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Research Article

The convexity induced by quasi-consistency and quasi-adjacency

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Abstract

In this paper, we introduce (quasi-)consistent spaces and (quasi-)adjacent spaces to characterize convexity spaces. Firstly, we show that convexity spaces can be characterized by quasi-consistent spaces. They can be induced by each other. In particular, each convexity space can be quasi-consistentizable. Every quasi-consistency $\mathcal{U}$ can induce two hull operators and thus determine different convexities $\mathcal{C}^{\mathcal{U}}$ and $\mathcal{C}_{\mathcal{U}}$. And $\mathcal{C}^{\mathcal{U}} = \mathcal{C}_{\mathcal{U}}$ holds when $\mathcal{U}$ is a consistency. Secondly, we use quasi-adjacent spaces to characterize convexity spaces. Each convexity space can be quasi-adjacentizable. In both of characterizations of convexity, remotehood systems play an important role in inducing convexity. Finally, we show there exists a close relation between a quasi-consistency and a quasi-adjacency. Furthermore, there exists a one-to-one correspondence between a quasi-adjacency and a fully ordered quasi-consistency. And we deeply study the relationships among these structures.

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1. Introduction

Uniformity is an important bridge between topology and metric, it is one of the most vital contents in topology and also a useful tool to investigate topology. So far, there has been a lot of work on (quasi-)uniformity in topology (see [4, 5]). At first, the concept of (quasi-)uniform space was introduced and studied by Weil [14]. He found every complete regular space can be uniformizable. Then Bourbaki [1] gave the systematic exposition of the theory of uniform spaces. Moreover, Fletcher and Lindgren [5] also collected and organized work in quasi-uniformities and quasi-proximities to show the usefulness in the study of general topology.

The concepts of a uniformity and of a proximity can be considered either as axiomatizations of some geometric notions, close to quite independent of the concept of a topological space, or as convenient tools for an investigation of topological spaces. Efremovič [3] firstly introduced proximity space, it is a natural generalization of a metric space and a topological group. Smirnov [9] indicated that every proximity can induce a topology and there is a close relation between uniformities and proximities. In [7], Naimpally and Warrack introduced the relative concept of proximity space in detail and got further developments.

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Aiming at the theory of convex structures, M.L.J. Van de Vel organized and perfected it in [10]. Convex structure theory has been penetrated into many branches of mathematics, such as vector space, poset, lattice, metric space and so on (see [6,8,10,11]). Further, convexity also has a close relation with algebra. Wei and Shi [12,13] proved all filters on effect algebras form a convexity. Dong and Shi [2] introduced convex structures on MV-algebras such that the MV-operations are convexity-preserving or weak convexity-preserving. It is a natural question to ask whether we can introduce some nice mathematical structures in convex structure theory to deeply study such as uniformity and proximity in topology.

In this paper, we give a positive answer about the above question. There are two basic approaches to define a quasi-uniformity: relation-based and uniform covering-based. Here we introduce a (quasi-)consistency by a relation and a (quasi-)adjacency by a map. And we use them to characterize convexity in terms of remotehood system. The structure of this paper is organized as follows. In section 2, we recall some preliminary concepts and properties of a convexity and a remotehood system. In section 3, We introduce a (quasi-)consistency and consistent map. we show quasi-consistency and convexity can be induced by each other. And each convexity space is quasi-consistentizable. We prove convexity spaces category **CS** can be embedded in quasi-consistent spaces category **QCS**. Further, a quasi-consistency $\mathcal{U}$ can induce two different convexities $\mathcal{C}^{\mathcal{U}}$ and $\mathcal{C}_{\mathcal{U}}$. And $\mathcal{C}^{\mathcal{U}} = \mathcal{C}_{\mathcal{U}}$ holds when $\mathcal{U}$ has a symmetric base. In section 4, we study the relation of quasi-adjacency and convexity. We show **CS** also can be embedded in quasi-adjacent spaces category **QAS**. In section 5, we prove there is a one-to-one correspondence between a (quasi-)adjacency and a fully ordered (quasi-)consistency. Moreover, there exists an isomorphism between **QAS** and **FQCS**–the category of fully ordered quasi-consistent spaces.

2. Preliminaries

In this section, we recall some basic concepts and notations which are be used in the paper (more details can be found in [10,15]).

Definition 2.1 ([10])**.** Let X be a nonempty set. A nonempty subset $\mathcal{C} \subseteq 2^X$ is called a convexity on X if it satisfies the following properties:

(C1) $\emptyset, X \in \mathcal{C}$;

(C2) if $\{A_i\}_{i\in I} \subseteq \mathcal{C}$ is nonempty, then $\bigcap_{i\in I} A_i \in \mathcal{C}$;

(C3) if $\{A_t\}_{t\in T} \subseteq \mathcal{C}$ is nonempty and totally ordered, then $\bigcup_{t\in t} A_t \in \mathcal{C}$.

The pair $(X, \mathcal{C})$ is called a convexity space.

Remark 2.2 ([10])**.** (C3) is equivalent to (C3′):

(C3′) If $\{A_t\} \subseteq \mathcal{C}$ is directed, then $\bigcup_{t\in T} A_t \in \mathcal{C}$.

For a convexity space $(X, \mathcal{C})$, the (convex) hull of $A \in 2^X$ is

$$co(A) = \bigcap\{B \mid A \subseteq B \in \mathcal{C}\}.$$

Definition 2.3 ([10])**.** A map $co : 2^X \longrightarrow 2^X$ is called a hull operator if it satisfies the following properties:

(H1) $co(\emptyset) = \emptyset$;

(H2) $A \subseteq co(A)$;

(H3) $co(co(A)) = co(A)$;

(H4) $co(A) = \bigcup_{B\in 2^A_{fin}} co(B)$, where $2^A_{fin} = \{B \in 2^X \mid B \subseteq A \text{ and } B \text{ is finite}\}$.

Remark 2.4. (H4) is equivalent to (H4′):

(H4′) $\forall\{A_t\}_{t\in T} \overset{dir}{\subseteq} 2^X$, $co(\bigcup_{t\in T} A_t) = \bigcup_{t\in T} co(A_t)$.

Lemma 2.5 ([10])**.** Let $co : 2^X \longrightarrow 2^X$ be a hull operator and define $\mathcal{C}^{co} \subseteq 2^X$ by

$$\mathcal{C}^{co} = \{A \in 2^X \mid A = co(A)\}.$$

Then $\mathcal{C}^{co}$ is a convexity on X.

Definition 2.6 ([15])**.** Let $(X,\mathcal{C})$ be a convexity space, $x\in X$ and $A\in 2^X$. A is called a remotehood of x if there exists $B\in\mathcal{C}$ such that $x\notin B\supseteq A$.

Set $R_x=\{A\in 2^X\mid \exists B\in\mathcal{C},\text{ such that } x\notin B\supseteq A\}$. Then $R=\{R_x\mid x\in X\}$ is called the remotehood system of $(X,\mathcal{C})$. It is easy to check $B\in\mathcal{C}\Leftrightarrow\forall x\notin B, B\in R_x$.

Lemma 2.7 ([15])**.** The remotehood system $R=\{R_x^{\mathcal{C}}\mid x\in X\}$ of $(X,\mathcal{C})$ satisfies the following properties:
(CR1) $\forall x\in X$, $\emptyset\in R_x^{\mathcal{C}}$;
(CR2) $A\in R_x^{\mathcal{C}}$ and $B\subseteq A\Rightarrow B\in R_x^{\mathcal{C}}$;
(CR3) $\forall\{A_t\}_{t\in T}\overset{dir}{\subseteq} R_x^{\mathcal{C}}$, $\bigcup_{t\in T}A_t\in R_x^{\mathcal{C}}$;
(CR4) $A\in R_x^{\mathcal{C}}\Rightarrow x\notin A$;
(CR5) $A\in R_x^{\mathcal{C}}\Rightarrow\exists B\in R_x^{\mathcal{C}}$ and $A\subseteq B$ such that $\forall y\notin B, B\in R_y^{\mathcal{C}}$.

Remark 2.8 ([15])**.** (1) (CR5) is equivalent to (CR5′):
(CR5′) $A\in R_x^{\mathcal{C}}\Rightarrow\exists B\in R_x^{\mathcal{C}}$ such that $\forall y\notin B, A\in R_y^{\mathcal{C}}$.
(2) $R=\{R_x\mid x\in X\}$ is called a remotehood system if it satisfies (CR1)–(CR5).

Lemma 2.9 ([15])**.** Let $R=\{R_x\mid x\in X\}$ be a remotehood system. Define

$$\mathcal{C}^R=\{A\in 2^X\mid\forall x\notin A, A\in R_x\}.$$

Then $\mathcal{C}^R$ is a convexity.

Definition 2.10 ([10])**.** A map $f:(X,\mathcal{C}_X)\longrightarrow(Y,\mathcal{C}_Y)$ between convexity spaces is called convexity-preserving (CP, in short) if

$$\forall B\in\mathcal{C}_Y,\quad f^{-1}(B)\in\mathcal{C}_X.$$

A map f is called CP at x means: $\forall V\in R_{f(x)}^{\mathcal{C}_Y}$, $f^{-1}(V)\in R_x^{\mathcal{C}_X}$.

The category of convexity spaces and CP maps is denoted by **CS**.

3. Quasi-consistent spaces characterize convexity spaces

In this section, we will show that convexity and quasi-consistency can be induced by each other and discuss the relation between CP maps and consistent maps. We will prove every convexity space can be quasi-consistentizable.

Here we recall the relative concepts of a relation. A relation U is a subset of $X\times X$, its complement is defined by $U'=\{(x,y)\mid(x,y)\notin U\}$ and its inverse is defined by $U^{-1}=\{(y,x)\mid(x,y)\in U\}$. U is called a symmetric element if $U=U^{-1}$. If U and V are two relations, its composition is given by $U\circ V=\{(x,y)\mid\exists y\in X\text{ such that }(x,y)\in V,(y,z)\in U\}$. Denote $U(x)=\{y\in X\mid(x,y)\in U\}$. Next we give the definition of a (quasi-)consistency.

Definition 3.1. A nonempty subset $\mathcal{U}\subseteq 2^{X\times X}$ is called a quasi-consistency on X if it satisfies the following properties:
(QC1) $\forall D\in\mathcal{U}, x\in X, (x,x)\in D$;
(QC2) $D\in\mathcal{U}$ and $D\subseteq E\Rightarrow E\in\mathcal{U}$;
(QC3) $\forall\{D_t\}_{t\in T}\overset{cdir}{\subseteq}\mathcal{U}$, then $\bigcap_{t\in T}D_t\in\mathcal{U}$;
(QC4) $\forall D\in\mathcal{U}\Rightarrow\exists V,W\in\mathcal{U}$ such that $V\circ W\subseteq D$.

$\mathcal{B}\subseteq\mathcal{U}$ is called a basis of $\mathcal{U}$ if for each $D\in U$, there exists $B\in\mathcal{B}$ such that $B\subseteq D$.

A quasi-consistency $\mathcal{U}$ is called consistency if $\mathcal{U}$ also has a basis of symmetric elements. If $\mathcal{U}$ is a (quasi-)consistency on X, then the pair $(X, \mathcal{U})$ is called a (quasi-)consistent space.

There is a one-to-one correspondence between a relation containing diagonal condition and a remotehood map. For each relation $D \subseteq X \times X$ with $(x, x) \in D$, we can define a map $f_D : X \longrightarrow 2^X$ as follows:

$$\forall x \in X, \quad f_D(x) = D'(x).$$

Then it satisfies

(1) $\forall x \in X,\ x \notin f_D(x)$;

(2) $D \subseteq E \Rightarrow f_E(x) \subseteq f_D(x)$;

(3) $\forall \{D_t\}_{t\in T} \subseteq X \times X$, then $f_{\bigcap_{t\in T} D_t} = \bigcup_{t\in T} f_{D_t}$;

(4) $f_{D\circ E}(x) = \bigcap_{z\notin f_E(x)} f_D(z)$.

f_D is called a remotehood map when it satisfies (1). In Section 3, we show $\mathcal{U}'_x = \{D'(x) \mid D \in \mathcal{U}\}$ is a remotehood system of x. So it is reasonable for f_D to be called a remotehood map. On the contrary, for a remotehood map $f : X \longrightarrow 2^X$, define $D_f \subseteq X \times X$ as follows:

$$D_f = \{(x, y) \mid y \in f'(x) \triangleq (f(x))'\}.$$

Then $(x, x) \in D_f$ and $f_{D_f} = f,\ D_{f_D} = D$.

Definition 3.2. A map $f : (X, \mathcal{U}_X) \longrightarrow (Y, \mathcal{U}_Y)$ between (quasi-)consistent spaces is called consistent if

$$\forall V \in \mathcal{U}_Y, \quad (f \times f)^{-1}(V) \in \mathcal{U}_X,$$

where $(f \times f)^{\leftarrow}(V)(x_1, x_2) = V(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$. The category of quasi-consistent spaces and consistent maps is denoted by **QCS**.

Lemma 3.3. Let $(X, \mathcal{U})$ be a quasi-consistent space. Define $R^{\mathcal{U}} = \{\mathcal{U}'_x \mid x \in X\}$, where $\mathcal{U}'_x = \{D'(x) \mid D \in \mathcal{U}\}$. Then $R^{\mathcal{U}}$ is a remotehood system.

Proof. It suffices to check $R^{\mathcal{U}}$ satisfies (CR1)–(CR5$'$).

(CR1) $X \times X \in \mathcal{U}$ since $\mathcal{U}$ is an up-set. So $X \in \mathcal{U}(x)$ for any $x \in X$. Hence $\emptyset \in \mathcal{U}'(x)$.

(CR2) Since $\mathcal{U}$ is an up-set, $\mathcal{U}_x = \{D(x) \mid D \in \mathcal{U}\}$ is an up-set. If $D(x) \in \mathcal{U}_x$ and $D(x) \subseteq B$, then we can construct $E = (\{x\} \times B) \cup (\{x\}' \times X)$. It is obvious that $D \subseteq E$ and $E(x) = B$, then $B \in \mathcal{U}_x$ holds. So $\mathcal{U}'_x$ is a down-set.

(CR3) Since $\mathcal{U}$ is closed for all codirected intersections, we can prove $\mathcal{U}_x$ is closed for all codirected intersections by the above constructive way. Then $\mathcal{U}'_x$ is closed for directed unions.

(CR4) For any $D \in \mathcal{U}$, $x \in D(x)$ implies $x \notin D'(x)$.

(CR5$'$) If $A'(x) \in \mathcal{U}'_x$, then there exists $C, B \in \mathcal{U}$ such that $C \circ B \subseteq A$. Next we prove $\forall y \notin B'(x)$, $A'(x) \in \mathcal{U}'_y$. $y \notin B'(x)$ implies $(x, y) \in B$. Then

$$z \in C(y) \Rightarrow (y, z) \in C \text{ and } (x, y) \in B \Rightarrow (x, z) \in C \circ B \subseteq A \Rightarrow z \in A(x).$$

This proves $C(y) \subseteq A(x)$, i.e., $A'(x) \subseteq C'(y)$. Since $\mathcal{U}'_y$ is a down-set, $A'(x) \in \mathcal{U}'_y$.

Thus, by the above proof, $R^{\mathcal{U}}$ is a remotehood system. □

Proposition 3.4. Let $(X, \mathcal{U})$ be a quasi-consistent space. Define

$$\mathcal{C}_{\mathcal{U}} = \{A \in 2^X \mid \forall x \notin A, \exists D \in \mathcal{U}, A \subseteq D'(x)\}.$$

Then $\mathcal{C}_{\mathcal{U}}$ is a convexity.

Proof. It is straightforward to check $\mathcal{C}_{\mathcal{U}}$ is a convexity by Lemma 2.9 and Lemma 3.3. □

Since $R^{\mathcal{U}}$ satisfies (CR5$'$), it follows that $R^{\mathcal{U}} = \{\mathcal{U}'_x \mid x \in X\}$ is the remotehood system of $(X, \mathcal{C}_{\mathcal{U}})$.

Proposition 3.5. For each $A \in 2^X$, $co(A) = \{x \in X \mid \forall D \in \mathcal{U}, \exists y \in A, \text{ s.t. } y \in D(x)\}$ is the convex hull of A in $(X, \mathcal{C}_{\mathcal{U}})$.

Proof. We first need to show $co(A) \in \mathcal{C}_{\mathcal{U}}$. If $x \notin co(A)$, then there exists $E \in \mathcal{U}$ such that $A \subseteq E'(x)$. $E \in \mathcal{U}$ implies $\exists C, D \in \mathcal{U}$ such that $C \circ D \subseteq E$. $co(A) \subseteq D'(x)$ can be obtained by the following implication.

$$y \in D(x) \Rightarrow C(y) \subseteq C \circ D(x) \subseteq E(x) \subseteq A' \Rightarrow y \notin co(A).$$

It follows that for any $x \notin co(A)$, we find $D \in \mathcal{U}$ such that $co(A) \subseteq D'(x)$. Hence $co(A) \in \mathcal{C}_{\mathcal{U}}$.

Next we need to show $co(A) \subseteq C$ for any $A \subseteq C \in \mathcal{C}_{\mathcal{U}}$. $C \in \mathcal{C}_{\mathcal{U}}$ implies there exists $D \in \mathcal{U}$ such that $D(x) \subseteq C' \subseteq A'$ for any $x \notin C$. Thus $x \notin co(A)$ and then $co(A) \subseteq C$ holds. □

Lemma 3.6. $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ is CP if and only if f is CP at each point $x \in X$.

Proof. Necessity: For each $x \in X$ and $V \in R_{f(x)}$, there exists $C \in \mathcal{C}_Y$ such that $f(x) \notin C \supseteq V$. So $x \notin f^{-1}(C) \supseteq f^{-1}(V)$. Since f is CP, $f^{-1}(C) \in \mathcal{C}_X$. Thus $f^{-1}(V) \in R_x$.
Sufficiency: For any $C \in \mathcal{C}_Y$ and $x \notin f^{-1}(C)$, then $C \in R_{f(x)}$. Since f is CP at $f(x)$, we have $f^{-1}(C) \in R_x$. Hence $f^{-1}(C) \in \mathcal{C}_X$. □

Proposition 3.7. If $f : (X, \mathcal{U}_X) \longrightarrow (Y, \mathcal{U}_Y)$ is consistent, then $f : (X, \mathcal{C}_{\mathcal{U}_X}) \longrightarrow (Y, \mathcal{C}_{\mathcal{U}_Y})$ is CP.

Proof. It suffices to show $f^{-1}(V) \in R_x$ for each $x \in X$ and $V \in R_{f(x)}$. Since $R_{f(x)} = \{D'(f(x)) \mid D \in \mathcal{U}_Y\}$, there exists $D \in \mathcal{U}_Y$ such that $V = D'(f(x))$. So we only need to check that $f^{-1}(D'(f(x))) \in R_x$. Next, we prove an equation $(f \times f)^{-1}(D) = f^{-1} \circ D \circ f$.

$$(x, y) \in (f \times f)^{-1}(D) \Rightarrow (f(x), f(y)) \in D \Rightarrow f(y) \in D(f(x)) \Rightarrow y \in f^{-1} \circ D \circ f(x).$$

Then

$$f^{-1}(D'(f(x))) = (f^{-1}(D(f(x))))' = ((f \times f)^{-1}(D))'(x).$$

We have $(f \times f)^{-1}(D) \in \mathcal{U}_X$ since f is a consistent map and $D \in \mathcal{U}_Y$. Hence, $(f \times f)^{-1}(D)'(x) \in R_x$. □

Lemma 3.8. Let $\{L_t\}_{t \in T}$ be a family of sets and $L_t = \{p_{t_i} \mid i \in I_t\}$ be codirected for all $t \in T$. Then $\prod_{t \in T} L_t$ is codirected.

Proof. Let $(l_t)_{t \in T}, (k_t)_{t \in T} \in \prod_{t \in T} L_t$ and $l_t,\ k_t \in L_t$ for all $t \in T$. Since L_t is codirected, there exists $m_t \in L_t$ such that $l_t, k_t \leq m_t$. Then $(m_t)_{t \in T} \geq (l_t)_{t \in T},\ (k_t)_{t \in T}$. □

Proposition 3.9. Let $(X, \mathcal{C})$ be a convexity space. Define

$$\mathcal{U}_{\mathcal{C}} = \{D \in 2^{X \times X} \mid \exists \{P_i\}_{i \in I} \subseteq \mathcal{C}, \text{ s.t. } \{D_{P_i}\}_{i \in I} \text{ is codirected and } \bigcap_{i \in I} D_{P_i} \subseteq D\},$$

where $D_P \subseteq X \times X$ is defined by $D_P(x) = \begin{cases} P', & x \in P', \\ X, & x \in P. \end{cases}$ Then $\mathcal{U}_{\mathcal{C}}$ is a quasi-consistency.

Proof. (QC1) and (QC2) are obvious.

(QC3) Take $\{D_t\}_{t \in T} \overset{cdir}{\subseteq} \mathcal{U}_{\mathcal{C}}$. Then for each $t \in T$, there exists $\{P_{t_i}\}_{i \in I_t} \subseteq \mathcal{C}$ such that $\{D_{P_{t_i}}\}_{i \in I_t}$ is codirected and $\bigcap_{i \in I_t} D_{P_{t_i}} \subseteq D_t$. So we have $\bigcap_{t \in T} \bigcap_{i \in I_t} D_{P_{t_i}} \subseteq D_t$. By Lemma 3.8, $\{D_{P_{t_i}}\}_{t \in T, i \in I_t}$ is codirected. Thus, $\bigcap_{t \in T} D_t \in \mathcal{U}_{\mathcal{C}}$.

(QC4) For any $D \in \mathcal{U}_{\mathcal{C}}$, there exists $\{P_i\}_{i \in I} \subseteq \mathcal{C}$, such that $\{D_{P_i}\}_{i \in I}$ is codirected and $\bigcap_{i \in I} D_{P_i} \subseteq D_t$. By $D_P \circ D_P = D_P$, it follows that $\bigcap_{i \in I} D_{P_i} \circ \bigcap_{i \in I} D_{P_i} \subseteq \bigcap_{i \in I} (D_{P_i} \circ D_{P_i}) = \bigcap_{i \in I} D_{P_i} \subseteq D$. Let $V = \bigcap_{i \in I} D_{P_i} \in \mathcal{U}_{\mathcal{C}}$. Then V satisfies $V \circ V \subseteq D$. □

Proposition 3.10. If $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ is CP, then $f : (X, \mathcal{U}_{\mathcal{C}_X}) \longrightarrow (Y, \mathcal{U}_{\mathcal{C}_Y})$ is a consistent map.

Proof. If $D \in \mathcal{U}_{\mathcal{C}_Y}$, then there exists $\{P_i\}_{i\in I} \subseteq \mathcal{C}_Y$ such that $\{D_{P_i}\}_{i\in I}$ is codirected and $\bigcap_{i\in I} D_{P_i} \subseteq D$. Since $(f\times f)^{-1}$ is inf-preserving(order-preserving), we have

$$\bigcap_{i\in I}(f\times f)^{-1}(D_{P_i}) = (f\times f)^{-1}(\bigcap_{i\in I} D_{P_i}) \subseteq (f\times f)^{-1}(D).$$

Next, we use two equations as follows

$$(1)\ (f\times f)^{-1}(D) = f^{-1}\circ D\circ f, \qquad (2)\ \forall P\in\mathcal{C}_X,\ (f\times f)^{-1}(D_P) = D_{f^{-1}(P)}.$$

The first equation is checked in the prove of Proposition 3.7. Here we check the second equation. Since $P \in \mathcal{C}_Y$ and f is CP, it follows that $f^{-1}(P) \in \mathcal{C}_X$ and

$$D_{f^{-1}(P)}(x) = \begin{cases} f^{-1}(P'), & x\in f^{-1}(P'), \\ X, & x\in f^{-1}(P). \end{cases}$$

Then we can prove $(f\times f)^{-1}(D_P)(x) = f^{-1}(D_P(f(x))) = D_{f^{-1}(P)}(x)$ for any $x\in X$. This implies that

$$\bigcap_{i\in I}(f\times f)^{-1}(D_{P_i}) = \bigcap_{i\in I} D_{f^{-1}(P_i)} \subseteq (f\times f)^{-1}(D).$$

Since $\{P_i\}_{i\in I} \subseteq \mathcal{C}_Y$ and f is CP, we know $\{f^{-1}(P_i)\}_{i\in I} \subseteq \mathcal{C}_X$. And $\{(f\times f)^{-1}(D_{P_i})\}_{i\in I}$ is codirected because $\{D_{P_i}\}_{i\in I}$ is codirected and $(f\times f)^{-1}$ is order-preserving. By equation (2), $\{D_{f^{-1}(P_i)}\}_{i\in I}$ is codirected. In general, $(f\times f)^{-1}(D) \in \mathcal{U}_{\mathcal{C}_X}$. Hence, $f : (X, \mathcal{U}_{\mathcal{C}_X}) \longrightarrow (Y, \mathcal{U}_{\mathcal{C}_Y})$ is a consistent map. □

By Proposition 3.4 and Proposition 3.9, we have the following theorem.

Theorem 3.11. Let $(X, \mathcal{C})$ be a convexity space. Then we have $\mathcal{C}_{\mathcal{U}_{\mathcal{C}}} = \mathcal{C}$. i.e., each convexity space is quasi-consistentizable.

Proof. Firstly, we prove $\mathcal{C}_{\mathcal{U}_{\mathcal{C}}} \subseteq \mathcal{C}$. Let $P \in \mathcal{C}_{\mathcal{U}_{\mathcal{C}}}$. Then for each $x \notin P$, there exists $D \in \mathcal{U}_{\mathcal{C}}$ such that $P \subseteq D'(x)$. This implies for each $x \notin P$, there exists $\{P_i\}_{i\in I} \subseteq \mathcal{C}$, such that $\{D_{P_i}\}_{i\in I}$ is codirected and $\bigcap_{i\in I} D_{P_i}(x) \subseteq D(x) \subseteq P'$. So we have $x \notin (\bigcap_{i\in I} D_{P_i})'(x) \supseteq P$. Furthermore, $\{D'_{P_i}(x)\}_{i\in I}$ is directed since $\{D_{P_i}\}_{i\in I}$ is codirected, which implies $\bigcup_{i\in I} D'_{P_i}(x) \in \mathcal{C}$. So $P \in \mathcal{C}$.

Secondly, we prove $\mathcal{C}_{\mathcal{U}_{\mathcal{C}}} \supseteq \mathcal{C}$. Let $P \in \mathcal{C}$. Then $D_p \in \mathcal{U}_{\mathcal{C}}$. By the definition of D_P, it follows that $D_P(x) = P'$ for any $x \notin P$. Thus, $P \in \mathcal{C}_{\mathcal{U}_{\mathcal{C}}}$. □

In conclusion, by Proposition 3.2–Theorem 3.11, the relation of their categories as follows.

Theorem 3.12. Let $F : \mathbf{CS} \longrightarrow \mathbf{QCS}$ be defined by $F((X, \mathcal{C})) = (X, \mathcal{U}_{\mathcal{C}})$. Then F is an embedding functor from **CS** to **QCS**.

At the end of this section, another way to induce convexity by quasi-adjacency is introduced. We further study the relation of two convexities induced by different ways.

Proposition 3.13. Let $(X, \mathcal{U})$ be a quasi-consistent space. Define

$$co^{\mathcal{U}}(A) = \bigcap_{D\in\mathcal{U}} \bigcup_{x\in A} D(x).$$

Then $co^{\mathcal{U}}(A)$ is a hull operator.

Proof. (H1) and (H2) are obvious.

(H3) Take $\{A_t\}_{t\in T} \overset{dir}{\subseteq} 2^X$. It is obvious that $co^{\mathcal{U}}$ is order-preserving. So $\bigcup_{t\in T} co^{\mathcal{U}}(A_t) \subseteq co^{\mathcal{U}}(\bigcup_{t\in T} A_t)$ holds. It suffices to show $x \notin co^{\mathcal{U}}(\bigcup_{t\in T} A_t)$ if $x \notin \bigcup_{t\in T} co^{\mathcal{U}}(A_t)$. This can be proved from the following implications.

$$x \notin \bigcup_{t\in T} co^{\mathcal{U}}(A_t) \Rightarrow \forall t \in T, x \notin co^{\mathcal{U}}(A_t)$$
$$\Rightarrow \forall t \in T,\ \exists D_t \in \mathcal{U},\ \text{s.t. } x \notin \bigcup_{y\in A_t} D_t(y).$$

Let

$$\mathcal{D}_t = \{D \in \mathcal{U} \mid x \notin \bigcup_{y\in A_t} D(y)\} \text{ and } D_t^* = \bigcup_{D\in\mathcal{D}_t} D.$$

Then $D_t^* \in \mathcal{U}$. Next, we prove $D_t^* \in \mathcal{D}_t$ and $\{D_t^*\}_{t\in T}$ is codirected.

For any $D \in \mathcal{D}_t$, we have $x \notin \bigcup_{y\in A_t} D(y)$. *i.e.*, $x \notin \bigcup_{D\in\mathcal{D}_t}\bigcup_{y\in A_t} D(y) = \bigcup_{y\in A_t}\bigcup_{D\in\mathcal{D}_t} D(y)$. This implies $x \notin \bigcup_{y\in A_t} D_t^*(y)$. Take any D_1^* and D_2^*, they correspond to A_1 and A_2 respectively. Since $\{A_t\}_{t\in T}$ is directed, there exists A_3 such that $A_1, A_2 \subseteq A_3$. It follows that $x \notin \bigcup_{y\in A_1} D(y)$ and $x \notin \bigcup_{y\in A_2} D(y)$ if $x \notin \bigcup_{y\in A_3} D(y)$. So we have $\mathcal{D}_3 \subseteq \mathcal{D}_1, \mathcal{D}_2$, which means $\mathcal{D}_3^* \subseteq \mathcal{D}_1^*, \mathcal{D}_2^*$. Hence, $\{D_t^*\}_{t\in T}$ is codirected. Let $D^* = \bigcap_{t\in T} D_t^*$. Then $D^* \in \mathcal{U}$. Finally, we have for any $t \in T$,

$$x \notin \bigcup_{y\in A_t} D_t^*(y) \supseteq \bigcup_{y\in A_t}\bigcap_{t\in T} D_t^*(y) = \bigcup_{y\in A_t} D^*(y).$$

Then $x \notin \bigcup_{t\in T}\bigcup_{y\in A_t} D^*(y) = \bigcup_{y\in\bigcup_{t\in T} A_t} D^*(y)$. In general, there exists $D^* \in \mathcal{U}$ such that $x \notin \bigcup_{y\in\bigcup_{t\in T} A_t} D^*(y)$. Thus $x \notin co^{\mathcal{U}}(\bigcup_{t\in T} A_t)$. □

So there is another way to induce convexity by quasi-consistency as follows:

$$\mathcal{C}^{\mathcal{U}} = \{A \in 2^X \mid co^{\mathcal{U}}(A) = A\}.$$

By the next example, we know $\mathcal{C}^{\mathcal{U}}$ and $\mathcal{C}_{\mathcal{U}}$ are not comparable generally.

Example 3.14. Let $X = \{0, 1\}$. Define two binary relations $U, V \subseteq X \times X$ as follows:

$$U = \{(0,0),(1,0),(1,1)\}, \quad V = \{(0,0),(0,1),(1,0),(1,1)\} = X \times X.$$

Then $\mathcal{U} = \{U, V\}$ is a quasi-consistency on X. And it is easy to show

$$\mathcal{C}^{\mathcal{U}} = \{\emptyset, \{0\}, X\}, \quad \mathcal{C}_{\mathcal{U}} = \{\emptyset, \{1\}, X\}.$$

Hence, $\mathcal{C}^{\mathcal{U}} \neq \mathcal{C}_{\mathcal{U}}$.

By Example 3.14, if $\mathcal{U}$ is a quasi-consistency, $\mathcal{C}^{\mathcal{U}} \neq \mathcal{C}_{\mathcal{U}}$. But if it is a consistency, then we have the following result.

Theorem 3.15. Let $(X, \mathcal{U})$ be a consistent space. Then $\mathcal{C}^{\mathcal{U}} = \mathcal{C}_{\mathcal{U}}$.

Proof. It is routine to check it by Proposition 3.5 and the definition of $co^{\mathcal{U}}$. □

4. Quasi-adjacency spaces characterize convexity spaces

In this section, we use quasi-adjacent spaces to characterize convexity spaces. In fact, a proximity in topology is a binary relation. Each relation can be seen as a map. So a proximity has a equivalent definition of maps. Here we introduce (quasi-)adjacency in terms of maps, which is similar to classical proximity.

Definition 4.1. A quasi-adjacency on 2^X is a function $\delta : 2^X \times 2^X \longrightarrow \{0, 1\}$ that satisfies the following conditions: for any $A, B, C \in 2^X$,

(A1) $\delta(\emptyset, X) = \delta(X, \emptyset) = 0$;

(A2) $\forall \{A_t\} \overset{dir}{\subseteq} 2^X$, $\delta(B, \bigcup_{t\in T} A_t) = \bigvee_{t\in T} \delta(B, A_t)$, $\delta(\bigcup_{t\in T} A_t, B) = \bigvee_{t\in T} \delta(A_t, B)$;

(A3) $\delta(A, B) = 0 \Rightarrow \exists C \in 2^X$ such that $\delta(A, C) = 0$ and $\delta(C', B) = 0$;

(A4) $\delta(A,B)=0\Rightarrow A\subseteq B'$.
A quasi-adjacency δ is called adjacency if it also satisfies
(A5) $\delta(A,B)=\delta(B,A)$.

If δ is a (quasi-)adjacency on 2^X, then the pair (X,δ) is called a (quasi-)adjacent space.

Remark 4.2. (A2) shows that $\delta(B,-):2^X\longrightarrow 2^X$ and $\delta(-,B):2^X\longrightarrow 2^X$ are order-preserving.

Definition 4.3. A map between quasi-adjacent spaces $f:(X,\delta_X)\longrightarrow(Y,\delta_Y)$ is called adjacent if

$$\forall A,B\in 2^Y,\quad \delta_Y(A,B)=0\Rightarrow \delta_X(f^{-1}(A),f^{-1}(B))=0.$$

The category of quasi-adjacent spaces and adjacent maps is denoted by **QAS**.

In the following content, we show a quasi-adjacent space can induce a convex hull operator, and then it can induce a convexity space.

Proposition 4.4. Let (X,δ) be a quasi-adjacenct space. For any $A\in 2^X$, define

$$co^\delta(A)=\bigcap\{B\in 2^X\mid \delta(A,B')=0\}.$$

Then
(1) $\delta(A,B')=0\Rightarrow co^\delta(A)\subseteq B$.
(2) $x\notin co^\delta(A)\Leftrightarrow \delta(A,\{x\})=0$.
(3) $co^\delta:2^X\longrightarrow 2^X$ is a hull operator.

Proof. (1) is obvious.

(2) Necessity. If $x\notin co^\delta(A)$, then there exists $B\in 2^X$ such that $\delta(A,B')=0$ and $x\notin B$. i.e., $\{x\}\subseteq B'$. This shows $\delta(A,\{x\})\leq\delta(A,B')=0$.

Sufficiency. If $\delta(A,\{x\})=0$, then $co^\delta(A)\subseteq\{x\}'$. This implies $x\notin co^\delta(A)$.

(3) It suffices to show co^δ satisfies (H1)–(H4′).

(H1) By $\delta(\emptyset,X)=0$ and (1), $co^\delta(\emptyset)=\emptyset$ holds.

(H2) If $\delta(A,B')=0$, then $A\subseteq B$. It implies $A\subseteq co^\delta(A)$.

(H3) Firstly, we prove co^δ is order-preserving. If $A\subseteq B$ and $\delta(B,C')=0$, then $\delta(A,C')=0$. This guarantees $co^\delta(A)\subseteq co^\delta(B)$. By (H2), $co^\delta(A)\subseteq co^\delta(co^\delta(A))$. Next we prove the inverse direction. If $x\notin co^\delta(A)$, then $\delta(A,\{x\})=0$ by (2). By (A3) there exists $B\in 2^X$ such that $\delta(A,B)=0$ and $\delta(B',\{x\})=0$. Hence, $co^\delta(A)\subseteq B'$ and $x\notin co^\delta(B')$. Since co^δ is order-preserving, $co^\delta(co^\delta(A))\subseteq co^\delta(B')$ holds. Thus $x\notin co^\delta(co^\delta(A))$.

(H4′) It suffices to show that co^δ preserves directed joins. Take $\{A_t\}_{t\in T}\overset{dir}{\subseteq}2^X$. $\bigcup_{t\in T}co^\delta(A_t)\subseteq co^\delta(\bigcup_{t\in T}A_t)$ is obvious. For the inverse direction, suppose that $x\notin\bigcup_{t\in T}co^\delta(A_t)$, then $x\notin co^\delta(A_t)$ for all $t\in T$. i.e., $\delta(A_t,\{x\})=0$ for all $t\in T$. By (A2), $\delta(\bigcup_{t\in T}A_t,\{x\})=\bigvee_{t\in T}\delta(A_t,\{x\})=0$. This means $x\notin co^\delta(\bigcup_{t\in T}A_t)$. □

Remark 4.5. In section 3, we know a quasi-consistency can induce two hull operators. Similar to it, a quasi-adjacency δ also can induce two hull operators. Another hull operator $co_\delta:2^X\longrightarrow 2^X$ is defined as follows:

$$co_\delta(A)=\bigcap\{B\in 2^X\mid \delta(B',A)=0\}.$$

If δ also satisfies symmetric condition, then $co^\delta=co_\delta$.

Proposition 4.6. Let (X,δ) be a quasi-adjacenct space. Thus $\mathcal{C}_\delta=\{A\in 2^X\mid co^\delta(A)=A\}$ is a convexity induced by δ. Then $R^\delta=\{R^\delta_x\mid x\in X\}$ is the remotehood system, where $R^\delta_x=\{A\in 2^X\mid \delta(A,\{x\})=0\}$.

Proof. The proof can be obtained by the following implications.

$$A\in\mathcal{C}_\delta\Leftrightarrow A=co^\delta(A)\Leftrightarrow\forall x\notin A,x\notin co^\delta(A)\Leftrightarrow\forall x\notin A,\delta(A,\{x\})=0\Leftrightarrow\forall x\notin A,A\in R^\delta_x.$$

□

Next, we prove a quasi-adjacency can be induced by a remotehood system, so a quasi-adjacent space can be induced by a convexity space in terms of its remotehood system.

Proposition 4.7. Let $R = \{R_x \mid x \in X\}$ be a remotehood system. Define $\delta_R : 2^X \times 2^X \longrightarrow \{0,1\}$ as follows:

$$\forall A, B \in 2^X, \quad \delta_R(A,B) = 0 \Leftrightarrow \forall x \in B, A \in R_x.$$

Then δ_R is a quasi-adjacency.

Proof. (A1) $\forall x \in X, \emptyset \in R_x \Rightarrow \delta_R(\emptyset, X) = 0$. $\delta_R(X, \emptyset) = 0$ is obvious.

(A2) Take $\{A_t\} \overset{dir}{\subseteq} 2^X$. We first show $\delta_R(B,-) : 2^X \longrightarrow \{0,1\}$ and $\delta_R(-,B) : 2^X \longrightarrow \{0,1\}$ are order-preserving. If $A \subseteq C$ and $\delta_R(B,C) = 0$, then we have

$$\forall x \in C, B \in R_x \Rightarrow \forall x \in A, B \in R_x \Rightarrow \delta_R(B,A) = 0.$$

This implies $\delta_R(B,A) \leq \delta_R(B,C)$. If $\delta_R(C,B) = 0$, then $\forall x \in B$, it holds $C \in R_x$. i.e., $\delta_R(A,B) = 0$. Since R_x is a down-set, it follows $A \in R_x$ for any $x \in B$. This implies $\delta_R(A,B) \leq \delta_R(A,C)$. Thus, $\bigvee_{t\in T} \delta_R(B, A_t) \leq \delta_R(B, \bigcup_{t\in T} A_t)$ and $\bigvee_{t\in T} \delta_R(A_t, B) \leq \delta_R(\bigcup_{t\in T} A_t, B)$ hold. Next we prove $\delta_R(B, \bigcup_{t\in T} A_t) \leq \bigvee_{t\in T} \delta_R(B, A_t)$. This can be proved by the following implications.

$$\begin{aligned}
\bigvee_{t\in T} \delta_R(B, A_t) = 0 &\Rightarrow \forall t \in T, \delta_R(B, A_t) = 0 \\
&\Rightarrow \forall t \in T,\ \forall x \in A_t,\ B \in R_x \\
&\Rightarrow \forall x \in \bigcup_{t\in T} A_t,\ B \in R_x \\
&\Rightarrow \delta_R(B, \bigcup_{t\in T} A_t) = 0.
\end{aligned}$$

Finally, $\delta_R(\bigcup_{t\in T} A_t, B) \leq \bigvee_{t\in T} \delta_R(A_t, B)$ can be obtained from the following implications.

$$\begin{aligned}
\bigvee_{t\in T} \delta_R(A_t, B) = 0 &\Rightarrow \forall t \in T, \delta_R(A_t, B) = 0 \\
&\Rightarrow \forall t \in T,\ \forall x \in B,\ A_t \in R_x \\
&\Rightarrow \forall x \in B,\ \forall t \in T,\ A_t \in R_x \\
&\Rightarrow \forall x \in B,\ \bigcup_{t\in T} A_t \in R_x \quad \text{(By (CR3))} \\
&\Rightarrow \delta_R(\bigcup_{t\in T} A_t, B) = 0.
\end{aligned}$$

(A3) can be proved by the following implications.

$$\begin{aligned}
\delta_R(A,B) = 0 &\Rightarrow \forall x \in B, A \in R_x \\
&\Rightarrow \forall x \in B, \exists F_x \in R_x, \text{ such that } \forall y \notin F_x,\ A \in R_y \text{ (By (CR5}'\text{))} \\
&\Rightarrow \forall x \in B,\ \exists F = \bigcap_{x\in B} F_x \in R_x, \text{ and } \forall y \notin F,\ A \in R_y \\
&\Rightarrow \delta_R(F,B) = 0,\ \delta_R(A,F') = 0.
\end{aligned}$$

Let $C = F'$. This shows that there exists $C \in 2^X$ such that $\delta_R(A,C) = 0,\ \delta_R(C',B) = 0$.

(A4) can be showed from the following implications.

$$\begin{aligned}
\delta_R(A,B) = 0 &\Rightarrow \forall x \in B,\ A \in R_x \\
&\Rightarrow \forall x \in B,\ x \notin A \text{ (By (CR4))} \\
&\Rightarrow B \subseteq A'.\ i.e., A \subseteq B'.
\end{aligned}$$

Therefore, δ_R is a quasi-adjacency. □

By Proposition 4.7, a convexity $\mathcal{C}$ can induce a quasi-adjacency $\delta_{\mathcal{C}} : 2^X \times 2^X \longrightarrow \{0,1\}$ in terms of remotehood system as follows:

$$\forall A, B \in 2^X, \quad \delta_{\mathcal{C}}(A,B) = 0 \Leftrightarrow \forall x \in B, A \in R_x^{\mathcal{C}} \Leftrightarrow \forall x \in B,\ \exists P \in \mathcal{C},\ \text{s.t. } x \notin P \supseteq A.$$

Theorem 4.8. Let $\mathcal{C}$ be a convexity. Then $\mathcal{C}_{\delta_{\mathcal{C}}} = \mathcal{C}$.

Proof. This proof obtained by the following implications.

$$A \in \mathcal{C}_{\delta_{\mathcal{C}}} \Leftrightarrow \forall x \notin A, \delta_{\mathcal{C}}(A, \{x\}) = 0 \Leftrightarrow \forall x \notin A, A \in R_x^{\mathcal{C}} \Leftrightarrow A \in \mathcal{C}.$$

□

Proposition 4.9. If $f : (X, \delta_X) \longrightarrow (Y, \delta_Y)$ is a adjacent map, then $f : (X, \mathcal{C}_{\delta_X}) \longrightarrow (Y, \mathcal{C}_{\delta_Y})$ is CP.

Proof. We need to check that $f^{-1}(P) \in \mathcal{C}_{\delta_X}$ for any $P \in \mathcal{C}_{\delta_Y}$. If $x \notin f^{-1}(P)$, then we have $f(x) \notin co^{\delta}(P)$ by $P \in \mathcal{C}_{\delta_Y}$. It follows that $\delta_Y(P, \{f(x)\}) = 0$. And since f is adjacent, $\delta_X(f^{-1}(P), \{x\}) = 0$ is true. Hence f is CP. □

Proposition 4.10. If $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ is a CP map, then $f : (X, \delta_{\mathcal{C}_X}) \longrightarrow (Y, \delta_{\mathcal{C}_Y})$ is adjacent.

Proof. We need to prove that $\delta_{\mathcal{C}_X}(f^{-1}(A), f^{-1}(B)) = 0$ if $\delta_{\mathcal{C}_Y}(A,B) = 0$ for any $A, B \in 2^Y$. If $\delta_{\mathcal{C}_Y}(A,B) = 0$ and $f(x) \in B$, then we have $A \in R_{f(x)}^{\mathcal{C}_Y}$. Since f is CP, $f^{-1}(A) \in R_x^{\mathcal{C}_X}$. Thus $\delta_{\mathcal{C}_X}(f^{-1}(A), f^{-1}(B)) = 0$. □

By Theorem 4.8, Proposition 4.9 and Proposition 4.10, we have the following conclusion.

Theorem 4.11. Let $G : \mathbf{CS} \longrightarrow \mathbf{QAS}$ be defined by $G((X, \mathcal{C})) = (X, \delta_{\mathcal{C}})$. Then G is an embedding functor from **CS** to **QAS**.

5. The relation between (quasi-)consistency and (quasi-)adjacency

In this section, we study the relation between (quasi-)consistency and (quasi-)adjacency. They can be induced by each other and there exists a one-to-one correspondence between quasi-adjacency and fully ordered quasi-consistency.

By Lemma 3.3, a quasi-consistency $\mathcal{U}$ can induce a remotehood system $R^{\mathcal{U}} = \{\mathcal{U}'_x \mid x \in X\}$, where $\mathcal{U}'_x = \{D'(x) \mid D \in \mathcal{U}\}$. And then by the Proposition 4.7, $R^{\mathcal{U}}$ can induce a quasi-adjacency $\delta_{\mathcal{U}} : 2^X \times 2^X \longrightarrow \{0,1\}$ as follows:

$$\forall A, B \in 2^X, \quad \delta_{\mathcal{U}}(A,B) = 0 \Leftrightarrow \forall x \in B,\ \exists D \in \mathcal{U},\ \text{s.t. } A \subseteq D'(x).$$

$\mathcal{U}$ can induce a convexity $\mathcal{C}_{\mathcal{U}}$ and a quasi-adjacency $\delta_{\mathcal{U}}$. Moreover, $\delta_{\mathcal{U}}$ can induce a convexity. This result shows $\mathcal{U}$ and $\delta_{\mathcal{U}}$ induce the same convexity.

Theorem 5.1. Let $\mathcal{U}$ be a quasi-consistency. Then $\mathcal{U}$ and $\delta_{\mathcal{U}}$ induce the same convexity. i.e., $\mathcal{C}_{\delta_{\mathcal{U}}} = \mathcal{C}_{\mathcal{U}}$.

Proof. The equality can be obtained by the following implications.

$$\begin{aligned} A \in \mathcal{C}_{\delta_{\mathcal{U}}} \Leftrightarrow \forall x \notin A, A \in R_x^{\delta_{\mathcal{U}}} &\Leftrightarrow \forall x \notin A, \delta_{\mathcal{U}}(A, \{x\}) = 0 \\ &\Leftrightarrow \forall x \notin A, \exists D \in \mathcal{U},\ \text{s.t. } A \subseteq D'(x) \\ &\Leftrightarrow A \in \mathcal{C}_{\mathcal{U}}. \end{aligned}$$

□

Proposition 5.2. If $f : (X, \mathcal{U}_X) \longrightarrow (Y, \mathcal{U}_Y)$ is a consistent map, then $f : (X, \delta_{\mathcal{U}_X}) \longrightarrow (Y, \delta_{\mathcal{U}_Y})$ is a adjacent map.

Proof. If $\delta_{\mathcal{U}_Y}(A,B)=0$ for $A,B\in 2^Y$, then for all $y\in B$, there exists $D\in\mathcal{U}_Y$ such that $A\subseteq D'(y)$. Next, we need to show that there exists $E\in\mathcal{U}_X$ such that $f^{-1}(A)\subseteq E'(x)$ for any $x\in f^{-1}(B)$. It follows that $A\subseteq D'(f(x))$. And

$$f^{-1}(A)\subseteq f^{-1}(D'(f(x)))=(f\times f)^{-1}(D')(x)=(f\times f)^{-1}(D)'(x)$$

holds because f^{-1} is order-preserving. Since $D\in\mathcal{U}_Y$ and f is a consistent map, $(f\times f)^{-1}(D)\in\mathcal{U}_X$. Let $E=(f\times f)^{-1}(D)$. Then $f^{-1}(A)\subseteq E'(x)$. □

Next we show that a quasi-adjacency δ can induce a quasi-consistency $\mathcal{U}_\delta$. In addition, if δ also satisfies symmetry condition, then $\mathcal{U}_\delta$ is also a consistency. We first prove the following lemma.

Lemma 5.3. Let (X,δ) be a quasi-adjacenct space. Consider the set $\mathcal{A}_\delta=\{(A,B)\in 2^X\times 2^X\mid \delta(A,B)=0\}$, define $D_{A,B}\subseteq X\times X$ for any pair $(A,B)\in\mathcal{A}_\delta$ as follows:

$$D_{A,B}(x)=\begin{cases} A', & x\in B,\\ X, & x\notin B.\end{cases}$$

Then (1) if $\{(A_t,B_t)\}_{t\in T}\subseteq\mathcal{A}_\delta$ and $\{A_t\}_{t\in T}$, $\{B_t\}_{t\in T}$ are directed respectively, then $D_{\bigcup_{t\in T}A_t,\bigcup_{t\in T}B_t}\subseteq\bigcap_{t\in T}D_{A_t,B_t}$;

(2) if $\{(A_t,B_t)\}_{t\in T}\subseteq\mathcal{A}_\delta$, then $D_{\bigcap_{t\in T}A_t,\bigcap_{t\in T}B_t}=\bigcup_{t\in T}D_{A_t,B_t}$.

Proof. (1) It is straightforward to show that if $(A_1,B_1),(A_2,B_2)\in\mathcal{A}_\delta$ and $(A_1,B_1)\subseteq(A_2,B_2)$, i.e., $A_1\subseteq A_2, B_1\subseteq B_2$, then $D_{A_2,B_2}\subseteq D_{A_1,B_1}$. Next we only need to prove $(\bigcup_{t\in T}A_t,\bigcup_{t\in T}B_t)\in\mathcal{A}_\delta$ if $\{A_t\}_{t\in T}$, $\{B_t\}_{t\in T}$ are directed respectively. Since

$$\delta(\bigcup_{t\in T}A_t,\bigcup_{t\in T}B_t)=\bigvee_{t\in T}\delta(A_t,B_t)=0\quad\text{(By (A2))}$$

holds, the proof is obvious.

(2) is obvious. □

Proposition 5.4. Let (X,δ) be a quasi-adjacenct space. Define

$$\mathcal{U}_\delta=\{D\subseteq X\times X\mid \exists(A,B)\in\mathcal{A}_\delta,\ \text{s.t. } D_{A,B}\subseteq D\}.$$

Then $\mathcal{U}_\delta$ is a quasi-consistency. Moreover, if δ is a adjacency, then $\mathcal{U}_\delta$ is a consistency.

Proof. (QC1) and (QC2) are obvious. It remains to show that (QC3) and (QC4).

(QC3) Take $\{D_t\}_{t\in T}\overset{cdir}{\subseteq}\mathcal{U}_\delta$. Then for all $t\in T$, there exists $(A_t,B_t)\in\mathcal{A}_\delta$ such that $D_{A_t,B_t}\subseteq D_t$. Let

$$\mathcal{D}_t=\{(A_t,B_t)\in\mathcal{A}_\delta\mid D_{A_t,B_t}\subseteq D_t\}$$

and

$$(A_t^*,B_t^*)=\bigcap_{(A_t,B_t)\in\mathcal{D}_t}(A_t,B_t)=(\bigcap_{A_t\in\mathcal{D}_t}A_t,\bigcap_{B_t\in\mathcal{D}_t}B_t).$$

Then $\delta(A_t^*,B_t^*)\le\delta(A_t,B_t)=0$. This implies $(A_t^*,B_t^*)\in\mathcal{A}_\delta$. Next we prove that $\{(A_t^*,B_t^*)\mid t\in T\}$ is directed. Take $(A_{t_1}^*,B_{t_1}^*)$ and $(A_{t_2}^*,B_{t_2}^*)$, they correspond to D_{t_1},D_{t_2} respectively. Since $\{D_t\}_{t\in T}$ is codirected, there exists $D_{t_3}\subseteq D_{t_1},D_{t_2}$. This guarantees $\mathcal{D}_{t_3}\subseteq\mathcal{D}_{t_1},\mathcal{D}_{t_2}$. And then $(A_{t_3}^*,B_{t_3}^*)\supseteq(A_{t_1}^*,B_{t_1}^*),(A_{t_2}^*,B_{t_2}^*)$. Furthermore, since $D_{A_t,B_t}\subseteq D_t$ for all $(A_t,B_t)\in\mathcal{D}_t$, by Lemma 5.3, we have

$$D_{A_t^*,B_t^*}=D_{\bigcap_{(A_t,B_t)\in\mathcal{D}_t}(A_t,B_t)}=\bigcup_{(A_t,B_t)\in\mathcal{D}_t}D_{A_t,B_t}\subseteq D_t.$$

By the above proof, we know for all $t\in T$, there exists $\{(A_t^*,B_t^*)\mid t\in T\}\overset{dir}{\subseteq}\mathcal{A}_\delta$, such that $D_{A_t^*,B_t^*}\subseteq D_t$. Then

$$D_{\bigcup_{t\in T}A_t^*,\bigcup_{t\in T}B_t^*}\subseteq\bigcap_{t\in T}D_{A_t^*,B_t^*}\subseteq\bigcap_{t\in T}D_t,$$

which implies $\bigcap_{t\in T} D_t \in \mathcal{U}_\delta$.

(QC4) If $D \in \mathcal{U}_\delta$, there exists $(A,B) \in \mathcal{A}_\delta$ such that $D_{A,B} \subseteq D$. By (A4), $\delta(A,B)=0$ implies that there exists C such that $\delta(A,C)=0$ and $\delta(C',B)=0$. We can check

$$D_{A,C} \circ D_{C',B}(x) = \bigcup_{z\in D_{C',B}(x)} D_{A,C}(z) = \begin{cases} A', & x \in B, \\ X, & x \notin B \end{cases} = D_{A,B}(x).$$

This shows $D_{A,C} \circ D_{C',B} = D_{A,B}$.

In addition, if δ satisfies symmetry condition (A5) and $D \in \mathcal{U}_\delta$, then there exists $(A,B) \in \mathcal{A}_\delta$ such that $D_{A,B} \subseteq D$. Therefore, we have $D^{-1}_{A,B} \subseteq D^{-1}$. By the definition of $D_{A,B}$, we know $D_{A,B} = (B \times A') \cup (B' \times X)$ and $D_{B,A} = (A \times B') \cup (A' \times X)$. So $D^{-1}_{A,B} = (A' \times B) \cup (X \times B')$ and $D_{B,A} = D^{-1}_{A,B}$. It follows that $\mathcal{B} = \{D_{A,B} \mid (A,B) \in \mathcal{A}_\delta\}$ is a symmetric base of $\mathcal{U}_\delta$ since $\delta(B,A) = \delta(A,B) = 0$. Hence, $\mathcal{U}_\delta$ is a adjacency. □

A quasi-adjacency δ can induce a convexity $\mathcal{C}_\delta$ and a quasi-consistency $\mathcal{U}_\delta$. Furthermore, $\mathcal{U}_\delta$ also can induce a convexity. The following theorem shows that δ and $\mathcal{U}_\delta$ induce the same convexity.

Theorem 5.5. Let δ be a quasi-adjacency. Then δ and $\mathcal{U}_\delta$ induce the same convexity. i.e., $\mathcal{C}_\delta = \mathcal{C}_{\mathcal{U}_\delta}$.

Proof. We first show $\mathcal{C}_{\mathcal{U}_\delta} \subseteq \mathcal{C}_\delta$. If $A \in \mathcal{C}_{\mathcal{U}_\delta}$, then $\forall x \notin A, \exists D \in \mathcal{U}_\delta$ such that $A \subseteq D'(x)$. By $D \in \mathcal{U}_\delta$, there exists $(E,F) \in \mathcal{A}_\delta$ such that $D_{E,F} \subseteq D$. Then we consider the following case.

(1) If $x \in F$, then $D_{E,F}(x) = E' \subseteq D(x) \subseteq A'$. This implies $A \subseteq E$. Then it follows that $\delta(A,\{x\}) \le \delta(E,F) = 0$.

(2) If $x \notin F$, then $D_{E,F}(x) = X$. $D(x) = X$ since $D_{E,F} \subseteq D$. This guarantees $A = \emptyset$, which means $\delta(A,\{x\}) = 0$. Thus, $\forall x \notin A$, $\delta(A,\{x\}) = 0$.

Next we prove $\mathcal{C}_\delta \subseteq \mathcal{C}_{\mathcal{U}_\delta}$. If $A \in \mathcal{C}_\delta$, then $\forall x \notin A$, $\delta(A,\{x\}) = 0$. i.e., $(A,\{x\}) \in \mathcal{A}_\delta$. It follows from the construction of $\mathcal{U}_\delta$ that $D_{A,\{x\}} \in \mathcal{U}_\delta$. Then $D'_{A,\{x\}}(x) = A$. □

Proposition 5.6. If $f : (X,\delta_X) \longrightarrow (Y,\delta_Y)$ is a adjacent map, then $(X,\mathcal{U}_{\delta_X}) \longrightarrow (Y,\mathcal{U}_{\delta_Y})$ is a consistent map.

Proof. Our aim is to show that $(f\times f)^{-1} \in \mathcal{U}_{\delta_X}$ for all $D \in \mathcal{U}_{\delta_Y}$. This means we need to find $(F,G) \in \mathcal{A}_{\delta_X}$ such that $D_{F,G} \subseteq (f\times f)^{-1}(D)$. If $D \in \mathcal{U}_{\delta_Y}$, then there exists $A,B \in 2^Y$ such that $\delta_Y(A,B)=0$ and $D_{A,B} \subseteq D$. Since $\delta_Y(A,B)=0$ and f is a adjacent map, we have $\delta_X(f^{-1}(A), f^{-1}(B)) = 0$. i.e., $(f^{-1}(A), f^{-1}(B)) \in \mathcal{A}_{\delta_X}$. Next, we consider the following cases to verify $D_{f^{-1}(A),f^{-1}(B)}(x) \subseteq (f\times f)^{-1}(D)(x)$ for all $x \in X$.

(1) If $x \in f^{-1}(B)$ and $z \in D_{f^{-1}(A),f^{-1}(B)}(x)$, then $D_{f^{-1}(A),f^{-1}(B)}(x) = f^{-1}(A')$ and $f(z) \in A'$. By $f(x) \in B$, we have $D_{A,B}(f(x)) = A'$. Since

$$f(z) \in D_{A,B}(f(x)) \Rightarrow (x,z) \in (f\times f)^{-1}(D_{A,B}) \subseteq (f\times f)^{-1}(D) \Rightarrow z \in (f\times f)^{-1}(D)(x),$$

we obtain $D_{f^{-1}(A),f^{-1}(B)}(x) \subseteq (f\times f)^{-1}(D)(x)$.

(2) If $x \notin f^{-1}(B)$, then $D_{f^{-1}(A),f^{-1}(B)}(x) = X$. It follows that $D_{A,B}(f(x)) = Y$ for any $f(x) \notin B$. For all $z \in X$, $f(z) \in Y = D_{A,B}(f(x))$. This implies $(f(x), f(z)) \in D_{A,B}$. And then $(x,z) \in (f\times f)^{-1}(D_{A,B}) \subseteq (f\times f)^{-1}(D)$. i.e., $z \in (f\times f)^{-1}(D)(x)$. Hence $(f\times f)^{-1}(D)(x) = X$.

In general, take $F = f^{-1}(A)$ and $G = f^{-1}(B)$. Then it satisfies what we want. □

Theorem 5.7. Let δ be a quasi-adjacency. Then $\delta_{\mathcal{U}_\delta} = \delta$.

Proof. We first need to show $\delta_{\mathcal{U}_\delta} \le \delta$. If $\delta(A,B) = 0$, then $D_{A,B} \in \mathcal{U}_\delta$ and

$$D_{A,B}(x) = \begin{cases} A', & x \in B, \\ X, & x \notin B. \end{cases}$$

This assures that for all $x \in B$, there exists $D_{A,B} \in \mathcal{U}_\delta$ such that $A = D'_{A,B}(x)$. This shows $\delta_{\mathcal{U}_\delta}(A, B) = 0$.

Next we prove the inverse direction $\geq$. Suppose that $\delta_{\mathcal{U}_\delta}(A, B) = 0$. Then for all $x \in B$, there exists $D \in \mathcal{U}_\delta$ such that $A \subseteq D'(x)$. For $D \in \mathcal{U}_\delta$, there exists $(E, F) \in \mathcal{A}_\delta$ such that $D_{E,F} \subseteq D$. Next, we have the following cases.

(1) If $B \subseteq F$ and $x \in B$, then $x \in F$ and $D'_{E,F}(x) = E$. This implies $A \subseteq D'(x) \subseteq D'_{E,F}(x) = E$. Since δ is order-preserving on each variables, then we have $\delta(A, B) \leq \delta(E, B) \leq \delta(E, F) = 0$.

(2) If $B \nsubseteq F$, then $\exists x \in B$ but $x \notin F$. Here $D'(x) \subseteq D'_{E,F}(x) = \emptyset$. This implies $A = \emptyset$. Then $\delta(A, B) = \delta(\emptyset, B) \leq \delta(\emptyset, X) = 0$. □

Theorem 5.8. Let $\mathcal{U}$ be a quasi-consistency. Then $\mathcal{U}_{\delta_\mathcal{U}} \subseteq \mathcal{U}$.

Proof. If $D \in \mathcal{U}_{\delta_\mathcal{U}}$, then there exists $(A, B) \in \mathcal{A}_{\delta_\mathcal{U}}$ such that $D_{A,B} \subseteq D$. $\delta_\mathcal{U}(A, B) = 0$ implies for all $x \in B$, there exists $E \in \mathcal{U}$ such that $A \subseteq E'(x)$. Then $E(x) \subseteq A' = D_{A,B}(x) \subseteq D(x)$ for any $x \in B$. However, if $x \notin B$, then $D_{A,B}(x) = X$ and $D(x) = X$. This shows $E \subseteq D$. By (QC2), $D \in \mathcal{U}$. □

Then we consider whether $\mathcal{U}_{\delta_\mathcal{U}} = \mathcal{U}$ holds if $\mathcal{U}$ satisfies extra condition. Next we introduce fully ordered quasi-consistency.

Definition 5.9. A quasi-consistency $\mathcal{U}$ is said to be fully ordered if there exists a basis $\mathcal{B}$ of $\mathcal{U}$ such that for any $D \in \mathcal{B}$, the family $\{D(x) \mid x \in X\}$ is totally ordered.

The category of fully ordered quasi-consistent spaces and consistent maps is denoted by **FQCS**.

Proposition 5.10. Let δ be a quasi-adjacency. Then $\mathcal{U}_\delta$ is fully ordered.

Proof. By the definition of $\mathcal{U}_\delta$, we know $\{D_{A,B} \mid (A, B) \in \mathcal{A}_\delta\}$ is a basis of $\mathcal{U}_\delta$. It holds $\{D_{A,B}(x) \mid x \in X\} = \{A', X\}$ for any $(A, B) \in \mathcal{A}_\delta$. □

Theorem 5.11. Let $\mathcal{U}$ be a fully ordered quasi-consistency. Then $\mathcal{U}_{\delta_\mathcal{U}} = \mathcal{U}$.

Proof. We only need to show that $\mathcal{U} \subseteq \mathcal{U}_{\delta_\mathcal{U}}$ from Proposition 5.8. Since $\mathcal{U}$ is a fully ordered quasi-consistency, there exists a basis $\mathcal{B}$ of $\mathcal{U}$ such that for any $D \in \mathcal{B}$, the family $\{D(x) \mid x \in X\}$ is fully ordered. It is easy to check that there exists $E, F \in \mathcal{B}$ such that $E \circ F \subseteq D$ for the given $D \in \mathcal{B}$. Let $A_x = D'(x)$ and $B_x = F(x)$. Next we prove $\delta_\mathcal{U}(A_x, B_x) = 0$. It suffices to verify $A_x = D'(x) \subseteq E'(z)$ for each $z \in B_x = F(x)$. This can be obtained by the following implications.

$$y \in E(z) \Rightarrow (z, y) \in E \text{ and } (x, z) \in F \Rightarrow (x, y) \in E \circ F \Rightarrow y \in D(x).$$

So $E(z) \subseteq D(x)$. Then $\delta_\mathcal{U}(A_x, B_x) = 0$ and $D_{A_x,B_x} \in \mathcal{U}_{\delta_\mathcal{U}}$. $\{A_x \mid x \in X\}$ and $\{B_x \mid x \in X\}$ are directed since they are fully ordered. By the proof of Lemma 5.3, we know $(\bigcup_{x\in X} A_x, \bigcup_{x\in X} B_x) \in \mathcal{A}_{\delta_\mathcal{U}}$. Then $D_{\bigcup_{x\in X} A_x, \bigcup_{x\in X} B_x} \in \mathcal{U}_{\delta_\mathcal{U}}$ and $D_{\bigcup_{x\in X} A_x, \bigcup_{x\in X} B_x} \subseteq \bigcap_{x\in X} D_{A_x,B_x}$. Since $\mathcal{U}_{\delta_\mathcal{U}}$ is an up-set, $\bigcap_{x\in X} D_{A_x,B_x} \in \mathcal{U}_{\delta_\mathcal{U}}$.

Let $H = \bigcap_{x\in X} D_{A_x,B_x}$. It follows that $z \in B_z = F(z)$ for each $z \in X$. This implies $\bigcap_{x\in X} D_{A_x,B_x}(z) \subseteq D_{A_z,B_z}(z) = A'_z = D(z)$. Thus, $H \subseteq D$ and $D \in \mathcal{U}_{\delta_\mathcal{U}}$. □

Theorem 5.12. Let $\mathbf{H} :\ \mathbf{QAS} \longrightarrow \mathbf{QCS}$ be defined by $\mathbf{H}((X, \delta)) = (X, \mathcal{U}_\delta)$. Then $\mathbf{H}$ is an embedding functor from **QAS** to **QS**. And the functor $\mathbf{K} :\ \mathbf{FQCS} \longrightarrow \mathbf{QAS}$ which is defined by $\mathbf{K}((X, \mathcal{U})) = (X, \delta_\mathcal{U})$ is an isomorphism, in this case its inverse is the restrict domain functor $\mathbf{H} :\ \mathbf{QAS} \longrightarrow \mathbf{FQCS}$.

Next we deeply consider the relation among convexity $\mathcal{C}$, quasi-consistency $\mathcal{U}$ and quasi-adjacency δ.

Theorem 5.13. Let $\mathcal{U}$ be a quasi-consistency. Then $\mathcal{U}$ and $\mathcal{C}_{\mathcal{U}}$ induce the same quasi-adjacency. i.e., $\delta_{\mathcal{U}} = \delta_{\mathcal{C}_{\mathcal{U}}}$.

Proof. $\delta_{\mathcal{U}} = \delta_{\mathcal{C}_{\mathcal{U}}}$ can be obtained by the following implications. For any $A, B \in 2^X$, we have

$$\delta_{\mathcal{C}_{\mathcal{U}}}(A,B) = 0 \Leftrightarrow \forall x \in B, A \in R_x^{\mathcal{C}_{\mathcal{U}}} \Leftrightarrow \forall x \in B, \exists D \in \mathcal{U}, \text{ s.t. } A \subseteq D'(x) \Leftrightarrow \delta_{\mathcal{U}}(A,B) = 0.$$

□

Theorem 5.14. Let $\mathcal{C}$ be a convexity. Then $\mathcal{C}$ and $\mathcal{U}_{\mathcal{C}}$ induce the same quasi-adjacency. i.e., $\delta_{\mathcal{C}} = \delta_{\mathcal{U}_{\mathcal{C}}}$.

Proof. It is obvious by Theorem 5.13 and Theorem 3.11. □

Theorem 5.15. Let $\mathcal{C}$ be a convexity. Then $\mathcal{U}_{\delta_{\mathcal{C}}} \subseteq \mathcal{U}_{\mathcal{C}}$.

Proof. If $D \in \mathcal{U}_{\delta_{\mathcal{C}}}$, then there exists $A, B \in 2^X$ such that $\delta_{\mathcal{C}}(A,B) = 0$ and $D_{A,B} \subseteq D$. Since $\delta_{\mathcal{C}}(A,B) = 0$, there exists $P_x \in \mathcal{C}$ and $x \notin P_x \supseteq A$ for any $x \in B$. Let $P = \bigcap_{x \in B} P_x$. Then we have $P \in \mathcal{C}$ and $x \notin P \supseteq A$. Furthermore,

$$D_P(x) = \begin{cases} P', & x \in P', \\ X, & x \in P. \end{cases} \quad \text{and} \quad D_{A,B}(x) = \begin{cases} A', & x \in B, \\ X, & x \notin B. \end{cases}$$

It is obvious that $x \in P'$ when $x \in B$. Further, $D_P(x) = P' \subseteq A' = D_{A,B}(x)$. Otherwise, $D_P(x) \subseteq X = D_{A,B}(x)$ when $x \notin B$. Hence, $D_P \subseteq D_{A,B}$ is true and then $D \in \mathcal{U}_{\mathcal{C}}$. □

Conclusions

In this paper, we introduce (quasi-)consistency and (quasi-)adjacency to characterize convexity. We study the relation among convexity $\mathcal{C}$, quasi-consistency $\mathcal{U}$ and quasi-adjacency δ. They can be induced by each other. In the mutual induction, remotehood systems play an vital role. Moreover, there is a close relation among their categories. These results show that there has many nice structures in convex spaces.

In general topology, every topological space can be uniformizable when it is complete regular. So let us end this paper with a question for further study. Whether convex space can be consistentizable when it satisfies some higher separation axioms.

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