that the inequality

$$h\left(\tilde{Y}_p(1;0,(S_0,E_0,I_0)),\widetilde{W}^{H(\varepsilon)}_{p,\Delta,\Delta_*(\varepsilon),\psi(\varepsilon)}(1;0,(S_0,E_0,I_0))\right)<\varepsilon$$

holds for every $\Delta\le\Delta^*(\varepsilon)$.

5. NUMERICAL SIMULATIONS

In this section, the possible impact of the use of available resources (vaccination and treatment) on the epidemic in a scenario where the epidemic spreads rapidly will be simulated. Vaccination and treatment resources are limited and also depleted as they are spent.

The model presented here is suitable for any disease model, such as Covid-19, H1N1 (influenza), measles etc. Using the algorithm outlined in [15], we determine the set $\widetilde{W}^H_{2,\Delta,\Delta_*,\sigma}\big(1;0,(S_0,E_0,I_0)\big)$, which approximates the set $\tilde{Y}_2(1;0,(S_0,E_0,I_0))$ that is reachable for the system (2.3) at time t = 1.

It is assumed that the acceptable control functions are belong to the space $L_2([0,1];\mathbb{R}^2)$ and their L_2-norms limited by the positive number μ_0. For various values of the control stock parameter μ_0, the set $\widetilde{W}^H_{2,\Delta,\Delta_*,\sigma}\big(1;0,(S_0,E_0,I_0)\big)$ is approximatively calculated. The full resource for vaccination and treatment in this case is μ_0, which can be used either continuously or intermittently.

As mentioned in the previous sections, $R(t)=1-S(t)-E(t)-I(t)$ can be used to calculate the percentage of recovered people at any point in time t.

In the model examined in the second section (Model 2.1) , an epidemic scenario was created in societies with high contact rate and low natural recovery immunity. As seen in the figure 2.1, in a short period of time, the proportion of individuals exposed to the virus and subsequently infected individuals increased rapidly in the population. The parameters and initial conditions used in the calculations here are as follows:

Table1. Parameters and initial conditions for the system.

S_0	E_0	I_0	β	ε	γ	ν	α
0.7	0.1	0,2	0.2	0.06	0.0001	0.0002	0.004

Therefore, in the designed scenario, it is clearly seen that there must be an external influence in order to control the epidemic, since almost the entire society becomes infected in a short time.

The figures below (Figure1, Figure 2, Figure 3) show the sections of approximated attainable sets of system (2.3) for various values of the control stocks μ_0. Let us interpret how the system is affected under the influence of control.

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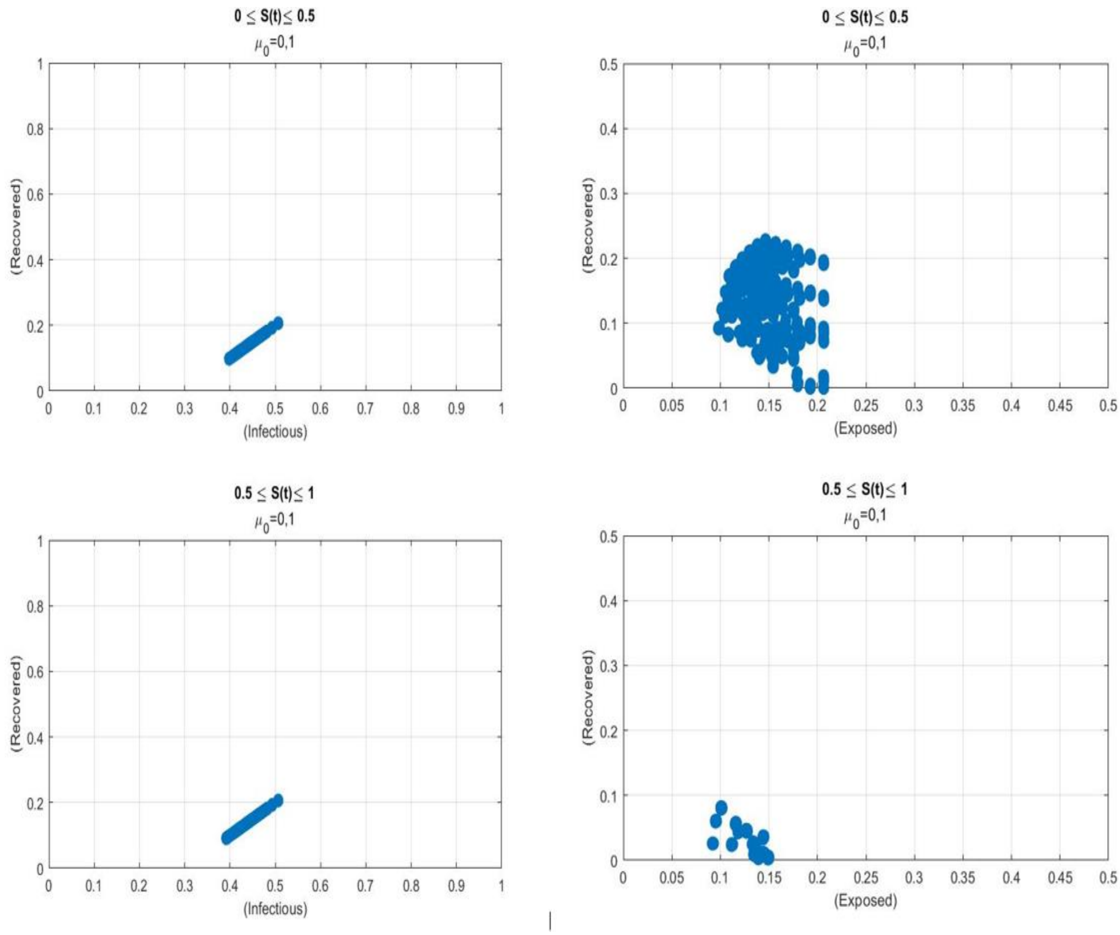

Figure 2. *Sections of Infectious-Recovered and Exposed-Recovered Individuals with respect to Susceptible fractions and control stock* $\mu_0 = 0.1$.

If $\mu_0 = 0.1$ and $(S_0, E_0, I_0) = (0.7, 0.1, 0.2)$, then according to the Figure 2, we get the conclusion that with this control stock, the proportion of infected individuals remains between 40 and 50 percent, while the proportion of those exposed to the virus varies little from the baseline value. So, this control stock is insufficient to produce a positive outcome. The number of persons who develop a permanent immunity to infection is insufficient, and infection rates are still high.

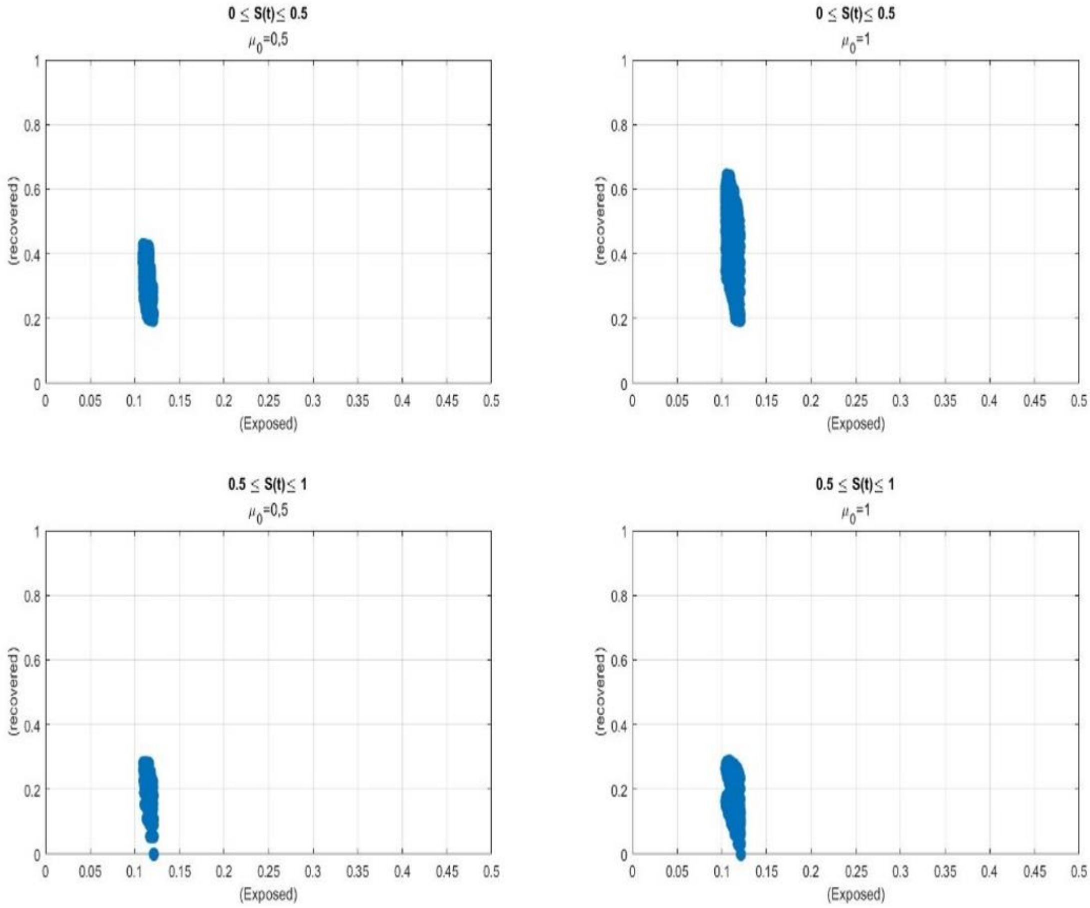

Figure 3. *Sections of Exposed-Recovered Individuals with respect to Susceptible fractions and control stocks* $\mu_0 = 0.5, \mu_0 = 1$.

For $\mu_0 = 0.5$ and $\mu_0 = 1$, in Figure 3, the proportions of individuals exposed to the virus and individuals immunized as treatment are shown in the population. As seen in the graphics, while the rate of individuals exposed to the virus has decreased by half, the rate of individuals who have acquired permanent immunity exceeds 60 percent. This shows that the controls implemented were successful in bringing the epidemic to the desired level.

Finally, in Figure 4 below, graphs are given for $\mu_0 = 0.5$ and $\mu_0 = 1$, at which points the fraction of infected individuals and individuals who have acquired permanent immunity as treatment in the population can reach under vaccination and treatment controls.

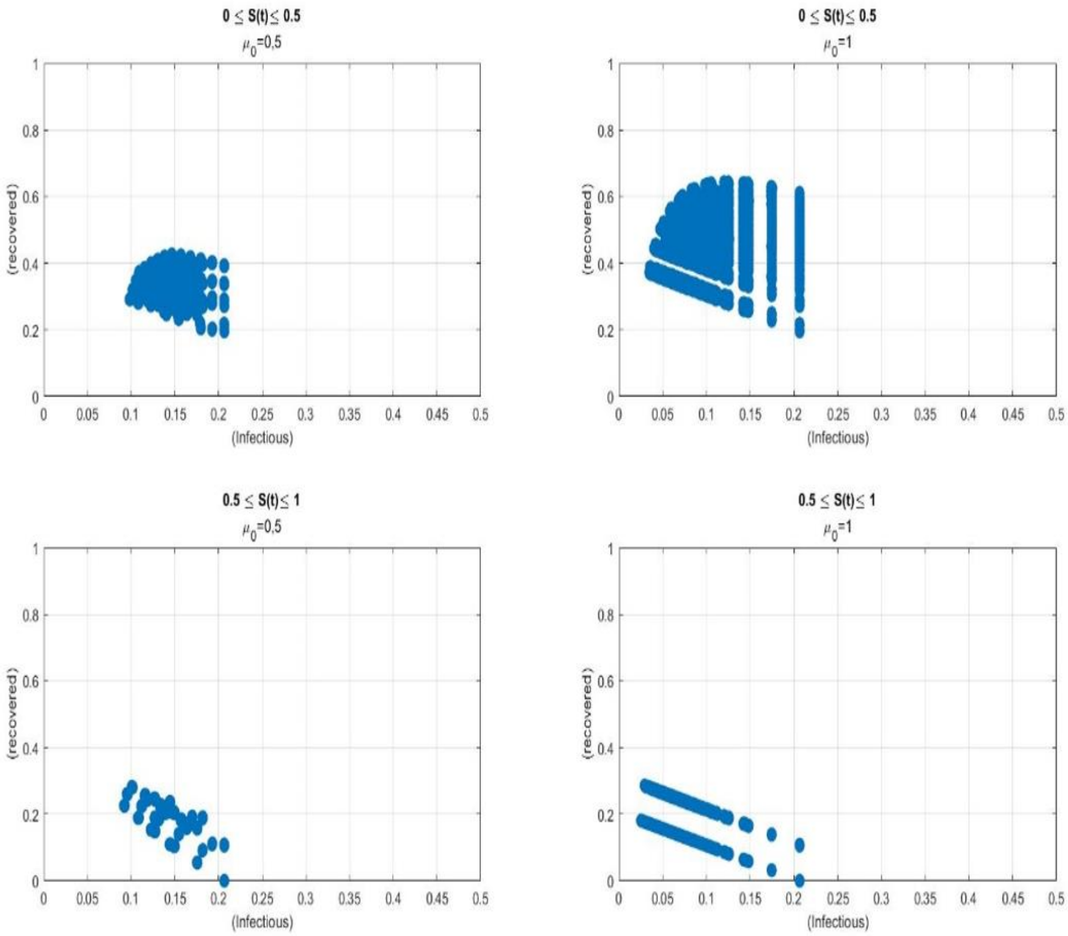

Figure 4. *Sections of Infectious-Recovered Individuals with respect to Susceptible fractions and control stocks* $\mu_0 = 0.5$, $\mu_0 = 1$.

6. CONCLUSION

In this study, the spread of a disease suitable for the SEIR model in a population with certain demographic conditions was simulated. It has been seen in the approximate calculations that the resources to be used for the control of the spread of the epidemic can stop the spread of the epidemic and bring it to the desired levels. Of course, the use of resources will have an economic cost for a society. However, the study carried out here only aims to determine where the system can reach with the use of existing resources, in other words, how the capacity of the system will be. In this way, resource allocation planning for the measures to be taken can be made in advance.

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